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Anna Florio

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ACADÉMIE D'AIX MARSEILLE  
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UFR SCIENCES  
ECOLE DOCTORALE "AGROSCIENCES ET SCIENCES"

# Asymptotic Maslov indices

THÈSE DE MATHÉMATIQUES

**Anna Florio**

JURY DE THÈSE :

Mme MARIE-CLAUDE ARNAUD, Professeure, Avignon Université (Directrice de thèse)

M. FRANÇOIS BÉGUIN, Professeur, Université Paris 13 Nord (Rapporteur)

M. PHILIPPE BOLLE, Professeur, Avignon Université (Examineur)

M. PATRICE LE CALVEZ, Professeur, Université Pierre et Marie Curie (Rapporteur)

M. JEAN-PIERRE MARCO, MCF HDR, Université Pierre et Marie Curie (Examineur)

Mme ANA RECHTMAN, MCF HDR, Université de Strasbourg (Examinatrice)

M. ANDREA VENTURELLI, MCF, Avignon Université (Co-directeur de thèse)

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## **Abstract**

We study the asymptotic Maslov index for surface diffeomorphisms. Roughly speaking, this quantity is the limit of the average rotational velocity of tangent vectors which evolve under the action of the differential of the diffeomorphism. For twist maps on the annulus, we prove that the set of points of zero index has Hausdorff dimension at least one. In the framework of conservative twist maps, we show that every bounded instability region has a positive Lebesgue measure set of points with non zero index. Finally, we study such index in the presence of periodic hyperbolic points with transverse homoclinic intersections, providing examples of points at which the asymptotic Maslov index does not exist.





## Résumé

Nous étudions l'indice de Maslov asymptotique pour de difféomorphismes de surface. En mots, cette quantité est la limite de la vitesse angulaire moyenne des vecteurs tangents qui évoluent sous l'action de la différentielle du difféomorphisme. Pour des applications déviant la verticale de l'anneau, nous montrons que l'ensemble des points d'indice nul a une dimension d'Hausdorff supérieure ou égale à 1. Dans le cadre des applications déviant la verticale conservatives, nous prouvons que chaque région d'instabilité bornée a un ensemble de mesure de Lebesgue positive de points d'indice non nul. Finalement, nous étudions cet indice en présence de points périodiques hyperboliques avec intersections homoclines transverses, en donnant des exemples de points auxquels l'indice de Maslov asymptotique n'existe pas.



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# Notations

$\mathbb{T}$  : 1-dimensional torus  $\mathbb{R}/\mathbb{Z}$

$\mathbb{T}^2$  : 2-dimensional torus  $\mathbb{R}^2/\mathbb{Z}^2$

$\mathbb{A}$  : unbounded annulus  $\mathbb{T} \times \mathbb{R}$

$p : \mathbb{R} \rightarrow \mathbb{T}$  : universal covering of the 1-dimensional torus

$p \times \text{Id} : \mathbb{R}^2 \rightarrow \mathbb{A}$  : universal covering of the annulus

$p_1, p_2$  : projections over  $\mathbb{R}^2$  on the first and second coordinates

$\bar{p}_1, \bar{p}_2$  : projections over  $\mathbb{A}$  on the first and second coordinates

$\text{Diff}^1(M, N)$  : set of  $\mathcal{C}^1$  diffeomorphisms from  $M$  to  $N$

$\mathcal{R}(a, \psi)$  : rotation in  $\mathbb{R}^2$  centered at  $a$  of angle  $\psi$

$\tau_v$  : translation in  $\mathbb{R}^2$  of vector  $v$

$\theta(u, v)$  : oriented angle between the non zero vectors  $u, v$

$I = (f_t)_t$  : isotopy joining the identity to  $f = f_1$

$X$  : reference continuous vector field to fix the trivialization

$v(I)(x, \xi, \cdot)$  : oriented angle function between  $X(f_t(x))$  and  $Df_t(x)\xi$  (Def. 1.1.1)

$\tilde{v}(I)(x, \xi, \cdot)$  : lift of the oriented angle function  $v(I)(x, \xi, \cdot)$

$\text{Torsion}_n(I, x, \xi)$  : torsion at finite time  $n$  of  $x$  with respect to the vector  $\xi$  (Def. 1.1.2)

$\text{Torsion}_n(f, x, \xi)$  : torsion at finite time  $n$  of  $x$  with respect to the vector  $\xi$  when it is independent from the chosen isotopy

$\text{Torsion}(I, x)$  : torsion of the orbit of  $x$  (Def. 1.1.3)

$\text{Torsion}(f, x)$  : torsion of the orbit of  $x$  when it is independent from the chosen isotopy

$\text{Torsion}(I, \mu)$  : torsion of the  $f$ -invariant measure  $\mu$  (Def. 1.1.4)

$\text{Torsion}(f, \mu)$  : torsion of the  $f$ -invariant measure  $\mu$  when it is independent from the chosen isotopy

$\text{GL}(2, \mathbb{R})$  : linear group of degree 2 of  $\mathbb{R}$

$\text{GL}^+(2, \mathbb{R})$  : subgroup of  $\text{GL}(2, \mathbb{R})$  of matrices with positive determinant

$\mathcal{H}$  : constant horizontal vector  $(1, 0)$

$\chi$  : constant vertical vector  $(0, 1)$

$\mathcal{P} : \mathbb{R}^2 \rightarrow \mathbb{T}^2$  : universal covering of the 2-dimensional torus

$CC(U, x)$  : connected component of  $U$  containing  $x$

$E_q^s, E_q^u$  : stable and unstable subspaces of a hyperbolic point  $q$

$W^s(q), W^u(q)$  : stable and unstable manifolds of the point  $q$

$W_{loc,\varepsilon}^s(q), W_{loc,\varepsilon}^u(q)$  : local stable and unstable manifolds of the point  $q$   
 $\mathcal{O}(x, f)$  : orbit of  $x$  with respect to  $f$   
 $B_r^n(x)$  :  $n$ -dimensional ball of radius  $r$  and centre  $x$   
 $\overline{B_r^n(x)}$  : closure of the  $n$ -dimensional ball of radius  $r$  and centre  $x$   
 $\text{Linking}_n(I, x, y)$  : linking number at finite time  $n$  of the points  $x, y \in \mathbb{R}^2, x \neq y$  (Def. 1.2.1)  
 $\text{Linking}(I, x, y)$  : linking number of  $x, y \in \mathbb{R}^2, x \neq y$  (Def. 1.2.1)  
 $\Delta$  : diagonal in  $\mathbb{R}^4$ , i.e.  $\{(z_1, z_2) \in \mathbb{R}^4 : z_1 = z_2\}$   
 $\dim_H(U)$  : Hausdorff dimension of the set  $U$   
 $\mathbb{I}_U(\cdot)$  : characteristic function of the set  $U$   
 $\mathcal{J}(f)$  : union of invariant continuous graphs of conservative twist map  $f$  on  $\mathbb{A}$  (Notation 3.1.2)  
 $\mathcal{N}(f)$  : complement set of  $\mathcal{J}(f)$  in  $\mathbb{A}$  (Notation 3.1.2)  
 $V_x$  : vertical line passing through the point  $x$   
 $\mathcal{V}(x)$  : vertical subspace in  $T_x\mathbb{R}^2$ , i.e.  $\ker(Dp_{1T_x\mathbb{R}^2})$   
 $(x_n)_{n \in \mathbb{Z}}$  : configuration for  $F$  lift of a conservative twist map (Def. 3.2.2)  
 $\mathcal{L}((x_n)_{n \in \mathbb{Z}})$  : Aubry diagram of  $(x_n)_{n \in \mathbb{Z}}$  (Def. 3.2.3)  
 $C(\mathcal{D})$  : set of configurations  $(x_n)_{n \in \mathbb{Z}}$  such that  $(x_n, x_{n+1}) \in \mathcal{D} \forall n$   
 $\mathcal{M}$  : set of minimizing configurations (Def. 3.2.7)  
 $\mathcal{M}(\mathcal{D})$  : set of minimizing configurations among  $C(\mathcal{D})$  (Def. 3.2.7)  
 $\rho((x_n)_{n \in \mathbb{Z}})$  : rotation number of the minimizing configuration  $(x_n)_{n \in \mathbb{Z}}$  (Prop. 3.2.7)  
 $\mathcal{M}_\rho$  : set of minimizing configurations with rotation number  $\rho$   
 $q$  : hyperbolic fixed point for  $f^N = \mathfrak{f}$   
 $p$  : transverse homoclinic point of  $q$  for  $f^N = \mathfrak{f}$   
 $O_\varepsilon$  : adapted neighborhood of  $q$  for  $\mathcal{O}(p, \mathfrak{f})$  with respect to  $\varepsilon$  (Def. 4.2.3)  
 $U_\varepsilon$  : adapted neighborhood of  $\{q\} \cup \mathcal{O}(p, \mathfrak{f})$  with respect to  $\varepsilon$  (Def. 4.2.4)  
 $H(U_\varepsilon, j)$  :  $f^N$ -invariant horseshoe in  $U_\varepsilon$  for  $p$  (Def. 4.3.2)  
 $\Lambda(U_\varepsilon)$  : maximal  $f$ -invariant set in  $U_\varepsilon$  (Notation 4.2.3)

# Introduction

Let us consider a symplectic smooth dynamical system. This thesis looks after relations between some properties of the dynamical system and the possible values of the so-called asymptotic Maslov index.

Roughly speaking, on surfaces this quantity describes how vectors asymptotically “turn” under the action of the differential of the dynamical system (see [BB13]).

In higher dimensions, the asymptotic Maslov index is defined in the symplectic framework, for example for Hamiltonian flows and when looking at action over Lagrangian subspaces (see [CGIP03] and [AF08]).

Although the notion of Maslov index was first introduced by V. I. Arnold in [Arn67], the definition of *asymptotic* Maslov index has first appeared in the work of D. Ruelle in [Rue85] in 1985.

In this thesis we are interested in asymptotic Maslov indices for surface diffeomorphism. Different names denote the same notion: asymptotic Maslov index, Ruelle’s rotation number, B guin and Boubaker’s torsion, . . . From now on, we refer to it as **torsion**.

Let  $S$  be a parallelizable (not necessarily compact) Riemannian surface, that is a Riemannian surface whose tangent bundle is trivial. Examples of parallelizable surfaces are the annulus  $\mathbb{A} = \mathbb{T} \times \mathbb{R}$ , the annulus with a finite number of holes, the torus  $\mathbb{T}^2$ , the disk  $\mathbb{D}^2$  (eventually with a finite number of holes), . . . Instead, neither the 2-dimensional sphere  $\mathbb{S}^2$  nor a compact surface without boundary with genus  $g \geq 2$  are parallelizable. Let  $I = (f_t)_{t \in \mathbb{R}}$  be an isotopy in  $\text{Diff}^1(S)$  joining the identity  $\text{Id}_S$  to  $f_1 = f$  and such that  $f_{1+t} = f_t \circ f$ . The tangent bundle inherits the dynamics through the differential  $Df_t : TS \rightarrow TS$ . Fix a Riemannian metric, an orientation and let  $X : S \rightarrow TS$  be a non vanishing continuous vector field.

For  $x \in S$  and  $v \in T_x S \setminus \{0\}$  we consider the continuous oriented angle function

$$\mathbb{R}_+ \ni t \mapsto v(I)(x, v, t) := \theta(X(f_t(x)), Df_t(x)v) \in \mathbb{T},$$

where  $\theta(u, v)$  denotes the oriented angle between the two non zero vectors  $u$  and  $v$ . Let

$$\mathbb{R}_+ \ni t \mapsto \tilde{v}(I)(x, v, t) \in \mathbb{R}$$

be a continuous determination of the oriented angle function  $\theta$ . For  $n \in \mathbb{N}$  the  $n$ -finite time torsion at  $(x, v) \in TS$  is

$$\text{Torsion}_n(I, x, v) = \frac{\tilde{v}(I)(x, v, n) - \tilde{v}(I)(x, v, 0)}{n}.$$

The torsion at the orbit of  $x$ , denoted as  $\text{Torsion}(I, x)$ , is the limit

$$\lim_{n \rightarrow +\infty} \text{Torsion}_n(I, x, v),$$



whenever it exists. The torsion at the orbit of  $x$  does not depend on the vector  $v \in T_x S$ . Moreover, it is independent of the point of the orbit at which we calculate it. In addition, when it exists, the torsion does not depend on the chosen Riemannian metric (see Proposition 1.1.4). The torsion a priori depends on the vector field  $X$  (see Proposition 1.1.5). In many cases, the torsion (already at finite time) is independent from the chosen isotopy  $I = (f_t)_t$ , see Remark 1.1.3 and Proposition 1.3.2. Whenever it is the case, we will denote the torsion (respectively finite time torsion) as  $\text{Torsion}(f, x)$  (respectively  $\text{Torsion}_n(f, x, v)$ ).

If  $\mu$  is a  $f$ -invariant Borel probability measure with compact support, then its torsion is

$$\text{Torsion}(I, \mu) = \int_S \text{Torsion}(I, x) d\mu(x).$$

Ruelle proved that for such a measure  $\mu$ , for almost every point, the torsion exists. Thus, the torsion of  $\mu$  is well-defined.

From now on, when not specified, we will consider a constant reference vector field  $X$ . The notion of torsion of measures has been studied by Gambaudo and Ghys in [GG97] in the framework of  $\mathcal{C}^1$  diffeomorphisms on the 2-dimensional disk  $\mathbb{D}^2$ . Let  $\mu$  be a Borel probability measure on  $\mathbb{D}^2$ . Gambaudo and Ghys have shown that the torsion of  $\mu$  is a homogeneous quasi-morphism on the set of diffeomorphisms of the disk that are the identity near the boundary  $\partial\mathbb{D}^2$  and preserve the measure  $\mu$  (see Proposition 2.8 in [GG97]). Moreover, they have also proved that the torsion is invariant by topological conjugacy, assuming that the measures are without atoms (see Theorem 2.11 in [GG97]).

The notion of torsion of  $f$ -invariant measures has also been discussed by Conejeros, who in his PhD thesis (see [Con15]) has compared it to his notion of fibered rotation number.

The set of zero torsion value points is useful to understand certain dynamical behaviors.

In [MN02] Matsumoto and Nakayama prove that for every  $\mathcal{C}^\infty$  diffeomorphism  $f$  of  $\mathbb{T}^2$  isotopic to the identity there exists a  $f$ -invariant probability measure  $\mu$  such that  $\text{Torsion}(f, \mu)$  is null.

For conservative twist maps on the annulus, the structure of some null torsion sets, called Aubry-Mather sets, has been studied by Mather (see [Mat82a] and [Mat91]) and Angenent (see [Ang88]) through a variational approach.

Through a more topological point of view in [Cro03], Crovisier has obtained results for non-conservative twist maps on  $\mathbb{A}$ . He has proved that for any rotation number there exists an Aubry-Mather set of null torsion (see Theorem 1.2 in [Cro03]).

In Chapter 2 we introduce the notion of negative-torsion map on the annulus. We study the zero torsion set for negative-torsion maps. On the annulus, the torsion does not depend on the chosen isotopy (see Proposition 1.3.2), so we omit it in the notation. We consider a constant reference vector field  $X$ .

**Definition.** A negative-torsion (respectively positive-torsion) map  $f : \mathbb{A} \rightarrow \mathbb{A}$  is a  $\mathcal{C}^1$  diffeomorphism isotopic to the identity such that for every  $z \in \mathbb{A}$  it holds

$$\text{Torsion}_1(f, z, \chi) < 0 \quad (\text{ respectively } > 0),$$

where  $\chi$  is the vertical vector  $(0, 1)$ .

The notion of negative-torsion map coincides with the definition of positive tilt map, as presented in [Hu98] and [GR13]. Moreover, the same negative-torsion maps can be defined through the notion of positive/negative paths presented in [Her83] and in [LC88].

Examples of negative-torsion maps are positive twist maps. A positive twist map  $f : \mathbb{A} \rightarrow \mathbb{A}$  is a  $\mathcal{C}^1$  diffeomorphism isotopic to the identity such that for any lift  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  of  $f$  and for any  $x \in \mathbb{R}$  the function

$$y \mapsto p_1 \circ F(x, y)$$

is an increasing diffeomorphism of  $\mathbb{R}$ , where  $p_1$  denotes the projection over the first coordinate.

The interest for twist maps has largely spread all along the literature (see for example [LC91], [Mat82a], [Mat91] and [Mos86]) and, as mentioned before, several authors have studied their connection with the notion of torsion.

By the twist property of  $f$ , at every point the image of the vertical vector through  $Df$  lies in the right half-plane. At every point the torsion at time 1 with respect to the vector  $\chi = (0, 1)$  is negative. This property is already remarked in [Cro03] and in [LC91]. Actually, a more precise estimation can be given.

**Theorem A.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a positive twist map. For any  $z \in \mathbb{A}$  and for any  $n \in \mathbb{N}, n \neq 0$  it holds*

$$Torsion_n(f, z, \chi) \in \left(-\frac{1}{2}, 0\right).$$

An immediate outcome is the following

**Corollary A.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a positive twist map. Then for any  $z \in \mathbb{A}$ , where the torsion exists, it holds*

$$Torsion(f, z) \in \left[-\frac{1}{2}, 0\right].$$

We will prove that for negative-torsion maps the Hausdorff dimension of the set of zero torsion points is greater or equal to one. This result follows from the following Theorem, for which we need the definition of essential curve.

**Definition.** An essential curve  $\gamma : \mathbb{T} \rightarrow \mathbb{A}$  is a  $\mathcal{C}^0$  embedding such that  $\gamma(\mathbb{T})$  is not homotopic to a point.

**Theorem B.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a negative-torsion map. Let  $\gamma : \mathbb{T} \rightarrow \mathbb{A}$  be a  $\mathcal{C}^1$  essential curve. Then there exists at least one point  $z \in \gamma(\mathbb{T})$  such that*

$$Torsion(f, z) = 0.$$

**Corollary B.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a negative-torsion map. Then*

$$\dim_H(\{z \in \mathbb{A} : Torsion(f, z) = 0\}) \geq 1.$$

The same results hold also for positive-torsion maps.

**Question:** can we give more precise results over the Hausdorff dimension of the set of zero torsion points?

We obtain as a by-product of the proof of Theorem B a version of Birkhoff's theorem for negative-torsion maps.

**Theorem C.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a negative-torsion (respectively positive-torsion) map. Let  $\gamma : \mathbb{T} \rightarrow \mathbb{A}$  be a  $\mathcal{C}^1$   $f$ -invariant essential curve such that  $f|_\gamma$  is non wandering. Then  $\gamma(\mathbb{T})$  is the graph of a Lipschitz function over  $\mathbb{T}$ .*

What about points of non zero torsion? Béguin and Boubaker in [BB13] have given conditions to assure the existence of orbits with non zero torsion. In particular, they have shown that if  $f$  is an area-preserving diffeomorphism of the disk with compact support (which is not the identity), then  $f$  has an orbit with non zero torsion (see Theorem A in [BB13]). Moreover, if  $f$  is a diffeomorphism of  $\mathbb{T}^2$  whose rotation number set has not empty interior, then  $f$  has an orbit with non zero torsion (see Theorem B in [BB13]). In their work, Béguin and Boubaker use the relation between the linking number and the torsion of points in the lifted framework.

In the setting of the plane  $\mathbb{R}^2$ , for any  $x, y \in \mathbb{R}^2, x \neq y$  the linking number of  $x$  and  $y$  is the asymptotic angular velocity of the vector  $f_t(y) - f_t(x)$ .

In Chapter 1 we study the link between these two quantities for a  $\mathcal{C}^1$  diffeomorphism on  $\mathbb{R}^2$ . We focus on the following question: for a given isotopy, assuming that the linking number of two points  $x, y$  is not zero, does there exist at least a point  $z$  on the segment connecting  $x$  and  $y$  such that its torsion is also not zero?

We state the following result, where the torsion is calculated with respect to a constant reference vector field.

**Theorem D.** *Let  $I = (f_t)_{t \in [0,1]}$  be an isotopy in  $\text{Diff}^1(\mathbb{R}^2)$  joining  $\text{Id}_{\mathbb{R}^2}$  to  $f_1 = f$ . Assume that there exist two points  $x, y \in \mathbb{R}^2, x \neq y$  such that*

$$\text{Linking}_1(I, x, y) = l \in \mathbb{R}.$$

*Then there exists a point  $z \in [x, y]$  so that*

$$\text{Torsion}_1(I, z, y - x) = l.$$

Passing to asymptotic quantities, we deduce the following

**Corollary C.** *Let  $I = (f_t)_{t \in [0,1]}$  be an isotopy in  $\text{Diff}^1(\mathbb{R}^2)$  joining  $\text{Id}_{\mathbb{R}^2}$  to  $f_1 = f$ . Assume that there exist two points  $x, y \in \mathbb{R}^2, x \neq y$  such that*

$$\text{Linking}(I, x, y) = l \in \mathbb{R}.$$

*Suppose that  $\bigcup_{n \in \mathbb{N}} f^n([x, y])$  is relatively compact, where  $[x, y]$  denotes the segment joining the two points.*

*Then there exists a  $f$ -invariant Borel probability measure  $\mu$  so that  $\text{Torsion}(I, \mu) = l$ .*

The following question is due to F. Béguin:

**Question:** let  $x_0$  be a fixed point. Assume that the set of points  $x$  such that the asymptotic linking number of  $(x_0, x)$  is not null has positive Lebesgue measure. Does the set of points with non zero torsion have positive Lebesgue measure?

The link between torsion and linking number, i.e. Theorem D, together with Theorem A in the framework of positive twist map, enables us to obtain results over linking number of points for lifts of twist maps.

**Corollary D.** *Let  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a lift of a positive twist map and let  $I = (F_t)_t$  be the isotopy joining the identity to  $F$ , obtained as a lift of an isotopy on  $\mathbb{A}$ . Let  $z_1, z_2 \in \mathbb{R}^2, z_1 \neq z_2$  be such that their linking number exists. Then*

$$\text{Linking}(I, z_1, z_2) \in \left[ -\frac{1}{2}, 0 \right].$$

This result was already known by Le Calvez for periodic orbits and then, through the  $\mathcal{C}^1$  closing Lemma, also for  $F$ -invariant measures, but our corollary generalizes it, holding true for any couple of points for which the (asymptotic) linking number exists.

A similar argument holds also for lifts of negative-torsion (respectively positive-torsion) maps, that is for any lift of a negative-torsion (respectively positive-torsion) map the linking number of any couple of points (whenever it exists) is non positive (respectively non negative).

We then consider conservative twist maps, that is

**Definition.** A twist map  $f$  is conservative if  $f^*\lambda - \lambda$  is an exact 1-form, where  $\lambda = ydx$ .

In particular, a conservative twist map preserves the Lebesgue measure. Concerning points with non zero torsion, in the framework of conservative twist maps, we show that bounded instability regions have sets of positive Lebesgue measure where the torsion is not null. In particular, in Chapter 3 we analyse the torsion of bounded connected components of the complementary set of  $\mathcal{I}(f)$ , where  $\mathcal{I}(f)$  denotes the union of all  $f$ -invariant essential curves of  $\mathbb{A}$ . We prove the following

**Theorem E.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a conservative (positive) twist map. Then any bounded connected component of  $\mathbb{A} \setminus \mathcal{I}(f)$  has a positive Lebesgue measure set of points of not zero torsion.*

More precisely, we discuss the two possible types of bounded connected components of  $\mathbb{A} \setminus \mathcal{I}(f)$ . In particular, for a bounded essential subannulus we prove the following result.

**Theorem F.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a conservative twist map. Let  $U \subset \mathbb{A}$  be a  $f$ -invariant essential subannulus which is the interior of its closure and which is bounded. Then the torsion is zero for almost every point in  $U$  if and only if  $f|_U$  is  $\mathcal{C}^0$ -integrable.*

We say that  $f|_U$  is  $\mathcal{C}^0$ -integrable if there exists a partition of  $U$  into continuous essential  $f$ -invariant curves. We wonder if a similar result could hold true in a different setting: the following question is due to J.-P. Marco.

**Question:** if the torsion is null on a dense  $G_\delta$  set, is the dynamics  $\mathcal{C}^0$ -integrable?

An analogous result can be obtained also for Tonelli Hamiltonian flows on  $T^*\mathbb{T}^n$  as an outcome of results in [CGIP03] and in [AABZ15].

The discussion for instability periodic disks largely relies on Green bundles techniques, as presented in [Gre58] and [Arn10].

We can then ask the following

**Question:** in bounded instability regions we always have not negligible (from a Lebesgue measure point of view) sets of not zero torsion. Is it a feature of almost every point of instability bounded regions? That is, is the set of points with non zero torsion of full Lebesgue measure within instability regions?

Another question concerns unbounded instability regions.

**Question:** are there examples of an unbounded instability region for a conservative twist map such that the torsion is zero at Lebesgue-almost every point of the region?

The notion of torsion is given through a limit. Through Ruelle's result, we have already remarked that the torsion exists at almost every point. It is so natural asking what about points at which the torsion does not exist.

In Chapter 4, we consider a  $\mathcal{C}^1$  diffeomorphism  $f$  isotopic to the identity with hyperbolic periodic points admitting transverse homoclinic intersections (on  $\mathbb{R}^2$ ,  $\mathbb{A}$  or  $\mathbb{T}^2$ ). We do not ask that  $f$  is either conservative or a twist map or a negative-torsion map. We are interested in the associated horseshoe.

**Definition.** Let  $f$  be a  $\mathcal{C}^1$  diffeomorphism. A horseshoe  $H$  is a uniformly hyperbolic set for  $f^N$  (for some  $N > 0$ ) such that the dynamics of  $f^N$  on the horseshoe is conjugated to a shift dynamics on  $\{0, 1\}^{\mathbb{Z}}$ . The orbit of the horseshoe is  $\bigcup_{i=0}^{N-1} f^i(H)$ .

After recalling the construction of the horseshoe dynamics for transverse homoclinic points of intersections, we prove that the torsion at points of the horseshoe can be calculated from the symbolic dynamics associated to it.

Denote as  $(\delta_i(x))_{i \in \mathbb{Z}}$  the sequence in  $\{0, 1\}^{\mathbb{Z}}$  associated to a point  $x$  in the horseshoe.

**Theorem G.** *There exist  $a, \Delta \in \mathbb{R}$  such that for any  $x$  in the horseshoe it holds<sup>1</sup>*

$$\{\text{limit points of } (Torsion_n(f, x))_{n \in \mathbb{N}}\} = \left\{ \text{limit points of } \left( a + \Delta \frac{\sum_{i=1}^n \delta_i(x)}{n} \right)_{n \in \mathbb{N}} \right\}.$$

Interesting outcomes follow when  $\Delta \neq 0$ . In particular, we can deduce that:

- (i) any value in  $[a, a + \Delta]$  is realized as torsion of some points in the horseshoe. Referring to [HH86], any irrational value within such an interval is the torsion value of uncountable many disjoint Cantor sets. Moreover, for any  $\alpha$  in  $[a, a + \Delta]$  the set of points of torsion value  $\alpha$  is dense in the horseshoe;

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1. In the notation of torsion we do not make explicit the choice of the tangent vector because the asymptotic torsion does not depend on it and for lightening the notation.

- (ii) the set of points at which the torsion does not exist contains a dense  $G_\delta$  subset of the horseshoe;
- (iii) any value in  $[a, a + \Delta]$  is the torsion of a  $f$ -invariant ergodic measure whose support is contained in the orbit of the horseshoe.

Using multifractal analysis (see [Pes97] and [BS00]), we can prove that the set of points of the horseshoe at which the torsion does not exist has positive Hausdorff dimension.

If  $f$  is  $\mathcal{C}^2$ , then the condition  $\Delta \neq 0$  is equivalent to the fact that a given finite time torsion is cohomologous to a constant.

Due to a recent result of Buzzi, Crovisier and Sarig (see [BCS]), for a  $\mathcal{C}^\infty$  diffeomorphism on  $\mathbb{A}$  or on  $\mathbb{R}^2$  the existence of a transverse homoclinic point of intersection always implies the existence of a horseshoe such that  $\Delta \neq 0$ .

The presence of transverse homoclinic intersections leads to the discussion of the topological entropy of the system. Actually, we can deduce that for a  $\mathcal{C}^\infty$  diffeomorphism on the bounded annulus or on the compact disc, by using Katok's result in [Kat80], if the torsion exists everywhere then the topological entropy has to be null. A natural question is thus the following.

**Question:** does the converse hold true? Can we characterise the positiveness of the topological entropy in terms of non existence of the torsion at some points?

Further natural questions concern the study of asymptotic Maslov index for (conformally) symplectic dynamics in higher dimensions. For example, can we obtain results over the Hausdorff dimension of zero torsion set in higher dimensions? What about asymptotic Maslov index and horseshoes in higher dimensions?



# Introduction

Nous allons considérer un système dynamique symplectique lisse. Cette thèse s'intéresse aux relations entre les propriétés du système dynamique et les possibles valeurs de l'indice de Maslov asymptotique.

Grosso modo sur une surface, l'indice de Maslov asymptotique décrit comment les vecteurs “tournent” sous l'action du système dynamique différentiel (voir [BB13]).

En dimension supérieure, l'indice de Maslov asymptotique est défini dans le cadre symplectique, par exemple pour des flots hamiltoniens et pour l'action sur des sous-espaces lagrangiens (voir [CGIP03] et [AF08]).

Bien que la notion d'indice de Maslov ait été introduite par V. I. Arnold in [Arn67], la définition d'indice de Maslov *asymptotique* est apparue pour la première fois dans le travail de D. Ruelle en [Rue85] en 1985.

Dans cette thèse on s'intéresse à l'indice de Maslov asymptotique pour des difféomorphismes de surfaces. Plusieurs noms indiquent la même notion : indice de Maslov asymptotique, nombre de rotation de Ruelle, torsion de Béguin et Boubaker, ... Dorénavant, on l'appellera **torsion**.

Soit  $S$  une surface riemannienne parallélisable (pas forcément compacte), c'est-à-dire une surface riemannienne dont le fibré tangent est trivial. Par exemple, l'anneau  $\mathbb{A} = \mathbb{T} \times \mathbb{R}$  (éventuellement avec un nombre fini de trous), le tore  $\mathbb{T}^2$ , le disque (éventuellement avec un nombre fini de trous) sont des surfaces parallélisables. Par contre, ni la sphère  $\mathbb{S}^2$  ni aucune surface compacte sans bord de genre  $g \geq 2$  ne sont parallélisables. Soit  $f$  un difféomorphisme de  $S$  de classe  $\mathcal{C}^1$  isotope à l'identité. Soit  $I = (f_t)_{t \in \mathbb{R}}$  une isotopie sur  $\text{Diff}^1(S)$  qui joint  $\text{Id}_S$  à  $f_1 = f$  et telle que  $f_{1+t} = f_t \circ f$ . Le fibré tangent hérite de la dynamique grâce à la différentielle  $Df_t : TS \rightarrow TS$ . Fixons une métrique riemannienne et une orientation sur  $S$ . Soit  $X : S \rightarrow TS$  un champ de vecteurs continu qui ne s'annule jamais.

Soit  $x \in S$  et  $v \in T_x S \setminus \{0\}$ . On considère la fonction d'angle orienté continue

$$\mathbb{R}_+ \ni t \mapsto v(I)(x, v, t) := \theta(X(f_t(x)), Df_t(x)v) \in \mathbb{T},$$

où  $\theta(u, v)$  est l'angle orienté entre les deux vecteurs non nuls  $u$  et  $v$ . Soit

$$\mathbb{R}_+ \ni t \mapsto \tilde{v}(I)(x, v, t) \in \mathbb{R}$$

une détermination continue de la fonction d'angle orienté  $\theta$ . Pour  $n \in \mathbb{N}$  la torsion au temps fini  $n$  de  $(x, v) \in TS$  est

$$\text{Torsion}_n(I, x, v) = \frac{\tilde{v}(I)(x, v, n) - \tilde{v}(I)(x, v, 0)}{n}.$$

La torsion de l'orbite de  $x$ , notée  $\text{Torsion}(I, x)$ , est la limite

$$\lim_{n \rightarrow +\infty} \text{Torsion}_n(I, x, v),$$



lorsqu'elle existe. La torsion de l'orbite de  $x$  ne dépend pas ni du vecteur  $v \in T_x S$  ni du point de l'orbite où elle est calculée. De plus, lorsqu'elle existe, la torsion ne dépend pas de la métrique riemannienne choisie (voir Proposition 1.1.4). La torsion ne dépend à priori que du champ de vecteur  $X$  (voir Proposition 1.1.5).

Dans de nombreux cas, la torsion (déjà en temps fini) est indépendante de l'isotopie choisie  $I = (f_t)_t$ , voir Remarque 1.1.3 et Proposition 1.3.2. Chaque fois que c'est le cas, nous notons la torsion (la torsion en temps fini) comme  $\text{Torsion}(f, x)$  ( $\text{Torsion}_n(f, x, v)$ ). Si  $\mu$  est une mesure de Borel  $f$ -invariante avec support compact, alors sa torsion est

$$\text{Torsion}(I, \mu) = \int_S \text{Torsion}(I, x) d\mu(x).$$

Ruelle a montré que pour cette mesure  $\mu$  et pour presque tous les points, la torsion existe. Donc, la torsion de  $\mu$  est bien définie.

Dorénavant, on fixe un champ de vecteur de référence  $X$  constant.

La notion de torsion des mesures a été étudiée par Gambaudo et Ghys en [GG97] pour des difféomorphismes  $\mathcal{C}^1$  du disque 2-dimensionnel  $\mathbb{D}^2$ . Soit  $\mu$  une mesure borélienne de probabilité de  $\mathbb{D}^2$ . Gambaudo et Ghys ont montré que la torsion de  $\mu$  est un quasi-morphisme homogène sur l'ensemble des difféomorphismes du disque qui sont l'identité proche du bord  $\partial\mathbb{D}^2$  et préservent la mesure  $\mu$  (voir Proposition 2.8 en [GG97]). De plus, ils ont aussi prouvé que la torsion est invariante par conjugaison topologique, en supposant que les mesures sont sans atomes (voir Théorème 2.11 en [GG97]).

La notion de torsion des mesures  $f$ -invariantes a été discuté par Conejeros qui, dans sa thèse (voir [Con15]), l'a comparé à sa notion de nombre de rotation fibré.

L'ensemble des points de torsion nulle est utile pour comprendre certaines caractéristiques dynamiques.

Matsumoto et Nakayama (voir [MN02]) montrent que pour tout difféomorphisme de classe  $\mathcal{C}^\infty$   $f$  de  $\mathbb{T}^2$  isotope à l'identité, il existe une mesure de probabilité  $f$ -invariante  $\mu$  telle que  $\text{Torsion}(f, \mu) = 0$ .

Pour des applications conservatives déviant la verticale de l'anneau, les ensembles de torsion nulle, appelés ensembles d'Aubry-Mather, ont été étudiés par Mather (voir [Mat82a] et [Mat91]) et par Angenent (voir [Ang88]) en utilisant une méthode variationnelle.

Grâce à un point de vue topologique en [Cro03], Crovisier a obtenu des résultats pour des applications non conservatives déviant la verticale. Il a montré que pour tout nombre de rotation, il existe un ensemble d'Aubry-Mather de torsion nulle (voir Théorème 1.2 en [Cro03]).

Dans le chapitre 2 on introduit la notion d'application de torsion négative dans l'anneau. On étudie l'ensemble de torsion nulle des applications de torsion négative. Sur l'anneau, la torsion ne dépend pas de l'isotopie choisie (voir Proposition 1.3.2), donc nous l'omettons dans la notation. Rappelons que nous considérons un champ de vecteur de référence  $X$  constant.

**Définition.** Une application de torsion négative (positive)  $f : \mathbb{A} \rightarrow \mathbb{A}$  est un difféomorphisme  $\mathcal{C}^1$  isotope à l'identité telle que pour tout  $z \in \mathbb{A}$

$$\text{Torsion}_1(f, z, \chi) < 0 \quad (> 0),$$

où  $\chi$  est le vecteur vertical  $(0, 1)$ .

La notion d'application de torsion négative correspond à celle d'application tilt positive (voir [Hu98] et [GZ04]). De plus, les applications de torsion négative peuvent être définies avec la notion de chemin positif/négatif présente en [Her83] et [LC88].

Des exemples d'applications de torsion négative sont les applications déviant la verticale à droite. Une application déviant la verticale à droite  $f : \mathbb{A} \rightarrow \mathbb{A}$  est un difféomorphisme  $\mathcal{C}^1$  isotope à l'identité telle que pour tout relevé  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  de  $f$  et pour tous  $x \in \mathbb{R}$  la fonction

$$y \mapsto p_1 \circ F(x, y)$$

est un difféomorphisme croissant de  $\mathbb{R}$ , où  $p_1$  est la projection sur la première coordonnée. L'intérêt pour les applications déviant la verticale s'est développé en littérature (voir [LC91], [Mat82a], [Mat91] et [Mos86]) et, comme mentionné ci-dessus, plusieurs auteurs ont étudié leurs connections avec la notion de torsion.

Pour la propriété déviant la verticale de  $f$ , en chaque point l'image du vecteur vertical par  $Df$  est dans le demi-plan à droite. En chaque point la torsion au temps 1 par rapport au vecteur  $\chi = (0, 1)$  est négative. Cette propriété a déjà été remarquée en [Cro03] et [LC91]. À vrai dire, cela donne une estimation plus précise.

**Théorème A.** *Soit  $f : \mathbb{A} \rightarrow \mathbb{A}$  une application déviant la verticale à droite. Pour tout  $z \in \mathbb{A}$  et pour tout  $n \in \mathbb{N}, n \neq 0$  on a*

$$Torsion_n(f, z, \chi) \in \left(-\frac{1}{2}, 0\right).$$

Une conséquence immédiate est

**Corollaire A.** *Soit  $f : \mathbb{A} \rightarrow \mathbb{A}$  une application déviant la verticale à droite. Alors pour tout  $z \in \mathbb{A}$ , quand elle est définie, on a*

$$Torsion(f, z) \in \left[-\frac{1}{2}, 0\right].$$

On montre que pour une application de torsion négative la dimension de Hausdorff de l'ensemble des points de torsion nulle est supérieure ou égale à 1. Ce résultat vient du théorème suivant pour lequel on introduit la définition de courbe essentielle.

**Définition.** Une courbe essentielle  $\gamma : \mathbb{T} \rightarrow \mathbb{A}$  est un plongement de classe  $\mathcal{C}^0$  tel que  $\gamma(\mathbb{T})$  n'est pas homotope à un point.

**Théorème B.** *Soit  $f : \mathbb{A} \rightarrow \mathbb{A}$  une application de torsion négative. Soit  $\gamma : \mathbb{T} \rightarrow \mathbb{A}$  une courbe essentielle  $\mathcal{C}^1$ . Alors il existe au moins un point  $z \in \gamma(\mathbb{T})$  tel que*

$$Torsion(f, z) = 0.$$

**Corollaire B.** *Soit  $f : \mathbb{A} \rightarrow \mathbb{A}$  une application de torsion négative. Alors*

$$\dim_H(\{z \in \mathbb{A} : Torsion(f, z) = 0\}) \geq 1.$$

Le même résultat est valable pour des applications de torsion positive.

**Question :** peut-on donner des résultats plus précis sur la dimension de Hausdorff de l'ensemble des points de torsion nulle ?

Comme sous-produit de la preuve du Théorème B on obtient une version du théorème de Birkhoff pour des applications de torsion négative.

**Théorème C.** *Soit  $f : \mathbb{A} \rightarrow \mathbb{A}$  une application de torsion négative (de torsion positive). Soit  $\gamma : \mathbb{T} \rightarrow \mathbb{A}$  une courbe essentielle  $\mathcal{C}^1$  invariante par  $f$  et telle que  $f|_\gamma$  est non errante. Alors  $\gamma(\mathbb{T})$  est le graph d'une fonction lipschitzienne sur  $\mathbb{T}$ .*

Que peut-on dire des points de torsion non nulle ? Béguin et Boubaker en [BB13] ont donné des conditions pour assurer l'existence des orbites de torsion non nulle. Ils ont montré que si  $f$  est un difféomorphisme du disque à support compact qui préserve l'aire (qui n'est pas l'identité), alors  $(f, \mathbb{D}^2)$  a une orbite de torsion non nulle (voir Théorème A in [BB13]). De plus, si  $f$  est un difféomorphisme de  $\mathbb{T}^2$  tel que son ensemble de nombre de rotation a un intérieur non vide, alors  $(f, \mathbb{T}^2)$  a une orbite avec torsion non nulle (voir Théorème B en [BB13]). Dans leur travail, Béguin et Boubaker utilisent la relation entre le nombre d'enlacement et la torsion des points dans le cadre du relevé.

Dans  $\mathbb{R}^2$ , pour tous  $x, y \in \mathbb{R}^2, x \neq y$  le nombre d'enlacement de  $x$  et  $y$  est la vitesse rotationnelle asymptotique du vecteur  $f_t(y) - f_t(x)$ .

Dans le chapitre 1 on étudie le lien entre ces deux quantités pour un difféomorphisme  $\mathcal{C}^1$  de  $\mathbb{R}^2$ . On considère la question suivante : pour une certaine isotopie et supposant que le nombre d'enlacement des deux points  $x, y$  est non nul, existe-t-il au moins un point  $z$  sur le segment qui joint  $x$  et  $y$  tel que sa torsion soit aussi non nulle ?

Rappelons que la torsion est calculée par rapport à un champ de vecteur de référence  $X$  constant.

**Théorème D.** *Soit  $I = (f_t)_{t \in [0,1]}$  une isotopie dans  $\text{Diff}^1(\mathbb{R}^2)$  qui joint  $\text{Id}_{\mathbb{R}^2}$  à  $f_1 = f$ . Supposons qu'il existe deux points  $x, y \in \mathbb{R}^2, x \neq y$  tels que*

$$\text{Linking}_1(I, x, y) = l \in \mathbb{R}.$$

*Alors il existe un point  $z \in [x, y]$  tel que*

$$\text{Torsion}_1(I, z, y - x) = l.$$

En considérant les quantités asymptotiques, on montre le corollaire suivant.

**Corollaire C.** *Soit  $I = (f_t)_{t \in [0,1]}$  une isotopie dans  $\text{Diff}^1(\mathbb{R}^2)$  qui joint  $\text{Id}_{\mathbb{R}^2}$  à  $f_1 = f$ . Supposons qu'il existe deux points  $x, y \in \mathbb{R}^2, x \neq y$  tels que*

$$\text{Linking}(I, x, y) = l \in \mathbb{R}.$$

*Supposons que  $\bigcup_{n \in \mathbb{N}} f^n([x, y])$  est relativement compact où  $[x, y]$  est le segment qui joint les deux points. Alors il existe une mesure borélienne de probabilité  $f$ -invariante  $\mu$  telle que  $\text{Torsion}(I, \mu) = l$ .*

La question suivante est due à F. Béguin :

**Question :** soit  $x_0$  un point fixe. Supposons que l'ensemble des points  $x$  tels que le nombre d'enlacement asymptotique de  $(x_0, x)$  est non nul, possède une mesure de Lebesgue positive. Est-ce que l'ensemble des points de torsion non nulle possède une mesure de Lebesgue positive ?

Le lien entre torsion et nombre d'enlacement, i.e. le théorème D et le théorème A dans le cadre des applications déviant la verticale à droite, nous permet d'obtenir le résultat suivant.

**Corollaire D.** *Soit  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  un relevé d'une application déviant la verticale à droite et soit  $I = (F_t)_t$  une isotopie qui joint l'identité à  $F$ , obtenue comme relevé d'une isotopie sur  $\mathbb{A}$ . Soient  $z_1, z_2 \in \mathbb{R}^2, z_1 \neq z_2$  tels que leur nombre d'enlacement existe. Alors*

$$\text{Linking}(I, z_1, z_2) \in \left[ -\frac{1}{2}, 0 \right].$$

Ce résultat était déjà connu par Le Calvez pour des orbites périodiques et donc, grâce au  $\mathcal{C}^1$  closing lemma, aussi connu pour des mesures  $F$ -invariantes. Notre corollaire est bien une généralisation car il est valable pour tout couples de points pour lesquels le nombre d'enlacement (asymptotique) existe.

Un résultat similaire est vrai pour des relevés d'applications de torsion négative (de torsion positive), c'est-à-dire pour un relevé d'une application de torsion négative (positive) le nombre d'enlacement de tout couple de points (où il existe) est non positif (non négatif).

Nous considérons des applications déviant la verticale conservatives, c'est-à-dire

**Définition.** Une application déviant la verticale  $f$  est conservative si  $f^*\lambda - \lambda$  est une 1-forme exacte où  $\lambda = ydx$ .

En particulier, une application déviant la verticale conservative préserve la mesure de Lebesgue. Concernant les points de torsion non nulle, dans le cadre des applications conservatives déviant la verticale, on montre que les régions d'instabilité bornées ont un ensemble de mesure de Lebesgue positive où la torsion est non nulle.

Dans le chapitre 3 on étudie la torsion des composantes connexes bornées du complémentaire de  $\mathcal{J}(f)$ , où  $\mathcal{J}(f)$  est l'union de toutes les courbes essentielles  $f$ -invariantes sur  $\mathbb{A}$ .

**Théorème E.** *Soit  $f : \mathbb{A} \rightarrow \mathbb{A}$  une application déviant la verticale (à droite) conservative. Alors chaque composante connexe bornée de  $\mathbb{A} \setminus \mathcal{J}(f)$  a un ensemble de mesure de Lebesgue positive de points de torsion non nulle.*

Plus précisément, on discute les deux types de composantes connexes bornées possibles de  $\mathbb{A} \setminus \mathcal{J}(f)$ . En particulier, pour un sous-anneau essentiel borné on prouve le résultat suivant.

**Théorème F.** *Soit  $f : \mathbb{A} \rightarrow \mathbb{A}$  une application déviant la verticale conservative. Soit  $U \subset \mathbb{A}$  un sous-anneau essentiel  $f$ -invariant qui est l'intérieur de son adhérence et qui est borné. Alors la torsion est nulle presque partout sur  $U$  si et seulement si  $f|_U$  est  $\mathcal{C}^0$ -*

*intégrable.*

On dit que  $f|_U$  est  $\mathcal{C}^0$ -intégrable s'il existe une partition de  $U$  en courbes essentielles continues  $f$ -invariantes. Nous nous demandons si un résultat similaire est valable dans un cadre différent. La question suivante est due à J.-P. Marco.

**Question :** si la torsion est nulle sur un ensemble  $G_\delta$  dense, est-ce que la dynamique est  $\mathcal{C}^0$ -intégrable ?

Un résultat similaire est valable pour des flots hamiltoniens Tonelli sur  $T^*\mathbb{T}^n$  comme conséquence des résultats en [CGIP03] et [AABZ15].

La preuve pour des disques d'instabilité périodiques s'inspire des fibrés de Green (voir [Gre58] et [Arn10]).

**Question :** le Théorème E affirme que dans toute région d'instabilité bornée, l'ensemble des points de torsion non nulle est Lebesgue-non négligeable. Sa mesure de Lebesgue est-elle égale à celle de toute la région d'instabilité bornée ?

Une autre question concerne les régions d'instabilité non bornées.

**Question :** y a-t-il des exemples de région d'instabilité non bornée pour une application déviant la verticale conservative telle que la torsion est nulle pour Lebesgue-presque tout point de la région ?

La notion de torsion est donnée par une limite. Grâce au résultat de Ruelle, on a déjà remarqué que la torsion existe presque partout. Il est donc naturel de se demander ce qu'on peut dire des points où la torsion n'existe pas.

Dans le chapitre 4, on considère un difféomorphisme  $f$  de classe  $\mathcal{C}^1$  isotope à l'identité avec des points périodiques hyperboliques qui ont des intersections homoclines transverses (sur  $\mathbb{R}^2$ ,  $\mathbb{A}$  ou  $\mathbb{T}^2$ ). On ne suppose pas que  $f$  est conservative ou une application de torsion négative. Nous nous intéressons au fer à cheval associé.

**Définition.** Soit  $f$  un difféomorphisme de classe  $\mathcal{C}^1$ . Un fer à cheval  $H$  est un ensemble uniformément hyperbolique pour  $f^N$  (pour quelque  $N > 0$ ) tel que la dynamique de  $f^N$  restreinte au fer à cheval est conjuguée à la dynamique du décalage sur  $\{0, 1\}^{\mathbb{Z}}$ . L'orbite du fer à cheval est  $\bigcup_{i=0}^{N-1} f^i(H)$ .

Après avoir rappelé la construction d'un fer à cheval pour des points d'intersection homocline transverse, on montre comment calculer la torsion des points du fer à cheval en utilisant la dynamique symbolique associée.

Notons  $(\delta_i(x))_{i \in \mathbb{Z}}$  la suite en  $\{0, 1\}^{\mathbb{Z}}$  associée au point  $x$  du fer à cheval.

**Théorème G.** Il existe  $a, \Delta \in \mathbb{R}$  tels que pour tout  $x$  dans le fer à cheval on a

$$\{ \text{valeurs d'adhérence de } (Torsion_n(f, x))_{n \in \mathbb{N}} \} = \left\{ \text{valeurs d'adhérence de } \left( a + \delta \frac{\sum_{i=1}^n \delta_i(x)}{n} \right)_{n \in \mathbb{N}} \right\}.$$

Il y a des conséquences intéressantes quand  $\Delta \neq 0$ . En particulier, on en déduit que :

- (i) chaque valeur en  $[a, a + \Delta]$  est réalisée comme torsion de certains points du fer à cheval. Grâce à [HH86], chaque valeur irrationnelle de cet intervalle est la torsion d'une famille non dénombrable d'ensembles de Cantor disjoints. En plus, pour tout  $\alpha \in [a, a + \Delta]$  l'ensemble des points de torsion  $\alpha$  est dense dans le fer à cheval ;
- (ii) l'ensemble des points où la torsion n'existe pas contient un  $G_\delta$  dense du fer à cheval ;
- (iii) chaque valeur dans  $[a, a + \Delta]$  est la torsion d'une mesure ergodique  $f$ -invariante dont le support est contenu dans l'orbite du fer à cheval.

Grâce à l'analyse multifractale (voir [Pes97] et [BS00]), on peut montrer que l'ensemble des points du fer à cheval où la torsion n'existe pas, a une dimension de Hausdorff positive. Si  $f$  est de classe  $\mathcal{C}^2$ , alors  $\Delta = 0$  mais si et seulement si une certaine torsion en temps fini est cohomologue à une constante.

En utilisant un résultat récent de Buzzi, Crovisier et Sarig (voir [BCS]), pour un difféomorphisme de classe  $\mathcal{C}^\infty$  de  $\mathbb{A}$  ou de  $\mathbb{R}^2$  l'existence d'un point d'intersection homocline transverse implique toujours l'existence d'un fer à cheval tel que  $\Delta \neq 0$ .

La présence des intersections homoclines transverses nous amène à parler de l'entropie topologique du système. En fait, on peut montrer pour un difféomorphisme de classe  $\mathcal{C}^\infty$  de l'anneau borné ou du disque compact, grâce à un résultat de Katok (voir [Kat80]), que si la torsion existe partout alors l'entropie topologique est nulle.

**Question :** est-ce que l'inverse est vrai ? Est-ce qu'on peut caractériser l'entropie topologique positive par le fait qu'il y ait des points où la torsion n'existe pas ?

D'autres questions concernant l'étude de l'indice de Maslov asymptotique pour des dynamiques (conformément) symplectiques en dimension supérieure se posent. Peut-on retrouver le résultat sur la dimension d'Hausdorff en dimension supérieure ? Que peut-on dire de l'indice de Maslov asymptotique des fers à cheval en dimension supérieure ?



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# Chapter 1

## Torsion and linking number

Denote as  $\mathbb{T}$  the quotient space  $\mathbb{R}/\mathbb{Z}$  and as

$$p : \mathbb{R} \rightarrow \mathbb{T}$$

$$x \mapsto x \bmod 1$$

the universal covering of the 1-dimensional torus  $\mathbb{T}$ .

We use the notation  $\mathbb{A}$  for the product space  $\mathbb{T} \times \mathbb{R}$  and

$$p \times \text{Id} : \mathbb{R}^2 \rightarrow \mathbb{A}$$

$$(x, y) \mapsto (x \bmod 1, y)$$

for the universal covering of the annulus  $\mathbb{A}$ . In the case of some possible ambiguity, a point of the annulus is denoted by  $\bar{z} = (\bar{x}, y) \in \mathbb{A}$ , while  $z = (x, y) \in \mathbb{R}^2$  refers to a lift of  $\bar{z}$  over  $\mathbb{R}^2$ .

The functions

$$\bar{p}_1 : \mathbb{A} \rightarrow \mathbb{T}, \quad (\bar{x}, y) \mapsto \bar{x} \tag{1.1}$$

$$\bar{p}_2 : \mathbb{A} \rightarrow \mathbb{R}, \quad (\bar{x}, y) \mapsto y \tag{1.2}$$

are the projections over the first and the second coordinates, respectively; the coordinate projections of  $\mathbb{R}^2$  are denoted as  $p_1, p_2$ .

The 2-dimensional torus is the quotient space

$$\mathbb{T}^2 := \mathbb{R}^2 / \mathbb{Z}^2.$$

All along the work, the counterclockwise orientation of the plane is chosen.

Once provided a Riemannian metric<sup>1</sup> and an orientation, the *oriented angle* between two non-zero vectors  $u, v \in \mathbb{R}^2$  is well defined as an element of  $\mathbb{T}$ . A *measure of the angle* is an element of  $\mathbb{R}$  whose image through  $p$  coincides with the oriented angle.

The notation  $\mathcal{R}(a, \psi)$  refers to the rotation of the plane  $\mathbb{R}^2$  of center  $a \in \mathbb{R}^2$  and angle  $\psi$ , while  $\tau_v$  denotes the translation on the plane by the vector  $v \in \mathbb{R}^2$ .

A fundamental notion will be that of *isotopy*:

**Definition 1.0.1.** Let  $M, N$  be differential manifolds and let  $f, g : M \rightarrow N$  be in  $\text{Diff}^1(M, N)$ .

An *isotopy*  $(\psi_t)_{t \in [0,1]}$  joining  $f$  to  $g$  is an arc in  $\text{Diff}^1(M, N)$  such that  $\psi_0 = f, \psi_1 = g$  and which is continuous with respect to the *weak* or *compact-open*  $\mathcal{C}^1$  topology on  $\text{Diff}^1(M, N)$ .

---

1. If not specified we are endowing  $\mathbb{R}^2$  with the standard Riemannian metric.

**Definition 1.0.2.** Let  $I \subseteq \mathbb{R}$  be an interval. A *continuous determination* of an angle function  $\theta : I \rightarrow \mathbb{T}$  is a continuous lift of  $\theta$ , i.e. a continuous function  $\tilde{\theta} : I \rightarrow \mathbb{R}$  such that  $\tilde{\theta}(s)$  is a measure of the oriented angle  $\theta(s)$  for any  $s \in I$ .

We remark that a necessary and sufficient condition for the existence of a continuous determination is the continuity of its angle function.

## 1.1 Definition of torsion and first properties

Let  $S$  be a connected Riemannian parallelizable surface, i.e. a connected Riemannian surface whose tangent bundle is trivial. Denote as  $TS_*$  the set  $\{(x, \xi) : x \in S, \xi \in T_x S \setminus \{0\}\}$ . We fix an orientation and we endow  $S$  with a Riemannian metric: the notion of oriented angle between two non zero vectors of the same tangent space is well-defined. The notation  $T^1 S$  refers to the unitary tangent bundle.

**Remark 1.1.1.** The choice of an orientation and of a reference continuous vector field  $X$  over  $S$  that never vanishes is equivalent to that of a trivialization diffeomorphism. Indeed, on every tangent space we define the endomorphism  $J$

$$J : T_x S \rightarrow T_x S$$

as a rotation of angle  $\frac{1}{4}$  according to the fixed orientation. It holds that  $J^2 = -\text{Id}$ . For any  $x \in S$ ,  $(X(x), JX(x))$  provides a direct basis of the tangent space. A trivialization diffeomorphism is so given by

$$TS \ni (x; \alpha X(x) + \beta JX(x)) \xrightarrow{\phi} (x; \alpha, \beta) \in S \times \mathbb{R}^2$$

where  $\alpha, \beta \in \mathbb{R}$  are the coordinates with respect to the basis  $(X(x), JX(x))$ .

Let  $I = (f_t)_{t \in [0,1]}$  be an isotopy joining the identity to  $f_1 = f$ . We then extend the isotopy for any positive time in the following way: let  $t \in \mathbb{R}_+$ , then the  $\mathcal{C}^1$  diffeomorphism  $f_t : S \rightarrow S$  is defined as

$$f_t := f_{\{t\}} \circ f^{\lfloor t \rfloor}$$

where  $\{\cdot\}, \lfloor \cdot \rfloor$  denote the fractionary and integer part of  $t$ , respectively.

**Notation 1.1.1.** With an abuse of notation, we also denote the extended isotopy as  $I = (f_t)_t$ .

In addition, we fix a reference vector field  $X$  that never vanishes (see Remark 1.1.1). Suppose that  $X(x)$  has unitary norm for any  $x \in S$ . We will make explicit the choice of  $X$  when needed. We recall the notation  $\theta(u, v)$  for the oriented angle between two non zero vectors  $v$  and  $u$ .

Our definition of *torsion*, the one given by Béguin and Boubaker in [BB13], actually coincides with Ruelle's notion of *rotation number* (see [Rue85]).

**Definition 1.1.1.** Let  $S$  be a parallelizable surface and let  $I = (f_t)_{t \in [0,1]}$  be an isotopy in  $\text{Diff}^1(S)$  joining the identity  $\text{Id}_S$  to  $f_1 = f$ . Then, we define the function  $v(I)$  as follows:

$$\begin{aligned} v(I) : TS_* \times \mathbb{R} &\rightarrow \mathbb{T} \\ (x, \xi, t) &\mapsto \theta(X(f_t(x)), Df_t(x)\xi). \end{aligned} \tag{1.3}$$

Fix then  $(x, \xi) \in TS_*$  and, since the angle function  $v(I)(x, \xi, \cdot)$  is continuous, consider a continuous determination  $\tilde{v}(I)(x, \xi, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$  of it.

**Definition 1.1.2.** Let  $S$  and  $f$  be as above. Let  $x \in S$  and  $\xi \in T_x S \setminus \{0\}$ . Consider  $v(I)(x, \xi, \cdot)$  and  $\tilde{v}(I)(x, \xi, \cdot)$  as in Definition 1.1.1. Then, for any  $n \in \mathbb{N}, n \neq 0$  the *torsion at finite time  $n$*  is

$$\text{Torsion}_n(I, x, \xi) := \frac{1}{n} (\tilde{v}(I)(x, \xi, n) - \tilde{v}(I)(x, \xi, 0)). \quad (1.4)$$

**Definition 1.1.3.** Let  $S$  and  $f$  be as above. Let  $x \in S$ . Assume that the quantity  $\text{Torsion}_n(I, x, \xi)$  converges as  $n \rightarrow +\infty$  for some  $\xi \in T_x S \setminus \{0\}$ . The *torsion of the orbit of  $x$*  is then

$$\text{Torsion}(I, x) := \lim_{n \rightarrow +\infty} \text{Torsion}_n(I, x, \xi). \quad (1.5)$$

Whenever the limit exists, the previous quantity does not depend on the chosen lift of the angle function (see Proposition 1.1.1) or on the non zero vector of the tangent space (see Proposition 1.1.3). Moreover, it does not depend on the point of the orbit at which we calculate it.

**Definition 1.1.4.** Let  $S$  and  $f$  be as above. Let  $\mu$  be an  $f$ -invariant Borel probability measure on  $S$ . Assume that  $\mu$  or  $I = (f_t)_t$  has compact support<sup>2</sup>. Then, the *torsion of the measure  $\mu$*  is

$$\text{Torsion}(I, \mu) := \int_S \text{Torsion}(I, x) d\mu(x). \quad (1.6)$$

**Remark 1.1.2.** This integral is well defined. Indeed

$$\text{Torsion}(I, x) = \lim_{n \rightarrow +\infty} \text{Torsion}_n(I, x, \xi)$$

and

$$\begin{aligned} \text{Torsion}_n(I, x, \xi) &= \frac{1}{n} \sum_{i=0}^{n-1} \text{Torsion}_1(I, f^i(x), Df^i(x)\xi) = \\ &= \frac{1}{n} \sum_{i=0}^{n-1} \text{Torsion}_1(I, \cdot, \cdot) \circ f_*^i(x, \xi) \end{aligned}$$

where we set

$$\begin{aligned} f_* : T^1 S &\rightarrow T^1 S \\ (x, \xi) &\mapsto \left( f(x), \frac{Df(x)\xi}{\|Df(x)\xi\|} \right). \end{aligned} \quad (1.7)$$

Lift  $\mu$  to  $\mu_*$ , a  $f_*$ -invariant Borel probability measure on  $T^1 S$  as follows. Denote as  $Leb$  the normalized Lebesgue measure on  $\mathbb{S}^1$ . Define for  $n \in \mathbb{N}$

$$\bar{\mu}_n = \frac{1}{n} \sum_{i=0}^{n-1} (f_*^i)^*(\mu \times Leb).$$

---

2. By asking that  $I = (f_t)_t$  has compact support, we demand that for any  $t \in [0, 1]$  the support of  $f_t$  is in a compact set, independent of  $t$ .

Each  $\bar{\mu}_n$  is a probability measure on  $T^1S$  whose projection on  $S$  is  $\mu$ . By the compact hypothesis on the support of  $\mu$  or of the isotopy, we can extract a subsequence  $(\bar{\mu}_{n_k})_{k \in \mathbb{N}}$  converging to  $\mu_*$ . Then  $\mu_*$  is a  $f_*$ -invariant Borel probability measure on  $T^1S$ . Notice that

$$\text{Torsion}_1(I, \cdot, \cdot) \in L^1(\mu_*)$$

thanks to the assumption on the support of  $\mu$  or of  $I = (f_t)_t$ . We deduce by Birkhoff's Ergodic Theorem that the function  $\text{Torsion}(I, \cdot)$  is defined  $\mu$ -a.e. and in  $L^1(\mu)$ .

The following propositions highlight some interesting properties of torsion concerning the choice of the continuous determination, of the tangent vector and of the isotopy.

**Proposition 1.1.1.** *For any  $(x, \xi) \in TS_*$  the quantities*

$$\text{Torsion}_n(I, x, \xi) \quad \forall n \in \mathbb{N}, n \neq 0$$

$$\text{Torsion}(I, x) = \lim_{n \rightarrow +\infty} \text{Torsion}_n(I, x, \xi) \quad \text{when it exists}$$

*do not depend on the choice of the continuous determination of the angle function  $v(I)(x, \xi, \cdot)$ . Let  $I' = (g_t)_t$  be another isotopy joining the identity to  $f$ . There exists an integer  $k \in \mathbb{Z}$  independent of  $x \in S$  and  $\xi \in T_x S \setminus \{0\}$  so that*

$$\text{Torsion}_n(I, x, \xi) = \text{Torsion}_n(I', x, \xi) + k \quad \forall n \in \mathbb{N}, n \neq 0$$

$$\text{Torsion}(I, x) = \text{Torsion}(I', x) + k.$$

The proof is an immediate consequence of the continuity of the involved functions and the property of  $f$  of being isotopic to the identity.

**Remark 1.1.3.** In many cases the torsion (already at finite time) does not depend on the chosen isotopy. If  $S = \mathbb{R}^2$  and  $f$  has compact support, up to consider compact-supported isotopies, the torsion does not depend on the chosen  $I = (f_t)_t$ , see Remark 1.3.2 or [BB13]. Actually, it is an outcome of the fact that the group of  $\mathcal{C}^1$  diffeomorphisms of  $\mathbb{R}^2$  homotopic to the identity with compact support is simply connected, see [Sma59], [Hir76] and [Kup19]. Also if  $S = \mathbb{T}^2$  or  $S = \mathbb{A}$ , then the torsion is independent from the chosen isotopy (see [BB13] or Proposition 1.3.2 and Remark 1.3.1). In the framework of  $\mathcal{C}^\infty$  diffeomorphisms, if  $S$  is a compact connected Riemannian parallelizable surface, eventually with boundary, which is neither the disk nor the bounded annulus nor the torus, then the torsion does not depend on the chosen isotopy. This is an outcome of Gramain's results in [Gra73]. With these hypothesis the group of  $\mathcal{C}^\infty$  diffeomorphisms that are homotopic to the identity is contractible. Therefore there is only one homotopy class of isotopies.

**Proposition 1.1.2.** *Fix  $x \in S$  and define the following functions*

$$\Pi : \mathbb{R} \rightarrow T_x S$$

$$s \mapsto \cos(2\pi s)X(x) + \sin(2\pi s)JX(x) \tag{1.8}$$

*and*

$$\begin{aligned} w(I, x) : \mathbb{R} \times \mathbb{R} &\rightarrow \mathbb{T} \\ (s, t) &\mapsto \theta(X(f_t(x)), Df_t(x)\Pi(s)). \end{aligned} \tag{1.9}$$

*Then, there exists a unique continuous determination of  $w(I, x)$ , denoted as  $W : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ , such that  $W(0, 0) = 0$ . Moreover*

- (i)  $W(\cdot, 0) = Id_{\mathbb{R}}(\cdot)$ .
- (ii) For any  $t \in \mathbb{R}$ ,  $W(\cdot, t)$  is an increasing homeomorphism of  $\mathbb{R}$ .
- (iii) For any  $s, t \in \mathbb{R}$ ,  $W(s + \frac{1}{2}, t) = W(s, t) + \frac{1}{2}$ .

*Proof.* By the continuity of the isotopy with respect to the compact-open  $\mathcal{C}^1$  topology, the function  $w(I, x)$  is continuous. There is a unique continuous determination  $W$  such that  $W(0, 0) = 0$ , since by fixing the value of  $W$  in a point we are selecting the lift.

$$\begin{array}{ccc} \mathbb{R} \times \mathbb{R} & \xrightarrow{W} & \mathbb{R} \\ & \searrow w \quad \swarrow p & \\ & \mathbb{T} & \end{array}$$

Notice that

$$\begin{aligned} W(\cdot, 0) : \mathbb{R} &\rightarrow \mathbb{R} \\ s &\mapsto W(s, 0) \end{aligned}$$

is a lift of  $\mathbb{R} \ni s \mapsto w(I, x)(s, 0) = \theta(X(x), \Pi(s)) = p(s)$ . Since  $W(0, 0) = 0$ ,  $W(\cdot, 0)$  is the identity of  $\mathbb{R}$ .

Let us introduce the following function

$$\bar{\Pi} : \mathbb{T} \rightarrow T_x S$$

$$\xi \mapsto \cos(2\pi\xi)X(x) + \sin(2\pi\xi)JX(x).$$

For any fixed  $t \in \mathbb{R}$ , the function  $W(\cdot, t) : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous lift of the angle function

$$\begin{aligned} m(\cdot, t) : \mathbb{T} &\rightarrow \mathbb{T} \\ \xi &\mapsto \theta(X(f_t(x)), Df_t(x)\bar{\Pi}(\xi)). \end{aligned}$$

As  $Df_t(x)$  is linear and preserves the orientation,  $m(\cdot, t)$  is an orientation preserving circle homeomorphism such that  $m(\xi + \frac{1}{2}, t) = m(\xi, t) + \frac{1}{2}$ . Hence, its lift  $W(\cdot, t)$  is an increasing homeomorphism of  $\mathbb{R}$ .

The functions  $(s, t) \mapsto W(s, t) + \frac{1}{2}$  and  $(s, t) \mapsto W(s + \frac{1}{2}, t)$  are two lifts of  $(s, t) \mapsto m(s, t) + \frac{1}{2}$  that coincide for  $(s, t) = (0, 0)$ , hence  $W(s + \frac{1}{2}, t) = W(s, t) + \frac{1}{2}$ , i.e.  $W(\cdot, t)$  commutes with the translation of  $\frac{1}{2}$  for any  $t \in \mathbb{R}$ . □

From Proposition 1.1.2 we deduce the following

**Lemma 1.1.1.** *Let  $f : S \rightarrow S$  be a  $\mathcal{C}^1$  diffeomorphism isotopic to the identity. Let  $x \in S$  and  $\xi_1, \xi_2 \in T_x S \setminus \{0\}$ . Let  $\tilde{v}(I)(x, \xi_1, \cdot), \tilde{v}(I)(x, \xi_2, \cdot)$  be continuous determinations of the angle functions  $v(I)(x, \xi_1, \cdot), v(I)(x, \xi_2, \cdot)$ , respectively.*

*If*

$$\tilde{v}(I)(x, \xi_1, 0) > \tilde{v}(I)(x, \xi_2, 0),$$

*then for any  $t \in \mathbb{R}$*

$$\tilde{v}(I)(x, \xi_1, t) > \tilde{v}(I)(x, \xi_2, t).$$

*Proof.* The definitions of  $\tilde{v}(I)(x, \xi_1, \cdot)$  and  $\tilde{v}(I)(x, \xi_2, \cdot)$  only depend on the vector directions. Consequently, we consider  $T_x^1 S$ , the set of unitary tangent vectors at  $x$ , and assume  $\xi_1, \xi_2$  are vectors of unitary norms.

Consider the function  $\Pi$  defined in (1.8) and the oriented angle function  $w(I, x)$  defined in (1.9) in Proposition 1.1.2. Observe that for any  $s, t \in \mathbb{R}$  we have  $w(I, x)(s, t) = v(I)(x, \Pi(s), t)$ . Let  $W : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be the continuous determination of  $w(I, x)$  given by Proposition 1.1.2.

Let  $s_1, s_2 \in \mathbb{R}$  be such that  $\Pi(s_1) = \xi_1, \Pi(s_2) = \xi_2$  and

$$\tilde{v}(I)(x, \xi_1, 0) = s_1 \quad \text{and} \quad \tilde{v}(I)(x, \xi_2, 0) = s_2.$$

The continuous functions  $t \mapsto \tilde{v}(I)(x, \xi_1, t)$  and  $t \mapsto W(s_1, t)$  are equal because they are lifts of the same angle function and coincide at  $t = 0$  from point (i) of Proposition 1.1.2. Similarly, the continuous functions  $t \mapsto \tilde{v}(I)(x, \xi_2, t)$  and  $t \mapsto W(s_2, t)$  coincide. From point (ii) of Proposition 1.1.2, since by hypothesis

$$\tilde{v}(I)(x, \xi_1, 0) = s_1 > s_2 = \tilde{v}(I)(x, \xi_2, 0)$$

and since

$$W(s_1, 0) = \tilde{v}(I)(x, \xi_1, 0) \quad \text{and} \quad W(s_2, 0) = \tilde{v}(I)(x, \xi_2, 0),$$

we conclude that for any  $t \in \mathbb{R}$  it holds

$$\tilde{v}(I)(x, \xi_1, t) = W(s_1, t) > W(s_2, t) = \tilde{v}(I)(x, \xi_2, t).$$

□

**Proposition 1.1.3.** *Let  $x \in S$ . Assume that for some  $\xi \in T_x S \setminus \{0\}$  the quantity  $\text{Torsion}_n(I, x, \xi)$  converges as  $n \rightarrow +\infty$ . Then, the torsion of the orbit of  $x$  does not depend on the choice of the tangent vector. In other words, for any vector  $\delta \in T_x S \setminus \{0\}$  it holds*

$$\text{Torsion}(I, x) = \lim_{n \rightarrow +\infty} \text{Torsion}_n(I, x, \xi) = \lim_{n \rightarrow +\infty} \text{Torsion}_n(I, x, \delta).$$

*Proof.* Consider  $\xi, \delta \in T_x S \setminus \{0\}$  and assume that  $\lim_{n \rightarrow +\infty} \text{Torsion}_n(I, x, \xi)$  exists. Then, we are going to prove that also  $\lim_{n \rightarrow +\infty} \text{Torsion}_n(I, x, \delta)$  exists and it coincides with the previous one.

The result easily follows once we prove that

$$\lim_{n \rightarrow +\infty} |\text{Torsion}_n(I, x, \xi) - \text{Torsion}_n(I, x, \delta)| = 0.$$

**Lemma 1.1.2.** *Fix  $x \in S$ . For  $n \in \mathbb{N}, n \neq 0$  and for  $\xi, \delta \in T_x S \setminus \{0\}$  it holds*

$$|\text{Torsion}_n(I, x, \xi) - \text{Torsion}_n(I, x, \delta)| < \frac{1}{2n}. \quad (1.10)$$

*Proof.* The quantity

$$|\text{Torsion}_n(I, x, \xi) - \text{Torsion}_n(I, x, \delta)|$$

can be written as

$$\frac{1}{n} |(\tilde{v}(I)(x, \xi, n) - \tilde{v}(I)(x, \delta, n)) - (\tilde{v}(I)(x, \xi, 0) - \tilde{v}(I)(x, \delta, 0))|.$$

These quantities do not depend on the chosen determination of the angle function  $v$ . Concerning the relative position of the vectors  $\xi, \delta$ , four cases can occur:

$$\tilde{v}(I)(x, \xi, 0) - \tilde{v}(I)(x, \delta, 0) \begin{cases} = k & \text{if } \xi, \delta \text{ are positively colinear} \\ = \frac{1}{2} + k & \text{if } \xi, \delta \text{ are negatively colinear} \\ \in (0, \frac{1}{2}) + k & \text{if } (\xi, \delta) \text{ is a direct basis} \\ \in (\frac{1}{2}, 1) + k & \text{if } (\xi, \delta) \text{ is an indirect basis.} \end{cases}$$

At any time, the same four cases can occur and

$$\tilde{v}(I)(x, \xi, t) - \tilde{v}(I)(x, \delta, t) \begin{cases} = k & \text{if } \xi, \delta \text{ are positively colinear} \\ = \frac{1}{2} + k & \text{if } \xi, \delta \text{ are negatively colinear} \\ \in (0, \frac{1}{2}) + k & \text{if } (\xi, \delta) \text{ is a direct basis} \\ \in (\frac{1}{2}, 1) + k & \text{if } (\xi, \delta) \text{ is an indirect basis,} \end{cases}$$

where the integer  $k \in \mathbb{Z}$  is the same for any  $t$ .

This holds in particular for  $t = n$  and, checking all the possible cases, we obtain

$$\frac{1}{n} |(\tilde{v}(I)(x, \xi, n) - \tilde{v}(I)(x, \delta, n)) - (\tilde{v}(I)(x, \xi, 0) - \tilde{v}(I)(x, \delta, 0))| < \frac{1}{2n}.$$

□

From Lemma 1.1.2 we conclude since

$$0 \leq \lim_{n \rightarrow +\infty} |\text{Torsion}_n(I, x, \xi) - \text{Torsion}_n(I, x, \delta)| \leq \lim_{n \rightarrow +\infty} \frac{1}{2n} = 0.$$

□

We discuss now the independence of the torsion from the choice of the Riemannian metric.

**Notation 1.1.2.** Fix an orientation of  $S$  and a reference continuous vector field  $X : S \rightarrow TS$  which never vanishes.

Let  $g$  be a Riemannian metric on  $S$  and on every tangent space denote as

$$J_g(x) : T_x S \rightarrow T_x S$$

the rotation of angle  $\frac{1}{4}$  with respect to the given Riemannian metric.

For any  $x \in S$ , denote as  $\text{Torsion}(g)(I, x)$  ( $\text{Torsion}_n(g)(I, x, \xi)$ ) the torsion at  $x$  for  $I = (f_t)_t$  (the  $n$  finite-time torsion at  $(x, \xi)$  for  $I = (f_t)_t$ ) with respect to the metric  $g$ .

**Proposition 1.1.4.** *Let  $g_1, g_2$  be two Riemannian metrics on  $S$ . Let  $x \in S$  and assume that  $\text{Torsion}(g_1)(I, x)$  exists. Then*

$$\text{Torsion}(g_1)(I, x) = \text{Torsion}(g_2)(I, x).$$

The proof of Proposition 1.1.4 follows immediately from the following

**Lemma 1.1.3.** *Let  $g_1, g_2$  be two Riemannian metrics on  $S$ . For any  $(x, \xi) \in TS, \xi \neq 0$  we have for any  $n \in \mathbb{N}^*$*

$$|\text{Torsion}_n(g_1)(I, x, \xi) - \text{Torsion}_n(g_2)(I, x, \xi)| < \frac{1}{n}.$$



*Proof of Lemma 1.1.3.* Fix  $(x, \xi) \in TS, \xi \neq 0$ . For any  $t \in \mathbb{R}_+$  consider the basis of vectors

$$(X(f_t(x)), J_{g_1}X(f_t(x))) \quad \text{and} \quad (X(f_t(x)), J_{g_2}X(f_t(x))).$$

Denote  $J_{g_2}X(f_t(x)) = \alpha(t)X(f_t(x)) + \beta(t)J_{g_1}X(f_t(x))$ . For any  $t \in \mathbb{R}_+$  the value  $\beta(t)$  is positive because the two basis determine the same orientation. Consider now

$$Df_t(x)\xi = u_1(t)X(f_t(x)) + u_2(t)J_{g_1}X(f_t(x)) = v_1(t)X(f_t(x)) + v_2(t)J_{g_2}X(f_t(x)),$$

that is express the vector  $Df_t(x)\xi$  in the two highlighted basis. In particular

$$\begin{aligned} u_1(t) &= v_1(t) + v_2(t)\alpha(t), \\ u_2(t) &= v_2(t)\beta(t). \end{aligned} \tag{1.11}$$

The oriented angle between  $X(f_t(x))$  and  $Df_t(x)\xi$  with respect to the metric  $g_1$  is

$$\theta(t) = \arg(u_1(t) + iu_2(t)),$$

while the oriented angle between  $X(f_t(x))$  and  $Df_t(x)\xi$  with respect to the metric  $g_2$  is

$$\Theta(t) = \arg(v_1(t) + iv_2(t)).$$

Let  $\mathbb{R}_+ \ni t \mapsto \tilde{\theta}(t) \in \mathbb{R}$  be the lift of the oriented angle function  $\mathbb{R}_+ \ni t \mapsto \theta(t) \in \mathbb{T}$  such that  $\tilde{\theta}(0) \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ . Let  $\mathbb{R}_+ \ni t \mapsto \tilde{\Theta}(t) \in \mathbb{R}$  be the lift of the oriented angle function  $\mathbb{R}_+ \ni t \mapsto \Theta(t) \in \mathbb{T}$  such that  $\tilde{\Theta}(0) \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ .

Observe that for any  $n \in \mathbb{N}^*$  it holds

$$n \text{ Torsion}_n(g_1)(I, x, \xi) = \tilde{\theta}(n) - \tilde{\theta}(0)$$

and

$$n \text{ Torsion}_n(g_2)(I, x, \xi) = \tilde{\Theta}(n) - \tilde{\Theta}(0).$$

Let us discuss the possible cases.

- (i) Assume that  $\tilde{\theta}(0) = 0$ . That is,  $\xi = X(x)$ . Consequently, we also have  $\tilde{\Theta}(0) = 0$ . From a similar argument we deduce that  $\tilde{\theta}(0) = \frac{1}{2}$  if and only if  $\tilde{\Theta}(0) = \frac{1}{2}$ , because  $\xi = -X(f_t(x))$ .
- (ii) Assume that  $\tilde{\theta}(0) \in (0, \frac{1}{2})$ , that is  $(X(x), \xi)$  determines a positive orientation. Then, since the two basis determine the same orientation, also  $\tilde{\Theta}(0) \in (0, \frac{1}{2})$ . From a similar argument we deduce that  $\tilde{\theta}(0) \in (-\frac{1}{2}, 0)$  if and only if  $\tilde{\Theta}(0) \in (-\frac{1}{2}, 0)$ .

In particular, it holds

$$\left| \tilde{\theta}(0) - \tilde{\Theta}(0) \right| < \frac{1}{2}. \tag{1.12}$$

**Claim 1.1.1.** For the choice of the lifts such that (1.12) holds, for any  $t \in \mathbb{R}_+$  we have

$$\left| \tilde{\theta}(t) - \tilde{\Theta}(t) \right| < \frac{1}{2}.$$

*Proof.* Argue by contradiction and assume there exists  $t \in \mathbb{R}_+$  such that  $|\tilde{\theta}(t) - \tilde{\Theta}(t)| = \frac{1}{2}$ . Without loss of generality suppose that  $\tilde{\Theta}(t) = \tilde{\theta}(t) + \frac{1}{2}$ . The case  $\tilde{\Theta}(t) = \tilde{\theta}(t) - \frac{1}{2}$  can be treated similarly. Recall that

$$Df_t(x)\xi = u_1(t)X(f_t(x)) + u_2(t)J_{g_1}X(f_t(x)) = v_1(t)X(f_t(x)) + v_2(t)J_{g_2}X(f_t(x))$$

and

$$u_1(t) + iu_2(t) = r(t)e^{2\pi i\tilde{\theta}(t)} \quad \text{and} \quad v_1(t) + iv_2(t) = R(t)e^{2\pi i\tilde{\Theta}(t)},$$

where  $r(t), R(t) > 0$ . From the absurd hypothesis and from (1.11) we have

$$\begin{aligned} v_1(t) + iv_2(t) &= R(t)e^{2\pi i\tilde{\Theta}(t)} = -R(t)e^{2\pi i\tilde{\theta}(t)} = -\frac{R(t)}{r(t)}r(t)e^{2\pi i\tilde{\theta}(t)} = -\frac{R(t)}{r(t)}(u_1(t) + iu_2(t)) = \\ &= -\frac{R(t)}{r(t)}(v_1(t) + v_2(t)\alpha(t) + iv_2(t)\beta(t)). \end{aligned}$$

Equivalently

$$v_2(t) = -\frac{R(t)}{r(t)}v_2(t)\beta(t) \quad \text{and} \quad v_1(t) = -\frac{R(t)}{r(t)}(v_1(t) + v_2(t)\alpha(t)).$$

Since  $r(t), R(t)$  and  $\beta(t)$  are all positive, we deduce that  $v_2(t) = 0$  and we obtain  $v_1(t) = -\frac{R(t)}{r(t)}v_1(t)$ , which is the required contradiction.  $\square$

We then conclude because

$$\begin{aligned} |n \text{Torsion}_n(g_1)(I, x, \xi) - n \text{Torsion}_n(g_2)(I, x, \xi)| &\leq \\ &\leq |\tilde{\theta}(n) - \tilde{\Theta}(n)| + |\tilde{\theta}(0) - \tilde{\Theta}(0)| < 1. \end{aligned}$$

$\square$

Let us discuss now the dependance from the choice of the trivialization. Recall that a trivialization on a parallelizable surface  $S$  is a diffeomorphism  $\phi : TS \rightarrow S \times \mathbb{R}^2$  which allows us to fix a coordinate system on the tangent bundle.

**Definition 1.1.5.** Let  $\phi_1, \phi_2$  be two trivializations. The two trivializations  $\phi_1, \phi_2$  are homotopic if there exists an homotopy  $(H_t)_{t \in [0,1]}$  such that  $H_0 = \phi_1, H_1 = \phi_2$  and  $H_t$  is a trivialization for any  $t \in [0, 1]$ .

**Fact 1.1.1.** Let  $\phi_1, \phi_2$  be two homotopic trivializations. For  $i = 1, 2$  denote as  $X_i$  the vector field such that  $X_i(x) = \phi_i^{-1}(x; 1, 0)$  for any  $x \in S$ . Then for any loop  $\gamma : [0, 1] \rightarrow S$ ,  $\gamma(0) = \gamma(1)$  it holds that

$$\tilde{\theta}(X_1(\gamma(1)), X_2(\gamma(1))) - \tilde{\theta}(X_1(\gamma(0)), X_2(\gamma(0))) = 0,$$

where  $\tilde{\theta}$  is a continuous determination of the oriented angle function

$$[0, 1] \ni t \mapsto \theta(X_1(\gamma(t)), X_2(\gamma(t))) \in \mathbb{T}.$$

**Proposition 1.1.5.** *Let  $f : S \rightarrow S$  be a  $C^1$  diffeomorphism isotopic to the identity. Let  $I = (f_t)_t$  be an isotopy joining the identity to  $f$ . Assume that  $I = (f_t)_t$  has compact support. Denote as  $\text{Torsion}(\phi_1)(I, \cdot)$ ,  $\text{Torsion}(\phi_2)(I, \cdot)$  the torsion of  $I = (f_t)_t$  with respect to the trivializations  $\phi_1, \phi_2$  respectively. Let  $x \in S$  and assume that  $\text{Torsion}(\phi_1)(I, x)$  exists. If the trivializations  $\phi_1, \phi_2$  are homotopic, then*

$$\text{Torsion}(\phi_1)(I, x) = \text{Torsion}(\phi_2)(I, x).$$

*Proof.* Fix  $0 < \varepsilon < \frac{1}{2}$ . For  $x \in \text{Supp}((f_t)_t) = \text{Supp}(I)$  consider a neighborhood  $U_x$  of  $x$  such that for any  $y \in U_x$  it holds that

$$\theta(X_1(x), X_2(x)) - \theta(X_1(y), X_2(y)) \in \mathbb{T}$$

admits a measure in  $(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ .

Since  $\text{Supp}(I)$  is compact by hypothesis, we can extract a finite open covering of such neighborhoods

$$\text{Supp}(I) \subset \bigcup_{i=1}^N U_{x_i}.$$

Consider then  $y \in \text{Supp}(I)$ . Let  $v \in T_y S, v \neq 0$ . Recall that the torsion does not depend on the choice of the tangent vector (see Proposition 1.1.3). Look now at the oriented angle function

$$\mathbb{R}_+ \ni t \mapsto \theta(X_1(f_t(y)), Df_t(y)v) - \theta(X_2(f_t(y)), Df_t(y)v) = \theta(X_1(f_t(y)), X_2(f_t(y))) \in \mathbb{T}.$$

Let  $t \mapsto \tilde{\theta}(X_1(f_t(y)), X_2(f_t(y)))$  be a continuous determination of such angle function. Fix  $n \in \mathbb{N}$ . Then

$$\begin{aligned} |n \text{Torsion}_n(X_1)(I, y, v) - n \text{Torsion}_n(X_2)(I, y, v)| &= \\ &= \left| \tilde{\theta}(X_1(f^n(y)), X_2(f^n(y))) - \tilde{\theta}(X_1(y), X_2(y)) \right|. \end{aligned} \quad (1.13)$$

Since the cover  $\bigcup_{i=1}^N U_{x_i}$  is finite, there exist  $\bar{i} \in \llbracket 1, N \rrbracket$  such that  $f^{j_1}(y), f^{n-j_2}(y) \in U_{x_{\bar{i}}}$  for some  $j_1, j_2 \in \llbracket 1, N \rrbracket$ .

Consider then the loop  $l$  obtained by concatenating  $(f_t(y))_{t \in [j_1, n-j_2]}$  and a path  $\gamma$  from  $\gamma(0) = f^{n-j_2}(y)$  to  $\gamma(1) = f^{j_1}(y)$  contained in  $U_{x_{\bar{i}}}$ .

Since the trivializations determined by  $X_1, X_2$  are homotopic, the variation of the angle between  $X_1$  and  $X_2$  along the loop  $l$  is null (see Fact 1.1.1). Such angle variation is

$$\begin{aligned} &\left( \tilde{\theta}(X_1(f^{n-j_2}(y)), X_2(f^{n-j_2}(y))) - \tilde{\theta}(X_1(f^{j_1}(y)), X_2(f^{j_1}(y))) \right) + \\ &+ \left( \tilde{\theta}(X_1(\gamma(1)), X_2(\gamma(1))) - \tilde{\theta}(X_1(\gamma(0)), X_2(\gamma(0))) \right). \end{aligned}$$

Since the path  $\gamma$  is contained in  $U_{x_{\bar{i}}}$  and since for any  $x \in U_{x_{\bar{i}}}$  we have that  $\theta(X_1(x_{\bar{i}}), X_2(x_{\bar{i}})) - \theta(X_1(x), X_2(x))$  admits a measure in  $(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ , we deduce that

$$\begin{aligned} &\left| \tilde{\theta}(X_1(\gamma(1)), X_2(\gamma(1))) - \tilde{\theta}(X_1(\gamma(0)), X_2(\gamma(0))) \right| = \\ &= \left| \tilde{\theta}(X_1(\gamma(1)), X_2(\gamma(1))) - \tilde{\theta}(X_1(x_{\bar{i}}), X_2(x_{\bar{i}})) + \tilde{\theta}(X_1(x_{\bar{i}}), X_2(x_{\bar{i}})) - \tilde{\theta}(X_1(\gamma(0)), X_2(\gamma(0))) \right| < \\ &< \varepsilon. \end{aligned}$$

Consequently

$$\begin{aligned}
& \left| ((n - j_2) \text{Torsion}_{n-j_2}(X_1)(I, y, v) - j_1 \text{Torsion}_{j_1}(X_1)(I, y, v)) - \right. \\
& \quad \left. - ((n - j_2) \text{Torsion}_{n-j_2}(X_2)(I, y, v) - j_1 \text{Torsion}_{j_1}(X_2)(I, y, v)) \right| = \\
& = \left| \tilde{\theta}(X_1(f^{n-j_2}(y)), X_2(f^{n-j_2}(y))) - \tilde{\theta}(X_1(f^{j_1}(y)), X_2(f^{j_1}(y))) \right| < \varepsilon. \tag{1.14}
\end{aligned}$$

Thus, from (1.13), we have that

$$\begin{aligned}
& |n \text{Torsion}_n(X_1)(I, y, v) - n \text{Torsion}_n(X_2)(I, y, v)| \leq \\
& \leq \left| \tilde{\theta}(X_1(f^{n-j_2}(y)), X_2(f^{n-j_2}(y))) - \tilde{\theta}(X_1(f^{j_1}(y)), X_2(f^{j_1}(y))) \right| + \\
& \quad + |j_1 \text{Torsion}_{j_1}(X_1)(I, y, v) - j_1 \text{Torsion}_{j_1}(X_2)(I, y, v)| + \\
& \quad + |j_2 \text{Torsion}_{j_2}(X_1)(I, f^{n-j_2}(y), Df^{n-j_2}(y)v) - j_2 \text{Torsion}_{j_2}(X_2)(I, f^{n-j_2}(y), Df^{n-j_2}(y)v)|.
\end{aligned}$$

Since  $I = (f_t)_t$  has compact support by hypothesis and since  $j_1, j_2 \leq N$ , we have that there exists a constant  $C > 0$  such that

$$|j_1 \text{Torsion}_{j_1}(X_1)(I, y, v) - j_1 \text{Torsion}_{j_1}(X_2)(I, y, v)| \leq C$$

and

$$|j_2 \text{Torsion}_{j_2}(X_1)(I, f^{n-j_2}(y), Df^{n-j_2}(y)v) - j_2 \text{Torsion}_{j_2}(X_2)(I, f^{n-j_2}(y), Df^{n-j_2}(y)v)| \leq C.$$

Consequently, from (1.14) we deduce that

$$|\text{Torsion}_n(X_1)(I, y, v) - \text{Torsion}_n(X_2)(I, y, v)| < \frac{\varepsilon}{n} + \frac{2C}{n}.$$

Passing to the limit for  $n \rightarrow +\infty$  we deduce that the limit  $\lim_{n \rightarrow +\infty} \text{Torsion}_n(X_1)(I, y, v)$  exists if and only if the limit  $\lim_{n \rightarrow +\infty} \text{Torsion}_n(X_2)(I, y, v)$  exists and in particular

$$\text{Torsion}(X_1)(I, y) = \text{Torsion}(X_2)(I, y).$$

The argument can be repeated for any  $y \in \text{Supp}(I)$  and we conclude. □

Proposition 1.1.5 enables us to give some conditions to assure invariance of the torsion for  $\mathcal{C}^1$  conjugacy.

Let us fix a Riemannian metric, an orientation and a never vanishing vector field  $X$ .

**Proposition 1.1.6.** *Let  $f : S \rightarrow S$  be a  $\mathcal{C}^1$  diffeomorphism isotopic to the identity. Let  $I = (f_t)_t$  be an isotopy joining the identity to  $f$  with compact support. Let  $h : S \rightarrow S$  be a  $\mathcal{C}^1$  diffeomorphism with compact support such that the trivializations  $\phi_1, \phi_2$  so that for any  $x \in S$*

$$\phi_1^{-1}(x; 1, 0) = (x, Dh(x)X(x)) \quad \text{and} \quad \phi_2^{-1}(x; 1, 0) = (x, X(h(x)))$$

*are homotopic. Denote as  $H = (h \circ f_t \circ h^{-1})_t$  the isotopy joining the identity to  $h \circ f \circ h^{-1}$ . Let  $x \in S$  and assume that  $\text{Torsion}(I, x)$  exists. Then:*

(i)  $Torsion(I, x) = Torsion(H, h(x))$ , if  $h$  is orientation preserving;

(ii)  $Torsion(I, x) = -Torsion(H, h(x))$  if  $h$  is orientation reversing.

*Proof.* Let  $x \in Supp(I)$  and let  $v \in T_x S, v \neq 0$ . Denote as  $\mathbb{R}_+ \ni t \mapsto \tilde{\theta}(t) \in \mathbb{R}$  a continuous determination of the oriented angle function

$$\mathbb{R}_+ \ni t \mapsto \theta(X(f_t(x)), Df_t(x)v) \in \mathbb{T}. \quad (1.15)$$

(i) If  $h$  is orientation preserving there exists a continuous determination  $\mathbb{R}_+ \ni t \mapsto \tilde{\Theta}(t) \in \mathbb{R}$  of the oriented angle function

$$\mathbb{R}_+ \ni t \mapsto \theta(Dh(f_t(x))X(f_t(x)), Dh(f_t(x))Df_t(x)v) \in \mathbb{T} \quad (1.16)$$

such that for any  $t \in \mathbb{R}_+$  it holds

$$|\tilde{\theta}(t) - \tilde{\Theta}(t)| < \frac{1}{2}.$$

We refer to Proposition 1.4.1 and Appendix 1.6 for a detailed proof of the last statement.

Consequently, for any  $n \in \mathbb{N}$  it holds

$$\left| n \text{Torsion}_n(I, x, v) - (\tilde{\Theta}(n) - \tilde{\Theta}(0)) \right| = \left| (\tilde{\theta}(n) - \tilde{\theta}(0)) - (\tilde{\Theta}(n) - \tilde{\Theta}(0)) \right| < 1.$$

Equivalently,  $\text{Torsion}(I, x)$  exists if and only if the limit  $\lim_{n \rightarrow +\infty} \frac{\tilde{\Theta}(n) - \tilde{\Theta}(0)}{n}$  exists and in particular

$$\text{Torsion}(I, x) = \lim_{n \rightarrow +\infty} \frac{\tilde{\Theta}(n) - \tilde{\Theta}(0)}{n}. \quad (1.17)$$

(ii) If  $h$  is orientation reversing there exists a continuous determination  $\mathbb{R}_+ \ni t \mapsto \tilde{\Theta}(t) \in \mathbb{R}$  of the oriented angle function

$$\mathbb{R}_+ \ni t \mapsto \theta(Dh(f_t(x))X(f_t(x)), Dh(f_t(x))Df_t(x)v) \in \mathbb{T}$$

such that for any  $t \in \mathbb{R}_+$  it holds

$$|\tilde{\theta}(t) + \tilde{\Theta}(t)| < \frac{1}{2}.$$

For the proof of this last inequality we refer to Proposition 1.6.1 in Appendix 1.6. Consequently, for any  $n \in \mathbb{N}$  it holds

$$\left| n \text{Torsion}_n(I, x, v) + (\tilde{\Theta}(n) - \tilde{\Theta}(0)) \right| = \left| (\tilde{\theta}(n) - \tilde{\theta}(0)) + (\tilde{\Theta}(n) - \tilde{\Theta}(0)) \right| < 1.$$

Thus,  $\text{Torsion}(I, x)$  exists if and only if the limit  $\lim_{n \rightarrow +\infty} \frac{\tilde{\Theta}(n) - \tilde{\Theta}(0)}{n}$  exists and

$$\text{Torsion}(I, x) = - \lim_{n \rightarrow +\infty} \frac{\tilde{\Theta}(n) - \tilde{\Theta}(0)}{n}. \quad (1.18)$$

We can so calculate the variation of the angle function

$$t \mapsto \theta(Dh(f_t(x))X(f_t(x)), Dh(f_t(x))Df_t(x)v). \quad (1.19)$$

Denote as  $y = h(x)$ ,  $\xi = Dh(x)v$ . Then the angle function in (1.19) is

$$t \mapsto \theta(Dh(f_t \circ h^{-1}(y))X(f_t \circ h^{-1}(y)), D(h \circ f_t \circ h^{-1})(y)\xi).$$

By hypothesis the trivializations determined by  $DhX$  and  $X \circ h$  are homotopic and  $f, h$  have compact support. Arguing as in Proposition 1.1.5 we can show that, whenever the limit exists,

$$\lim_{n \rightarrow +\infty} \frac{\tilde{\Theta}(n) - \tilde{\Theta}(0)}{n} = \lim_{n \rightarrow +\infty} \text{Torsion}_n(H, y, \xi), \quad (1.20)$$

where  $H = (h \circ f_t \circ h^{-1})_t$ . Indeed, let us fix  $n \in \mathbb{N}$ . It holds

$$\begin{aligned} & \left| \left( \tilde{\Theta}(n) - \tilde{\Theta}(0) \right) - n \text{Torsion}_n(H, y, \xi) \right| = \\ & = \left| \tilde{\psi}(Dh(f^n \circ h^{-1}(y))X(f^n \circ h^{-1}(y)), X(h \circ f^n \circ h^{-1}(y))) - \tilde{\psi}(Dh(h^{-1}(y))X(h^{-1}(y)), X(y)) \right|, \end{aligned}$$

where  $\tilde{\psi}$  denotes a continuous determination of the oriented angle function

$$t \mapsto \theta(Dh(f_t \circ h^{-1}(y))X(f_t \circ h^{-1}(y)), X(h \circ f_t \circ h^{-1}(y))).$$

Since  $\text{Supp}(I) \cup \text{Supp}(h)$  is compact, we can proceed as in the proof of Proposition 1.1.5. Follow the path  $f_t \circ h^{-1}(y)$  for  $t \in [0, n]$  and then close it up in a suitable way. By the fact that the involved trivializations are homotopic (see Definition 1.1.5 and Fact 1.1.1), passing to the limit for  $n \rightarrow +\infty$  we derive then equality (1.20).

Finally, from (1.17), (1.18) and (1.20), we conclude that

$$\text{Torsion}(I, x, v) = \begin{cases} \text{Torsion}(H, h(x)) & \text{if } h \text{ is orientation preserving,} \\ -\text{Torsion}(H, h(x)) & \text{if } h \text{ is orientation reversing.} \end{cases}$$

□

**Remark 1.1.4.** The following remark is due to P. Le Calvez. In the sequel we largely refer to [Sch57].

Let  $S$  be a connected parallelizable Riemannian surface. Any continuous function  $\phi : TS_* \rightarrow \mathbb{T}$  determines a cohomology class  $\alpha \in H^1(TS_*, \mathbb{Z})$  as follows. Recall that (see for example [God71] or [Hat02]) the first cohomology group  $H^1(TS_*, \mathbb{Z})$  is isomorphic to  $\text{Hom}(H_1(TS_*), \mathbb{Z})$  which is isomorphic to  $\text{Hom}(\pi_1(TS_*), \mathbb{Z})$ .

Think so at  $\alpha$  as the following homomorphism on the first fundamental group of  $TS_*$ . Let  $\gamma$  be a loop in  $TS_*$  and let  $[\gamma] \in \pi_1(TS_*)$  be its homotopic class. Let  $\mathcal{F} : [0, 1] \rightarrow \mathbb{R}$  be a continuous lift of

$$[0, 1] \ni t \mapsto \phi \circ \gamma(t) \in \mathbb{T}.$$

Then  $\alpha([\gamma]) = \mathcal{F}(1) - \mathcal{F}(0) \in \mathbb{Z}$ . Observe that it does not depend on the choice of the element  $\gamma$  in the homotopic class  $[\gamma]$ .

Two continuous functions  $\phi, \phi'$  of  $TS_*$  in  $\mathbb{T}$  are cohomologous if  $\phi - \phi' : TS_* \rightarrow \mathbb{T}$  admits

a real-valued continuous lift. If two functions  $\phi, \phi'$  are homotopic then they are also cohomologous (see [God71]).

Let us come back to the framework of the torsion. Given a Riemannian metric  $g$  on  $S$  and a non singular vector field  $X$  on  $S$ , we define a continuous function

$$\begin{aligned}\phi_{g,X} : TS_* &\rightarrow \mathbb{T} \\ (x, \xi) &\mapsto \theta_g(X(x), \xi),\end{aligned}$$

where  $\theta_g(u, v)$  is the oriented angle (with respect to the metric  $g$ ) between the non-zero vectors  $u$  and  $v$ . Remark that if  $g, g'$  are Riemannian metric on  $S$  and if  $X, X'$  are homotopic non singular vector fields on  $S$ , then  $\phi_{g,X}$  and  $\phi_{g',X'}$  are homotopic and, consequently, cohomologous.

Let  $f : S \rightarrow S$  be a  $\mathcal{C}^1$  diffeomorphism isotopic to the identity and let  $I = (f_t)_{t \in \mathbb{R}_+}$  be an isotopy joining  $\text{Id}_S$  to  $f$ . Assume that  $I$  has compact support and denote  $\mathcal{I}$  the support of the isotopy.

Let  $\phi_{g,X}, \phi_{g',X'}$  be cohomologous. Denote the torsion calculated with respect to  $\phi_{g,X}$  as  $\text{Torsion}(g, X)$  and with respect to  $\phi_{g',X'}$  as  $\text{Torsion}(g', X')$ . Then, whenever the torsion exists,  $\text{Torsion}(g, X)(I, x)$  and  $\text{Torsion}(g', X')(I, x)$  will be the same. Indeed, the continuous function  $\phi_{g,X} - \phi_{g',X'}$  has a real-valued continuous lift  $\mathcal{F} : T\mathcal{I}_* \rightarrow \mathbb{R}$ . In particular,  $\mathcal{F}$  is bounded. Consequently

$$|\text{Torsion}(g, X)(I, x) - \text{Torsion}(g', X')(I, x)| = \lim_{n \rightarrow +\infty} \frac{1}{n} \left| \mathcal{F}(f^n(x), Df^n(x)\xi) - \mathcal{F}(x, \xi) \right| = 0.$$

For a deeper discussion, we recall that we have shown that the torsion does not depend on the Riemannian metric  $g$  on  $S$  (see Proposition 1.1.4). Proposition 1.1.5 shows that, for compact-supported isotopies, the torsion depends only on the homotopy class of the non singular vector field  $X$ .

If we drop the compactness assumption, then the torsion does not depend only on the cohomology class of the function  $\phi_{g,X}$ , as shown by the next example.

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be  $(x, y) \mapsto f(x, y) = (x, y + 1)$  and consider the isotopy  $I = (f_t)_{t \in \mathbb{R}_+}$  such that for any  $t$  it holds  $f_t(x, y) = (x, y + t)$ . Fix the standard Riemannian metric and the standard orientation. Let  $X_1(x, y) = (1, 0)$  and  $X_2(x, y) = (\cos(2\pi y), \sin(2\pi y))$ . Then

$$\phi_{X_1} : T^1\mathbb{R}^2 \rightarrow \mathbb{T}, \quad ((x, y), \xi) \mapsto \theta((1, 0), \xi)$$

and

$$\phi_{X_2} : T^1\mathbb{R}^2 \rightarrow \mathbb{T}, \quad ((x, y), \xi) \mapsto \theta((\cos(2\pi y), \sin(2\pi y)), \xi)$$

are cohomologous, but for any  $(x, y) \in \mathbb{R}^2$  the torsion calculated with respect to  $X_1$  at  $(x, y)$  is null, while the torsion with respect to  $X_2$  at  $(x, y)$  is  $-1$ .

In the sequel we will be interested also in non compact-supported isotopies.

Finally, we remark that on  $\mathbb{T}^2$  the torsion depends only on the cohomology class associated to  $\phi_{g,X}$ . On  $\mathbb{T}^2$  the choice of the standard trivialization corresponds to the choice of the cohomology class  $(0, d\xi)$ , where we denote as  $((x, y), \xi)$  a point in  $T^1\mathbb{T}^2$ , where we are identifying  $T^1\mathbb{T}^2$  with  $\mathbb{T}^2 \times \mathbb{S}^1$ .

## 1.2 Notion of linking number

In the setting of  $\mathbb{R}^2$  we refer to [BB13] to introduce the notion of *linking number*.

**Notation 1.2.1.** The counterclockwise orientation of  $\mathbb{R}^2$  is considered. Moreover, we fix the constant vector field  $X = (1, 0)$  and we denote it as  $\mathcal{H}$ .

**Definition 1.2.1.** Let  $I = (F_t)_t$  be an isotopy in  $\text{Diff}^1(\mathbb{R}^2)$  joining the identity to  $F_1 = F$ . Let us denote  $\Delta := \{(z_1, z_2) \in \mathbb{R}^4 : z_1 = z_2\}$  and define the function

$$\begin{aligned} u(I) : (\mathbb{R}^4 \setminus \Delta) \times \mathbb{R} &\rightarrow \mathbb{T} \\ (z_1, z_2, t) &\mapsto \theta(\mathcal{H}, F_t(z_2) - F_t(z_1)). \end{aligned} \quad (1.21)$$

Fix  $(z_1, z_2) \in \mathbb{R}^4 \setminus \Delta$  and consider  $\tilde{u}(I)(z_1, z_2, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ , a continuous determination of the angle function  $u(I)(z_1, z_2, \cdot)$ .

For any  $n \in \mathbb{N}, n \neq 0$ , the *linking number of  $z_1$  and  $z_2$  at finite time  $n$*  is

$$\text{Linking}_n(I, z_1, z_2) := \frac{1}{n} (\tilde{u}(I)(z_1, z_2, n) - \tilde{u}(I)(z_1, z_2, 0)). \quad (1.22)$$

The *linking number of  $z_1$  and  $z_2$*  is

$$\text{Linking}(I, z_1, z_2) := \lim_{n \rightarrow +\infty} \text{Linking}_n(I, z_1, z_2) \quad (1.23)$$

whenever the limit exists.

**Remark 1.2.1.** Let  $F$  be as in Definition 1.2.1. Let  $\mu$  be a  $F$ -invariant Borel probability measure on  $\mathbb{R}^2$  with compact support. Then, for  $\mu$ -almost every  $x \in \mathbb{R}^2$ , the linking number  $\text{Linking}(I, x, y)$  exists for  $\mu$ -almost every  $y \in \mathbb{R}^2 \setminus \{x\}$ . Indeed

$$\begin{aligned} \text{Linking}(I, x, y) &= \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \text{Linking}_1(I, F^i(x), F^i(y)) = \\ &= \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \text{Linking}_1(I, \cdot, \cdot) \circ F_*^i(x, y) \end{aligned}$$

where

$$\begin{aligned} F_* : \mathbb{R}^4 \setminus \Delta &\rightarrow \mathbb{R}^4 \setminus \Delta \\ (x, y) &\mapsto (F(x), F(y)). \end{aligned}$$

Considering the product measure  $\mu \times \mu$  on  $\mathbb{R}^4 \setminus \Delta$ , which is  $F_*$ -invariant, observe that

$$\text{Linking}_1(I, \cdot, \cdot) \in L^1(\mu \times \mu)$$

since  $\mu$  has compact support. Then, Birkhoff's Ergodic Theorem tells us that the function  $\text{Linking}(I, \cdot, \cdot)$  is defined  $\mu \times \mu$ -almost everywhere and it is in  $L^1(\mu \times \mu)$ . By Fubini's theorem, for  $\mu$ -almost every  $x \in \mathbb{R}^2$  the function  $\text{Linking}(I, x, \cdot)$  is defined  $\mu$ -almost everywhere.

Properties analogous of those described in Proposition 1.1.3 for the torsion hold true for the linking number.



**Proposition 1.2.1.** *Let  $I = (F_t)_{t \in \mathbb{R}}$  be an isotopy in  $\text{Diff}^1(\mathbb{R}^2)$  joining the identity  $F_0 = \text{Id}_{\mathbb{R}^2}$  to  $F_1 = F$ . For any points  $z_1, z_2 \in \mathbb{R}^2, z_1 \neq z_2$  the quantities*

$$\text{Linking}_n(I, z_1, z_2) \quad \forall n \in \mathbb{N}, n \neq 0$$

$$\text{Linking}(I, z_1, z_2) \quad \text{when it exists}$$

*do not depend on the choice of the continuous determination of the angle function  $u(I)(z_1, z_2, \cdot)$ . Let  $I' = (G_t)_t$  be another isotopy joining the identity to  $F$ . Then, there exists an integer  $k \in \mathbb{Z}$  independent of the points  $z_1, z_2 \in \mathbb{R}^2$  so that*

$$\text{Linking}_n(I, z_1, z_2) = \text{Linking}_n(I', z_1, z_2) + k \quad \forall n \in \mathbb{N}, n \neq 0$$

$$\text{Linking}(I, z_1, z_2) = \text{Linking}(I', z_1, z_2) + k.$$

As for Proposition 1.1.3, the results of Proposition 1.2.1 follow from the continuity of the involved functions and from the property of  $F$  of being isotopic to the identity.

## 1.3 On torsion of $\mathcal{C}^1$ diffeomorphism of $\mathbb{A}$

### 1.3.1 Independence of the torsion from the choice of the isotopy on $\mathbb{A}$

In [BB13], Béguin and Boubaker show that the torsion is independent of the choice of the isotopy both for an isotopy with compact support and for a diffeomorphism on the 2-dimensional torus  $\mathbb{T}^2$ . In this Section, we prove the independence of the torsion from the isotopy for a  $\mathcal{C}^1$  diffeomorphism over the annulus (with no further hypothesis on its support).

**Notation 1.3.1.** Consider the unbounded annulus  $\mathbb{A} = \mathbb{T} \times \mathbb{R}$ . Let  $I = (f_t)_t$  be an isotopy in  $\text{Diff}^1(\mathbb{A})$  joining  $\text{Id}_{\mathbb{A}}$  to  $f_1 = f$ . Let us fix the counterclockwise orientation and consider as continuous never-vanishing vector field  $X$  the constant one  $\mathcal{H} = (1, 0)$ . Let  $\tilde{I} = (F_t)_t$  be the isotopy obtained as the lift of  $I = (f_t)_t$  such that  $F_0 = \text{Id}_{\mathbb{R}^2}$ . It joins the identity  $\text{Id}_{\mathbb{R}^2}$  to  $F$ , where  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a lift of  $f$ . We then remark that for any time  $t$  and for any  $z = (x, y) \in \mathbb{R}^2$  it holds

$$F_t(x + 1, y) = F_t(x, y) + (1, 0). \quad (1.24)$$

As an intermediate step, we first show that the linking number in the lifted setting does not depend on the choice of the annulus isotopy.

**Proposition 1.3.1.** *Let  $I = (f_t)_t, I' = (g_t)_t$  be two different isotopies in  $\text{Diff}^1(\mathbb{A})$  joining  $\text{Id}_{\mathbb{A}}$  to  $f_1 = g_1 = f$ . Let  $\tilde{I} = (F_t)_t, \tilde{I}' = (G_t)_t$  in  $\text{Diff}^1(\mathbb{R}^2)$  be lifts of the isotopies  $I = (f_t)_t, I' = (g_t)_t$  such that  $F_0 = G_0 = \text{Id}_{\mathbb{R}^2}$ . Then for any  $z_1, z_2 \in \mathbb{R}^2, z_1 \neq z_2$  it holds*

$$\text{Linking}_1(\tilde{I}, z_1, z_2) = \text{Linking}_1(\tilde{I}', z_1, z_2)$$

*and hence, whenever the limit exists,  $\text{Linking}(\tilde{I}, z_1, z_2) = \text{Linking}(\tilde{I}', z_1, z_2)$ .*

*Proof.* Recalling the definition of the diagonal in  $\mathbb{R}^4$ , that is

$$\Delta := \{((x, y), (x', y')) \in \mathbb{R}^4 : (x, y) = (x', y')\},$$

we define the following functions

$$\begin{aligned} \text{Linking}_1(\tilde{I}) : (\mathbb{R}^2 \times \mathbb{R}^2) \setminus \Delta &\rightarrow \mathbb{R} \\ (z, z') &\mapsto \text{Linking}_1(\tilde{I}, z, z') \end{aligned}$$

and

$$\begin{aligned} \text{Linking}_1(\tilde{I}') : (\mathbb{R}^2 \times \mathbb{R}^2) \setminus \Delta &\rightarrow \mathbb{R} \\ (z, z') &\mapsto \text{Linking}_1(\tilde{I}', z, z'). \end{aligned}$$

Both these functions are continuous ones. Moreover, for any  $(z, z') \in (\mathbb{R}^2 \times \mathbb{R}^2) \setminus \Delta$  there exists  $k = k_{z, z'} \in \mathbb{Z}$  such that

$$\text{Linking}_1(\tilde{I}, z, z') = \text{Linking}_1(\tilde{I}', z, z') + k.$$

Since  $(\mathbb{R}^2 \times \mathbb{R}^2) \setminus \Delta$  is connected, the integer  $k \in \mathbb{Z}$  does not depend on the points  $(z, z')$  of  $(\mathbb{R}^2 \times \mathbb{R}^2) \setminus \Delta$ .

Consider then points  $z \neq z'$  such that  $z' = z + (1, 0)$ . To fix the ideas, let us choose  $z = (0, 0)$ ,  $z' = (1, 0)$ . Because of (1.24), it holds that

$$\text{Linking}_1(\tilde{I}, z, z') = \text{Linking}_1(\tilde{I}', z, z') = 0. \quad (1.25)$$

By this observation, we conclude that  $k = 0$ , i.e. the linking number does not depend on the chosen isotopy. □

The next proposition proves that the definition of torsion for a  $\mathcal{C}^1$  diffeomorphism  $f : \mathbb{A} \rightarrow \mathbb{A}$  isotopic to the identity is independent of the choice of the isotopy.

**Proposition 1.3.2.** *Let  $I = (f_t)_t$ ,  $I' = (g_t)_t$  be two different isotopies in  $\text{Diff}^1(\mathbb{A})$  joining the identity  $\text{Id}_{\mathbb{A}}$  to  $f_1 = g_1 = f$  and let consider the standard trivialization. Then for any  $\bar{z} \in \mathbb{A}$  and for any  $\xi \in T_{\bar{z}}\mathbb{A} \setminus \{0\}$*

$$\text{Torsion}_1(I, \bar{z}, \xi) = \text{Torsion}_1(I', \bar{z}, \xi). \quad (1.26)$$

Moreover

$$\text{Torsion}(I, \bar{z}) = \text{Torsion}(I', \bar{z}) \quad (1.27)$$

whenever the limit exists.

*Proof.* Let  $\tilde{I} = (F_t)_t$  and  $\tilde{I}' = (G_t)_t$  be the corresponding lifts of the isotopies  $I = (f_t)_t$  and  $I' = (g_t)_t$  to the plane  $\mathbb{R}^2$  such that  $F_0 = G_0 = \text{Id}_{\mathbb{R}^2}$ . Let  $z \in \mathbb{R}^2$  and  $\xi \in T_z\mathbb{R}^2 \setminus \{0\} \cong T_{\bar{z}}\mathbb{A} \setminus \{0\}$ . Thanks to the choice of the trivialization, denoting as  $\bar{z} \in \mathbb{A}$  the projection of  $z$  on the annulus, it holds

$$\text{Torsion}_1(\tilde{I}, z, \xi) = \text{Torsion}_1(I, \bar{z}, \xi)$$

and

$$\text{Torsion}_1(\tilde{I}', z, \xi) = \text{Torsion}_1(I', \bar{z}, \xi).$$

By Proposition 1.1.1 it holds

$$\text{Torsion}_1(\tilde{I}, z, \xi) = \text{Torsion}_1(\tilde{I}', z, \xi) + k \quad (1.28)$$

where  $k \in \mathbb{Z}$  does not depend on the point or on the vector since  $\mathbb{R}^2$  is connected. Recall the functions  $v, u$ , used in Definitions 1.1.1 and 1.2.1:

$$\begin{aligned} v(\tilde{I})(z, \xi, \cdot) : [0, 1] &\rightarrow \mathbb{T} \\ t &\mapsto \theta(\mathcal{H}, DF_t(z)\xi) \end{aligned}$$

and

$$\begin{aligned} u(\tilde{I})(z, z', \cdot) : [0, 1] &\rightarrow \mathbb{T} \\ t &\mapsto \theta(\mathcal{H}, F_t(z') - F_t(z)), \end{aligned}$$

where  $\mathcal{H} = (1, 0)$ . Let us look at  $z' = z + \xi$ . Parametrize the segment  $[z, z + \xi]$  by setting for any  $s \in [0, 1]$

$$z(s) := z + s\xi.$$

Modify now the definitions of functions  $u, v$  in the following way:

$$\begin{aligned} u(\tilde{I}) : [0, 1] \times [0, 1] &\rightarrow \mathbb{T} \\ (s, t) &\mapsto \theta(\mathcal{H}, F_t(z(s)) - F_t(z)) \quad s \neq 0 \\ (0, t) &\mapsto \theta(\mathcal{H}, DF_t(z)\xi) \end{aligned} \quad (1.29)$$

and

$$\begin{aligned} v(\tilde{I}) : [0, 1] \times [0, 1] &\rightarrow \mathbb{T} \\ (s, t) &\mapsto \theta(\mathcal{H}, DF_t(z(s))\xi). \end{aligned} \quad (1.30)$$

Observe that both  $v(s, t)$  and  $u(s, t)$  are continuous functions, by the continuity of the isotopy with respect to the weak  $\mathcal{C}^1$  topology in  $\text{Diff}^1(\mathbb{R}^2)$ .

Since the definition of  $u(\tilde{I})$  coincides with that of  $v(\tilde{I})$  for  $s = 0$  and since  $u(\tilde{I})$  is continuous, for any time  $t$  we have that  $v(\tilde{I})(0, t) = u(\tilde{I})(0, t) = \lim_{s \rightarrow 0^+} u(\tilde{I})(s, t)$ .

The definitions of torsion and linking number do not depend on the chosen lift. So we select continuous determinations  $\tilde{v}(\tilde{I})$  and  $\tilde{u}(\tilde{I})$  such that  $\tilde{v}(\tilde{I})(0, t) = \tilde{u}(\tilde{I})(0, t)$  for any time  $t$  and similarly  $\tilde{v}(\tilde{I}')(0, t) = \tilde{u}(\tilde{I}')(0, t)$ .

By Proposition 1.3.1 for any  $s \in [0, 1]$  it holds

$$\tilde{u}(\tilde{I})(s, 1) - \tilde{u}(\tilde{I})(s, 0) = \tilde{u}(\tilde{I}')(s, 1) - \tilde{u}(\tilde{I}')(s, 0).$$

Passing to the limit for  $s$  going to  $0^+$ , we obtain

$$\tilde{v}(\tilde{I})(0, 1) - \tilde{v}(\tilde{I})(0, 0) = \tilde{v}(\tilde{I}')(0, 1) - \tilde{v}(\tilde{I}')(0, 0),$$

that is

$$\text{Torsion}_1(\tilde{I}, z, \xi) = \text{Torsion}_1(\tilde{I}', z, \xi).$$

We conclude that the integer  $k$  in (1.28) is null. □

**Remark 1.3.1.** With the same techniques, it can be shown that also for a  $\mathcal{C}^1$  diffeomorphism over the torus  $\mathbb{T}^2$  isotopic to the identity the torsion is independent of the choice of the isotopy. Actually, this independence has been already remarked by Béguin and Boubaker in Section 2 in [BB13].

**Remark 1.3.2.** Using the same strategy of the proofs of Proposition 1.3.1 and Proposition 1.3.2, we can show that both the torsion and the linking number (already at finite time) for a  $\mathcal{C}^1$  diffeomorphism of  $\mathbb{R}^2$  with compact support do not depend on the choice of the isotopy, up to consider compact-supported isotopies. Indeed, for  $z_1, z_2 \in \mathbb{R}^2, z_1 \neq z_2$  not belonging to the support of the isotopy, the linking number (already at finite-time) of  $(z_1, z_2)$  is zero with respect to any compact-supported isotopy.

### 1.3.2 Invariance under $\mathcal{C}^1$ conjugacy

The following result concerns the invariance of the torsion for conjugation through  $\mathcal{C}^1$  diffeomorphisms of the annulus isotopic to the identity with compact support. With respect to Proposition 1.1.6, this result does not require that  $I = (f_t)_t$  has compact support, but it holds for an isotopic-to-identity  $\mathcal{C}^1$  conjugation  $h$  with compact support. See also Section 2.9 in [Bou12]. Because of Proposition 1.3.2, the torsion does not depend on the chosen isotopy on the annulus. Therefore, in the following, we will omit the isotopy in the notation.

**Proposition 1.3.3.** *Let  $f, h : \mathbb{A} \rightarrow \mathbb{A}$  be  $\mathcal{C}^1$  diffeomorphisms isotopic to the identity. Assume that  $h$  has compact support. Let  $\bar{z} \in \mathbb{A}$  and assume that  $\text{Torsion}(f, h^{-1}(\bar{z}))$  exists. Then*

$$\text{Torsion}(h \circ f \circ h^{-1}, \bar{z}) = \text{Torsion}(f, h^{-1}(\bar{z})).$$

We start by proving the following lemma which will be used in the proof of Proposition 1.3.3.

**Lemma 1.3.1.** *Let  $h : \mathbb{A} \rightarrow \mathbb{A}$  be a  $\mathcal{C}^1$  diffeomorphism isotopic to the identity. Then for any  $\bar{z} \in \mathbb{A}$  and any  $\xi \in T_{\bar{z}}\mathbb{A}$  it holds*

$$\text{Torsion}_1(h^{-1}, \bar{z}, \xi) = -\text{Torsion}_1(h, h^{-1}(\bar{z}), Dh^{-1}(\bar{z})\xi).$$

*Proof.* By Proposition 1.3.2, the time-one torsion of  $h^{-1}$  does not depend on the choice of the isotopy. Therefore, let  $H = (h_t)_t$  be an isotopy joining the identity to  $h$  and consider the isotopy  $H^{-1} = (h_t^{-1})_{t \in [0,1]}$  where

$$h_t^{-1} = h_{1-t} \circ h^{-1} \quad \forall t \in [0, 1].$$

The isotopy  $H^{-1} = (h_t^{-1})_t$  joins the identity to  $h^{-1}$ . Fix  $\bar{z} \in \mathbb{A}$  and  $\xi \in T_{\bar{z}}\mathbb{A}$ .

The function  $t \mapsto \tilde{v}(H^{-1})(\bar{z}, \xi, t)$  is a continuous determination of the oriented angle function

$$t \mapsto \theta(X(h_t^{-1}(\bar{z})), Dh_t^{-1}(\bar{z})\xi) = \theta(X(h_{1-t}(h^{-1}(\bar{z}))), Dh_{1-t}(h^{-1}(\bar{z}))Dh^{-1}(\bar{z})\xi).$$

This last oriented angle function is  $t \mapsto v(H)(h^{-1}(\bar{z}), Dh^{-1}(\bar{z})\xi, 1-t)$ . Choosing a continuous determination such that  $\tilde{v}(H^{-1})(\bar{z}, \xi, 0) = \tilde{v}(H)(h^{-1}(\bar{z}), Dh^{-1}(\bar{z})\xi, 1)$ , we conclude that for any  $t \in [0, 1]$  it holds

$$\tilde{v}(H^{-1})(\bar{z}, \xi, t) = \tilde{v}(H)(h^{-1}(\bar{z}), Dh^{-1}(\bar{z})\xi, 1-t).$$

Consequently

$$\begin{aligned}
& \text{Torsion}_1(h^{-1}, \bar{z}, \xi) = \\
& = \tilde{v}(H^{-1})(\bar{z}, \xi, 1) - \tilde{v}(H^{-1})(\bar{z}, \xi, 0) = \\
& = \tilde{v}(H)(h^{-1}(\bar{z}), Dh^{-1}(\bar{z})\xi, 0) - \tilde{v}(H)(h^{-1}(\bar{z}), Dh^{-1}(\bar{z})\xi, 1) = \\
& = -\text{Torsion}_1(h, h^{-1}(\bar{z}), Dh^{-1}(\bar{z})\xi).
\end{aligned}$$

□

*Proof of Proposition 1.3.3.* Let  $H = (h_t)_t, I = (f_t)_t$  be isotopies joining the identity to  $h, f$  respectively. Denote as  $H^{-1} = (h_{1-t} \circ h^{-1})_{t \in [0,1]}$ . To calculate the torsion for  $h \circ f \circ h^{-1}$ , thanks to Proposition 1.3.2, we can use the following isotopy  $G = (g_t)_{t \in [0,1]}$  joining the identity to  $h \circ f \circ h^{-1}$ :

$$g_t := \begin{cases} h_{1-3t} \circ h^{-1} & \text{for } t \in [0, \frac{1}{3}] \\ f_{3t-1} \circ h^{-1} & \text{for } t \in [\frac{1}{3}, \frac{2}{3}] \\ h_{3t-2} \circ f \circ h^{-1} & \text{for } t \in [\frac{2}{3}, 1] \end{cases}$$

Fix now  $\bar{z} \in \mathbb{A}$  and  $\xi \in T_{\bar{z}}\mathbb{A}$ . We have

$$\begin{aligned}
& \text{Torsion}_1(h \circ f \circ h^{-1}, \bar{z}, \xi) = \tilde{v}(G)(\bar{z}, \xi, 1) - \tilde{v}(G)(\bar{z}, \xi, 0) = \\
& = \tilde{v}(H)(f \circ h^{-1}(\bar{z}), D(f \circ h^{-1})(\bar{z})\xi, 1) - \tilde{v}(H)(f \circ h^{-1}(\bar{z}), D(f \circ h^{-1})(\bar{z})\xi, 0) + \\
& \quad + \tilde{v}(I)(h^{-1}(\bar{z}), Dh^{-1}(\bar{z})\xi, 1) - \tilde{v}(I)(h^{-1}(\bar{z}), Dh^{-1}(\bar{z})\xi, 0) + \\
& \quad + \tilde{v}(H^{-1})(\bar{z}, \xi, 1) - \tilde{v}(H^{-1})(\bar{z}, \xi, 0).
\end{aligned}$$

Since  $\tilde{v}(H^{-1})(\bar{z}, \xi, 1) - \tilde{v}(H^{-1})(\bar{z}, \xi, 0) = \text{Torsion}_1(h^{-1}, \bar{z}, \xi)$ , using Lemma 1.3.1, we obtain

$$\begin{aligned}
& \text{Torsion}_1(h \circ f \circ h^{-1}, \bar{z}, \xi) = \text{Torsion}_1(h, f \circ h^{-1}(\bar{z}), D(f \circ h^{-1})(\bar{z})\xi) + \\
& \quad + \text{Torsion}_1(f, h^{-1}(\bar{z}), Dh^{-1}(\bar{z})\xi) - \text{Torsion}_1(h, h^{-1}(\bar{z}), Dh^{-1}(\bar{z})\xi). \quad (1.31)
\end{aligned}$$

Consequently for  $n \in \mathbb{N}$

$$\text{Torsion}_n(h \circ f \circ h^{-1}, \bar{z}, \xi) = \frac{1}{n} \sum_{i=0}^{n-1} \text{Torsion}_1(h \circ f \circ h^{-1}, h \circ f^i \circ h^{-1}(\bar{z}), D(h \circ f^i \circ h^{-1})(\bar{z})\xi).$$

Using (1.31) and erasing the corresponding terms, we have

$$\begin{aligned}
& \text{Torsion}_n(h \circ f \circ h^{-1}, \bar{z}, \xi) = \frac{1}{n} \sum_{i=0}^{n-1} \text{Torsion}_1(f, f^i \circ h^{-1}(\bar{z}), D(f^i \circ h^{-1})(\bar{z})\xi) + \\
& \quad + \frac{1}{n} \left( \text{Torsion}_1(h, f^n \circ h^{-1}(\bar{z}), Df^n(h^{-1}(\bar{z}))Dh^{-1}(\bar{z})\xi) - \text{Torsion}_1(h, h^{-1}(\bar{z}), Dh^{-1}(\bar{z})\xi) \right).
\end{aligned}$$

Since  $x \mapsto \text{Torsion}_1(h, x, v)$  is continuous, since the support of  $h$  is compact and since outside its support the time-one torsion of  $h$  is null, we deduce that

$$\text{Torsion}_1(h, f^n \circ h^{-1}(\bar{z}), Df^n(h^{-1}(\bar{z}))Dh^{-1}(\bar{z})\xi) - \text{Torsion}_1(h, h^{-1}(\bar{z}), Dh^{-1}(\bar{z})\xi)$$

is bounded (uniformly in  $n$ ). Passing to the limit for  $n \rightarrow +\infty$ , we conclude that

$$\begin{aligned} \text{Torsion}(h \circ f \circ h^{-1}, \bar{z}) &= \lim_{n \rightarrow +\infty} \text{Torsion}_n(h \circ f \circ h^{-1}, \bar{z}, \xi) = \\ &= \lim_{n \rightarrow +\infty} \text{Torsion}_n(f, h^{-1}(\bar{z}), Dh^{-1}(\bar{z})\xi) + \\ &+ \lim_{n \rightarrow +\infty} \frac{1}{n} \left( \text{Torsion}_1(h, f^n \circ h^{-1}(\bar{z}), Df^n(h^{-1}(\bar{z}))Dh^{-1}(\bar{z})\xi) - \text{Torsion}_1(h, h^{-1}(\bar{z}), Dh^{-1}(\bar{z})\xi) \right) = \\ &= \text{Torsion}(f, h^{-1}(\bar{z})). \end{aligned}$$

□

## 1.4 Link between torsion and linking number

In [BB13], the authors provide conditions for which the existence of two points with non-zero linking number implies the existence of a point with non-zero torsion. However, the value and even the sign of the linking number and of the torsion can be different.

Let  $x, y \in \mathbb{R}^2$  be points with linking value  $l$ . In Theorem 1.4.1, we claim the existence of a point with torsion value exactly  $l$ . In addition we locate such a point on the segment joining  $x$  and  $y$ . We remark that this result can be applied also to the zero value case, since it does not depend on the value of the linking number. Throughout the section, we consider torsion with respect to a constant reference vector field (i.e. with respect to the standard trivialization).

**Theorem 1.4.1.** *Let  $I = (F_t)_{t \in [0,1]}$  be an isotopy in  $\text{Diff}^1(\mathbb{R}^2)$  joining  $\text{Id}_{\mathbb{R}^2}$  to  $F_1 = F$ . Assume that there exist two points  $x, y \in \mathbb{R}^2$ ,  $x \neq y$  such that*

$$\text{Linking}_1(I, x, y) = l \in \mathbb{R}.$$

*Then there exists a point  $z \in [x, y]$  so that*

$$\text{Torsion}_1(I, z, y - x) = l.$$

At first sight, this result could recall a mean value theorem but the arguments and the strategies needed in the proof are much more sophisticated and subtle.

First of all, through continuous modifications of the isotopy, we bring ourselves in a rotation frame of reference, reducing then the discussion to the case  $l = 0$ .

In order to avoid self-intersections of the curve, a passage to the universal covering of the punctured plane is required: the strategy in doing so is using a polar coordinate frame, with respect to which one of the endpoints of the segment coincides with the singularity. Finally, we carefully study the behavior of points in a neighborhood of the singularity  $x$ , the point previously blown up which corresponds to the origin of the polar coordinate framework. We then apply the Turning Tangent Theorem (see [DC76], Chapter 4, Section 5).

**Notation 1.4.1.** Consider an isotopy  $I = (F_t)_t : [0, 1] \rightarrow \text{Diff}^1(\mathbb{R}^2)$  joining the identity to  $F_1 = F$ . With the notation  $I = (F_t)_t$  we refer also to the extended isotopy. We refer to the setting presented in Notation 1.2.1: we fix the counterclockwise orientation and we are going to measure angles with respect to the vector field  $\mathcal{H} = (1, 0)$ .

Given two points  $x, y \in \mathbb{R}^2$ ,  $x \neq y$ , the notation  $[x, y]$  refers to the segment joining the points.

Denote a point of the segment as  $z(s) := sy + (1 - s)x$  for  $s \in [0, 1]$ .

*Sketch of the proof of Theorem 1.4.1.* By contradiction, we assume that there is no point  $z \in [x, y]$  such that  $\text{Torsion}_1(I, z, y - x) = l$ . Then, by the continuity of the function  $z \mapsto \text{Torsion}_1(I, z, y - x)$  and by the connectedness of the segment, one of the following cases occur:

- (i) for any  $z \in [x, y]$  it holds  $\text{Torsion}_1(I, z, y - x) < l$ ;
- (ii) for any  $z \in [x, y]$  it holds  $\text{Torsion}_1(I, z, y - x) > l$ .

In Subsection 1.4.2 we show that case (i) leads to a contradiction. Similarly, case (ii) cannot even occur. □

A modification of the involved isotopy and the use of Theorem 1.4.1 easily adapt this result for any finite time  $n \in \mathbb{N}$ . We keep the same notation of Theorem 1.4.1.

**Corollary 1.4.1.** *Assume that there exist  $n \in \mathbb{N}, n \neq 0$  and  $x, y \in \mathbb{R}^2, x \neq y$ , such that  $\text{Linking}_n(I, x, y) = l \in \mathbb{R}$ .*

*Then there exists a point  $z \in [x, y]$  such that*

$$\text{Torsion}_n(I, z, y - x) = l. \quad (1.32)$$

*Proof.* We are interested in the time interval  $[0, n]$ . Define the isotopy  $I^n = (G_t)_{t \in [0, 1]} := (F_{nt})_{t \in [0, 1]}$ .

Hence, we are time-reparametrizing the initial isotopy. It holds

$$u(I^n, x, y)(t) = u(I, x, y)(nt).$$

Then,  $\tilde{u}(I^n, x, y)(t)$  and  $\tilde{u}(I, x, y)(nt)$  denote continuous determinations of the same angle function. Since the (finite time) linking number is independent of the choice of the lift (see Proposition 1.2.1), we refer to  $\tilde{u}(I^n, x, y)(t)$ .

The hypothesis  $\text{Linking}_n(I, x, y) = l$  is then equivalent to ask that  $\text{Linking}_1(I^n, x, y) = nl$ . By Theorem (1.4.1), there exists  $z \in [x, y]$  such that  $\text{Torsion}_1(I^n, z, y - x) = nl$ . For such a  $z$  it also holds

$$\text{Torsion}_n(I, z, y - x) = l \quad (1.33)$$

and this concludes the proof. □

We wonder if any such relation is satisfied between asymptotic torsion and asymptotic linking number: can any results as above hold true even when considering (1.5) in Definition 1.1.3 and (1.23) in Definition (1.2.1)?

The answer is positive looking at torsion of  $F$ -invariant measures, instead of orbits. Passing to asymptotic quantities, we are going to prove the existence of  $f$ -invariant Borel probability measures  $\mu$  whose torsion, i.e.  $\int_S \text{Torsion}((f_t)_t, x) d\mu(x)$ , equals  $l \in \mathbb{R}$ , where now  $l$  is the asymptotic linking number of two points.

**Corollary 1.4.2.** *Assume that there exist two points  $x, y \in \mathbb{R}^2, x \neq y$  such that*

$$\text{Linking}(I, x, y) = l \in \mathbb{R}.$$

*Suppose that  $\bigcup_{n \in \mathbb{N}} F^n([x, y])$  is relatively compact.*

*Then there exists a  $F$ -invariant probability measure  $\mu$  such that*

$$\text{Torsion}(I, \mu) = l.$$

*Moreover, there exist points with torsion greater or equal  $l$  and also points with torsion smaller or equal  $l$ .*

**Remark 1.4.1.** If  $F$  has compact support, then  $\bigcup_{n \in \mathbb{N}} F^n([x, y])$  is always relatively compact.

*Proof.* From our hypothesis

$$l = \text{Linking}(I, x, y) = \lim_{n \rightarrow +\infty} \text{Linking}_n(I, x, y)$$

For any fixed  $n \in \mathbb{N}$ , denote  $l_n := \text{Linking}_n(I, x, y)$ . By Corollary (1.4.1) there exists  $z_n \in [x, y]$  such that

$$\text{Torsion}_n(I, z_n, y - x) = l_n.$$

The notation  $\xi$  refers to the vector  $y - x$ . Consider the following probability measures on the unitary tangent bundle  $T^1\mathbb{R}^2$ :

$$\tilde{\mu}_n := \frac{1}{n} \sum_{i=0}^{n-1} \delta_{\left(F^i(z_n), \frac{DF^i(z_n)\xi}{\|DF^i(z_n)\xi\|}\right)} \quad (1.34)$$

where  $\delta_{(x,v)}$  denotes the Dirac measure centered on  $(x, v)$  in  $T^1\mathbb{R}^2$ . All the supports of these measures  $\tilde{\mu}_n$  are contained in the same set

$$T_{\mathcal{K}}^1\mathbb{R}^2$$

where

$$\mathcal{K} := \overline{\bigcup_{i \in \mathbb{N}} F^i([x, y])}.$$

From the hypothesis,  $\mathcal{K}$  is compact and so is  $T_{\mathcal{K}}^1\mathbb{R}^2$ .

Up to subsequences, the sequence  $(\tilde{\mu}_n)_n$  converges to a probability measure  $\tilde{\mu}$  on  $T^1\mathbb{R}^2$  which is invariant with respect to the dynamics on the unitary tangent bundle inherited from  $F$ . The projection  $\mu$  of  $\tilde{\mu}$  on  $\mathbb{R}^2$  is  $F$ -invariant as well.

Finally, referring to Definition (1.1.4) with respect to  $\mu$ , we have

$$\begin{aligned} \text{Torsion}(I, \mu) &= \int_{\mathbb{R}^2} \text{Torsion}(I, x) d\mu(x) = \\ &= \int_{T^1\mathbb{R}^2} \text{Torsion}(I, x) d\tilde{\mu}(x, v) \stackrel{*}{=} \int_{T^1\mathbb{R}^2} \text{Torsion}_1(I, x, v) d\tilde{\mu}(x, v) = \\ &= \lim_{n \rightarrow +\infty} \int_{T^1\mathbb{R}^2} \text{Torsion}_1(I, x, v) d\tilde{\mu}_n(x, v) = \\ &= \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \text{Torsion}_1 \left( I, F^i(z_n), \frac{DF^i(z_n)\xi}{\|DF^i(z_n)\xi\|} \right) = \\ &= \lim_{n \rightarrow +\infty} \text{Torsion}_n(I, z_n, \xi) = \lim_{n \rightarrow +\infty} l_n = l. \end{aligned}$$

Equality  $*$  is a consequence of Birkhoff's Ergodic Theorem applied to the framework where

$$\begin{aligned} F^* : (T^1\mathbb{R}^2, \tilde{\mu}) &\rightarrow (T^1\mathbb{R}^2, \tilde{\mu}) \\ (x, \xi) &\mapsto F^*(x, \xi) = \left( F(x), \frac{DF(x)\xi}{\|DF(x)\xi\|} \right) \end{aligned}$$

is a measure-preserving transformation and  $\text{Torsion}_1(I, \cdot, \cdot) \in L^1(T^1\mathbb{R}^2, \tilde{\mu})$ . The time average  $\text{Torsion}(I, \cdot)$  does not depend on the choice of the tangent vector (see Proposition



1.1.3) and, by Birkhoff's Ergodic Theorem (see Theorem 4.1.2 in [KH95]), it exists  $\tilde{\mu}$ -a.e., is measurable,  $F^*$ -invariant and such that

$$\int_{T^1\mathbb{R}^2} \text{Torsion}(I, x) d\tilde{\mu}(x, v) = \int_{T^1\mathbb{R}^2} \text{Torsion}_1(I, x, v) d\tilde{\mu}(x, v).$$

As an outcome, there exist points with torsion greater or equal  $l$  and also points with torsion smaller or equal  $l$ .

Arguing by contradiction, suppose that every  $x \in \mathbb{R}^2$  has  $\text{Torsion}(I, x)$  strictly greater than  $l$ . Then

$$l = \text{Torsion}(I, \mu) = \int_{\mathbb{R}^2} \text{Torsion}(I, x) d\mu(x) > \int_{\mathbb{R}^2} l d\mu(x) = l.$$

This provides the required contradiction. Analogous argument holds assuming that every point has torsion strictly less than  $l$ . □

### 1.4.1 Some consequences for the torus $\mathbb{T}^2$

Any diffeomorphism of the torus has an invariant measure with zero torsion: this result was already known by Matsumoto and Nakayama for  $\mathcal{C}^\infty$  diffeomorphisms. We present here a simpler proof which works also with  $\mathcal{C}^1$  diffeomorphisms. Therefore, we weaken the hypothesis required in [MN02].

**Notation 1.4.2.** Let

$$\mathcal{P} : \mathbb{R}^2 \rightarrow \mathbb{T}^2$$

$$(x, y) \mapsto \mathcal{P}(x, y) = (x \bmod 1, y \bmod 1)$$

be the universal covering of  $\mathbb{T}^2$ . Denote as  $\mathcal{P}(\mathbb{T}^2)$  the set of Borel probability measures over the torus  $\mathbb{T}^2$ . Fix the counterclockwise orientation and consider as reference vector field  $X$  the constant one  $\mathcal{H}$ .

Let us start by observing that in the case of torus diffeomorphisms the hypothesis of Corollary 1.4.2 are too strong. Therefore, we state the following

**Corollary 1.4.3.** *Let  $I = (f_t)_t$  be an isotopy in  $\text{Diff}^1(\mathbb{T}^2)$  joining  $\text{Id}_{\mathbb{T}^2}$  to  $f_1 = f$ . Let  $\tilde{I} = (F_t)_t$  in  $\text{Diff}^1(\mathbb{R}^2)$  be the lift of the isotopy  $I = (f_t)_t$  such that  $F_0 = \text{Id}_{\mathbb{R}^2}$ . Assume that there exist two points  $x, y \in \mathbb{R}^2, x \neq y$  such that  $\text{Linking}(\tilde{I}, x, y) = l \in \mathbb{R}$ . Then there exists a  $f$ -invariant probability measure  $\mu \in \mathcal{P}(\mathbb{T}^2)$  such that  $\text{Torsion}(I, \mu) = l$ . Moreover, there exist points in  $\mathbb{T}^2$  with torsion greater or equal  $l$  and also points with torsion smaller or equal  $l$ .*

The proof of Corollary 1.4.3 retraces the ideas of the proof of Corollary 1.4.2.

*Proof.* As in the proof of Corollary 1.4.2, denote  $l_n = \text{Linking}_n(\tilde{I}, x, y)$  and by hypothesis it holds  $\lim_{n \rightarrow +\infty} l_n = l = \text{Linking}(\tilde{I}, x, y)$ . By Corollary 1.4.1 for any  $n \in \mathbb{N}, n \neq 0$  there exists  $z_n \in [x, y]$  such that  $\text{Torsion}_n(\tilde{I}, z_n, y - x) = l_n$ .

Thanks to the choice of the trivialization we have that

$$\text{Torsion}_n(I, \mathcal{P}(z_n), y - x) = \text{Torsion}_n(\tilde{I}, z_n, y - x) = l_n.$$

For simplicity denote  $\mathcal{P}(z_n)$  as  $\hat{z}_n$ . Consider now the probability measures on the unitary tangent bundle  $T^1\mathbb{T}^2$ :

$$\tilde{\mu}_n := \frac{1}{n} \sum_{i=0}^{n-1} \delta \left( f^i(\hat{z}_n), \frac{Df^i(\hat{z}_n)(y-x)}{\|Df^i(\hat{z}_n)(y-x)\|} \right).$$

Being  $T^1\mathbb{T}^2$  compact, up to subsequences,  $(\tilde{\mu}_n)_n$  converges to  $\tilde{\mu}$  which is a probability measure on  $T^1\mathbb{T}^2$ . The measure  $\tilde{\mu}$  is invariant with respect to the dynamics on  $T^1\mathbb{T}^2$  and its projection on  $\mathbb{T}^2$   $\mu \in \mathcal{P}(\mathbb{T}^2)$  is  $f$ -invariant.

Repeating the ideas in the proof of Corollary 1.4.2, we have

$$\begin{aligned} \text{Torsion}(I, \mu) &= \int_{\mathbb{T}^2} \text{Torsion}(I, x) d\mu(x) = \int_{T^1\mathbb{T}^2} \text{Torsion}_1(I, x, v) d\tilde{\mu}(x, v) = \\ &= \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \text{Torsion}_1 \left( I, f^i(\hat{z}_n), \frac{Df^i(\hat{z}_n)(y-x)}{\|Df^i(\hat{z}_n)(y-x)\|} \right) = \\ &= \lim_{n \rightarrow +\infty} \text{Torsion}_n(I, \hat{z}_n, y-x) = l. \end{aligned}$$

We easily deduce the existence of points in  $\mathbb{T}^2$  with torsion greater or equal  $l$  (respectively smaller or equal  $l$ ).

□

We then deduce as a corollary the result by Matsumoto and Nakayama discussed above.

**Corollary 1.4.4.** *Let  $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$  be a  $\mathcal{C}^1$  diffeomorphism isotopic to the identity. Then, there exists a  $f$ -invariant Borel probability measure  $\mu \in \mathcal{P}(\mathbb{T}^2)$  of null torsion.*

*Proof.* Let  $I = (f_t)_t$  be an isotopy in  $\text{Diff}^1(\mathbb{T}^2)$  joining the identity to  $f$ . Recall that on  $\mathbb{T}^2$  the torsion does not depend on the chosen isotopy, see Remark 1.3.1. Let  $\tilde{I} = (F_t)_t$  be the isotopy obtained as the lift of the isotopy  $I = (f_t)_t$  such that  $F_0 = \text{Id}_{\mathbb{R}^2}$ . For any point  $(x, y) \in \mathbb{R}^2$

$$F_t(x + k_1, y + k_2) = F_t(x, y) + (k_1, k_2) \quad \forall (k_1, k_2) \in \mathbb{Z}^2, \forall t \in \mathbb{R}_+. \quad (1.35)$$

Consider now the points  $z_1 = (0, 0), z_2 = (1, 0) \in \mathbb{R}^2$ . For a fixed  $n \in \mathbb{N}, n \neq 0$  look at

$$\text{Linking}_n(\tilde{I}, z_1, z_2) = \frac{1}{n} \left( \tilde{u}(\tilde{I})(z_1, z_2, n) - \tilde{u}(\tilde{I})(z_1, z_2, 0) \right).$$

Since (1.35) holds for every  $t \geq 0$ , the vector  $F_t((1, 0)) - F_t((0, 0))$  (in whose direction we are interested) remains horizontal and so  $\text{Linking}_n(\tilde{I}, z_1, z_2) \equiv \frac{0}{n}$ . By the arbitrariness of  $n \in \mathbb{N}$  we deduce that  $\text{Linking}(\tilde{I}, z_1, z_2) = 0$ . Applying Corollary 1.4.3 to the points  $z_1, z_2$ , we conclude that there exists  $\mu \in \mathcal{P}(\mathbb{T}^2)$  which is  $f$ -invariant and such that  $\text{Torsion}(I, \mu) = {}^3\text{Torsion}(f, \mu) = 0$ .

□

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3. Recall that on  $\mathbb{T}^2$  the torsion does not depend on the chosen isotopy (see Remark 1.3.1).

### 1.4.2 Proof of case (i) of Theorem 1.4.1

In this section we assume that case (i) of the sketch of the proof of Theorem 1.4.1 (presented in Section 1.4) holds, that is for any  $z \in [x, y]$  we have

$$\text{Torsion}_1(I, z, y - x) < l = \text{Linking}_1(I, x, y).$$

We are going to find a contradiction, deducing that this case cannot occur and concluding so the proof of Theorem 1.4.1.

By continuity of the function and by compactness of the segment, we assume that there exists  $\varepsilon > 0$  such that for any point in  $[x, y]$

$$\text{Torsion}_1(I, z, y - x) < l - \varepsilon.$$

**Notation 1.4.3.** Denote

$$\xi := y - x$$

and parametrize the segment  $[x, y]$  as follows:

$$[0, 1] \ni s \mapsto z(s) := sy + (1 - s)x \in [x, y] \subset \mathbb{R}^2.$$

**Notation 1.4.4.** We use the notation introduced in (1.29) and (1.30) in order to modify the angle functions  $u, v$ . From these, we define linking number and torsion just along points of the segment  $[x, y]$ .

Since  $u, v$  are continuous and for any  $t$  it holds  $u(0, t) = v(0, t)$ , there exist continuous lifts  $\tilde{u}, \tilde{v}$  of the functions  $u, v$ , respectively, such that  $\tilde{u}(0, t) = \tilde{v}(0, t)$ .

By hypothesis for any  $s \in [0, 1]$

$$\text{Torsion}_1(I, z(s), \xi) < l - \varepsilon = \text{Linking}_1(I, x, y) - \varepsilon. \quad (1.36)$$

Referring to definitions (1.29) and (1.30), inequality (1.36) becomes

$$\tilde{v}(s, 1) - \tilde{v}(s, 0) < \tilde{u}(1, 1) - \tilde{u}(1, 0) - \varepsilon \quad (1.37)$$

for any  $s \in [0, 1]$ .

**Modification of the isotopy**  $I = (F_t)_t$

First, we modify the given isotopy  $(F_t)_t$  to obtain an isotopy  $H = (H_t)_t$  such that:

- the point  $x$  is fixed for  $H = (H_t)_t$ , that is  $H_t(x) = x$  for any  $t$ ;
- the linking number of  $x, y$  with respect to  $H$  is positive, while the torsion of any point of  $[x, y]$  with respect to  $H$  is negative.

In other words, we want to pass in a rotated and translated frame.

**Lemma 1.4.1.** *Let  $(F_t)_{t \in [0, 1]}$  be an isotopy in  $\text{Diff}^1(\mathbb{R}^2)$  joining  $\text{Id}_{\mathbb{R}^2}$  to  $F_1 = F$ . Consider  $x, y \in \mathbb{R}^2$ ,  $x \neq y$  such that, for a fixed  $\varepsilon > 0$ , for any  $s \in [0, 1]$*

$$\text{Torsion}_1((F_t)_t, z(s), \xi) < \text{Linking}_1((F_t)_t, x, y) - \varepsilon. \quad (1.38)$$

*Then, there exists an isotopy  $H = (H_t)_{t \in [0, 1]}$  in  $\text{Diff}^1(\mathbb{R}^2)$ , such that:*

- $H_0 = Id_{\mathbb{R}^2}$  and  $H := H_1$ ;
- for any  $s \in [0, 1]$

$$Torsion_1(H, z(s), \xi) \leq -\frac{\varepsilon}{2} < 0 < \frac{\varepsilon}{2} \leq Linking_1(H, x, y);$$

- $H_t(x) = x$  for any  $t \in [0, 1]$ .

*Proof.* Define the following continuous function

$$\begin{aligned} \Theta : [0, 1] &\rightarrow \mathbb{R} \\ \Theta(t) &:= \sup_{s \in [0, 1]} (\tilde{v}(s, t) - \tilde{v}(s, 0)) = \max_{s \in [0, 1]} (\tilde{v}(s, t) - \tilde{v}(s, 0)) \end{aligned} \quad (1.39)$$

We remark that  $\Theta(0) = 0$ . The new isotopy is then obtained as follows:

$$H = (H_t)_{t \in [0, 1]} := \mathcal{R}\left(x, -\Theta(t) - t\frac{\varepsilon}{2}\right) \circ \tau_{x - F_t(x)} \circ (F_t)_t$$

where  $\mathcal{R}(x, \psi)$  denotes the rotation of angle  $\psi$  centered at  $x$  and  $\tau_v$  denotes the translation of vector  $v$ .

The point  $x$  is fixed for the isotopy  $H = (H_t)_t$ . Denote as  $U, V$  the functions defined in (1.29) and (1.30) with respect to  $H$ , that is

$$U : [0, 1] \times \mathbb{R} \rightarrow \mathbb{T}$$

$$(s, t) \mapsto \theta(\mathcal{H}, H_t(z(s)) - H_t(x)) = \theta(\mathcal{H}, H_t(z(s)) - x) \quad \text{for } s \neq 0, \quad (1.40)$$

$$(0, t) \mapsto \theta(\mathcal{H}, DH_t(x)\xi) \quad (1.41)$$

and

$$V : [0, 1] \times \mathbb{R} \rightarrow \mathbb{T}$$

$$(s, t) \mapsto \theta(\mathcal{H}, DH_t(z(s))\xi) \quad (1.42)$$

where  $\theta$  denotes the oriented angle between the two vectors.

Observe that  $U, V$  are continuous and that, for any  $t$ ,  $U(0, t) = V(0, t)$ .

Define then the quantities  $\tilde{U}, \tilde{V}$  from  $\tilde{u}, \tilde{v}$  as:

$$\tilde{U}(s, t) = \tilde{u}(s, t) - \Theta(t) - t\frac{\varepsilon}{2} \quad (1.43)$$

$$\tilde{V}(s, t) = \tilde{v}(s, t) - \Theta(t) - t\frac{\varepsilon}{2}. \quad (1.44)$$

These functions are continuous determinations of the angle functions  $U$  and  $V$ , respectively.

From the definition of  $\Theta$  in (1.39), for every  $s \in [0, 1]$  and for every  $t \in (0, 1]$ , it follows

$$\tilde{V}(s, t) - \tilde{V}(s, 0) \leq -t\frac{\varepsilon}{2} < 0.$$

On the other hand, by hypothesis (1.37), for any  $s \in [0, 1]$  it holds

$$\tilde{V}(s, 1) - \tilde{V}(s, 0) \leq \tilde{U}(1, 1) - \tilde{U}(1, 0) - \varepsilon. \quad (1.45)$$

Let  $S \in [0, 1]$  be a point at which the *maximum*  $\Theta(1)$  is achieved (see (1.39)), i.e.

$$\Theta(1) = \tilde{v}(S, 1) - \tilde{v}(S, 0).$$

For such  $S$  we have  $\tilde{V}(S, 1) - \tilde{V}(S, 0) = -\frac{\varepsilon}{2}$  and (1.45) still holds true. Therefore

$$-\frac{\varepsilon}{2} \leq \tilde{U}(1, 1) - \tilde{U}(1, 0) - \varepsilon \quad \Rightarrow \quad \tilde{U}(1, 1) - \tilde{U}(1, 0) \geq \frac{\varepsilon}{2} > 0.$$

Hence, for any  $s \in [0, 1]$

$$\tilde{V}(s, 1) - \tilde{V}(s, 0) \leq -\frac{\varepsilon}{2} < 0 < \frac{\varepsilon}{2} \leq \tilde{U}(1, 1) - \tilde{U}(1, 0). \quad (1.46)$$

□

**Notation 1.4.5.** We will conserve this notation of  $U, V, \tilde{U}, \tilde{V}$  throughout the whole subsection, until the conclusion of the proof.

### Sign concordance of Linking and Torsion for small $s$

**Lemma 1.4.2.** *Let  $\tilde{U}$  and  $\tilde{V}$  be the functions introduced in (1.43) and (1.44). There exists  $s_0 \in (0, 1)$  such that for all  $s \in [0, s_0]$  it holds*

$$\tilde{U}(s, 1) - \tilde{U}(s, 0) \leq -\frac{\varepsilon}{4} < 0 < \tilde{U}(1, 1) - \tilde{U}(1, 0). \quad (1.47)$$

*Proof.* By definition of  $\tilde{U}, \tilde{V}$  (see (1.43) and (1.44)) it holds

$$\tilde{U}(0, 1) - \tilde{U}(0, 0) = \tilde{V}(0, 1) - \tilde{V}(0, 0).$$

Recalling the first inequality of (1.46), we have

$$\tilde{V}(0, 1) - \tilde{V}(0, 0) \leq -\frac{\varepsilon}{2}.$$

By the continuity of the function  $s \mapsto \tilde{U}(s, 1) - \tilde{U}(s, 0)$ , we conclude that there exists  $s_0 \in (0, 1)$  small enough such that

$$\tilde{U}(s, 1) - \tilde{U}(s, 0) \leq -\frac{\varepsilon}{4}$$

for any  $s \in [0, s_0]$ .

□

### Contradiction by using the Turning Tangent Theorem

To sum up, we are considering an isotopy  $H = (H_t)_{t \in [0, 1]}$  in  $\text{Diff}^1(\mathbb{R}^2)$  such that:

- $H_0 = \text{Id}_{\mathbb{R}^2}$  and  $H_1 = H$ ;
- the point  $x \in \mathbb{R}^2$  is fixed with respect to  $H = (H_t)_{t \in [0, 1]}$ ;
- for any  $s \in [0, 1]$ ,

$$\tilde{V}(s, 1) - \tilde{V}(s, 0) < 0 < \tilde{U}(1, 1) - \tilde{U}(1, 0); \quad (1.48)$$

— for any  $s < s_0$ ,

$$\tilde{U}(s, 1) - \tilde{U}(s, 0) < -\frac{\varepsilon}{4} < 0 < \tilde{U}(1, 1) - \tilde{U}(1, 0). \quad (1.49)$$

By eventually changing the reference system on the plane, assume that  $x$  is the origin and that the first vector of the canonical basis coincides with  $\xi = y - x$ .

Denote

$$\bar{s} := \min_{s \in (0,1)} \{s : \tilde{U}(s, 1) - \tilde{U}(s, 0) = 0\}. \quad (1.50)$$

The corresponding  $z(\bar{s}) \in [x, y]$  is the first point of the segment for which the lift of the angle associated to  $H_1(z(s))$  is zero, i.e.  $\tilde{U}(\bar{s}, 1) - \tilde{U}(\bar{s}, 0) = 0$ .

Such  $\bar{s}$  exists by inequality (1.49) and by continuity of  $\tilde{U}$ .

Recall that  $\tilde{U}(s, 1) - \tilde{U}(s, 0)$  does not depend on the chosen lift. It is important considering  $\bar{s}$  as the *first* point of intersection of the image of the segment at time  $t = 1$  with the first coordinate axis (which is the segment at time  $t = 0$ ). Otherwise, we could have no control on the image of the tangent vector through the isotopy.

The proof is divided into 3 cases: starting with the simpler one, we then move on to the most general case.

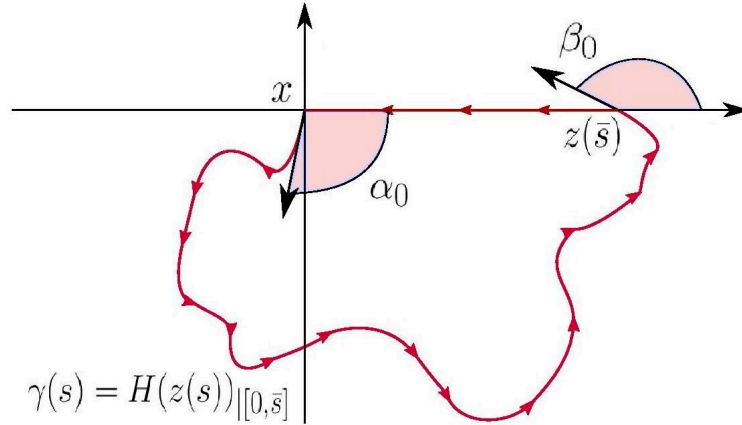


Figure 1.1 – The first case.

*First case:* As a first simpler case, consider the situation presented in Figure (1.1). That is to say, suppose that

- $\alpha_0 = \tilde{U}(0, 1) - \tilde{U}(0, 0) = \tilde{V}(0, 1) - \tilde{V}(0, 0) \in (-1, 0)$ ;
- the curve made up of  $H(z(s))|_{[0, \bar{s}]}$  and the segment  $[z(\bar{s}), x]$  is a simple, closed, piecewise regular, parametrized curve.

Denote

$$\gamma(s) = H(z(s))_{s \in [0, \bar{s}]}.$$

According to this notation, the quantity  $\tilde{V}(s, 1) - \tilde{V}(s, 0)$  is a measure of the angle between the first coordinate axis direction vector and  $\gamma'(s)$ .

By hypothesis -the first inequality of (1.48)- for any  $s \in [0, 1]$  we have

$$\tilde{V}(s, 1) - \tilde{V}(s, 0) < 0. \quad (1.51)$$

The angle  $V(\bar{s}, 1) - V(\bar{s}, 0)$  admits a measure  $\beta_0 \in [0, \frac{1}{2}]$ . Indeed, in a neighborhood of  $z(\bar{s})$ , for  $s < \bar{s}$ , the curve  $\gamma(s)$  crosses the first coordinate axis from the bottom up. So, the tangent vector  $\gamma'(\bar{s})$  has a non negative second coordinate and lies in the upper half-plane.

Look at the continuous determination  $\tilde{V}(\bar{s}, 1) - \tilde{V}(\bar{s}, 0)$ : we have

$$\tilde{V}(\bar{s}, 1) - \tilde{V}(\bar{s}, 0) = \beta_0 + k \quad k \in \mathbb{Z}. \quad (1.52)$$

By inequality (1.51), necessarily

$$k \leq -1. \quad (1.53)$$

Since the curve made up of  $\gamma(s)$  and  $[z(\bar{s}), x]$  is simple, closed and piecewise regular, we can apply the Turning Tangent Theorem on it (see Chapter 4, Section 5 in [DC76]). We obtain

$$\begin{aligned} & \left( (\tilde{V}(\bar{s}, 1) - \tilde{V}(\bar{s}, 0)) - (\tilde{V}(0, 1) - \tilde{V}(0, 0)) \right) + \\ & + \left( \frac{1}{2} - \beta_0 \right) + \left( \alpha_0 + \frac{1}{2} \right) = 1 \end{aligned}$$

that is

$$\beta_0 + k - \alpha_0 + \frac{1}{2} - \beta_0 + \alpha_0 + \frac{1}{2} = 1 + k = 1.$$

This last equality implies  $k = 0$  and contradicts (1.53).

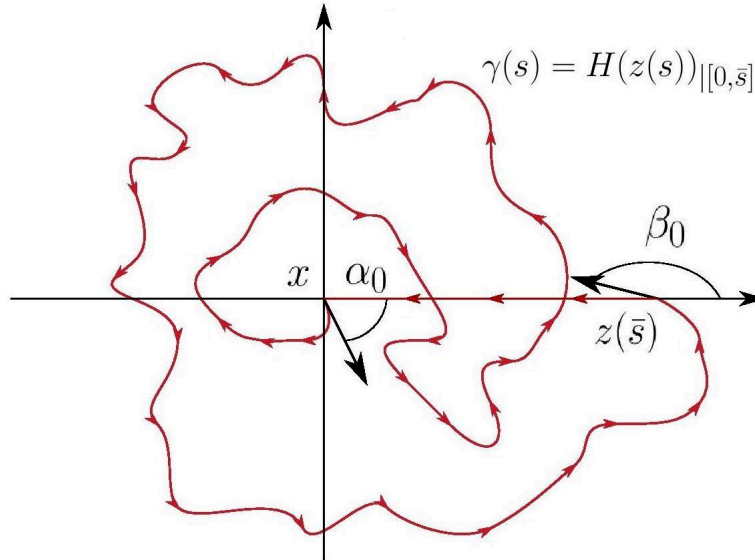


Figure 1.2 – The second case.

*Second case:* Consider the case presented in Figure (1.2). We allow the curve made up of  $\gamma(s) := H(z(s))|_{[0, \bar{s}]}$  and the segment  $[z(\bar{s}), x]$  to have self-intersections, but we require some regularity conditions at the origin.<sup>4</sup>

Define the function  $\Gamma : [0, \bar{s}] \rightarrow \mathbb{R}^+ \times \mathbb{R}$  as

$$\Gamma(s) = (\Gamma_1(s), \Gamma_2(s)) = (r(s), \tilde{U}(s, 1) - \tilde{U}(s, 0)) \quad (1.54)$$

---

4. These conditions will be precised later.

where  $r(s) = \|H(z(s)) - x\| \in \mathbb{R}^+$ .

Denote

$$\begin{aligned} P : \mathbb{R}^+ \times \mathbb{R} &\rightarrow \mathbb{R}^2 \\ (r, \theta) &\mapsto (r \cos(2\pi\theta), r \sin(2\pi\theta)). \end{aligned}$$

Notice that  $P_{|\mathbb{R}_*^+ \times \mathbb{R}}$  is the universal covering of  $\mathbb{R}^2 \setminus \{(0, 0)\}$ . Since  $P \circ \Gamma = \gamma$ , then  $\Gamma$  is a lift of  $\gamma$  through  $P$ . Identifying the plane  $\mathbb{R}^2$  with the complex one  $\mathbb{C}$ , we have

$$\gamma(s) = \Gamma_1(s)e^{i2\pi\Gamma_2(s)}.$$

In other words,  $(\Gamma_1(s), \Gamma_2(s))$  provide some polar “coordinates”. By hypothesis (1.49) and by definition of  $\bar{s}$  in (1.50), it holds

$$\tilde{U}(s, 1) - \tilde{U}(s, 0) = \Gamma_2(s) \leq 0 \quad \forall s \in [0, \bar{s}]. \quad (1.55)$$

Therefore, the curve  $\Gamma$  lies on the low quarter of the half-plane  $\mathbb{R}^+ \times \mathbb{R}$ . Precisely

$$\begin{aligned} \Gamma_1(s) &> 0, & \Gamma_2(s) &< 0 & \forall s \in (0, \bar{s}), \\ \Gamma_1(0) &= 0, & \Gamma_2(0) &< 0, \\ \Gamma_1(\bar{s}) &= \|z(\bar{s}) - x\|, & \Gamma_2(\bar{s}) &= 0. \end{aligned}$$

**Assumption 1.4.1.** Throughout this second case, assume that  $\Gamma$  is sufficiently regular at the origin, that is there exists

$$\Gamma'(0) := \lim_{s \rightarrow 0^+} \Gamma'(s) \neq 0.$$

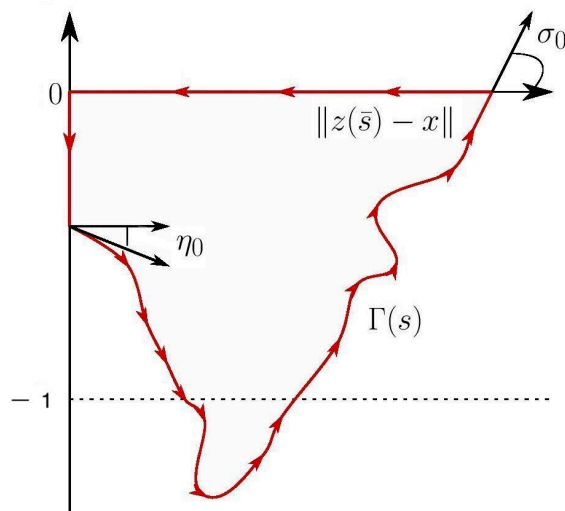


Figure 1.3 – The function  $\Gamma(s)$  in the second case.

**Notation 1.4.6.** Consider the curve in  $\mathbb{R}^+ \times \mathbb{R}$ , made up of



- (i)  $\Gamma(s)$  for  $s \in [0, \bar{s}]$ ;
- (ii) the horizontal segment  $\{0\} \times [r(\bar{s}), 0]$ , followed with decreasing radius;
- (iii) the vertical segment  $[0, \tilde{U}(0, 1) - \tilde{U}(0, 0)] \times \{0\}$ , followed downward.

This curve, thanks to Assumption 1.4.1 and thanks to the definition of  $\bar{s}$  in (1.50), is a simple, closed, piecewise regular curve (see Figure 1.3).

The vector  $\Gamma'(0)$  is oriented to the right in the plane  $\mathbb{R}^+ \times \mathbb{R}$ . Hence, the angle between the first coordinate axis direction vector and  $\Gamma'(0)$  admits a measure  $\eta_0 \in [-\frac{1}{4}, \frac{1}{4}]$ . Denote as  $\sigma_0$  the measure contained in the interval  $[0, \frac{1}{2}]$  of the angle between the first coordinate axis direction vector and  $\Gamma'(\bar{s})$ . Such a measure exists since in a neighborhood of  $\Gamma(\bar{s})$  the curve  $\Gamma$  crosses the first coordinate axis from the bottom up and so the tangent vector  $\Gamma'(\bar{s})$  lies then in the upper half-plane.

**Notation 1.4.7.** Denote as

$$\prec(\Gamma') : [0, \bar{s}] \rightarrow \mathbb{T}$$

the oriented angle function between the first coordinate axis direction vector and the vector  $\Gamma'(s)$ .

The notation  $\tilde{\prec}(\Gamma') : [0, \bar{s}] \rightarrow \mathbb{R}$  refers to the continuous determination of the angle function  $\prec(\Gamma')$  such that  $\tilde{\prec}(\Gamma'(0)) = \eta_0 \in [-\frac{1}{4}, \frac{1}{4}]$ .

Since  $\sigma_0$  and  $\tilde{\prec}(\Gamma'(\bar{s}))$  are lifts of the same oriented angle, we have

$$\tilde{\prec}(\Gamma'(\bar{s})) = \sigma_0 + j \quad j \in \mathbb{Z}.$$

Apply now the Turning Tangent Theorem to the closed curve highlighted in Notation 1.4.6. We obtain

$$(\sigma_0 + j - \eta_0) + \left(\frac{1}{2} - \sigma_0\right) + \frac{1}{4} + \left(\eta_0 + \frac{1}{4}\right) = 1$$

and so

$$1 + j = 1 \quad \Leftrightarrow \quad j = 0. \tag{1.56}$$

Hence

$$\tilde{\prec}(\Gamma'(\bar{s})) = \sigma_0. \tag{1.57}$$

Let us look now at the relation between the tangent vectors of  $\Gamma(\cdot)$  and the tangent ones of  $\gamma(\cdot) = H(z(\cdot))$ .

By hypothesis (1.48), it holds

$$\tilde{V}(s, 1) - \tilde{V}(s, 0) < 0 \quad \forall s \in [0, \bar{s}] \tag{1.58}$$

where  $\tilde{V}(s, 1) - \tilde{V}(s, 0)$  is a continuous determination of the angle function between the first coordinate axis direction vector and  $\gamma'(s)$ .

Denote

$$\tilde{V}(\bar{s}, 1) - \tilde{V}(\bar{s}, 0) = \beta_0 + k \quad \text{for some } k \in \mathbb{Z} \tag{1.59}$$

where  $\beta_0$  is the measure of the angle  $V(\bar{s}, 1) - V(\bar{s}, 0)$  in  $[0, \frac{1}{2}]$ . Such a measure exists since in a neighborhood of  $z(\bar{s})$  the curve  $\gamma$  crosses the first coordinate axis from the bottom up. Hence, the vector  $\gamma'(\bar{s})$  has non negative second coordinate. From (1.58), it holds then

$$k \leq -1. \quad (1.60)$$

We need now the following:

**Proposition 1.4.1.** *Let  $I \subset \mathbb{R}$  be an interval and let  $M, N$  be two 2-dimensional oriented Riemannian manifolds. Denote the tangent projections as  $\pi_M : TM \rightarrow M$ ,  $\pi_N : TN \rightarrow N$ . Let  $f : M \rightarrow N$  be a local diffeomorphism which preserves the orientation and let  $J_1 : I \rightarrow TM$ ,  $J_2 : I \rightarrow TM$  be continuous functions such that*

$$\pi_M \circ J_1 = \pi_M \circ J_2. \quad (1.61)$$

*Suppose that, for any  $t \in I$ ,  $J_i(t) \neq 0$ ,  $i = 1, 2$  and let  $\theta : I \rightarrow \mathbb{R}$  be a continuous determination of the angle function between the image vectors  $J_1, J_2$ .*

*Then, there exists a continuous determination  $\Theta : I \rightarrow \mathbb{R}$  of the angle function between the image vectors  $Df \circ J_1, Df \circ J_2$  such that*

$$|\theta(s) - \Theta(s)| < \frac{1}{2} \quad \forall s \in I \quad (1.62)$$

We postpone the proof of this Proposition to Appendix 1.6.

We apply Proposition 1.4.1 to  $I = (0, \bar{s}] \subset \mathbb{R}$ ,  $M = (\mathbb{R}^+ \setminus \{0\}) \times \mathbb{R}$ ,  $N = \mathbb{R}^2 \setminus \{0\}$  and

$$f : (\mathbb{R}^+ \setminus \{0\}) \times \mathbb{R} \rightarrow \mathbb{R}^2 \setminus \{0\}$$

$$(r, \theta) \mapsto (r \cos(2\pi\theta), r \sin(2\pi\theta)).$$

Observe that the determinant of  $Df(r, \theta)$  is equal to  $r$  and so always positive: this assures us that  $f$  is a local diffeomorphism which preserves the orientation. Consider

$$\begin{aligned} J_1 : (0, \bar{s}] &\rightarrow TM = (\mathbb{R}^+ \setminus \{0\}) \times \mathbb{R} \times \mathbb{R}^2 \\ s &\mapsto (\Gamma(s), (1, 0)) \end{aligned}$$

and

$$\begin{aligned} J_2 : (0, \bar{s}] &\rightarrow TM = (\mathbb{R}^+ \setminus \{0\}) \times \mathbb{R} \times \mathbb{R}^2 \\ s &\mapsto (\Gamma(s), \Gamma'(s)). \end{aligned}$$

Then

$$\begin{aligned} Df \circ J_1 : (0, \bar{s}] &\rightarrow TN = (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2 \\ s &\mapsto \left( \gamma(s), \frac{\gamma(s)}{\|\gamma(s)\|} \right) \end{aligned}$$

and

$$\begin{aligned} Df \circ J_2 : (0, \bar{s}] &\rightarrow TN = (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2 \\ s &\mapsto (\gamma(s), \gamma'(s)). \end{aligned}$$

The function  $\tilde{\angle}(\Gamma')$  introduced in Notation 1.4.7 is a continuous determination of the angle function between  $J_1(s)$  and  $J_2(s)$ . By Assumption 1.4.1, the function  $\tilde{\angle}(\Gamma')$  is continuous at  $s = 0$ .

Remind that, by our choice

$$\tilde{\angle}(\Gamma'(0)) = \eta_0 \in \left[-\frac{1}{4}, \frac{1}{4}\right]. \quad (1.63)$$

Observe that  $s \mapsto \left(\tilde{V}(s, 1) - \tilde{V}(s, 0)\right) - \left(\tilde{U}(s, 1) - \tilde{U}(s, 0)\right)$  is a continuous determination of the angle function between  $Df \circ J_1(s)$  and  $Df \circ J_2(s)$ . By our choice of  $\tilde{V}, \tilde{U}$ , for any  $t$  we have  $\tilde{V}(0, t) = \tilde{U}(0, t)$  and in particular

$$\left(\tilde{V}(0, 1) - \tilde{V}(0, 0)\right) - \left(\tilde{U}(0, 1) - \tilde{U}(0, 0)\right) = 0. \quad (1.64)$$

From (1.63), (1.64) and the continuity of the involved functions, there exists  $S > 0$  small enough such that

$$\left|\tilde{\angle}(\Gamma'(S)) - \left(\left(\tilde{V}(S, 1) - \tilde{V}(S, 0)\right) - \left(\tilde{U}(S, 1) - \tilde{U}(S, 0)\right)\right)\right| < \frac{1}{2}.$$

By Proposition 1.4.1, we deduce that for any  $s \in (0, \bar{s}]$

$$\left|\tilde{\angle}(\Gamma'(s)) - \left(\left(\tilde{V}(s, 1) - \tilde{V}(s, 0)\right) - \left(\tilde{U}(s, 1) - \tilde{U}(s, 0)\right)\right)\right| < \frac{1}{2}.$$

In particular, at  $s = \bar{s}$  by (1.57), (1.59) and (1.50)

$$\begin{aligned} \left|\tilde{\angle}(\Gamma'(\bar{s})) - \left(\left(\tilde{V}(\bar{s}, 1) - \tilde{V}(\bar{s}, 0)\right) - \left(\tilde{U}(\bar{s}, 1) - \tilde{U}(\bar{s}, 0)\right)\right)\right| &= \\ &= |\sigma_0 - \beta_0 - k| < \frac{1}{2}. \end{aligned} \quad (1.65)$$

**Claim 1.4.1.** The quantity  $\sigma_0 - \beta_0$  is in the open interval  $(-\frac{1}{2}, \frac{1}{2})$ .

*Proof of the claim.* Because  $\sigma_0, \beta_0 \in [0, \frac{1}{2}]$ , the difference  $\sigma_0 - \beta_0$  is in  $[-\frac{1}{2}, \frac{1}{2}]$ . Arguing by contradiction, suppose that  $\sigma_0 - \beta_0 = \frac{1}{2}$ , that is  $\sigma_0 = \frac{1}{2}, \beta_0 = 0$ . The measure  $\sigma_0$  is a lift of the angle between  $\mathcal{H}$  and  $\Gamma'(\bar{s})$ , while  $\beta_0$  is a lift of the angle between  $\gamma(\bar{s})/\|\gamma(\bar{s})\|$  and  $\gamma'(\bar{s})$ , which are the vectors  $Df(\Gamma(\bar{s}))\mathcal{H}$  and  $Df(\Gamma(\bar{s}))\Gamma'(\bar{s})$ . Since  $Df(\Gamma(\bar{s}))$  is a linear function and by inequality (1.65), this case cannot occur. Similarly, the case  $\sigma_0 - \beta_0 = -\frac{1}{2}$  is excluded.  $\square$

Since  $\sigma_0 - \beta_0 \in (-\frac{1}{2}, \frac{1}{2})$  and by (1.65), we deduce that  $k = 0$ . This inequality contradicts condition (1.60) and we conclude.

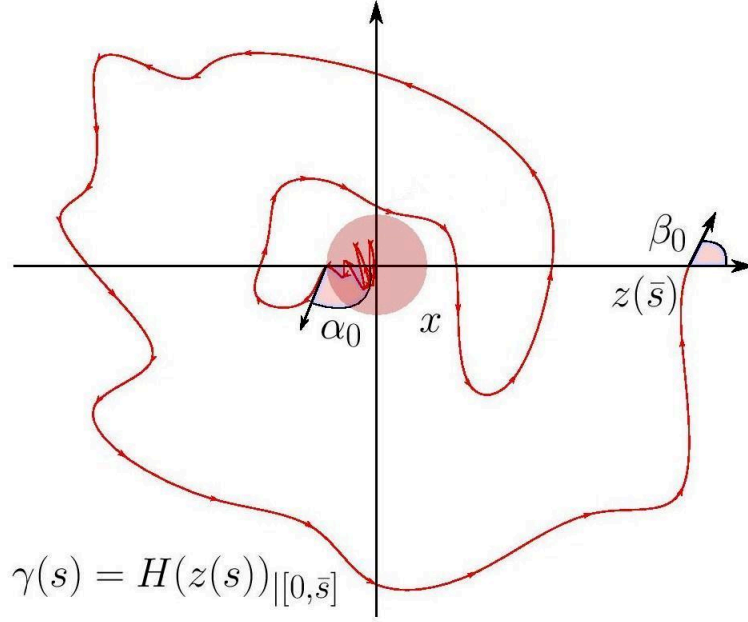


Figure 1.4 – The most general case.

*Third case:* Finally, consider the most general case, presented in Figure (1.4). We allow now the vector  $\Gamma'(0)$  not to exist or to be null.

The Turning Tangent Theorem can no more be applied on the curve used in the second case.

Fix  $\rho \in (0, \Gamma_1(\bar{s}))$  and consider the vertical line  $r \equiv \rho$  in  $\mathbb{R}^+ \times \mathbb{R}$ .

The notations  $\Gamma_1(\cdot), \Gamma_2(\cdot)$  refer to the first and second coordinates, respectively, of the curve  $\Gamma$  in  $\mathbb{R}^+ \times \mathbb{R}$ . Define then

$$s_\rho := \max_{s \in [0, \bar{s}]} \{s : \Gamma_1(s) = \rho\}. \quad (1.66)$$

This is a *maximum* since  $\Gamma_1$  is a continuous function considered on a compact interval  $[0, \bar{s}]$  where  $\Gamma_1(0) = 0$  and  $\Gamma_1(\bar{s}) > \rho$ .

Observe that

$$\lim_{\rho \rightarrow 0} s_\rho = 0$$

by the continuity of the function  $\Gamma_1(\cdot)$ , the compactness of the interval involved and the fact that  $s = 0$  is the only point for which the first coordinate projection of the curve vanishes.

Denote as  $\eta_0$  the measure of the angle between the first coordinate axis direction vector and  $\Gamma'(s_\rho)$  in the interval  $[-\frac{1}{4}, \frac{1}{4}]$ . This choice is possible since by definition of  $s_\rho$  the vector  $\Gamma'(s_\rho)$  is oriented to the right.

Let

$$\prec(\Gamma') : [s_\rho, \bar{s}] \rightarrow \mathbb{T}$$

denote the oriented angle between the first coordinate axis direction vector and the vector  $\Gamma'(s)$  and denote

$$\tilde{\prec}(\Gamma') : [s_\rho, \bar{s}] \rightarrow \mathbb{R}$$

the continuous determination of the angle function such that  $\tilde{\prec}(\Gamma'(s_\rho)) = \eta_0 \in [-\frac{1}{4}, \frac{1}{4}]$ .

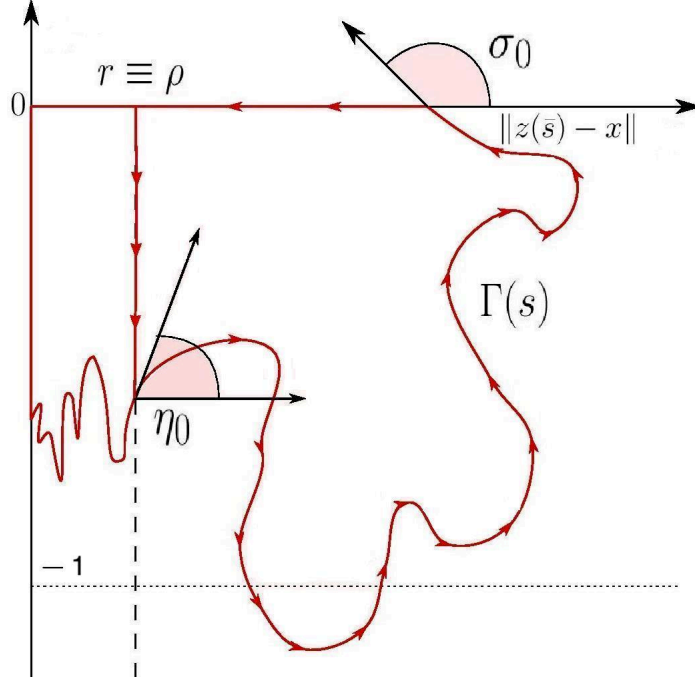


Figure 1.5 – The function  $\Gamma(s)$  in the general case.

**Claim 1.4.2.** If  $\rho$  is small enough, for any  $s \in [s_\rho, \bar{s}]$

$$\left| \prec(\Gamma'(s)) - \left( \left( \tilde{V}(s, 1) - \tilde{V}(s, 0) \right) - \left( \tilde{U}(s, 1) - \tilde{U}(s, 0) \right) \right) \right| < \frac{1}{2}. \quad (1.67)$$

*Proof of the claim.* Recall that  $\lim_{\rho \rightarrow 0^+} s_\rho = 0$  and functions  $\tilde{V}, \tilde{U}$  are continuous. Moreover, since  $\tilde{V}(0, t) = \tilde{U}(0, t)$  for any  $t$ ,

$$\left( \tilde{V}(0, 1) - \tilde{V}(0, 0) \right) - \left( \tilde{U}(0, 1) - \tilde{U}(0, 0) \right) = 0.$$

Then, for any  $\varepsilon > 0$ , there exists  $\rho > 0$  small enough such that

$$\left| \left( \tilde{V}(s_\rho, 1) - \tilde{V}(s_\rho, 0) \right) - \left( \tilde{U}(s_\rho, 1) - \tilde{U}(s_\rho, 0) \right) \right| < \varepsilon.$$

So, it holds

$$\left| \prec(\Gamma'(s_\rho)) - \left( \left( \tilde{V}(s_\rho, 1) - \tilde{V}(s_\rho, 0) \right) - \left( \tilde{U}(s_\rho, 1) - \tilde{U}(s_\rho, 0) \right) \right) \right| < \frac{1}{4} + \varepsilon.$$

By selecting  $\varepsilon > 0$  small enough such that  $\frac{1}{4} + \varepsilon < \frac{1}{2}$ , we have

$$\left| \prec(\Gamma'(s_\rho)) - \left( \left( \tilde{V}(s_\rho, 1) - \tilde{V}(s_\rho, 0) \right) - \left( \tilde{U}(s_\rho, 1) - \tilde{U}(s_\rho, 0) \right) \right) \right| < \frac{1}{2}.$$

By applying Proposition 1.4.1, inequality (1.67) holds for any  $s \in [s_\rho, \bar{s}]$ . □

Denote as  $\sigma_0$  the measure contained in  $[0, \frac{1}{2}]$  of the angle  $\prec(\Gamma'(\bar{s}))$ : again, this is possible because in a neighborhood of  $\Gamma(\bar{s})$ , the curve  $\Gamma$  crosses the first coordinate axis from the bottom up.

Let  $\beta_0$  be the measure of the angle  $V(\bar{s}, 1) - V(\bar{s}, 0)$  contained in  $[0, \frac{1}{2}]$ . Since  $\sigma_0$  and  $\tilde{\prec}(\Gamma'(\bar{s}))$  are continuous lifts of the angle  $\prec(\Gamma'(\bar{s}))$ , we have

$$\tilde{\prec}(\Gamma'(\bar{s})) = \sigma_0 + l \quad \text{for some } l \in \mathbb{Z},$$

$$\begin{aligned} \tilde{V}(\bar{s}, 1) - \tilde{V}(\bar{s}, 0) &= {}^5 \left( \tilde{V}(\bar{s}, 1) - \tilde{V}(\bar{s}, 0) \right) - \left( \tilde{U}(\bar{s}, 1) - \tilde{U}(\bar{s}, 0) \right) = \\ &= \beta_0 + j \quad \text{for some } j \in \mathbb{Z}. \end{aligned}$$

By inequality (1.67) it holds

$$\begin{aligned} \left| \tilde{\prec}(\Gamma'(\bar{s})) - \left( (\tilde{V}(\bar{s}, 1) - \tilde{V}(\bar{s}, 0)) - (\tilde{U}(\bar{s}, 1) - \tilde{U}(\bar{s}, 0)) \right) \right| = \\ = |\sigma_0 + l - \beta_0 - j| < \frac{1}{2}. \end{aligned} \quad (1.68)$$

By hypothesis (1.48),  $j \leq -1$ .

**Claim 1.4.3.** The quantity  $\sigma_0 - \beta_0$  is in the open interval  $(-\frac{1}{2}, \frac{1}{2})$ .

The argument is the same as Claim 1.4.1 in the second case.

Therefore  $l = j$  and so

$$l \leq -1. \quad (1.69)$$

Let us now consider the curve made up of

- (i)  $\Gamma_{|[s_\rho, \bar{s}]}$ , positively oriented;
- (ii) the horizontal segment  $\{0\} \times \{r : \rho \leq r \leq \|z(\bar{s}) - x\|\}$ , followed with decreasing radius;
- (iii) the vertical segment  $[\Gamma_2(s_\rho), 0] \times \{r \equiv \rho\}$ , followed downward.

This curve is a simple, closed, piecewise regular, parametrized one thanks to the regularity of the polar coordinates away from the origin and to the absence of self-intersections by the definition of  $s_\rho$  (see Figure 1.5).

Apply the Turning Tangent Theorem to this curve. We obtain then

$$(\sigma_0 + l - \eta_0) + \left( \frac{1}{2} - \sigma_0 \right) + \frac{1}{4} + \left( \eta_0 + \frac{1}{4} \right) = 1,$$

that is

$$1 + l = 1.$$

This implies  $l = 0$ , contradicting inequality (1.69).

□

---

5. By definition of  $\bar{s}$ ,  $\tilde{U}(\bar{s}, 1) - \tilde{U}(\bar{s}, 0)$  is null.

## 1.5 Examples

**Example 1.5.1** (Time-one flow of pendulum). Fix the standard Riemannian metric and the standard trivialization on  $\mathbb{A}$ . Let  $\mathcal{H} = (1, 0)$  be the reference constant vector field. Consider the dynamical system of the simple pendulum obtained from the Hamiltonian

$$H : \mathbb{A} \rightarrow \mathbb{R}$$

$$(\psi, r) \mapsto H(\psi, r) = \frac{r^2}{2} - \frac{\cos(2\pi\psi)}{4\pi^2}.$$

Let  $\phi : \mathbb{A} \times \mathbb{R} \rightarrow \mathbb{A}$  be the associated flow (see Figure 1.6). We calculate the torsion of the time-one flow of the pendulum, i.e.  $f = \phi(\cdot, 1) = \phi_1$ . We consider as isotopy the flow itself  $I = (\phi_t)_t = (\phi(\cdot, t))_t$ . Recall that from Proposition 1.3.2, the torsion (already at finite-time) does not depend on the choice of the isotopy.

At the fixed point  $(\frac{1}{2}, 0)$  for any  $t$  the vector  $(1, -1)$  is an eigenvector of  $D\phi_t(\frac{1}{2}, 0)$  with respect to a real positive eigenvalue. That is, the half-line  $\mathbb{R}_+(1, -1)$  is preserved by  $D\phi_t(\frac{1}{2}, 0)$  for any  $t$ . Therefore, recalling that the torsion does not depend on the chosen vector (see Proposition 1.1.3), we have

$$\text{Torsion}\left(\phi_1, \left(\frac{1}{2}, 0\right)\right) = \lim_{n \rightarrow +\infty} \text{Torsion}_n\left(\phi_1, \left(\frac{1}{2}, 0\right), (1, -1)\right) = 0.$$

At the elliptic fixed point  $(0, 0)$ , the differential  $D\phi_t(0, 0)$  is a clockwise rotation of angle  $t$ . Therefore, for any vector  $v \in T_{(0,0)}\mathbb{A}$ ,

$$\text{Torsion}(\phi_1, (0, 0)) = \lim_{n \rightarrow +\infty} \text{Torsion}_n(\phi_1, (0, 0), v) = -1.$$

For any other point  $\bar{z}$  of  $\mathbb{A}$  we are going to calculate the finite-time torsion with respect to  $X_H(\bar{z}) = \left(\frac{\partial}{\partial r}H(\bar{z}), -\frac{\partial}{\partial \psi}H(\bar{z})\right)$ , where  $X_H$  denotes the Hamiltonian vector field. Denote as  $U$  the open region contained between the two separatrices of the pendulum system.

Let  $\bar{z} \notin U$ . Denote as  $\phi_t(\bar{z}) = (\psi(t), r(t))$ . Observe that for any  $t \in \mathbb{R}$  the coordinate  $r(t)$  never changes sign. Equivalently, the vector

$$X_H(\phi_t(\bar{z})) = \left(r(t), -\frac{\sin(2\pi\psi(t))}{2\pi}\right)$$

is always contained either in the right half-plane (if  $r(t)$  is positive) or in the left half-plane (if  $r(t)$  is negative). In particular, for any  $t$  we have

$$\tilde{v}(I)(\bar{z}, X_H(\bar{z}), t) - \tilde{v}(I)(\bar{z}, X_H(\bar{z}), 0) \in \left(-\frac{1}{2}, \frac{1}{2}\right).$$

Since the angle variation is bounded for any  $t$ , we conclude that for any  $\bar{z} \notin U$

$$\text{Torsion}(\phi_1, \bar{z}) = 0.$$

Let  $\bar{z} \in U \setminus \{(0, 0)\}$ . The point  $\bar{z}$  is periodic and we denote as  $T(\bar{z})$  the period of  $\bar{z}$ . Observe that the vector  $X_H(\phi_t(\bar{z}))$  is turning clockwise once over a time interval of length  $T(\bar{z})$ . Therefore

$$\tilde{v}(I)(\bar{z}, X_H(\bar{z}), T(\bar{z})) - \tilde{v}(I)(\bar{z}, X_H(\bar{z}), 0) = -1$$

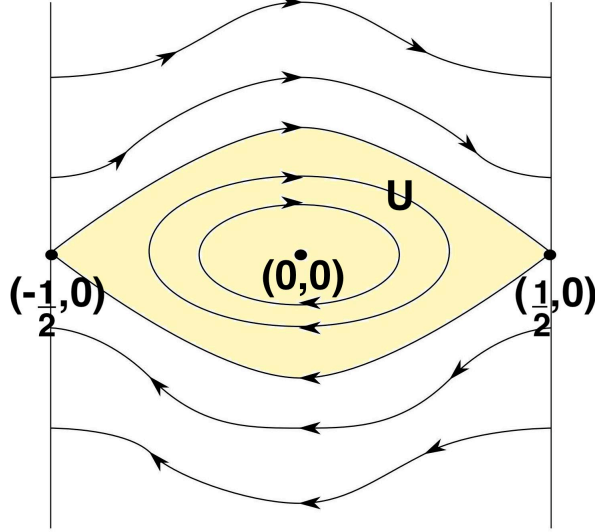


Figure 1.6 – The dynamical system of the pendulum of Example 1.5.1.

and for any  $n \in \mathbb{N}^*$

$$\tilde{v}(I)(\bar{z}, X_H(\bar{z}), nT(\bar{z})) - \tilde{v}(I)(\bar{z}, X_H(\bar{z}), 0) = -n.$$

By the compactness of the orbit of  $\bar{z}$  and of the interval  $[0, T(\bar{z})]$ , we conclude that

$$\begin{aligned} \text{Torsion}(\phi_1, \bar{z}) &= \lim_{n \rightarrow +\infty} \text{Torsion}_n(\phi_1, \bar{z}, X_H(\bar{z})) = \\ &= \lim_{n \rightarrow +\infty} \frac{1}{n} \left( \tilde{v}(I) \left( \bar{z}, X_H(\bar{z}), \left\lfloor \frac{n}{T(\bar{z})} \right\rfloor T(\bar{z}) \right) - \tilde{v}(I)(\bar{z}, X_H(\bar{z}), T(\bar{z})) \right) = -\frac{1}{T(\bar{z})}. \end{aligned}$$

**Example 1.5.2** (Example 2 in [MS17]). We discuss the example of the dissipative pendulum presented in [MS17]. That is, the system is obtained by adding a dissipative term to the classical pendulum system. Its flow is denoted as  $\phi : \mathbb{A} \times \mathbb{R} \rightarrow \mathbb{A}, (t, \psi_0, r_0) \mapsto \phi(t; \psi_0, r_0) = (\psi(t), r(t))$  and it is defined by

$$\begin{cases} \frac{d}{dt}\psi(t) = r(t) \\ \frac{d}{dt}r(t) = -\frac{\sin(2\pi\psi(t))}{2\pi} - \lambda r(t) \end{cases}$$

for  $\lambda > 0$ . Denote  $X_H(\psi(t), r(t))$  the vector  $\left( r(t), -\frac{\sin(2\pi\psi(t))}{2\pi} - \lambda r(t) \right)$  belonging to  $T_{(\psi(t), r(t))}\mathbb{A}$ .

We ask that  $\lambda < 2$  to assure that  $D\phi_1(0, 0)$  is conjugated to a rotation (see the discussion at the point  $(0, 0)$  below).

The phase portrait is sketched in Figure 1.7. As in Example 1.5.1, we are going to discuss the torsion of the time-one flow  $f = \phi_1$ .

At the fixed point  $(\frac{1}{2}, 0)$  there exists a vector  $v$  such that for any  $t$  it is an eigenvector of  $D\phi_t(\frac{1}{2}, 0)$  with respect to a real positive eigenvalue. Therefore

$$\text{Torsion} \left( \phi_1, \left( \frac{1}{2}, 0 \right) \right) = \lim_{n \rightarrow +\infty} \text{Torsion}_n \left( \phi_1, \left( \frac{1}{2}, 0 \right), v \right) = 0.$$



At the fixed point  $(0,0)$ , the differential  $D\phi_t(0,0)$  is conjugated to the matrix

$$\begin{pmatrix} e^{-\frac{\lambda}{2}t} & 0 \\ 0 & e^{-\frac{\lambda}{2}t} \end{pmatrix} \mathcal{R} \left( (0,0), -t \frac{\sqrt{4-\lambda^2}}{2} \right), \quad (1.70)$$

that is to the composition of a contraction and a rotation. We consider a  $\mathcal{C}^1$  diffeomorphism  $h$  isotopic to the identity with compact support  $U$  such that in a neighborhood of  $(0,0)$  it is a change of coordinates so that:

- $h(0,0) = (0,0)$ ;
- for any  $t$  it holds that  $D(h \circ \phi_t \circ h^{-1})((0,0))$  is the matrix in (1.70).

Consequently, we have for any  $v \in T_{(0,0)}\mathbb{A}$

$$\text{Torsion}_1(h \circ \phi_1 \circ h^{-1}, (0,0), v) = -\frac{\sqrt{4-\lambda^2}}{2}$$

and so

$$\text{Torsion}(h \circ \phi_1 \circ h^{-1}, (0,0)) = -\frac{\sqrt{4-\lambda^2}}{2}.$$

From Proposition 1.3.3 (i.e. the invariance of the torsion for diffeomorphisms isotopic to the identity with compact support on the unbounded annulus), we conclude that

$$\text{Torsion}(\phi_1, (0,0)) = \text{Torsion}(h \circ \phi_1 \circ h^{-1}, (0,0)) = -\frac{\sqrt{4-\lambda^2}}{2}.$$

Denote as  $\alpha$  the value  $-\frac{\sqrt{4-\lambda^2}}{2}$ .

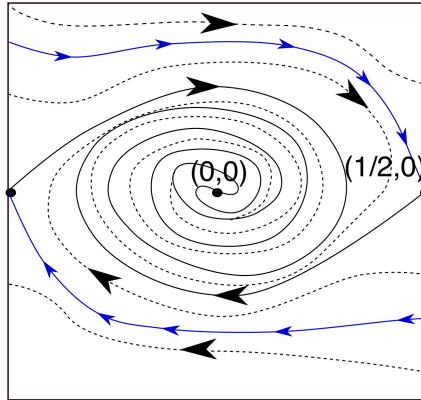


Figure 1.7 – The dynamical system of the dissipative pendulum of Example 1.5.2.

Let us consider now a point  $\bar{x} \in \mathbb{A} \setminus \{(0,0), (\frac{1}{2}, 0)\}$ . Let  $\bar{x}$  belong to the stable manifold of  $(0,0)$ , i.e.

$$W^s((0,0)) = \{\bar{x} \in \mathbb{A} : \lim_{t \rightarrow +\infty} \phi_t(\bar{x}) = (0,0)\}.$$

Recall that  $U$  denotes the neighborhood of  $(0,0)$  which is the support of the change of coordinates  $h$  introduced above. The time-one torsion of  $h \circ \phi_1 \circ h^{-1}$  is continuous with

respect to the point in  $\mathbb{A}$  where we calculate the time-one torsion. Fix  $\varepsilon > 0$ . Let  $V$  be a neighborhood of  $(0,0)$  contained in  $U$  such that for any  $x \in V$  for any  $v \in T_x^1\mathbb{A}$  it holds<sup>6</sup>

$$|\text{Torsion}_1(h \circ \phi_1 \circ h^{-1}, x, v) - \text{Torsion}_1(h \circ \phi_1 \circ h^{-1}, (0,0), v)| < \varepsilon. \quad (1.71)$$

Recall that  $\text{Torsion}_1(h \circ \phi_1 \circ h^{-1}, (0,0), v) = \alpha$  for any  $v \in T_{(0,0)}^1\mathbb{A}$ .

Denote as  $\bar{z} = h(\bar{x})$ . Observe that

$$\lim_{n \rightarrow +\infty} h \circ \phi_n \circ h^{-1}(\bar{z}) = \lim_{n \rightarrow +\infty} h \circ \phi_n(\bar{x}) = h((0,0)) = (0,0).$$

Hence, the point  $\bar{z}$  belongs to the stable manifold of  $(0,0)$  with respect to  $h \circ \phi_1 \circ h^{-1}$ . Consequently, there exists  $N \in \mathbb{N}$  such that for any  $n \geq N$  it holds

$$(h \circ \phi_1 \circ h^{-1})^n(\bar{z}) = (h \circ \phi_n \circ h^{-1})(\bar{z}) \in V. \quad (1.72)$$

Denote as  $\bar{y} = h \circ \phi_N \circ h^{-1}(\bar{z}) \in V$ .

**Claim 1.5.1.** For  $n \in \mathbb{N}$  we have

$$\text{Torsion}_n(h \circ \phi_1 \circ h^{-1}, \bar{y}, v) \in (\alpha - \varepsilon, \alpha + \varepsilon).$$

*Proof.* For  $n = 1$  the claim holds because  $\bar{y} \in V$  and from (1.71). Assume the statement holds true for  $n - 1$ . Then

$$\begin{aligned} n \text{Torsion}_n(h \circ \phi_1 \circ h^{-1}, \bar{y}, v) &= (n - 1) \text{Torsion}_{n-1}(h \circ \phi_1 \circ h^{-1}, \bar{y}, v) + \\ &\text{Torsion}_1(h \circ \phi_1 \circ h^{-1}, h \circ \phi_{n-1} \circ h^{-1}(\bar{y}), D(h \circ \phi_{n-1} \circ h^{-1})(\bar{y})v). \end{aligned}$$

From inductive hypothesis it holds

$$(n - 1) \text{Torsion}_{n-1}(h \circ \phi_1 \circ h^{-1}, \bar{y}, v) \in ((n - 1)(\alpha - \varepsilon), (n - 1)(\alpha + \varepsilon)). \quad (1.73)$$

Because of (1.72), the point  $h \circ \phi_{n-1} \circ h^{-1}(\bar{y})$  belongs to  $V$ . From (1.71), it holds (up to renormalize the vector)

$$\text{Torsion}_1(h \circ \phi_1 \circ h^{-1}, h \circ \phi_{n-1} \circ h^{-1}(\bar{y}), D(h \circ \phi_{n-1} \circ h^{-1})(\bar{y})v) \in (\alpha - \varepsilon, \alpha + \varepsilon). \quad (1.74)$$

Consequently, from (1.73) and (1.74),

$$n \text{Torsion}_n(h \circ \phi_1 \circ h^{-1}, \bar{y}, v) \in (n\alpha - n\varepsilon, n\alpha + n\varepsilon)$$

and so

$$\text{Torsion}_n(h \circ \phi_1 \circ h^{-1}, \bar{y}, v) \in (\alpha - \varepsilon, \alpha + \varepsilon).$$

□

We so deduce that

$$\text{Torsion}(h \circ \phi_1 \circ h^{-1}, \bar{y}) \in (\alpha - \varepsilon, \alpha + \varepsilon).$$

By the arbitrariness of  $\varepsilon$  and from the invariance of the torsion along the orbit of  $\bar{z}$ , it holds

$$\text{Torsion}(h \circ \phi_1 \circ h^{-1}, \bar{z}) = \text{Torsion}(h \circ \phi_1 \circ h^{-1}, \bar{y}) = \alpha.$$

---

6. We are identifying the unitary tangent spaces  $T_x^1\mathbb{A}$  and  $T_{(0,0)}^1\mathbb{A}$  through the standard trivialization.

Thanks to the invariance of the torsion for  $\mathcal{C}^1$  conjugation isotopic to the identity with compact support on the annulus (see Proposition 1.3.3), we conclude that

$$\text{Torsion}(\phi_1, \bar{x}) = \text{Torsion}(h \circ \phi_1 \circ h^{-1}, \bar{z}) = \alpha.$$

Let  $\bar{x} \in \mathbb{A} \setminus \{(0, 0), (\frac{1}{2}, 0)\}$  be a point of the stable manifold of  $(\frac{1}{2}, 0)$ , i.e.  $\lim_{t \rightarrow +\infty} \phi_t(\bar{x}) = (\frac{1}{2}, 0)$ . Recall that a vector  $v$  of the stable subspace of  $(\frac{1}{2}, 0)$  is an eigenvector with respect to a positive eigenvalue of  $D\phi_t(\frac{1}{2}, 0)$  for any  $t$ . Hence

$$\text{Torsion}_1 \left( \phi_1, \left( \frac{1}{2}, 0 \right), v \right) = 0.$$

Fix  $\varepsilon > 0$ . By the continuity of the torsion at finite-time and by the local stable manifold theorem, there exists a neighborhood  $U$  of  $(\frac{1}{2}, 0)$  such that for any  $x$  belonging to the connected component of  $U \cap W^s(\frac{1}{2}, 0)$  containing  $(\frac{1}{2}, 0)$  (i.e. the local stable manifold) it holds

$$|\text{Torsion}_1(\phi_1, x, w_x)| < \varepsilon, \quad (1.75)$$

where  $w_x \in T_x W^s(\frac{1}{2}, 0)$ . There exists  $n = n(\bar{x}) \in \mathbb{N}$  such that for any  $m \geq n$  we have that  $\phi_m(\bar{x})$  belongs to the local stable manifold. Hence, by (1.75), for any  $l > 0$

$$|\text{Torsion}_l(\phi_1, \phi_n(\bar{x}), w_{\phi_n(\bar{x})})| < \varepsilon.$$

That is

$$\begin{aligned} -\varepsilon &< \liminf_{l \rightarrow +\infty} \text{Torsion}_l(\phi_1, \phi_n(\bar{x}), w_{\phi_n(\bar{x})}) = \liminf_{l \rightarrow +\infty} \text{Torsion}_l(\phi_1, \bar{x}, w_{\bar{x}}) \leq \\ &\leq \limsup_{l \rightarrow +\infty} \text{Torsion}_l(\phi_1, \bar{x}, w_{\bar{x}}) = \limsup_{l \rightarrow +\infty} \text{Torsion}_l(\phi_1, \phi_n(\bar{x}), w_{\phi_n(\bar{x})}) < \varepsilon. \end{aligned}$$

By the arbitrariness of  $\varepsilon$ , we conclude that the torsion at  $\bar{x}$  exists and  $\text{Torsion}(\phi_1, \bar{x}) = 0$ .

**Remark 1.5.1.** Looking at the dynamical system of Example 1.5.2 with reversed time, we obtain an example where the torsion exists everywhere, the point  $(0, 0)$  has torsion value  $\alpha$ , while all the other points have null torsion.

Indeed, if a point  $x$  belongs to the stable manifold (for  $\phi_t^{-1}$ ) of  $(\frac{1}{2}, 0)$ , then we can show that  $\text{Torsion}(\phi_1^{-1}, x) = 0$ . The details are the same as those of the torsion of points belonging to the stable manifold of  $(\frac{1}{2}, 0)$  for  $\phi_1$ .

Let  $x = (\psi(0), r(0))$  have unbounded orbit. We calculate the torsion with respect to the vector field

$$X_H(\phi_{-t}(x)) = \left( r(-t), -\frac{\sin(2\pi\psi(-t))}{2\pi} - \lambda r(-t) \right),$$

where for any  $s \in \mathbb{R}$  we denote  $\phi_s(x) = (\psi(s), r(s))$ . There exists  $T \geq 0$  such that for any  $t \geq T$  the point  $\phi_{-t}(x)$  has positive (respectively negative) second coordinate  $r(-t)$ . Hence, the vector  $X_H(\phi_{-t}(x))$  is contained in the right half-plane (respectively left half-plane) for any  $t \geq T$ . Consequently, its angle variation is bounded and we conclude that  $\text{Torsion}(\phi_1^{-1}, x) = 0$ .

**Example 1.5.3** (Morse-Smale diffeomorphisms). We are going to calculate the torsion for a Morse-Smale diffeomorphism on the 2-dimensional torus  $\mathbb{T}^2$ . We start by recalling the definition of Morse-Smale diffeomorphism on  $\mathbb{T}^2$  and we refer to [Pal68].

**Definition 1.5.1.** A diffeomorphism  $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$  is called Morse-Smale if:

- (1) the non-wandering set  $\Omega(f)$  is finite (this implies that  $\Omega(f) = \text{Per}(f)$  where  $\text{Per}(f)$  is the set of periodic points for  $f$ );
- (2) all points in  $\text{Per}(f)$  are hyperbolic;
- (3) for any couple of points  $x, y \in \text{Per}(f)$  the stable manifold of  $x$   $W^s(x)$  intersects transversally the unstable manifold of  $y$   $W^u(y)$ .

**Proposition 1.5.1.** *Let  $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$  be a  $\mathcal{C}^1$  Morse-Smale diffeomorphism isotopic to the identity. Then, the torsion exists at every point and the set  $\{\text{Torsion}(f, z) : z \in \mathbb{T}^2\}$  is finite.*

Denote as  $\Omega(f) = \text{Per}(f) = \{P_1, \dots, P_m\}$ . We first calculate the torsion at periodic points.

**Lemma 1.5.1.** *Let  $P_i \in \text{Per}(f)$  have period  $N_i \in \mathbb{N}$ . If  $Df^{N_i}(P_i)$  has real eigenvalues, then there exists  $k \in \mathbb{Z}$  such that*

$$\text{Torsion}(f, P_i) = \frac{k}{2N_i}.$$

*Proof.* Let  $v \in T_{P_i}\mathbb{T}^2$  be an eigenvector of  $Df^{N_i}(P_i)$ . Since its eigenvalue is real, the subspace  $\mathbb{R}v$  is invariant by  $Df^{N_i}(P_i)$ . Consequently, there exists  $k \in \mathbb{Z}$  such that

$$N_i \text{Torsion}_{N_i}(f, P_i, v) = \frac{k}{2}.$$

Observe that  $k \in 2\mathbb{Z}$  if the corresponding eigenvalue is positive, while  $k \in 2\mathbb{Z} + 1$  if the corresponding eigenvalue is negative. We conclude that

$$\text{Torsion}(f, P_i) = \lim_{n \rightarrow +\infty} \text{Torsion}_{nN_i}(f, P_i, v) = \lim_{n \rightarrow +\infty} \frac{nk}{2nN_i} = \frac{k}{2N_i}.$$

□

**Lemma 1.5.2.** *Let  $P_i \in \text{Per}(f)$  have period  $N_i \in \mathbb{N}$ . If  $Df^{N_i}(P_i)$  has eigenvalues  $\lambda, \bar{\lambda} \in \mathbb{C} \setminus \mathbb{R}$ , then there exists  $k \in \mathbb{Z}$  such that*

$$\text{Torsion}(f, P_i) = \frac{\alpha + k}{N_i},$$

where  $\alpha = \pm \arg(\lambda)$ .

*Proof.* Consider a linear change of coordinates  $h$  isotopic to the identity so that the point  $P_i = h(P_i)$  and  $D(h \circ f^{N_i} \circ h^{-1})(P_i)$  is the composition of either a dilatation or a contraction and a rotation centered at the origin of angle  $\alpha = \pm \arg(\lambda)$ . In such a framework for any  $v \in T_{P_i}\mathbb{T}^2$  it holds

$$N_i \text{Torsion}_{N_i}(h \circ f \circ h^{-1}, P_i, v) = \text{Torsion}_1(h \circ f^{N_i} \circ h^{-1}, P_i, v) = \alpha + k,$$

for some  $k \in \mathbb{Z}$ . From the invariance of the torsion by conjugation of  $\mathcal{C}^1$  diffeomorphisms orientation preserving (see Proposition 1.1.6), we conclude that

$$\text{Torsion}(f, P_i) = \frac{1}{N_i} \text{Torsion}(f^{N_i}, P_i) = \frac{1}{N_i} \text{Torsion}(h \circ f^{N_i} \circ h^{-1}, P_i) = \frac{\alpha + k}{N_i}.$$

□

Actually, we are going to show that

$$\{\text{Torsion}(f, z) : z \in \mathbb{T}^2\} = \{\text{Torsion}(f, P_i) : P_i \in \text{Per}(f)\}.$$

Since the set of periodic points is finite from the definition of Morse-Smale diffeomorphisms, we deduce that the set of torsion values is finite too. For a Morse-Smale diffeomorphism we have (see Theorem 2.3 in [Sma67])

$$\mathbb{T}^2 = \bigcup_{i=1}^m W^s(P_i),$$

which is a disjoint union.

We are going to discuss two possible cases.

- (i)  $x$  belongs to the stable manifold of a periodic point  $P_i$  such that  $P_i$  is a saddle point (with respect to  $f^{N_i}$ ). In particular,  $Df^{N_i}(P_i)$  has real eigenvalues  $\lambda_1, \lambda_2$  such that  $|\lambda_1| < 1 < |\lambda_2|$ .
- (ii)  $x$  belongs to the stable manifold of a periodic point  $P_i$  such that  $P_i$  is a sink (with respect to  $f^{N_i}$ ).

**Lemma 1.5.3.** *Let  $x \in \mathbb{T}^2$ . Assume that  $x \in W^s(P_i)$  for some  $N_i$ -periodic hyperbolic point  $P_i$  such that  $P_i$  is a saddle point. Then*

$$\text{Torsion}(f, x) = \text{Torsion}(f, P_i) = \frac{k}{2N_i}.$$

*Proof.* Since  $P_i$  is a saddle fixed point for  $f^{N_i}$ , the tangent space  $T_{P_i}\mathbb{T}^2$  admits a hyperbolic splitting  $E^s \oplus E^u$ . In particular,  $E^s(P_i)$  denotes the stable subspace and it is  $Df^{N_i}$ -invariant. Let  $k \in \mathbb{Z}$  be such that

$$\text{Torsion}_{N_i}(f, P_i, v) = \frac{k}{2},$$

where  $v \in E^s(P_i)$  (see Lemma 1.5.1). Let  $\phi : U \rightarrow \mathbb{R}^2$  be a chart so that  $P_i \in U$ . From the stable manifold theorem we find a neighborhood  $U' \subset U$  such that the connected component of  $U' \cap W^s(P_i)$  containing  $P_i$ , denoted as  $W_{loc}^s(P_i)$ , is a  $\mathcal{C}^1$  submanifold, it is  $f^{N_i}$ -forward invariant and  $T_{P_i}W_{loc}^s(P_i) = E^s(P_i)$ .

Fix  $\varepsilon > 0$ . By the continuity of the torsion at finite time  $N_i$  and by the continuity of

$$W_{loc}^s(P_i) \ni x \mapsto T_x W^s(P_i),$$

there exists a neighborhood  $U'' \subset U'$  such that for any  $x \in W_{loc}^s(P_i) \cap U''$  we have

$$\left| \text{Torsion}_{N_i}(f, x, w) - \frac{k}{2} \right| < \varepsilon, \tag{1.76}$$

where  $w \in T_x W^s(P_i)$ .

Let  $x \in W^s(P_i)$ . There exists  $\bar{n} \in \mathbb{N}$  such that for any  $n \geq \bar{n}$  the image  $f^{N_i n}(x)$  belongs to  $U'' \cap W_{loc}^s(P_i)$ . Denote as  $z = f^{N_i \bar{n}}(x)$ .

**Claim 1.5.2.** For any  $n \in \mathbb{N}$  it holds

$$\left| \text{Torsion}_{N_i n}(f, z, w) - \frac{k}{2} \right| < \varepsilon,$$

where  $w \in T_z W^s(P_i)$ .

*Proof.* Let us argue by induction. The case  $n = 1$  is given by condition (1.76). Assume the claim holds for  $n - 1$ , that is

$$\left| N_i(n-1) \text{Torsion}_{N_i(n-1)}(f, z, w) - \frac{N_i(n-1)k}{2} \right| < N_i(n-1)\varepsilon.$$

Then

$$\begin{aligned} \left| N_i n \text{Torsion}_{N_i n}(f, z, w) - \frac{N_i n k}{2} \right| &\leq \left| N_i(n-1) \text{Torsion}_{N_i(n-1)}(f, z, w) - \frac{N_i(n-1)k}{2} \right| + \\ &+ \left| N_i \text{Torsion}_{N_i}(f, f^{N_i(n-1)}(z), Df^{N_i(n-1)}(z)w) - \frac{N_i k}{2} \right| < \\ &< N_i(n-1)\varepsilon + \left| N_i \text{Torsion}_{N_i}(f, f^{N_i(n-1)}(z), Df^{N_i(n-1)}(z)w) - \frac{N_i k}{2} \right|. \end{aligned}$$

Since  $f^{N_i(n-1)}(z)$  belongs to  $U'' \cap W_{loc}^s(P_i)$  and since  $Df^{N_i(n-1)}(z)w$  belongs to  $T_{f^{N_i(n-1)}(z)} W^s(P_i)$ , we apply hypothesis (1.76) and deduce that

$$\left| N_i \text{Torsion}_{N_i}(f, f^{N_i(n-1)}(z), Df^{N_i(n-1)}(z)w) - \frac{N_i k}{2} \right| < N_i \varepsilon.$$

We conclude that

$$\left| \text{Torsion}_{N_i n}(f, z, w) - \frac{k}{2} \right| < \varepsilon,$$

as desired.  $\square$

From Claim 1.5.2 and from the invariance of the torsion along the orbit of a point, we have

$$\left| \text{Torsion}(f, z) - \frac{k}{2} \right| = \left| \text{Torsion}(f, x) - \frac{k}{2} \right| < \varepsilon.$$

By the arbitrariness of  $\varepsilon > 0$ , we conclude that  $\text{Torsion}(f, x) = \text{Torsion}(f, P_i) = \frac{k}{2}$ .  $\square$

**Lemma 1.5.4.** Let  $x \in \mathbb{T}^2$ . Assume that  $x \in W^s(P_i)$  for some  $N_i$ -periodic point  $P_i$  such that  $P_i$  is a sink. Then

$$\text{Torsion}(f, x) = \text{Torsion}(f, P_i).$$

In order to prove Lemma 1.5.4, it is more convenient using Ruelle's definition of torsion and then recall that B guin and Boubaker's notion is equivalent. In the sequel we then refer to [Rue85].

Consider a diffeomorphism on  $\mathbb{T}^2$  where we fix the standard Riemannian metric and the standard trivialization. The arguments can be adapted for a parallelizable Riemannian surface.

**Notation 1.5.1.** Denote as  $GL^+(2, \mathbb{R})$  the subgroup of  $GL(2, \mathbb{R})$  of matrices with positive determinant.

For any  $A \in GL^+(2, \mathbb{R})$  we consider its polar decomposition and we refer to it as

$$A = U(\theta(A)) S(A),$$

where  $U(\theta(A))$  is the rotation of angle  $\theta(A) \in \mathbb{T}$  and  $S(A)$  is a symmetric positive definite matrix.

**Definition 1.5.2.** Let  $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$  be a  $\mathcal{C}^1$  diffeomorphism isotopic to the identity and let  $I = (f_t)_t$  be an isotopy joining the identity to  $f$ . For  $x \in \mathbb{T}^2$  consider the continuous angle function

$$\mathbb{R}_+ \ni t \mapsto \theta(Df_t(x)) \in \mathbb{T},$$

where

$$Df_t(x) = U(\theta(Df_t(x))) S(Df_t(x)).$$

Denote as  $\mathbb{R}_+ \ni t \mapsto \tilde{\theta}(Df_t(x)) \in \mathbb{R}$  a continuous determination of this angle function. Ruelle's torsion at  $x$  is, whenever it exists, the limit

$$\omega_I(x) := \lim_{n \rightarrow +\infty} \frac{\tilde{\theta}(Df^n(x)) - \tilde{\theta}(\text{Id})}{n}.$$

As remarked by Ruelle in [Rue85], for a  $f$ -invariant probability measure  $\mu$  (on  $\mathbb{T}^2$ ), Ruelle's torsion exists  $\mu$ -almost everywhere. Moreover, as for B guin and Boubaker's torsion, it does not depend on the choice of the continuous determination.

**Claim 1.5.3.** Let  $x \in \mathbb{T}^2$ . Ruelle's torsion  $\omega_I(x)$  exists if and only if  $\text{Torsion}(I, x)$  exists and, when they exist,  $\omega_I(x) = \text{Torsion}(I, x)$ .

*Proof.* Fix  $x \in \mathbb{T}^2$ . Let  $v \in T_x \mathbb{T}^2 \setminus \{0\}$ : recall that the asymptotic torsion does not depend on the choice of the tangent vector (see Proposition 1.1.3). For any  $t \in \mathbb{R}_+$  we have that  $\tilde{\theta}(Df_t(x))$  is a measure of the oriented angle between the fixed reference vector field  $X$  and the vector

$$U(\theta(Df_t(x)))v = Df_t(x) S(Df_t(x))^{-1}v$$

since  $U(\theta(Df_t(x)))$  is a rotation of angle  $\theta(Df_t(x))$ . Consider then the oriented angle function

$$\mathbb{R}_+ \ni t \mapsto \theta(Df_t(x) S(Df_t(x))^{-1}v, Df_t(x)v) \in \mathbb{T}$$

and denote as  $\mathbb{R}_+ \ni t \mapsto \tilde{\Theta}(t) \in \mathbb{R}$  a continuous determination of it. Consequently (referring to the notations used to introduce Ruelle's torsion and B guin and Boubaker's torsion), for any  $n \in \mathbb{N}$  we have that

$$\left| \left( \tilde{\theta}(Df_n(x)) - \tilde{\theta}(\text{Id}) \right) - (\tilde{v}(I)(x, v, n) - \tilde{v}(I)(x, v, 0)) \right| = \left| \tilde{\Theta}(n) - \tilde{\Theta}(0) \right|. \quad (1.77)$$

We are going now to show that

$$\left| \tilde{\Theta}(t) - \tilde{\Theta}(0) \right| < \frac{1}{2} \quad (1.78)$$

for any  $t \in \mathbb{R}_+$ .

Argue by contradiction and assume there exists  $\bar{t} \in \mathbb{R}_+$  such that  $\left| \tilde{\Theta}(\bar{t}) - \tilde{\Theta}(0) \right| = \frac{1}{2}$ .

Observe that  $\tilde{\Theta}(0) \in \mathbb{Z}$ . Thus, by contradiction hypothesis, we are assuming that the oriented angle  $\theta(Df_{\bar{t}}(x) S(Df_{\bar{t}}(x))^{-1}v, Df_{\bar{t}}(x)v)$  admits a measure equal to  $\frac{1}{2}$ . Equivalently, the vectors  $Df_{\bar{t}}(x) S(Df_{\bar{t}}(x))^{-1}v$  and  $Df_{\bar{t}}(x)v$  are negatively colinear. Since  $f_{\bar{t}}$  is invertible, it holds that  $S(Df_{\bar{t}}(x))^{-1}v$  and  $v$  are negatively colinear too. This contradicts the fact that the matrix  $S(Df_{\bar{t}}(x))^{-1}$  is positive definite and we conclude the proof of (1.78).

From (1.77) and (1.78) we conclude that for any  $n \in \mathbb{N}$

$$\left| \frac{\tilde{\theta}(Df^n(x)) - \tilde{\theta}(\text{Id})}{n} - \frac{\tilde{v}(I)(x, v, n) - \tilde{v}(I)(x, v, 0)}{n} \right| = \frac{1}{n} |\tilde{\Theta}(n) - \tilde{\Theta}(0)| < \frac{1}{2n}.$$

Passing to the limit for  $n \rightarrow +\infty$  we obtain the claimed result. □

*Proof of Lemma 1.5.4.* Consider a compact neighborhood  $U$  of  $P_i$  that is a basin of attraction of  $P_i$  for  $f^{N_i}$ . In particular the omega-limit set (with respect to  $f^{N_i}$ )  $\omega(U) = \{P_i\}$ . The restricted dynamical system  $f^{N_i} : U \rightarrow U$  is uniquely ergodic. The unique ergodic  $f^{N_i}$ -invariant measure is the Dirac's delta measure  $\delta_{P_i}$  concentrated at the point  $P_i$ .

Thus, for every  $\phi \in \mathcal{C}(U)$ , the limit of the Birkhoff sum of  $\phi$  converges for every point to the constant

$$\int_U \phi(x) d\delta_{P_i}(x) = \phi(P_i),$$

see Theorem 6.19 in [Wal82].

Let  $x \in W^s(P_i)$ . There exists  $\bar{n} \in \mathbb{N}$  such that for any  $n \geq \bar{n}$  the point  $f^{N_i n}(x)$  belongs to  $U$ . Denote as  $z = f^{N_i \bar{n}}(x)$ . According to Notation 1.5.1 and to Claim 1.77, in order to calculate the torsion at  $z$  we can consider Ruelle's torsion

$$\lim_{n \rightarrow +\infty} \frac{\tilde{\theta}(Df^{N_i n}(z)) - \tilde{\theta}(\text{Id})}{n}.$$

In order to simplify the notation, up to replace  $f$  with  $f^{N_i}$ , assume that  $P_i$  is a fixed point for  $f$ .

Retracing the proof of Ruelle in [Rue85], we observe that for any  $A, B \in \text{GL}^+(2, \mathbb{R})$  it holds that  $|\tilde{\theta}(BA) - \tilde{\theta}(B) - \tilde{\theta}(A)| < \frac{1}{2}$ , where  $\tilde{\theta}$  denotes the universal covering of the angle obtained from the polar decomposition. Fix now  $m > 0$  and write  $n = km + r$  for  $k > 0, 0 \leq r < m$ . Observe that

$$\left| \tilde{\theta}(Df^n(z)) - \tilde{\theta}(Df^m(z)) - \dots - \tilde{\theta}(Df^m(f^{m(k-1)}(z))) - \tilde{\theta}(Df^r(f^{mk}(z))) \right| < \frac{k}{2}.$$

In particular, applying Birkhoff's Ergodic Theorem and since  $f|_U$  is uniquely ergodic, we deduce that for every  $z \in U$

$$\lim_{n \rightarrow +\infty} \frac{\tilde{\theta}(Df^n(z)) - \tilde{\theta}(\text{Id})}{n} = 0. \tag{1.79}$$

Again because of Birkhoff's Ergodic Theorem and because of the unique ergodicity of the system, for every  $z \in U$  we also have that (we are using Claim 1.5.3 at  $P_i$ )

$$\lim_{k \rightarrow +\infty} \frac{\sum_{i=0}^{k-1} \tilde{\theta}(Df^m(f^{mi}(z)))}{km} - \frac{\tilde{\theta}(\text{Id})}{m} = \frac{\tilde{\theta}(Df^m(P_i)) - \tilde{\theta}(\text{Id})}{m} =$$



$$= \text{Torsion}_m(f, P_i, w) = \text{Torsion}(f, P_i), \quad (1.80)$$

where  $w \in T_{P_i} \mathbb{T}^2$ .

From (1.79) and (1.80), showing that the sequence  $\left( \frac{\tilde{\theta}(Df^n(z)) - \tilde{\theta}(\text{Id})}{n} \right)_{n \in \mathbb{N}}$  is a Cauchy sequence, we conclude that  $\left( \frac{\tilde{\theta}(Df^n(z)) - \tilde{\theta}(\text{Id})}{n} \right)_{n \in \mathbb{N}}$  converges for every  $z \in U$  (thanks to the unique ergodicity) to the same limit as (1.80). That is

$$\omega_f(z) = \text{Torsion}(f, z) = \text{Torsion}(f, P_i) = \frac{k}{2}.$$

By the invariance of the torsion along the orbit of a point, we finally conclude that for every  $x \in W^s(P_i)$  (assuming that  $P_i$  is a fixed point for  $f$ ) it holds

$$\text{Torsion}(f, x) = \text{Torsion}(f, z) = \text{Torsion}(f, P_i) = \frac{k}{2}.$$

□

From Lemmas 1.5.3 and 1.5.4 we conclude that the torsion exists at every point and that the set of torsion values is finite and coincides with the set of torsion values of periodic points.

## 1.6 Appendix of Chapter 1

We now present the proof of the technical Proposition 1.4.1, used in the discussion of case (i) of Theorem 1.4.1 (see Subsection 1.4.2). Consider  $J_1(s), J_2(s)$  for  $s \in I$ . By hypothesis  $\pi_M \circ J_1 = \pi_M \circ J_2$ , so they lie on the same tangent space.

Four different cases can occur:

- (1)  $J_1(s), J_2(s)$  are positively colinear, i.e.  $J_1(s) = \lambda J_2(s)$  for some  $\lambda > 0$ . Hence, the associated angle function satisfies  $\bar{\theta}(s) = 0 \bmod 1$  and any continuous determination  $\theta$  verifies  $\theta(s) = k$  for some  $k \in \mathbb{Z}$ .
- (2)  $J_1(s), J_2(s)$  are negatively colinear, i.e.  $J_1(s) = \lambda J_2(s)$  for some  $\lambda < 0$ . Hence, the associated angle function satisfies  $\bar{\theta}(s) = \frac{1}{2} \bmod 1$  and any continuous determination  $\theta$  verifies  $\theta(s) = \frac{1}{2} + k$  for some  $k \in \mathbb{Z}$ .
- (3)  $J_1(s), J_2(s)$  are linearly independent and  $(J_1(s), J_2(s))$  is a direct basis. Therefore, the associated angle function satisfies  $\bar{\theta}(s) \in (0, \frac{1}{2}) \bmod 1$  and any continuous determination  $\theta$  verifies  $\theta(s) \in (k, \frac{1}{2} + k)$  for some  $k \in \mathbb{Z}$ .
- (4)  $J_1(s), J_2(s)$  are linearly independent and  $(J_1(s), J_2(s))$  is a non-direct basis. Therefore, the associated angle function satisfies  $\bar{\theta}(s) \in (\frac{1}{2}, 1) \bmod 1$  and any continuous determination  $\theta$  verifies  $\theta(s) \in (\frac{1}{2} + k, k + 1)$  for some  $k \in \mathbb{Z}$ .

We denote as  $\bar{\Theta}(s)$  the oriented angle between  $Df \circ J_1(s)$  and  $Df \circ J_2(s)$ .

**Lemma 1.6.1.** *Let  $I \subset \mathbb{R}$  and let  $M, N$  be 2-dimensional oriented Riemannian manifolds. Let  $f : M \rightarrow N$  be a local diffeomorphism which preserves the orientation and let  $J_1, J_2 : I \rightarrow TM$  be continuous functions that never vanish. Assume also that  $\pi_M \circ J_1 = \pi_M \circ J_2$ .*

Let  $\bar{\theta}, \bar{\Theta} : I \rightarrow \mathbb{T}$  be the oriented angles, respectively, between the image vectors  $J_1, J_2$  and the image vectors  $Df \circ J_1, Df \circ J_2$ .

Then, for any  $s \in I$

$$\bar{\theta}(s) - \bar{\Theta}(s) \neq \frac{1}{2} \mod 1. \quad (1.81)$$

We postpone the proof of this lemma and we now prove Proposition 1.4.1.

*Proof of Proposition 1.4.1.* Let  $\theta$  be a chosen continuous determination of the angle  $\bar{\theta}$ , i.e. the angle between  $J_1, J_2$ . Let fix  $s_0 \in I$ . Depending on the cases, we have

$$\theta(s_0) \begin{cases} = k & \text{if } \bar{\theta}(s_0) = 0 \mod 1 \\ = k + \frac{1}{2} & \text{if } \bar{\theta}(s_0) = \frac{1}{2} \mod 1 \\ \in (0, \frac{1}{2}) + k & \text{if } \bar{\theta}(s_0) \in (0, \frac{1}{2}) \mod 1 \\ \in (\frac{1}{2}, 1) + k & \text{if } \bar{\theta}(s_0) \in (\frac{1}{2}, 1) \mod 1 \end{cases}$$

where  $k \in \mathbb{Z}$ .

Choose a measure  $\Theta(s_0)$  of the angle  $\bar{\Theta}(s_0)$  (i.e. the angle between  $Df \circ J_1(s_0)$  and  $Df \circ J_2(s_0)$ ) such that

$$|\theta(s_0) - \Theta(s_0)| < \frac{1}{2}.$$

By the continuity of the chosen determination  $\Theta$ , from the relation holding in  $s_0$  and from Lemma 1.6.1, for any  $s \in I$  we conclude

$$|\theta(s) - \Theta(s)| < \frac{1}{2}.$$

□

*Proof of Lemma 1.6.1.* As remarked above, only four cases can occur concerning the relative positions of vectors  $J_1(s), J_2(s)$  for any fixed  $s \in I$ .

We then show that for any  $s$

$$\bar{\theta}(s) - \bar{\Theta}(s) \neq \frac{1}{2} \mod 1.$$

Arguing by contradiction, assume that there exists  $s$  so that  $\bar{\theta}(s) - \bar{\Theta}(s) = \frac{1}{2} \mod 1$ . Then the couples of vectors  $(J_1(s), J_2(s))$  and  $(Df \circ J_1(s), Df \circ J_2(s))$  belong to different cases and this is a contradiction. Indeed, since  $f$  is a local diffeomorphism which preserves the orientation, looking at the relative position of vectors  $Df \circ J_1(s), Df \circ J_2(s)$ , the same four cases presented above can occur and for any fixed  $s \in I$  we remain in the same case as  $J_1(s), J_2(s)$ .

□

Observe that a similar (adapted) result could be obtained in the case of a local diffeomorphism which reverses the orientation.

**Lemma 1.6.2.** Let  $I \subset \mathbb{R}$  and let  $M, N$  be 2-dimensional oriented Riemannian manifolds. Let  $f : M \rightarrow N$  be a local diffeomorphism which inverts the orientation and let  $J_1, J_2 : I \rightarrow TM$  be continuous functions that never vanish. Assume also that  $\pi_M \circ J_1 = \pi_M \circ J_2$ . Let  $\bar{\theta}, \bar{\Theta} : I \rightarrow \mathbb{T}$  be the oriented angles, respectively, between the image vectors  $J_1, J_2$  and the image vectors  $Df \circ J_1, Df \circ J_2$ .

Then, for any  $s \in I$

$$\bar{\theta}(s) + \bar{\Theta}(s) \neq \frac{1}{2} \mod 1. \quad (1.82)$$

*Proof.* Only four cases can occur concerning the relative positions of vectors  $J_1(s), J_2(s)$  for any fixed  $s \in I$ . Argue by contradiction and assume that there exists  $s \in I$  such that  $\bar{\theta}(s) + \bar{\Theta}(s) = \frac{1}{2} \mod 1$ .

Let us discuss the possible cases. If  $J_1(s)$  and  $J_2(s)$  are positive colinear, then also their image through  $Df$  are positive colinear. That is, both  $\bar{\theta}(s)$  and  $\bar{\Theta}(s)$  are null: in particular  $\bar{\theta}(s) + \bar{\Theta}(s)$  cannot be equal to  $\frac{1}{2} \mod 1$ .

A similar argument excludes the case of  $J_1(s)$  and  $J_2(s)$  being negative colinear.

Assume now that  $\bar{\theta}(s) \in (0, \frac{1}{2})$ . Consequently, since  $f$  reverses the orientation, the angle  $\bar{\Theta}(s)$  is in  $(\frac{1}{2}, 1)$ . Thus

$$\bar{\theta}(s) + \bar{\Theta}(s) \in \left(\frac{1}{2}, 1 + \frac{1}{2}\right) \mod 1.$$

In particular,  $\bar{\theta}(s) + \bar{\Theta}(s)$  cannot be equal to  $\frac{1}{2} \mod 1$ . A similar argument excludes also the case of  $(J_1(s), J_2(s))$  giving a non-direct basis and we conclude.  $\square$

From Lemma 1.6.2 we can deduce the following

**Proposition 1.6.1.** *Let  $I \subset \mathbb{R}$  be an interval and let  $M, N$  be two-dimensional oriented Riemannian manifolds. Denote the tangent projections as  $\pi_M : TM \rightarrow M, \pi_N : TN \rightarrow N$ . Let  $f : M \rightarrow N$  be a local diffeomorphism which reverses the orientation and let  $J_1 : I \rightarrow TM, J_2 : I \rightarrow TM$  be continuous functions such that  $\pi_M \circ J_1 = \pi_M \circ J_2$ .*

*Suppose that, for any  $t \in I$ ,  $J_i(t) \neq 0$ ,  $i = 1, 2$  and let  $\theta : I \rightarrow \mathbb{R}$  be a continuous determination of the angle function between the image vectors  $J_1, J_2$ .*

*Then, there exists a continuous determination  $\Theta : I \rightarrow \mathbb{R}$  of the angle function between the image vectors  $Df \circ J_1, Df \circ J_2$  such that*

$$|\theta(s) + \Theta(s)| < \frac{1}{2} \quad \forall s \in I. \quad (1.83)$$

*Proof.* The proof is almost the same as Proposition 1.4.1. Let  $\theta$  be a continuous determination of  $\bar{\theta}$  and let fix  $s_0 \in I$ . Four cases can occur:

$$\theta(s_0) \begin{cases} = k & \text{if } \bar{\theta}(s_0) = 0 \mod 1 \\ = k + \frac{1}{2} & \text{if } \bar{\theta}(s_0) = \frac{1}{2} \mod 1 \\ \in (0, \frac{1}{2}) + k & \text{if } \bar{\theta}(s_0) \in (0, \frac{1}{2}) \mod 1 \\ \in (\frac{1}{2}, 1) + k & \text{if } \bar{\theta}(s_0) \in (\frac{1}{2}, 1) \mod 1 \end{cases}$$

where  $k \in \mathbb{Z}$ .

Choose then a measure  $\Theta(s_0)$  of the angle  $\bar{\Theta}(s_0)$ , the angle between  $Df \circ J_1(s_0)$  and  $Df \circ J_2(s_0)$ , so that  $|\theta(s_0) + \Theta(s_0)| < \frac{1}{2}$ . By the continuity of the chosen determinations and by Lemma 1.6.2, we conclude that for any  $s \in I$  it holds

$$|\theta(s) + \Theta(s)| < \frac{1}{2}.$$

$\square$

# Chapter 2

## Results on negative-torsion maps

In this Chapter we introduce negative-torsion maps and discuss properties of the set of points of zero torsion for such maps. We will start by focusing on positive (negative) twist maps, showing that they are examples of negative-torsion (positive-torsion) maps. We will state results over the Hausdorff dimension of the set of points of zero torsion for negative-torsion maps.

We also show that the notion of negative-torsion (positive-torsion) maps coincide with the notion of positive (negative) tilt maps.

**Notation 2.0.1.** In the following, the annulus  $\mathbb{A}$  is endowed with the standard Riemannian metric and trivialization. We fix the counterclockwise orientation and consider as reference vector field the vertical constant one  $\chi = (0, 1)$ . The notation  $\mathcal{H}$  refers to the constant horizontal vector  $(1, 0)$ .

**Definition 2.0.1.** A  $\mathcal{C}^1$  diffeomorphism isotopic to the identity  $f : \mathbb{A} \rightarrow \mathbb{A}$  is a negative-torsion map (respectively positive-torsion map) if for any  $\bar{z} \in \mathbb{A}$  it holds

$$\text{Torsion}_1(f, \bar{z}, \chi) < 0 \quad (\text{respectively } > 0).$$

**Remark 2.0.1.** We do not make explicit the choice of the isotopy (and so write  $\text{Torsion}_1(f, \bar{z}, \chi)$ ) because the torsion is independent of the choice of the isotopy on  $\mathbb{A}$  (see Proposition 1.3.2).

### 2.1 Torsion for twist maps

We introduce now the definition of twist map on the annulus  $\mathbb{A}$ . We refer to [LC91] and [Cro03]. In addition, other interesting references are [Mat82a], [Mat91] and [Mat82b].

**Definition 2.1.1.** A *positive twist map* (respectively *negative*)  $f : \mathbb{A} \rightarrow \mathbb{A}$  is a  $\mathcal{C}^1$  diffeomorphism isotopic to the identity such that for any lift  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  and for any  $x \in \mathbb{R}$  the function

$$\mathbb{R} \ni y \mapsto p_1 \circ F(x, y) \in \mathbb{R} \tag{2.1}$$

is a strictly increasing (respectively decreasing) diffeomorphism.

**Remark 2.1.1.** All over the literature (see [LC91] and [Cro03]), the definition of *positive twist map* asks also the further condition that for any lift  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  and for any  $x \in \mathbb{R}$  the function

$$\mathbb{R} \ni y \mapsto p_1 \circ F^{-1}(x, y) \in \mathbb{R} \tag{2.2}$$

is a decreasing diffeomorphism of  $\mathbb{R}$ . Actually, Definition 2.1.1 implies this condition and we omit it. Indeed, from (2.1) we immediately deduce, looking at the differential  $DF(F^{-1}(x, y))$  and at its inverse, that for any lift  $F$  and for any  $x \in \mathbb{R}$  the function  $y \mapsto p_1 \circ F^{-1}(x, y)$  is a diffeomorphism onto its image. Its image is actually the whole  $\mathbb{R}$  otherwise there would exist  $\xi \in \mathbb{R}$  such that the image of the vertical  $p_1 \circ F(V_{(\xi, 0)})$  would not be the whole  $\mathbb{R}$ , contradicting condition (2.1).

**Example 2.1.1.** The  $\mathcal{C}^1$  diffeomorphism

$$\mathbb{A} \ni (x, y) \mapsto (x + y, y) \in \mathbb{A}$$

is the first simplest example of positive twist map. The annulus is foliated by invariant circles.

**Example 2.1.2.** For any  $\alpha \in \mathbb{R}$ , the standard map  $f_\alpha$ , where

$$f_\alpha(x, y) = (x + y + \alpha \sin(2\pi x), y + \alpha \sin(2\pi x)),$$

is an example of positive twist map.

### 2.1.1 Limitedness of torsion for twist maps

The following result provides an estimation for finite-time torsion of positive twist maps. Recall that on  $\mathbb{A}$  the torsion does not depend on the chosen isotopy (see Proposition 1.3.2), so we will omit it in the notation.

**Theorem 2.1.1.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a positive twist map. Then, for any  $\bar{z} \in \mathbb{A}$  and for any  $n \in \mathbb{N}, n \neq 0$ , it holds*

$$\text{Torsion}_n(f, \bar{z}, \chi) \in \left(-\frac{1}{2}, 0\right).$$

**Remark 2.1.2.** In the framework of negative twist maps, an adapted version of Theorem 2.1.1 holds true. Indeed, if  $f$  is a negative twist map, then for any  $\bar{z} \in \mathbb{A}$  and for any  $n \in \mathbb{N}, n \neq 0$  we have

$$\text{Torsion}_n(f, \bar{z}, \chi) \in \left(0, \frac{1}{2}\right).$$

**Remark 2.1.3.** The torsion at any point for a positive twist map  $f$  is independent of the choice of the isotopy  $I = (f_t)_t$ , thanks to Proposition 1.3.2.

In Section 2 of [LC91], Patrice Le Calvez proved that any positive twist map  $f : \mathbb{A} \rightarrow \mathbb{A}$  can be joined to the identity  $\text{Id}_{\mathbb{A}}$  through an isotopy  $I = (f_t)_{t \in [0, 1]}$  in  $\text{Diff}^1(\mathbb{A})$  such that  $f_0 = \text{Id}_{\mathbb{A}}$ ,  $f_1 = f$  and for any  $t \in (0, 1]$  each  $f_t$  is a positive twist map. We are going to calculate torsion with respect to this isotopy.

In order to prove Theorem 2.1.1, we first introduce some preliminary steps.

**Proposition 2.1.1.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a positive twist map. For any  $\bar{z} \in \mathbb{A}$  it holds*

$$\text{Torsion}_1(f, \bar{z}, \chi) \in \left(-\frac{1}{2}, 0\right). \quad (2.3)$$

**Remark 2.1.4.** Proposition 2.1.1 implies that any positive twist map is a negative-torsion map according to Definition 2.0.1.

*Proof.* From Proposition 1.3.2, the torsion does not depend on the choice of the isotopy. Therefore, we use the isotopy given by P. Le Calvez (see Remark 2.1.3): for any  $t \in (0, 1]$  the  $\mathcal{C}^1$  diffeomorphism  $f_t$  is a positive twist map.

Let  $\tilde{I} = (F_t)_t$  be the lifted isotopy of  $I = (f_t)_t$  such that  $F_0 = \text{Id}_{\mathbb{R}^2}$ . It joins the identity to  $F_1 = F$ , a lift of  $f$ . The point  $z = (x, y) \in \mathbb{R}^2$  denotes a lift of the point  $\bar{z} \in \mathbb{A}$ .

Look then at

$$\text{Torsion}_1(\tilde{I}, z, \chi).$$

It is the variation of a continuous determination  $\tilde{v}(\tilde{I})(z, \chi, \cdot)$  of the oriented angle function between  $\chi$  and  $DF_t(z)\chi$ . Recall that it is independent of the choice of the continuous determination of the angle function (see Proposition 1.1.1).

By the choice of the isotopy, for any  $t \in (0, 1]$ ,  $f_t$  is a positive twist map. Then, since  $F_t$  is a lift of  $f_t$ , for any  $x \in \mathbb{R}$  the function

$$\mathbb{R} \ni y \mapsto p_1 \circ F_t(x, y) \in \mathbb{R}$$

is an increasing diffeomorphism of  $\mathbb{R}$ . In particular, its derivative is always positive, that is

$$D(p_1 \circ F_t)(z)\chi > 0. \quad (2.4)$$

For any  $t \in (0, 1]$  the first component of the image vector  $DF_t(z)\chi$  is positive. The vector remains in the right half-plane and it cannot cross the vertical any more. Thus, the variation

$$\tilde{v}(\tilde{I})(z, \chi, t) - \tilde{v}(\tilde{I})(z, \chi, 0)$$

has to stay in the interval  $(-\frac{1}{2}, 0)$  for any  $t \in (0, 1]$ , thanks also to the continuity of the lift. We then conclude that

$$\begin{aligned} \tilde{v}(\tilde{I})(z, \chi, 1) - \tilde{v}(\tilde{I})(z, \chi, 0) &= \text{Torsion}_1(\tilde{I}, z, \chi) = \\ &= \text{Torsion}_1(f, \bar{z}, \chi) \in \left(-\frac{1}{2}, 0\right). \end{aligned} \quad (2.5)$$

□

**Proposition 2.1.2.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a positive twist map. Let  $\bar{z} \in \mathbb{A}$  and let  $\xi \in T_{\bar{z}}\mathbb{A} \setminus \{0\}$ . Then it holds*

$$\text{Torsion}_1(f, \bar{z}, \xi) \in \left(-1, \frac{1}{2}\right). \quad (2.6)$$

*Proof.* Let  $I = (f_t)_t$  be an isotopy in  $\text{Diff}^1(\mathbb{A})$  joining the identity to  $f$ . We use the notations of Proposition 1.1.2. Then  $W(0, \cdot)$  and  $W(-\frac{1}{2}, \cdot)$  are continuous determinations of  $v(I)(\bar{z}, \chi, \cdot)$  and  $v(I)(\bar{z}, -\chi, \cdot)$  respectively, such that  $W(0, 0) = 0$  and  $W(-\frac{1}{2}, 0) = -\frac{1}{2}$ . We assume that  $\xi$  is in the right half-plane. Let us denote  $\tilde{v}(I)(\bar{z}, \xi, \cdot)$  a continuous determination of  $v(I)(\bar{z}, \xi, \cdot)$  such that  $\tilde{v}(I)(\bar{z}, \xi, 0) = \alpha \in [-\frac{1}{2}, 0]$ . Since  $\tilde{v}(I)(\bar{z}, \xi, \cdot) = W(\alpha, \cdot)$ , by point (ii) of Proposition 1.1.2 it holds

$$W\left(-\frac{1}{2}, 1\right) \leq \tilde{v}(I)(\bar{z}, \xi, 1) = W(\alpha, 1) \leq W(0, 1).$$

This implies

$$W\left(-\frac{1}{2}, 1\right) - W(0, 0) \leq \tilde{v}(I)(\bar{z}, \xi, 1) - \tilde{v}(I)(\bar{z}, \xi, 0) \leq W(0, 1) - W\left(-\frac{1}{2}, 0\right).$$

Because of (iii) of Proposition (1.1.2), we have that  $W(-\frac{1}{2}, 1) = W(0, 1) - \frac{1}{2}$  and  $W(-\frac{1}{2}, 0) = W(0, 0) - \frac{1}{2}$ . From these equalities and since

$$\text{Torsion}_1(f, \bar{z}, \xi) = \tilde{v}(I)(\bar{z}, \xi, 1) - \bar{v}(I)(\bar{z}, \xi, 0),$$

we obtain

$$W(0, 1) - W(0, 0) - \frac{1}{2} \leq \text{Torsion}_1(f, \bar{z}, \xi) \leq W(0, 1) - W(0, 0) + \frac{1}{2}.$$

By Proposition (2.1.1),  $W(0, 1) - W(0, 0)$  is in  $(-\frac{1}{2}, 0)$ , hence

$$-1 < \text{Torsion}_1(f, \bar{z}, \xi) < \frac{1}{2}.$$

If  $\xi$  is in the left half-plane, then  $-\xi$  is in the right half-plane and we know that

$$\text{Torsion}_1(f, \bar{z}, \xi) = \text{Torsion}_1(f, \bar{z}, -\xi) \in \left(-1, \frac{1}{2}\right).$$

□

*Proof of Theorem 2.1.1.* The proof of the Theorem is made by induction. The base case, that is the case with  $n = 1$ , is Proposition 2.1.1. Concerning the inductive step, assume that the statement holds true for  $n \in \mathbb{N}$ .

We use the notation of Proposition 1.1.2, but we add the dependence on the point  $W_{\bar{z}}(s, t)$ . Then  $W_{\bar{z}}(0, \cdot)$  is a continuous determination of the angle function  $v(I)(\bar{z}, \chi, \cdot)$  that satisfies (by Proposition 2.1.1)  $W_{\bar{z}}(0, 1) = \beta \in (-\frac{1}{2}, 0)$ , that is

$$W_{f(\bar{z})}\left(-\frac{1}{2}, 0\right) = -\frac{1}{2} < W_{\bar{z}}(0, 1) = W_{f(\bar{z})}(\beta, 0) < W_{f(\bar{z})}(0, 0) = 0.$$

Remark that  $t \mapsto W_{\bar{z}}(0, 1 + t)$  and  $t \mapsto W_{f(\bar{z})}(\beta, t)$  are continuous determinations of the same oriented angle function and they coincide at  $t = 0$ . Thus, the functions  $t \mapsto W_{\bar{z}}(0, 1 + t)$  and  $t \mapsto W_{f(\bar{z})}(\beta, t)$  are equal.

By point (ii) of Proposition 1.1.2 we have for any  $t$

$$W_{f(\bar{z})}\left(-\frac{1}{2}, t\right) < W_{f(\bar{z})}(\beta, t) = W_{\bar{z}}(0, 1 + t) < W_{f(\bar{z})}(0, t).$$

Using (iii) of Proposition 1.1.2 it holds

$$W_{f(\bar{z})}(0, t) - \frac{1}{2} < W_{\bar{z}}(0, 1 + t) < W_{f(\bar{z})}(0, t).$$

For  $t = n$  we have

$$W_{f(\bar{z})}(0, n) - \frac{1}{2} - W_{f(\bar{z})}(0, 0) < W_{\bar{z}}(0, n + 1) - W_{\bar{z}}(0, 0) < W_{f(\bar{z})}(0, n) - W_{f(\bar{z})}(0, 0).$$

By induction hypothesis, we have

$$W_{f(\bar{z})}(0, n) - W_{f(\bar{z})}(0, 0) \in \left(-\frac{n}{2}, 0\right)$$

and then

$$W_{\bar{z}}(0, n + 1) - W_{\bar{z}}(0, 0) \in \left(-\frac{n + 1}{2}, 0\right).$$

□

Theorem 2.1.1 and Remark 2.1.2 imply the following

**Corollary 2.1.1.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a positive twist map (respectively a negative twist map). Let  $\bar{z} \in \mathbb{A}$  be a point at which the torsion exists.*

*Then*

$$\text{Torsion}(f, \bar{z}) \in \left[-\frac{1}{2}, 0\right] \quad \left(\text{respectively } \left[0, \frac{1}{2}\right]\right). \quad (2.7)$$

**Remark 2.1.5.** The independence of the torsion from the chosen isotopy is assured by Proposition 1.3.2.

**Example 2.1.3.** Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a positive twist map. Any point of an Aubry-Mather set with irrational rotation number has zero torsion. This result has been proved by S. Crovisier in [Cro03] (see Theorem 1.2).

**Example 2.1.4.** Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a positive twist map. If  $z \in \mathbb{A}$  is a hyperbolic fixed point such that  $Df(z)$  has a negative real eigenvalue, then we have  $\text{Torsion}(f, z) = -\frac{1}{2}$ . To find an example of such a dynamics, consider the fixed point  $(0, 0) \in \mathbb{A}$  of the map  $(x, y) \mapsto f_\lambda(x, y) = (x + y - \frac{\lambda}{2\pi} \sin(2\pi x), y - \frac{\lambda}{2\pi} \sin(2\pi x))$  for  $\lambda \in \mathbb{R}, \lambda \geq 4$ .

## 2.1.2 Properties of linking number for lifts of twist maps

Thanks to Theorems 1.4.1 and 2.1.1, we can estimate also the linking number of any two points in the lifted framework  $\mathbb{R}^2$ . Indeed:

**Corollary 2.1.2.** *Let  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a lift of a positive twist map (respectively negative twist map) on  $\mathbb{A}$ . Let  $(F_t)_t$  be the isotopy joining the identity to  $F$ , obtained as a lift of an isotopy on  $\mathbb{A}$  joining  $\text{Id}_{\mathbb{A}}$  to the twist map. Let  $z_1, z_2 \in \mathbb{R}^2, z_1 \neq z_2$  be such that their linking number exists. Then*

$$\text{Linking}((F_t), z_1, z_2) \in \left[-\frac{1}{2}, 0\right] \quad \left(\text{respectively } \left[0, \frac{1}{2}\right]\right).$$

This result holds true for any couples of points for which the (asymptotic) linking number exists.

**Notation 2.1.1.** On  $\mathbb{R}^2$  we fix the counterclockwise orientation and we consider as reference vector field the vertical constant one  $\chi = (0, 1)$ .

*Proof of Corollary 2.1.2.* We are going to prove the result for positive twist maps. Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a positive twist map and let  $I = (f_t)_t$  be an isotopy in  $\text{Diff}^1(\mathbb{A})$  joining the identity to  $f_1 = f$ . Let  $\tilde{I} = (F_t)_t$  be the lift in  $\text{Diff}^1(\mathbb{R}^2)$  of  $I = (f_t)_t$  such that  $F_0 = \text{Id}_{\mathbb{R}^2}$ . So it joins the identity to  $F_1 = F$ , which is a lift of  $f$ .

Let  $z_1, z_2 \in \mathbb{R}^2, z_1 \neq z_2$  and assume that the limit

$$\text{Linking}(\tilde{I}, z_1, z_2) = \lim_{n \rightarrow +\infty} \text{Linking}_n(\tilde{I}, z_1, z_2)$$

exists.

For any  $n \in \mathbb{N}$  denote  $l_n$  as the quantity  $\text{Linking}_n(\tilde{I}, z_1, z_1)$ . Fix now  $n \in \mathbb{N}$ . From Corollary 1.4.1, there exists a point  $z$  lying on the segment joining  $z_1$  and  $z_2$ , such that

$$\text{Torsion}_n(\tilde{I}, z, z_2 - z_1) = l_n.$$



We have

$$\text{Torsion}_n(\tilde{I}, z, \chi) = \text{Torsion}_n(f, \bar{z}, \chi)$$

where  $\bar{z} \in \mathbb{A}$  is the projection on the annulus of the point  $z \in \mathbb{R}^2$ . Therefore Theorem 2.1.1 tells us that

$$\text{Torsion}_n(\tilde{I}, z, \chi) \in \left(-\frac{1}{2}, 0\right). \quad (2.8)$$

By Lemma 1.1.2, it holds

$$\left| \text{Torsion}_n(\tilde{I}, z, z_2 - z_1) - \text{Torsion}_n(\tilde{I}, z, \chi) \right| < \frac{1}{2n}$$

and then, by (2.8),

$$l_n = \text{Torsion}_n(\tilde{I}, z, z_2 - z_1) \in \left(-\frac{1}{2} - \frac{1}{2n}, \frac{1}{2n}\right).$$

We deduce that

$$l_n = \text{Linking}_n(\tilde{I}, z_1, z_2) \in \left(-\frac{1}{2} - \frac{1}{2n}, \frac{1}{2n}\right).$$

Since this holds for any fixed  $n \in \mathbb{N}, n \neq 0$ , passing to the limit, we conclude that

$$\text{Linking}(\tilde{I}, z_1, z_2) \in \left[-\frac{1}{2}, 0\right].$$

□

We also give an estimation of finite-time linking number under some further assumptions.

**Proposition 2.1.3.** *Let  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a lift of a positive twist map and let  $\tilde{I} = (F_t)_t$  be an isotopy joining the identity to  $F$  obtained as a lift of an isotopy on  $\mathbb{A}$ . Let  $p_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$  be the projection over the first coordinate. Let  $z_1, z_2 \in \mathbb{R}^2, z_1 \neq z_2$  be such that  $p_1(z_1) = p_1(z_2)$ . Then for any  $n \in \mathbb{N}$*

$$\text{Linking}_n(\tilde{I}, z_1, z_2) \in \left(-\frac{1}{2}, 0\right). \quad (2.9)$$

*Proof.* Arguing by contradiction, assume that there exist points  $z_1, z_2 \in \mathbb{R}^2, z_1 \neq z_2$  with  $p_1(z_1) = p_1(z_2)$  and  $n \in \mathbb{N}, n \neq 0$  such that

$$\text{Linking}_n(\tilde{I}, z_1, z_2) = l \quad (2.10)$$

with either  $l$  smaller or equal  $-\frac{1}{2}$  or  $l$  greater or equal 0. From the condition over the first coordinate projection, the vector  $z_2 - z_1$  joining the two points is vertical. By Corollary 1.4.1, there exists a point  $z \in \mathbb{R}^2$  lying on the segment joining  $z_2$  and  $z_1$  such that

$$\text{Torsion}_n(\tilde{I}, z, z_2 - z_1) = \text{Torsion}_n(\tilde{I}, z, \chi) = l. \quad (2.11)$$

The value  $l$  does not belong to the interval  $(-\frac{1}{2}, 0)$ . This contradicts Theorem 2.1.1 and we conclude.

□

Remark that if two points  $z_1, z_2$  do not have the same first coordinate projection, then the result of Proposition 2.1.3 does not hold, as shown by the following two examples. Moreover, Examples 2.1.5 and 2.1.6 show us that the extremal values 0 and  $-\frac{1}{2}$  of the admissible interval for the linking number in Corollary 2.1.2 can actually be attained.

**Example 2.1.5.** Consider a lift of a  $\mathcal{C}^1$  diffeomorphism on  $\mathbb{A}$  (not only lifts of twist maps): the linking number of any two points  $z_1, z_2 = z_1 + (1, 0)$  is null.

**Example 2.1.6.** Let  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a lift of a positive twist map on  $\mathbb{A}$ . Assume that  $z_0$  is a hyperbolic (saddle) fixed point such that  $DF(z_0)$  has a negative real eigenvalue of modulus strictly smaller than 1. Let  $z_1$  be a point lying on one of the stable branches of  $z_0$ . Then  $\text{Linking}(F, z_0, z_1) = -\frac{1}{2}$ .

### 2.1.3 Crovisier's torsion for twist maps: definition and comparison

In [Cro03] S. Crovisier gives another definition of torsion for a positive twist map. It seems natural to compare the two definitions: we prove that the two definitions are equivalent and so we deduce that Crovisier's results hold also referring to our Definition 1.1.3.

As before, we fix the counterclockwise orientation and we consider as reference vector field on  $\mathbb{A}$  the constant one  $\chi = (0, 1)$ .

**Notation 2.1.2.** Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a positive twist map. Let  $\bar{z} \in \mathbb{A}$ . We denote as

$$\begin{aligned} \theta^0 : T_{\bar{z}}\mathbb{A} \setminus \{0\} &\rightarrow (-1, 0] \subset \mathbb{R} \\ \xi &\mapsto \theta^0(\xi) \end{aligned}$$

the measure of the oriented angle between the vertical vector  $\chi$  and  $\xi$  contained in the interval  $(-1, 0]$ . The quantity  $\theta^0(Df(\bar{z})\chi)$  is then the measure of the oriented angle between  $\chi$  and  $Df(\bar{z})\chi$  contained in the interval  $(-1, 0]$ .

We denote as

$$\begin{aligned} \theta^1 : T_{\bar{z}}\mathbb{A} \setminus \{0\} &\rightarrow (\theta^0(Df(\bar{z})\chi) - 1, \theta^0(Df(\bar{z})\chi)] \subset \mathbb{R} \\ \xi &\mapsto \theta^1(\xi) \end{aligned}$$

the measure of the oriented angle between  $\chi$  and  $Df(\bar{z})\xi$  contained in the real interval  $(\theta^0(Df(\bar{z})\chi) - 1, \theta^0(Df(\bar{z})\chi)]$ .

**Definition 2.1.2** (Crovisier's definition in [Cro03]). Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a positive twist map. Let  $\bar{z} \in \mathbb{A}$ . According to Notation 2.1.2, we define the following function

$$\begin{aligned} \theta : T_{\bar{z}}\mathbb{A} \setminus \{0\} &\rightarrow \mathbb{R} \\ \xi &\mapsto \theta(\xi) := \theta^1(\xi) - \theta^0(\xi) \end{aligned}$$

which is a measure of the oriented angle between  $\xi$  and  $Df(\bar{z})\xi$ .

For a given  $n \in \mathbb{Z}$  define

$$\theta_n(\xi) := \begin{cases} \sum_{0 \leq k \leq n-1} \theta(Df^k(\bar{z})\xi) & n \geq 0 \\ -\theta_{-n}(Df^n(\bar{z})\xi) & n < 0. \end{cases} \quad (2.12)$$

Observe that for  $k \in \mathbb{N}$ , the quantity  $\theta(Df^k(\bar{z})\xi)$  is the difference between  $\theta^1(Df^k(\bar{z})\xi)$  and  $\theta^0(Df^k(\bar{z})\xi)$ , where  $\theta^0(Df^k(\bar{z})\xi)$  is the measure, contained in  $(-1, 0]$ , of the oriented angle between the vectors  $\chi$  and  $Df^k(\bar{z})\xi$ . These vectors lie in the tangent space  $T_{f^k(\bar{z})}\mathbb{A}$ . On the other hand,  $\theta^1(Df^k(\bar{z})\xi)$  is the measure, contained in the interval  $(\theta^0(Df(f^k(\bar{z}))\chi) - 1, \theta^0(Df(f^k(\bar{z}))\chi)]$ , of the oriented angle between  $\chi$  and  $Df^{k+1}(\bar{z})\xi$ . These vectors lie in the tangent space  $T_{f^{k+1}(\bar{z})}\mathbb{A}$ .

**Proposition 2.1.4.** *Let  $\bar{z} \in \mathbb{A}$  and  $\xi \in T_{\bar{z}}\mathbb{A} \setminus \{0\}$ . Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a positive twist map. Then (see (2.12) in Definition 2.1.2 and (1.4) in Definition 1.1.2)*

$$n \text{ Torsion}_n(f, \bar{z}, \xi) = \theta_n(\xi). \quad (2.13)$$

*Proof.* Let  $I = (f_t)_t$  be an isotopy in  $\text{Diff}^1(\mathbb{A})$  joining the identity to  $f$ . Remark that the torsion does not depend on the chosen  $I$ . Recall that (see (2.12))

$$\theta_n(\xi) = \sum_{0 \leq k \leq n-1} \theta(Df^k(\bar{z})\xi) = \sum_{0 \leq k \leq n-1} (\theta^1(Df^k(\bar{z})\xi) - \theta^0(Df^k(\bar{z})\xi)).$$

On the other hand, we have that (see (1.4))

$$\begin{aligned} n \text{ Torsion}_n(f, \bar{z}, \xi) &= \sum_{0 \leq k \leq n-1} \text{Torsion}_1(f, f^k(\bar{z}), Df^k(\bar{z})\xi) = \\ &= \sum_{0 \leq k \leq n-1} (\tilde{v}(I)(f^k(\bar{z}), Df^k(\bar{z})\xi, 1) - \tilde{v}(I)(f^k(\bar{z}), Df^k(\bar{z})\xi, 0)). \end{aligned}$$

We prove that for any  $0 \leq k \leq n-1$

$$\theta^1(Df^k(\bar{z})\xi) - \theta^0(Df^k(\bar{z})\xi) = \tilde{v}(I)(f^k(\bar{z}), Df^k(\bar{z})\xi, 1) - \tilde{v}(I)(f^k(\bar{z}), Df^k(\bar{z})\xi, 0)$$

and this concludes the proof.

We show it for  $k = 0$  since the proof of the equality of the other terms is the same.

The oriented angles involved are the same. Indeed,  $\theta^0(\xi)$  is a measure of the oriented angle between  $\chi$  and  $\xi$ , that is the angle  $v(I)(\bar{z}, \xi, 0)$ ;  $\theta^1(\xi)$  is a measure of the oriented angle between  $\chi$  and  $Df(\bar{z})\xi$ , that is the angle  $v(I)(\bar{z}, \xi, 1)$ .

The quantity  $\tilde{v}(I)(\bar{z}, \xi, 1) - \tilde{v}(I)(\bar{z}, \xi, 0)$  does not depend on the chosen lift. We show that, by choosing the lift so that  $\tilde{v}(I)(\bar{z}, \xi, 0) = \theta^0(\xi)$ , it holds  $\tilde{v}(I)(\bar{z}, \xi, 1) = \theta^1(\xi)$ . This implies the required equality.

We refer to the notation of Proposition 1.1.2. We choose the lift so that

$$\tilde{v}(I)(\bar{z}, \xi, 0) = \theta^0(\xi) \in (-1, 0].$$

Denote  $s = \theta^0(\xi)$ . Then  $W_{\bar{z}}(s, \cdot) = \tilde{v}(I)(\bar{z}, \xi, \cdot)$ .

Observe that  $W_{\bar{z}}(-1, 0) < W_{\bar{z}}(s, 0) \leq W_{\bar{z}}(0, 0)$ . By point (ii) of Proposition 1.1.2 it holds

$$W_{\bar{z}}(-1, 1) < W_{\bar{z}}(s, 1) = \tilde{v}(I)(\bar{z}, \xi, 1) \leq W_{\bar{z}}(0, 1).$$

Point (iii) of Proposition 1.1.2 implies that  $W_{\bar{z}}(-1, 1) = W_{\bar{z}}(0, 1) - 1$ , so we have

$$W_{\bar{z}}(0, 1) - 1 < \tilde{v}(I)(\bar{z}, \xi, 1) \leq W_{\bar{z}}(0, 1).$$

By Theorem 2.1.1 for  $n = 1$  it holds  $W_{\bar{z}}(0, 1) \in (-\frac{1}{2}, 0) \subset (-1, 0]$ . Being lifts of the same angle both contained in  $(-1, 0]$ , we have  $W_{\bar{z}}(0, 1) = \theta^0(Df(\bar{z})\chi)$ .

We then conclude that  $W_{\bar{z}}(s, 1) = \tilde{v}(I)(\bar{z}, \xi, 1) \in (-1 + \theta^0(Df(\bar{z})\chi), \theta^0(Df(\bar{z})\chi)]$  and so  $\tilde{v}(I)(\bar{z}, \xi, 1) = \theta^1(\xi)$  being lifts of the same angle both contained in the interval  $(-1 + \theta^0(Df(\bar{z})\chi), \theta^0(Df(\bar{z})\chi)]$ .

□

**Remark 2.1.6.** Thanks to Proposition 2.1.4, our torsion (see Definition 1.1.3) at a point exists if and only if Crovisier's torsion at that point exists (in particular they are equal).

We recall the result obtained by S. Crovisier in [Cro03]. Since the two definitions of torsion are equivalent, this result holds true also referring to the torsion presented in Definition 1.1.3. For the definition of *well-ordered sets* we refer to [Che85] and [Cro03].

**Definition 2.1.3** (Well-ordered set). A set  $\bar{E} \subset \mathbb{A}$ , not empty and invariant for  $f$ , and its lift  $E \subset \mathbb{R}^2$  are said *well-ordered* if

- (i)  $\bar{p}_1 : \bar{E} \rightarrow \mathbb{T}$  is injective;
- (ii) for any  $z, z' \in E$ , lifts of points  $\bar{z}, \bar{z}' \in \bar{E}$ , such that  $p_1(z) < p_1(z')$ , it holds that  $p_1(F(z)) < p_1(F(z'))$ .

Let  $z \in \mathbb{R}^2$ . The rotation number of  $z$  for  $F$  is, whenever it exists, the limit

$$\lim_{n \rightarrow +\infty} \left( \frac{p_1 \circ F^n(z) - p_1(z)}{n} \right).$$

Let  $E \subset \mathbb{R}^2$  be a well-ordered set. Then the rotation number of  $E$  is the rotation number of any  $z \in E$ . It is well-defined and it does not depend on  $z \in E$  (see [LC91]). From Proposition 2.1.4, the following result by Crovisier holds with respect to our torsion too.

**Theorem 2.1.2** (Theorem 1.2 in [Cro03]). *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a positive twist map. Then, every point of any well-ordered set with irrational rotation number has zero torsion.*

## 2.2 Set of points of zero torsion for negative-torsion maps

We start by defining an essential curve of the annulus.

**Definition 2.2.1.** An essential curve is a  $\mathcal{C}^0$  embedded circle in  $\mathbb{A}$  not homotopic to a point.

The main results of this section will be the following

**Theorem 2.2.1.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a negative-torsion map. Then for any  $\mathcal{C}^1$  essential curve  $\gamma : \mathbb{T} \rightarrow \mathbb{A}$  there exists  $\bar{z} \in \gamma(\mathbb{T})$  such that  $\text{Torsion}(f, \bar{z}) = 0$ .*

**Corollary 2.2.1.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a negative-torsion map. Then*

$$\dim_H(\{\bar{z} \in \mathbb{A} : \text{Torsion}(f, \bar{z}) = 0\}) \geq 1,$$

where  $\dim_H$  denotes the Hausdorff dimension of a set.

We refer to [Fal86] for the definition of Hausdorff dimension of a set. Concerning Theorem 2.2.1, we will first prove the following simpler version.

**Theorem 2.2.2.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a negative-torsion map. Then for any  $r \in \mathbb{R}$  there exists  $\bar{z}(r) \in \mathbb{T} \times \{r\}$  such that  $\text{Torsion}(f, \bar{z}(r)) = 0$ .*

After it, we will present the proof of Theorem 2.2.1 (which generalizes Theorem 2.2.2).

### 2.2.1 Points of zero torsion on simple circle curves

We start by showing some lemmas that will lead us to the proof of Theorem 2.2.2.

**Lemma 2.2.1.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a  $\mathcal{C}^1$  diffeomorphism isotopic to the identity. Let  $r \in \mathbb{R}$  and  $n \in \mathbb{N}^*$ . Then there exists  $\bar{z}(r, n) \in \mathbb{T} \times \{r\}$  such that*

$$\text{Torsion}_n(f, \bar{z}(r, n), \mathcal{H}) = 0, \quad (2.14)$$

where  $\mathcal{H}$  is the horizontal vector  $(1, 0)$ .

*Proof.* Let  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a lift of  $f$ . Let  $(F_t)_t \in \text{Diff}^1(\mathbb{R}^2)$  be the isotopy joining the identity of  $\mathbb{R}^2$  to  $F$ , obtained as lift of an isotopy on  $\mathbb{A}$  joining  $\text{Id}_{\mathbb{A}}$  to  $f$ .

Observe that for any  $t \in \mathbb{R}_+$  the function  $F_t$  commutes with the translation by  $(1, 0)$ . Consequently, for a fixed  $r \in \mathbb{R}$ , for any  $n \in \mathbb{N}^*$  it holds (see Definition 1.2.1)

$$\text{Linking}_n((F_t)_t, (0, r), (1, r)) = 0.$$

By Corollary 1.4.1 there exists  $z(r, n) \in [0, 1] \times \{r\}$  such that

$$\text{Torsion}_n((F_t)_t, z(r, n), \mathcal{H}) = 0,$$

where  $\mathcal{H}$  is the positive horizontal vector of norm one. Denoting as  $\bar{z}(r, n) \in \mathbb{T} \times \{r\}$  the projection on the annulus of the point  $z(r, n)$ , we conclude that

$$\text{Torsion}_n(f, \bar{z}(r, n), \mathcal{H}) = 0.$$

□

**Remark 2.2.1.** Let  $\bar{z}(r, n) \in \mathbb{T} \times \{r\}$  be the point given by Lemma 2.2.1 applied to a negative-torsion map  $f$ . Then, by Lemma 1.1.2, it holds

$$n \text{Torsion}_n(f, \bar{z}(r, n), \chi) \in \left(-\frac{1}{2}, 0\right). \quad (2.15)$$

**Lemma 2.2.2.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a negative-torsion map. Let  $m \in \mathbb{N}^*$  and let  $\bar{z} \in \mathbb{A}$  be such that  $m \text{Torsion}_m(f, \bar{z}, \chi) < -\frac{k}{2}$  for some  $k \in \mathbb{N}^*$ . Then for any  $n \geq m$  it holds  $n \text{Torsion}_n(f, \bar{z}, \chi) < -\frac{k}{2}$ .*

The statement is a consequence of the negative-torsion condition and of the following

**Lemma 2.2.3.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a  $\mathcal{C}^1$  diffeomorphism isotopic to the identity. Let  $a \in \mathbb{A}$ . Let  $\mathcal{N} \in \mathbb{N}^*$ ,  $(\mathcal{K}_i)_{i \in \llbracket 0, \mathcal{N}-1 \rrbracket} \in \mathbb{N}^{\mathcal{N}}$  and  $l_0 = 0 < l_1 < \dots < l_{\mathcal{N}}$  with  $l_i \in \mathbb{N}$  for any  $i$ . Assume that for all  $i \in \llbracket 0, \mathcal{N}-1 \rrbracket$  it holds*

$$(l_{i+1} - l_i) \text{Torsion}_{l_{i+1}-l_i}(f, f^{l_i}(a), \chi) < -\frac{\mathcal{K}_i}{2}.$$

Then for any vector  $\xi \in T_a \mathbb{A} \setminus \{0\}$  we have

$$l_{\mathcal{N}} \text{Torsion}_{l_{\mathcal{N}}}(f, a, \xi) < -\frac{\sum_{i=0}^{\mathcal{N}-1} \mathcal{K}_i}{2} + \frac{1}{2}.$$

In particular, when  $\xi = \chi$  we have

$$l_{\mathcal{N}} \text{Torsion}_{l_{\mathcal{N}}}(f, a, \chi) < -\frac{\sum_{i=0}^{\mathcal{N}-1} \mathcal{K}_i}{2}.$$

We postpone the proof of Lemma 2.2.3 to Appendix 2.5.

*Proof of Lemma 2.2.2.* For  $n = m$  there is nothing to prove. Fix  $n > m$  and apply Lemma 2.2.3 for  $f$  at the point  $z \in \mathbb{A}$  with respect to  $\mathcal{N} = 2, l_1 = m, l_2 = n, \mathcal{K}_1 = k, \mathcal{K}_2 = 0$ . We can use Lemma 2.2.3 because by hypothesis

$$m \text{ Torsion}_m(f, z, \chi) < -\frac{k}{2},$$

and because, since  $f$  is a negative-torsion map and  $n - m > 0$ , it holds

$$(n - m) \text{ Torsion}_{n-m}(f, f^m(z), \chi) < 0.$$

We conclude that

$$n \text{ Torsion}_n(f, z, \chi) < -\frac{k}{2}.$$

□

**Lemma 2.2.4.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a negative-torsion map. Let  $\bar{z}(r, n) \in \mathbb{T} \times \{r\}$  be the point given by Lemma 2.2.1 applied at  $f$ . Then for any  $m \in (0, n]$  it holds*

$$m \text{ Torsion}_m(f, \bar{z}(r, n), \chi) \in \left[-\frac{1}{2}, 0\right).$$

*Proof.* Argue by contradiction and assume there exists  $m \in (0, n]$  such that

$$m \text{ Torsion}_m(f, \bar{z}(r, n), \chi) \notin \left[-\frac{1}{2}, 0\right).$$

In particular, since  $f$  is a negative-torsion map, it holds

$$m \text{ Torsion}_m(f, \bar{z}(r, n), \chi) < -\frac{1}{2}.$$

By Remark 2.2.1 we have that  $n \text{ Torsion}_n(f, \bar{z}(r, n), \chi) \in (-\frac{1}{2}, 0)$ .

If  $m = n$ , then we immediately obtain the required contradiction. If  $m < n$ , then by Lemma 2.2.2 it holds that  $n \text{ Torsion}_n(f, \bar{z}(r, n), \chi) < -\frac{1}{2}$ , which provides again an absurd and we conclude.

□

**Notation 2.2.1.** Fix  $r \in \mathbb{R}$ . Consider the sequence of points  $(\bar{z}(r, n))_{n \in \mathbb{N}} \in \mathbb{T} \times \{r\}$  given by Lemma 2.2.1. Denote as  $\bar{z}(r) \in \mathbb{T} \times \{r\}$  a limit point of  $(\bar{z}(r, n))_{n \in \mathbb{N}^*}$ . Such a point exists since  $\mathbb{T} \times \{r\}$  is compact.

**Lemma 2.2.5.** *Let  $\bar{z}(r) \in \mathbb{T} \times \{r\}$  be a limit point of  $(\bar{z}(r, n))_{n \in \mathbb{N}}$ . Then for any  $N \in \mathbb{N}^*$  it holds*

$$N \text{ Torsion}_N(f, \bar{z}(r), \chi) \in \left[-\frac{1}{2}, 0\right)$$

*Proof.* Since  $f$  is a negative-torsion map, we have that for any  $N \in \mathbb{N}^*$

$$N \text{ Torsion}_N(f, \bar{z}(r), \chi) < 0.$$

Fix now  $N \in \mathbb{N}^*$ . Let  $\varepsilon > 0$ . Since  $\bar{z}(r)$  is a limit point of the sequence  $(\bar{z}(r, n))_{n \in \mathbb{N}}$  and by the continuity of the function  $x \mapsto N \text{ Torsion}_N(f, x, \chi)$ , there exists  $\bar{n} \in \mathbb{N}, \bar{n} > N$  such that

$$|N \text{ Torsion}_N(f, \bar{z}(r), \chi) - N \text{ Torsion}_N(f, \bar{z}(r, \bar{n}), \chi)| < \varepsilon. \quad (2.16)$$

Consequently we obtain, from (2.16) and from Lemma 2.2.4,

$$\begin{aligned} & N \text{Torsion}_N(f, \bar{z}(r), \chi) = \\ &= (N \text{Torsion}_N(f, \bar{z}(r), \chi) - N \text{Torsion}_N(f, \bar{z}(r, \bar{n}), \chi)) + N \text{Torsion}_N(f, \bar{z}(r, \bar{n}), \chi) > -\varepsilon - \frac{1}{2}. \end{aligned}$$

By the arbitrariness of  $\varepsilon$  we conclude that  $N \text{Torsion}_N(f, \bar{z}(r), \chi) \in [-\frac{1}{2}, 0)$ .  $\square$

*Proof of Theorem 2.2.2.* Fix  $r \in \mathbb{R}$  and consider a point  $\bar{z}(r) \in \mathbb{T} \times \{r\}$  which is a limit point of the sequence  $(\bar{z}(r, n))_{n \in \mathbb{N}}$  defined in Lemma 2.2.1. By Lemma 2.2.5 for any  $N \in \mathbb{N}^*$  it holds

$$\text{Torsion}_N(f, \bar{z}(r), \chi) \in \left[ -\frac{1}{2N}, 0 \right).$$

Consequently, as  $N$  goes to  $+\infty$ , we have that  $\text{Torsion}(f, \bar{z}(r)) = 0$ .  $\square$

*Proof of Corollary 2.2.1.* By Theorem 2.2.2 for any  $r \in \mathbb{R}$  there exists  $\bar{z}(r) \in \mathbb{T} \times \{r\}$  such that  $\text{Torsion}(f, \bar{z}(r)) = 0$ . Thus, looking at the projection  $\bar{p}_2$  over the second coordinate, we deduce that

$$\bar{p}_2(\{z \in \mathbb{A} : \text{Torsion}(f, z) = 0\}) = \mathbb{R}.$$

We consider now the Hausdorff dimension, denoted as  $\dim_H$ . Recall that if  $g$  is Lipschitz, then, for any set  $U$ ,  $\dim_H(U) \geq \dim_H(g(U))$  (see Lemma 1.8 in [Fal86]).

Since the projection  $\bar{p}_2$  is Lipschitz and since the Hausdorff dimension of  $\mathbb{R}$  is  $\dim_H(\mathbb{R}) = 1$ , we conclude that  $\dim_H(\{\bar{z} \in \mathbb{A} : \text{Torsion}(f, \bar{z}) = 0\}) \geq 1$ .  $\square$

## 2.2.2 Angle variation along $\gamma$ along a $\mathcal{C}^1$ essential curve

In this Subsection we explain how to calculate the angle variation of a vector along a  $\mathcal{C}^1$  essential curve. Such angle variation will be used in the proof of Theorem 2.2.1 and in Section 2.4.

Let  $\gamma : \mathbb{T} \rightarrow \mathbb{A}$  be a  $\mathcal{C}^1$  essential curve and let  $x, y \in \gamma(\mathbb{T})$ . Let  $s_1, s_2 \in \mathbb{T}$  be such that  $\gamma(s_1) = x, \gamma(s_2) = y$ . Fix  $S_1 \in \mathbb{R}$  a lift of  $s_1$  and let  $S_2 \in \mathbb{R}$  be the lift of  $s_2$  such that  $S_2 \in (S_1, S_1 + 1]$ .

Define the oriented angle function

$$\mathbb{R}_+ \ni t \mapsto \Theta(\gamma, S_1)(t) := \theta(\mathcal{H}, \gamma' \circ p(S_1 + t)) \in \mathbb{T},$$

where  $p : \mathbb{R} \rightarrow \mathbb{T}$  is the covering map of  $\mathbb{T}$ . Equivalently,  $\Theta(\gamma, S_1)(t)$  is the oriented angle between  $\mathcal{H}$  and the vector tangent to  $\gamma$  at  $\gamma(p(S_1 + t))$ .

Denote as  $\tilde{\Theta}(\gamma, S_1) : \mathbb{R}_+ \rightarrow \mathbb{R}$  a continuous determination of the previous angle function.

**Definition 2.2.2.** The angle variation along  $\gamma$  between  $x$  and  $y$  is

$$\text{Var}_\gamma(x, y) := \tilde{\Theta}(\gamma, S_1)(S_2 - S_1) - \tilde{\Theta}(\gamma, S_1)(0).$$

**Remark 2.2.2.** The angle variation along  $\gamma$  between  $x$  and  $y$  does not depend on the choice of the continuous determination  $\tilde{\Theta}(\gamma, S_1)$ .

**Proposition 2.2.1.** *Let  $\gamma$  be a  $\mathcal{C}^1$  essential curve. Let  $x, y, z \in \gamma(\mathbb{T})$ .*

- (1)  *$\text{Var}_\gamma(x, y)$  does not depend on the choice of the lift  $S_1$  of  $s_1 \in \mathbb{T}$  such that  $\gamma(s_1) = x$ .*
- (2)  *$\text{Var}_\gamma(x, x) = 0$ .*
- (3)  *$\text{Var}_\gamma(x, y) + \text{Var}_\gamma(y, z) = \text{Var}_\gamma(x, z)$ .*

*Proof.* (1) Let  $s_1, s_2 \in \mathbb{T}$  be such that  $\gamma(s_1) = x, \gamma(s_2) = y$ . Consider two lifts  $S_1, S_1 + 1$  of  $s_1 \in \mathbb{T}$ . We want to compare

$$\tilde{\Theta}(\gamma, S_1)(S_2 - S_1) - \tilde{\Theta}(\gamma, S_1)(0) \quad \text{and} \quad \tilde{\Theta}(\gamma, S_1 + 1)(S_2 - S_1) - \tilde{\Theta}(\gamma, S_1 + 1)(0),$$

where  $S_2 \in \mathbb{R}$  is the lift of  $s_2$  contained in  $(S_1, S_1 + 1]$ . Observe that  $S_2 + 1$  is the lift of  $s_2$  contained in  $(S_1 + 1, S_1 + 2]$ .

Since the angles  $\Theta(\gamma, S_1 + 1)(0)$  and  $\Theta(\gamma, S_1)(0)$  are equal, choose the continuous determinations such that  $\tilde{\Theta}(\gamma, S_1 + 1)(0) = \tilde{\Theta}(\gamma, S_1)(0)$ . Since  $\xi \mapsto \tilde{\Theta}(\gamma, S_1 + 1)(\xi)$  and  $\xi \mapsto \tilde{\Theta}(\gamma, S_1)(\xi)$  are lifts of the same angle function that coincide at  $\xi = 0$ , they are equal. In particular,  $\tilde{\Theta}(\gamma, S_1 + 1)(S_2 - S_1) = \tilde{\Theta}(\gamma, S_1)(S_2 - S_1)$ . Consequently

$$\tilde{\Theta}(\gamma, S_1 + 1)(S_2 - S_1) - \tilde{\Theta}(\gamma, S_1 + 1)(0) = \tilde{\Theta}(\gamma, S_1)(S_2 - S_1) - \tilde{\Theta}(\gamma, S_1)(0)$$

and we conclude.

- (2) Let  $s_1 \in \mathbb{T}$  be such that  $\gamma(s_1) = x$  and fix a lift  $S_1 \in \mathbb{R}$  of  $s_1$ . Then, the angle variation  $\text{Var}_\gamma(x, x)$  is

$$\tilde{\Theta}(\gamma, S_1)(1) - \tilde{\Theta}(\gamma, S_1)(0).$$

Consider the change of coordinates

$$\begin{aligned} g : \mathbb{A} &\rightarrow \mathbb{R}^2 \setminus \{(0, 0)\} \\ (x, y) &\mapsto e^y(\sin(x), \cos(x)). \end{aligned}$$

It is a local diffeomorphism that preserves the orientation and it is conformal. The image  $g \circ \gamma(\mathbb{T})$  is a simple closed  $\mathcal{C}^1$  curve. Let us parametrize it as

$$[0, 1] \ni t \mapsto g \circ \gamma \circ p(S_1 + t) \in \mathbb{R}^2 \setminus \{(0, 0)\}.$$

Apply now the Turning Tangent Theorem to  $g \circ \gamma(\mathbb{T})$  and obtain that the variation of the angle

$$\theta(\mathcal{H}, Dg(\gamma(p(S_1 + t)))\gamma'(p(S_1 + t))) \tag{2.17}$$

between  $t = 0$  and  $t = 1$  is equal to  $\delta \in \{\pm 1\}$ , where  $\delta$  depends on the orientation of the curve  $g \circ \gamma(\mathbb{T})$ . Consider now the vector  $Dg(\gamma(p(S_1 + t)))\mathcal{H}$  and calculate the variation of the angle

$$\theta(\mathcal{H}, Dg(\gamma(p(S_1 + t)))\mathcal{H})$$

between  $t = 0$  and  $t = 1$ . In particular

$$Dg(\gamma(p(S_1 + t)))\mathcal{H} = e^{\gamma_2} \begin{pmatrix} \cos(\gamma_1) & \sin(\gamma_1) \\ -\sin(\gamma_1) & \cos(\gamma_1) \end{pmatrix} (1, 0) = e^{\gamma_2}(\cos(2\pi\gamma_1), -\sin(2\pi\gamma_1)),$$

where  $(\gamma_1, \gamma_2) = (\gamma_1(t), \gamma_2(t)) = \gamma \circ p(S_1 + t)$ . The angle  $\theta(\mathcal{H}, Dg(\gamma(p(S_1 + t)))\mathcal{H})$  is so  $-\gamma_1(t)$ . Let  $\Gamma_1$  be a lift of  $\gamma_1$ . Since  $\gamma$  is an essential curve, it holds that



$\Gamma_1(t+1) = \Gamma_1(t) + \delta$ , where  $\delta \in \{\pm 1\}$  depends on the orientation of the curve  $g \circ \gamma(\mathbb{T})$  (see Lemma 2.4.1).

Therefore, the angle variation of the vector  $Dg(\gamma(p(S_1+t)))\mathcal{H}$  between  $t=0$  and  $t=1$  is  $\delta \in \{\pm 1\}$ . Such angle variation is the same as the angle variation of (2.17) between 0 and 1.

Consequently, the variation of the angle

$$\theta(Dg(\gamma(p(S_1+t)))\mathcal{H}, Dg(\gamma(p(S_1+t)))\gamma'(p(S_1+t)))$$

between  $t=0$  and  $t=1$  is null. Since  $g$  is conformal we deduce that the angle variation  $\theta(\mathcal{H}, \gamma'(p(S_1+t)))$  between  $t=0$  and  $t=1$  is also null. Such angle variation is  $\tilde{\Theta}(\gamma, S_1)(1) - \tilde{\Theta}(\gamma, S_1)(0)$  and so we conclude that  $Var_\gamma(x, x) = 0$ .

- (3) Let  $x, y, z \in \gamma(\mathbb{T})$ . Fix  $S_1 \in \mathbb{R}$  such that  $\gamma \circ p(S_1) = x$ . Let  $S_2 \in \mathbb{R}, S_2 \in (S_1, S_1+1]$  be a lift of  $s_2 \in \mathbb{T}$  such that  $\gamma(s_2) = y$  and let  $S_3 \in \mathbb{R}, S_3 \in (S_2, S_2+1]$  be a lift of  $s_3 \in \mathbb{T}$  such that  $\gamma(s_3) = z$ . Consider so the angle functions

$$\mathbb{R}_+ \ni t \mapsto \Theta(\gamma, S_1)(t) = \theta(\mathcal{H}, \gamma'(p(S_1+t))),$$

$$\mathbb{R}_+ \ni t \mapsto \Theta(\gamma, S_2)(t) = \theta(\mathcal{H}, \gamma'(p(S_2+t))).$$

Now

$$Var_\gamma(x, y) = \tilde{\Theta}(\gamma, S_1)(S_2 - S_1) - \tilde{\Theta}(\gamma, S_1)(0),$$

$$Var_\gamma(y, z) = \tilde{\Theta}(\gamma, S_2)(S_3 - S_2) - \tilde{\Theta}(\gamma, S_2)(0).$$

Choose the lift such that  $\tilde{\Theta}(\gamma, S_2)(0) = \tilde{\Theta}(\gamma, S_1)(S_2 - S_1)$ . Then, the lifts  $\xi \mapsto \tilde{\Theta}(\gamma, S_2)(\xi)$  and  $\xi \mapsto \tilde{\Theta}(\gamma, S_1)(S_2 - S_1 + \xi)$  are measures of the same angle function that coincide at  $\xi = 0$ , hence they are equal. In particular

$$\tilde{\Theta}(\gamma, S_2)(S_3 - S_2) = \tilde{\Theta}(\gamma, S_1)(S_3 - S_1).$$

We so obtain

$$\begin{aligned} Var_\gamma(x, y) + Var_\gamma(y, z) &= \\ &= \tilde{\Theta}(\gamma, S_1)(S_2 - S_1) - \tilde{\Theta}(\gamma, S_1)(0) + \tilde{\Theta}(\gamma, S_2)(S_3 - S_2) - \tilde{\Theta}(\gamma, S_2)(0) = \\ &= \tilde{\Theta}(\gamma, S_1)(S_3 - S_1) - \tilde{\Theta}(\gamma, S_1)(0). \end{aligned}$$

If  $S_3 \in (S_1, S_1+1]$ , then this last term is exactly  $Var_\gamma(x, z)$ . If  $S_3 \in (S_1+1, S_1+2]$ , then

$$\begin{aligned} &\tilde{\Theta}(\gamma, S_1)(S_3 - S_1) - \tilde{\Theta}(\gamma, S_1)(0) = \\ &= \tilde{\Theta}(\gamma, S_1)(S_3 - S_1) - \tilde{\Theta}(\gamma, S_1)(1) + \tilde{\Theta}(\gamma, S_1)(1) - \tilde{\Theta}(\gamma, S_1)(0) = \\ &= \tilde{\Theta}(\gamma, S_1)(S_3 - S_1) - \tilde{\Theta}(\gamma, S_1)(1) + Var_\gamma(x, x) = \tilde{\Theta}(\gamma, S_1)(S_3 - S_1) - \tilde{\Theta}(\gamma, S_1)(1). \end{aligned}$$

Since, from point (1), the angle variation does not depend on the lift of  $s_1 \in \mathbb{T}$ , it holds

$$\begin{aligned} &\tilde{\Theta}(\gamma, S_1)(S_3 - S_1) - \tilde{\Theta}(\gamma, S_1)(1) = \\ &= \tilde{\Theta}(\gamma, S_1+1)(S_3 - S_1 - 1) - \tilde{\Theta}(\gamma, S_1+1)(0) = Var_\gamma(x, z) \end{aligned}$$

and we conclude. □

**Remark 2.2.3.** Fix  $\gamma(s) \in \gamma(\mathbb{T})$ . We observe that the function  $\mathbb{R}_+ \ni t \mapsto \text{Var}_\gamma(\gamma(s), \gamma(s+p(t))) \in \mathbb{R}$  is 1-periodic.

**Remark 2.2.4.** An essential curve  $\gamma$  on the annulus is isotopic to either

$$\mathbb{T} \ni t \mapsto \mathbf{c}_1(t) = (t, 0) \quad \text{or} \quad \mathbb{T} \ni t \mapsto \mathbf{c}_{-1}(t) = (-t, 0).$$

See Proposition 2.4.1 for a deeper discussion.

**Proposition 2.2.2.** Let  $\gamma : \mathbb{T} \rightarrow \mathbb{A}$  be a  $\mathcal{C}^1$  essential curve. Let  $s_0, s_1 \in \mathbb{T}, s_0 \neq s_1$  correspond to points of maximal height on  $\gamma$ , that is

$$\bar{p}_2 \circ \gamma(s_0) = \bar{p}_2 \circ \gamma(s_1) = \max_{s \in \mathbb{T}} \bar{p}_2 \circ \gamma(s).$$

Then

$$\text{Var}_\gamma(\gamma(s_0), \gamma(s_1)) = 0.$$

*Proof.* Let  $s_0, s_1 \in \mathbb{T}, s_0 \neq s_1$  be such that  $\bar{p}_2 \circ \gamma(s_0) = \bar{p}_2 \circ \gamma(s_1) = \max_{s \in \mathbb{T}} \bar{p}_2 \circ \gamma(s)$ . Let  $S_0 \in \mathbb{R}$  be a lift of  $s_0 \in \mathbb{T}$  and let  $S_1 \in (S_0, S_0 + 1)$  be the lift of  $s_1$  (contained in  $(S_0, S_0 + 1)$ ).

Look now at the lifted framework and denote as  $\Gamma : \mathbb{R} \rightarrow \mathbb{R}^2$  a lift of  $\gamma$ . Consider the points  $\Gamma(S_0), \Gamma(S_1)$  and build the piecewise  $\mathcal{C}^1$  closed curve  $\mathcal{C}$  by concatenating the following ones (see Figure 2.1):

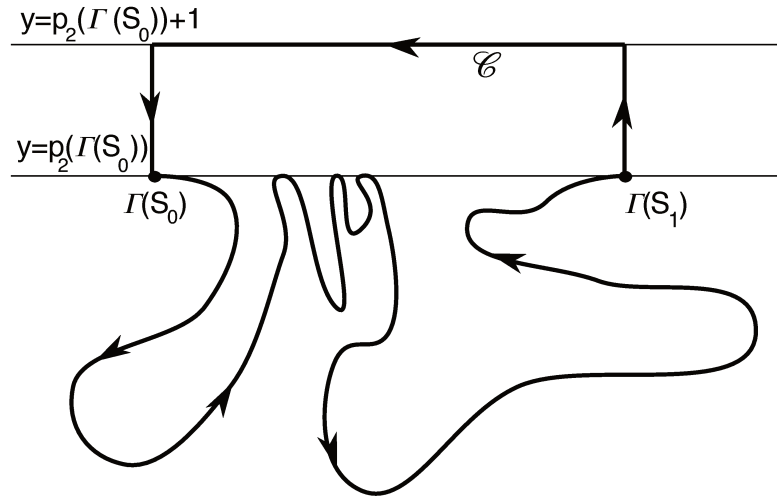


Figure 2.1 – The simple curve built in the proof of Proposition 2.2.2.

- $\{\Gamma(s) : s \in [S_0, S_1]\}$ ;
- the vertical segment  $\{(p_1 \circ \Gamma(S_0), p_2 \circ \Gamma(S_0) + \xi) : \xi \in [0, 1]\}$ ;
- the horizontal segment  $\{(\xi p_1 \circ \Gamma(S_0) + (1 - \xi)p_1 \circ \Gamma(S_1), p_2 \circ \gamma(S_0) + 1) : \xi \in [0, 1]\}$ ;
- the vertical segment  $\{(p_1 \circ \Gamma(S_1), p_2 \circ \Gamma(S_1) + 1 - \xi) : \xi \in [0, 1]\}$ .

Such a piecewise  $\mathcal{C}^1$  closed curve does not have self-intersections because both  $\Gamma(S_0)$  and  $\Gamma(S_1)$  are points of maximal height. We are then interested in

$$Var_\gamma(\gamma(s_0), \gamma(s_1)) = Var_\Gamma(\Gamma(S_0), \Gamma(S_1)) = \tilde{\Theta}(\Gamma, S_0)(S_1 - S_0) - \tilde{\Theta}(\Gamma, S_0)(0).$$

**Claim 2.2.1.** If  $\gamma$  is homotopic to  $\mathbf{c}_1$ , then

$$p_1 \circ \Gamma(S_0) < p_1 \circ \Gamma(S_1) \quad \text{and} \quad \Gamma'(S_0), \Gamma'(S_1) \in \mathbb{R}_+ \mathcal{H}.$$

If  $\gamma$  is homotopic to  $\mathbf{c}_{-1}$ , then

$$p_1 \circ \Gamma(S_0) > p_1 \circ \Gamma(S_1) \quad \text{and} \quad \Gamma'(S_0), \Gamma'(S_1) \in \mathbb{R}_- \mathcal{H}.$$

*Proof.* Since both  $S_0$  and  $S_1$  are points of maximal height of  $\Gamma$  and since  $\Gamma$  is  $\mathcal{C}^1$ , both  $\Gamma'(S_0)$  and  $\Gamma'(S_1)$  are in  $\mathbb{R}\mathcal{H}$ .

By Jordan's theorem, the closed curve  $\mathcal{C}$  (see Figure 2.1) separates the plane into two regions, a bounded one and an unbounded one.

The curve  $\Gamma(\mathbb{R})$  does not have self-intersections. Moreover, since  $\Gamma(S_0), \Gamma(S_1)$  are points of maximal height of  $\Gamma$ , the curve  $\Gamma$  cannot lie in  $\{(x, y) \in \mathbb{R}^2 : y > p_2 \circ \Gamma(S_0) = p_2 \circ \Gamma(S_1)\}$ . Thus,  $\Gamma(\mathbb{R} \setminus [S_0, S_1])$  cannot intersect  $\mathcal{C}$ .

Assume that  $\gamma$  is homotopic to  $\mathbf{c}_1$ . Argue by contradiction and assume that  $p_1 \circ \Gamma(S_1) < p_1 \circ \Gamma(S_0)$ <sup>1</sup>.

If  $\Gamma'(S_0) \in \mathbb{R}_+ \mathcal{H}$ , then, since  $\Gamma(\mathbb{R} \setminus [S_0, S_1])$  cannot intersect  $\mathcal{C}$ ,  $\Gamma((-\infty, S_0))$  is contained in the bounded region determined by the closed curve  $\mathcal{C}$ , which is a contradiction. Indeed, since  $\gamma$  is homotopic to  $\mathbf{c}_1$ , for any  $S \in \mathbb{R}$  and any  $n \in \mathbb{N}$  it holds  $\Gamma(S+n) = \Gamma(S) + (n, 0)$  (see case (a) in Figure 2.2).

Thus,  $\Gamma'(S_0) \in \mathbb{R}_- \mathcal{H}$ . Since  $\gamma$  is homotopic to  $\mathbf{c}_1$ , there exists  $n \in \mathbb{N}$  such that

$$p_1 \circ \Gamma(S_0 - n) < p_1 \circ \Gamma(S)$$

for every  $S \in [S_0, S_1]$ . In particular,  $\Gamma([S_0 - n, S_0])$  lies in the unbounded region determined by  $\mathcal{C}$ . Since  $p_2 \circ \Gamma(S_0 - n) = p_2 \circ \Gamma(S_0)$ , we can build a closed curve  $\mathcal{C}'$  as done for  $S_0, S_1$ , starting from  $S_0 - n, S_0$  (see case (b) in Figure 2.2). The point  $\Gamma(S_1)$  is contained in the bounded region determined by  $\mathcal{C}'$ . Thus, it follows that  $\Gamma((S_0, +\infty))$  is bounded, which is a contradiction. We conclude that  $p_1 \circ \Gamma(S_0) < p_1 \circ \Gamma(S_1)$ .

Since  $p_1 \circ \Gamma(S_0) < p_1 \circ \Gamma(S_1)$  and since both  $\Gamma((-\infty, S_0))$  and  $\Gamma((S_1, +\infty))$  cannot intersect  $\mathcal{C}$ , if  $\Gamma'(S_0) \in \mathbb{R}_- \mathcal{H}$  (respectively  $\Gamma'(S_1) \in \mathbb{R}_- \mathcal{H}$ ), then  $\Gamma((-\infty, S_0))$  (respectively  $\Gamma((S_1, +\infty))$ ) would be contained in the bounded region determined by the closed curve  $\mathcal{C}$ , providing the required contradiction.

The result for  $\gamma$  homotopic to  $\mathbf{c}_{-1}$  can be deduced similarly. □

In particular, if  $\gamma$  is homotopic to  $\mathbf{c}_1$  (respectively to  $\mathbf{c}_{-1}$ ) then the closed curve  $\mathcal{C}$  is oriented counterclockwisely (respectively clockwisely).

Apply then the Turning Tangent Theorem to the simple piecewise  $\mathcal{C}^1$  closed curve  $\mathcal{C}$  described above (see Figure 2.1). Using Claim 2.2.1, we discuss the two possible cases.

If  $\gamma$  is homotopic to  $\mathbf{c}_1$ , then we have

$$\tilde{\Theta}(\Gamma, S_0)(S_1 - S_0) - \tilde{\Theta}(\Gamma, S_0)(0) + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1.$$

---

1. Observe that the case  $p_1 \circ \Gamma(S_0) = p_1 \circ \Gamma(S_1)$  is not possible because  $S_0 \neq S_1$  and  $\Gamma$  is injective.

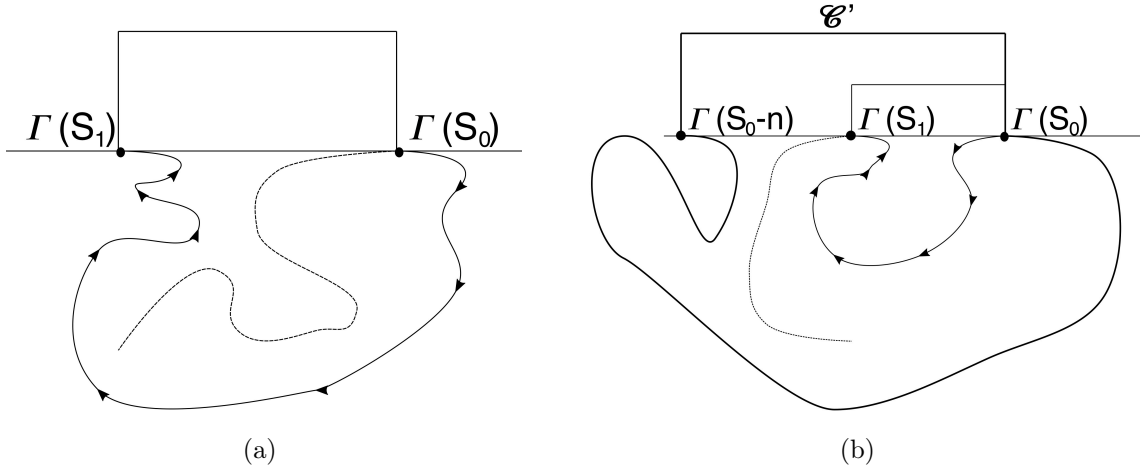


Figure 2.2 – The contradictions in Claim 2.2.1.

If  $\gamma$  is homotopic to  $\mathbf{c}_{-1}$ , then we have

$$\tilde{\Theta}(\Gamma, S_0)(S_1 - S_0) - \tilde{\Theta}(\Gamma, S_0)(0) - \frac{1}{4} - \frac{1}{4} - \frac{1}{4} - \frac{1}{4} = -1.$$

In both cases we deduce that

$$\tilde{\Theta}(\Gamma, S_0)(S_1 - S_0) - \tilde{\Theta}(\Gamma, S_0)(0) = \text{Var}_{\Gamma}(\Gamma(S_0), \Gamma(S_1)) = \text{Var}_{\gamma}(\gamma(s_0), \gamma(s_1)) = 0.$$

□

### 2.2.3 Points of zero torsion on $\mathcal{C}^1$ essential curves

Let  $\gamma : \mathbb{T} \rightarrow \mathbb{A}$  be a  $\mathcal{C}^1$  essential curve. The aim of this subsection is the proof of Theorem 2.2.1.

Let us begin by introducing some definitions. Let  $s_0 \in \mathbb{T}$  be a point of maximal height, that is such that  $\bar{p}_2 \circ \gamma(s_0) = \max_{s \in \mathbb{T}} \bar{p}_2 \circ \gamma(s)$ . Fix  $S_0 \in \mathbb{R}$  a lift of  $s_0$ .

**Definition 2.2.3** (Complexity of a  $\mathcal{C}^1$  essential curve). The complexity of the curve  $\gamma$  is

$$C(\gamma) := \sup_{t \in \mathbb{R}_+} |\text{Var}_{\gamma}(\gamma(p(S_0)), \gamma(p(S_0 + t)))| = \max_{t \in [0,1]} |\text{Var}_{\gamma}(\gamma(p(S_0)), \gamma(p(S_0 + t)))|,$$

where  $p : \mathbb{R} \rightarrow \mathbb{T}$  is the covering map of  $\mathbb{T}$ .

**Lemma 2.2.6.** *The definition of  $C(\gamma)$  is independent of the choice of  $s_0 \in \mathbb{T}$  such that*

$$\bar{p}_2 \circ \gamma(s_0) = \max_{s \in \mathbb{T}} \bar{p}_2 \circ \gamma(s).$$

*Proof.* Let  $s_0, \bar{s} \in \mathbb{T}$ ,  $s_0 \neq \bar{s}$  be such that  $\bar{p}_2 \circ \gamma(s_0) = \bar{p}_2 \circ \gamma(\bar{s}) = \max_{s \in \mathbb{T}} \bar{p}_2 \circ \gamma(s)$ . Let  $S_0 \in \mathbb{R}$  be a lift of  $s_0$  and let  $\bar{S}$  be the lift of  $\bar{s}$  contained in  $(S_0, S_0 + 1)$ . From Proposition 2.2.2 it holds that

$$\text{Var}_{\Gamma}(\Gamma(S_0), \Gamma(\bar{S})) = \text{Var}_{\gamma}(\gamma(s_0), \gamma(\bar{s})) = 0. \quad (2.18)$$

The complexity of the curve  $\gamma$  calculated with respect to  $\bar{s}$  is

$$\max_{t \in [0,1]} |\text{Var}_{\gamma}(\gamma(p(\bar{S})), \gamma(p(\bar{S} + t)))|.$$

From (2.18) and by properties (2) and (3) of Proposition 2.2.1, it holds

$$\begin{aligned}
\max_{t \in [0,1]} |Var_\gamma(\gamma(p(\bar{S})), \gamma(p(\bar{S} + t)))| &= \max_{t \in [0,1]} |Var_\gamma(\gamma(s_0), \gamma(\bar{s})) + Var_\gamma(\gamma(p(\bar{S})), \gamma(p(\bar{S} + t)))| = \\
&= \max_{t \in [0,1]} |Var_\gamma(\gamma(p(S_0)), \gamma(p(\bar{S} + t)))| = \\
&= \max \left( \max_{t \in [0, 1 - \bar{S} + S_0]} |Var_\gamma(\gamma(p(S_0)), \gamma(p(\bar{S} + t)))|, \max_{t \in [1 - \bar{S} + S_0, 1]} |Var_\gamma(\gamma(p(S_0)), \gamma(p(\bar{S} + t)))| \right) = \\
&= \max \left( \max_{\tau \in [\bar{S} - S_0, 1]} |Var_\gamma(\gamma(p(S_0)), \gamma(p(S_0 + \tau)))|, \max_{\tau \in [0, \bar{S} - S_0]} |Var_\gamma(\gamma(p(S_0)), \gamma(p(S_0 + 1 + \tau)))| \right) = \\
&= \max_{\tau \in [0,1]} |Var_\gamma(\gamma(p(S_0)), \gamma(p(S_0 + \tau)))| = C(\gamma).
\end{aligned}$$

□

The key step of the proof of Theorem 2.2.1 is the following Proposition.

**Proposition 2.2.3.** *Let  $\gamma : \mathbb{T} \rightarrow \mathbb{A}$  be a  $\mathcal{C}^1$  essential curve of complexity  $C(\gamma)$ . Let  $n \in \mathbb{N}^*$ . Then there exists  $\bar{z}(n) = \gamma(s_n) \in \gamma(\mathbb{T})$  such that*

$$|n \text{ Torsion}_n(f, \gamma(s_n), \gamma'(s_n))| \leq C(\gamma). \quad (2.19)$$

We postpone the proof of Proposition 2.2.3 and we will now show how to deduce Theorem 2.2.1 from Proposition 2.2.3.

A first outcome of Proposition 2.2.3 and of Lemma 2.2.3 is the following

**Lemma 2.2.7.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a negative-torsion map. Let  $K = \lfloor 2C(\gamma) \rfloor + 2$ . Let  $\bar{z}(n) = \gamma(s_n) \in \gamma(\mathbb{T})$  be a point given by Proposition 2.2.3 applied at  $f$ . Then for any  $m \in (0, n]$  it holds*

$$m \text{ Torsion}_m(f, \bar{z}(n), \chi) \in \left[ -\frac{K}{2}, 0 \right).$$

*Proof.* We start by recalling that, since  $f$  is a negative-torsion map, we have that for any  $m$  it holds  $m \text{ Torsion}_m(f, \bar{z}(n), \chi) < 0$ .

Argue then by contradiction and assume there exists  $m \in (0, n]$  such that

$$m \text{ Torsion}_m(f, \bar{z}(n), \chi) < -\frac{K}{2}.$$

If  $m = n$  then we contradict (2.19) because we would have, using Lemma 1.1.2 and since  $K = \lfloor 2C(\gamma) \rfloor + 2$ ,

$$n \text{ Torsion}_n(f, \gamma(s_n), \gamma'(s_n)) < n \text{ Torsion}_n(f, \bar{z}(n), \chi) + \frac{1}{2} < -\frac{K-1}{2} < -C(\gamma).$$

Suppose so that  $m < n$ . Again because  $f$  is a negative-torsion map, we have that

$$(n - m) \text{ Torsion}_{n-m}(f, f^m(\bar{z}(n)), \chi) < 0.$$

Apply so Lemma 2.2.3 for  $f$  at  $\bar{z}(n)$  with respect to  $\mathcal{N} = 2, l_1 = m, l_2 = n, \mathcal{K}_1 = K, \mathcal{K}_2 = 0$ . We obtain

$$n \text{ Torsion}_n(f, \gamma(s_n), \gamma'(s_n)) < -\frac{K-1}{2} = -\frac{\lfloor 2C(\gamma) \rfloor + 1}{2} < -C(\gamma),$$

contradicting (2.19). This concludes the proof.

□

The candidate point of zero torsion on the curve  $\gamma$  is then a limit point of the sequence  $(\bar{z}(n))_{n \in \mathbb{N}}$ , built in Proposition 2.2.3. We first estimate the finite-time torsion of such limit point.

**Lemma 2.2.8.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a negative-torsion map. Let  $K = \lfloor 2C(\gamma) \rfloor + 2$ . Let  $\bar{z}(\infty) \in \gamma(\mathbb{T})$  be a limit point of the sequence  $(\bar{z}(n))_{n \in \mathbb{N}}$  built in Proposition 2.2.3. Then for any  $N \in \mathbb{N}$  it holds*

$$N \text{ Torsion}_N(f, \bar{z}(\infty), \chi) \in \left[ -\frac{K}{2}, 0 \right).$$

The proof of Lemma 2.2.8 uses the same ideas as the proof of Lemma 2.2.5.

*Proof.* Since  $f$  is a negative-torsion map, we already know that for any  $N \in \mathbb{N}^*$  it holds  $N \text{ Torsion}_N(f, \bar{z}(\infty), \chi) < 0$ . Fix now  $N \in \mathbb{N}^*$ . Let  $\varepsilon > 0$ . Since there exists a subsequence of  $(\bar{z}(n))_{n \in \mathbb{N}}$  (that we will still denote as  $(\bar{z}(n))_{n \in \mathbb{N}}$ ) which converges to  $\bar{z}(\infty)$  and by the continuity of the function  $x \mapsto N \text{ Torsion}_N(f, x, \chi)$ , there exists  $\bar{n} \in \mathbb{N}, \bar{n} > N$  such that

$$|N \text{ Torsion}_N(f, \bar{z}(\infty), \chi) - N \text{ Torsion}_N(f, \bar{z}(\bar{n}), \chi)| < \varepsilon.$$

Consequently we have, using Lemma 2.2.7,

$$\begin{aligned} & N \text{ Torsion}_N(f, \bar{z}(\infty), \chi) = \\ &= (N \text{ Torsion}_N(f, \bar{z}(\infty), \chi) - N \text{ Torsion}_N(f, \bar{z}(\bar{n}), \chi)) + N \text{ Torsion}_N(f, \bar{z}(\bar{n}), \chi) > -\varepsilon - \frac{K}{2}. \end{aligned}$$

By the arbitrariness of  $\varepsilon$  we conclude that  $N \text{ Torsion}_N(f, \bar{z}(\infty), \chi) \in \left[ -\frac{K}{2}, 0 \right)$ . □

We can finally prove Theorem 2.2.1.

*Proof of Theorem 2.2.1.* Consider a  $\mathcal{C}^1$  essential curve and a point  $\bar{z}(\infty) \in \gamma(\mathbb{T})$  which is a limit point of the sequence  $(\bar{z}(n))_{n \in \mathbb{N}}$  defined in Proposition 2.2.3. By Lemma 2.2.8 for any  $N \in \mathbb{N}^*$  it holds

$$\text{Torsion}_N(f, \bar{z}(\infty), \chi) \in \left[ -\frac{K}{2N}, 0 \right),$$

where  $K = \lfloor 2C(\gamma) \rfloor + 2$  (in particular  $K$  is independent of  $N$ ). Consequently, as  $N$  goes to  $+\infty$ , we have that  $\text{Torsion}(f, \bar{z}(\infty)) = 0$ . □

### Proof of Proposition 2.2.3

We recall that the complexity of a curve  $\gamma$  is independent of the choice of the point of maximal height of  $\gamma$  (see Lemma 2.2.6).

Let  $(f_t)_{t \in \mathbb{R}_+}$  be an isotopy joining the identity  $\text{Id}_{\mathbb{A}}$  to  $f$ . Recall that the torsion does not depend on the choice of the isotopy (see Proposition 1.3.2). The following notations will be largely used throughout the proof of Proposition 2.2.3.

**Notation 2.2.2.** For any  $t \in \mathbb{R}_+$  denote as  $\gamma_t$  the curve

$$\mathbb{T} \ni s \mapsto \gamma_t(s) := f_t(\gamma(s)) \in \mathbb{A}.$$

Consider the function

$$M_\gamma^h : \mathbb{R}_+ \rightarrow \mathbb{R}$$

$$t \mapsto M_\gamma^h(t) := \max_{s \in \mathbb{T}} \bar{p}_2 \circ \gamma_t(s).$$

For any  $t \in \mathbb{R}_+$  denote

$$\text{Argmax}(\bar{p}_2 \circ \gamma_t) = \{s \in \mathbb{T} : \bar{p}_2 \circ \gamma_t(s) = M_\gamma^h(t)\}, \quad (2.20)$$

that is the set of  $s \in \mathbb{T}$  whose image through  $\gamma_t$  achieves the maximal height among  $\gamma_t(\mathbb{T})$ . Observe that, since each  $\gamma_t$  is  $\mathcal{C}^1$ , for any  $s \in \text{Argmax}(\bar{p}_2 \circ \gamma_t)$  the tangent vector  $\gamma_t'(s)$  belongs to  $\mathbb{R}\mathcal{H}$ .

For any  $t \in \mathbb{R}_+$  denote as  $s_t$  an element of  $\text{Argmax}(\bar{p}_2 \circ \gamma_t)$ .

**Notation 2.2.3.** The function  $\Phi : \mathbb{R}_+ \rightarrow \mathbb{Z}$  is

$$\mathbb{R}_+ \ni t \mapsto t \text{Torsion}_t(f, \gamma(s_t), \gamma'(s_t)) + \text{Var}_\gamma(\gamma(s_0), \gamma(s_t)) \in \mathbb{R}. \quad (2.21)$$

The function  $\Phi$  takes values in  $\mathbb{Z}$  because, thanks to Claim 2.2.1, if  $\gamma$  is homotopic to  $\mathbf{c}_1$  (respectively to  $\mathbf{c}_{-1}$ ) then both  $Df_t(\gamma(s_t))\gamma'(s_t)$  and  $\gamma'(s_0)$  belongs to  $\mathbb{R}_+\mathcal{H}$  (respectively  $\mathbb{R}_-\mathcal{H}$ ).

The idea of considering points of maximal (respectively minimal) height on a curve is due to P. Le Calvez (see Section 5 in [LC91]).

We need now to discuss some properties of the function  $\Phi$ : in particular, we will see that it is the constant null function.

**Lemma 2.2.9.** *For any  $t \in \mathbb{R}_+$ , the value  $\Phi(t)$  does not depend on the choice of  $s_t \in \text{Argmax}(\bar{p}_2 \circ \gamma_t)$ .*

*Proof.* Let  $s_t, \bar{s}_t \in \text{Argmax}(\bar{p}_2 \circ \gamma_t)$ ,  $s_t \neq \bar{s}_t$ . From Proposition 2.2.2 it holds that

$$\text{Var}_{\gamma_t}(s_t, \bar{s}_t) = 0. \quad (2.22)$$

Look now at  $t \text{Torsion}_t(f, \gamma(s_t), \gamma'(s_t))$  and  $t \text{Torsion}_t(f, \gamma(\bar{s}_t), \gamma'(\bar{s}_t))$ . First, remark that

$$\mathbb{R}_+ \ni \tau \mapsto \text{Var}_{\gamma_\tau}(\gamma_\tau(s_t), \gamma_\tau(\bar{s}_t)) \in \mathbb{R}$$

is continuous because  $\tau \mapsto f_\tau$  is continuous in the  $\mathcal{C}^1$  compact-open topology.

In particular, since the torsion at finite-time does not depend on the chosen lift, we calculate the torsion at  $\gamma(\bar{s}_t)$  using the continuous lift

$$\mathbb{R}_+ \ni \tau \mapsto \tilde{v}(f, \gamma(s_t), \gamma'(s_t))(\tau) + \text{Var}_{\gamma_\tau}(\gamma_\tau(s_t), \gamma_\tau(\bar{s}_t)) \in \mathbb{R}, \quad (2.23)$$

where  $\tilde{v}(f, \gamma(s_t), \gamma'(s_t))(\cdot)$  is a continuous lift of the angle function  $\tau \mapsto \theta(\mathcal{H}, Df_\tau(\gamma(s_t))\gamma'(s_t))$ .

Therefore, the function in (2.23) is a continuous determination of the angle function  $\tau \mapsto \theta(\mathcal{H}, \gamma'_\tau(\bar{s}_t))$ .

Consequently the value  $\Phi(t)$  calculated with respect to  $\bar{s}_t$  is

$$t \text{Torsion}_t(f, \gamma(\bar{s}_t), \gamma'(\bar{s}_t)) + \text{Var}_\gamma(\gamma(s_0), \gamma(\bar{s}_t)).$$

Let us write  $t \text{Torsion}_t(f, \gamma(\bar{s}_t), \gamma'(\bar{s}_t))$  using the continuous determination in (2.23) and obtain

$$\tilde{v}(f, \gamma(s_t), \gamma'(s_t))(t) + \text{Var}_{\gamma_t}(\gamma_t(s_t), \gamma_t(\bar{s}_t)) - \tilde{v}(f, \gamma(s_t), \gamma'(s_t))(0) - \text{Var}_\gamma(\gamma(s_t), \gamma(\bar{s}_t)) +$$

$$\begin{aligned}
& +Var_{\gamma}(\gamma(s_0), \gamma(\bar{s}_t)) = \\
& = t \text{Torsion}_t(f, \gamma(s_t), \gamma'(s_t)) + \\
& +Var_{\gamma_t}(\gamma_t(s_t), \gamma_t(\bar{s}_t)) - Var_{\gamma}(\gamma(s_t), \gamma(\bar{s}_t)) + Var_{\gamma}(\gamma(s_0), \gamma(\bar{s}_t)) = \\
& = t \text{Torsion}_t(f, \gamma(s_t), \gamma'(s_t)) + Var_{\gamma_t}(\gamma_t(s_t), \gamma_t(\bar{s}_t)) + Var_{\gamma}(\gamma(s_0), \gamma(s_t)),
\end{aligned}$$

where in the last equality we have used property (3) of Proposition 2.2.1. Finally, since from (2.22) we have that  $Var_{\gamma_t}(\gamma_t(s_t), \gamma_t(\bar{s}_t)) = 0$ , we conclude that

$$t \text{Torsion}_t(f, \gamma(\bar{s}_t), \gamma'(\bar{s}_t)) + Var_{\gamma}(\gamma(s_0), \gamma(\bar{s}_t)) = t \text{Torsion}_t(f, \gamma(s_t), \gamma'(s_t)) + Var_{\gamma}(\gamma(s_0), \gamma(s_t)),$$

that is  $\Phi(t)$  does not depend on the choice of  $s_t \in \text{Argmax}(\bar{p}_2 \circ \gamma_t)$ .

□

**Lemma 2.2.10.** *The function  $\Phi : \mathbb{R}_+ \rightarrow \mathbb{Z}$  is the constant zero function.*

*Proof.* We are going to show that  $\Phi$  is continuous: since  $\Phi$  takes values in  $\mathbb{Z}$  and since  $\Phi(0) = 0$ , we will conclude that  $\Phi$  is the constant zero function.

As a first step we are going to consider the function  $\Phi_{|[0,1]} : [0,1] \rightarrow \mathbb{R}$  and to show that its graph is compact. Since a function from a compact space (here  $[0,1]$ ) into an Hausdorff space (here  $\mathbb{R}$ ) is continuous if and only if its graph is compact (see Theorem 5.6.34 in [Soh03]), we will conclude that  $\Phi_{|[0,1]}$  is continuous.

Denote for any  $t \in [0,1]$

$$\mathbb{K}_t = \{s \in \mathbb{T} : s \in \text{Argmax}(\bar{p}_2 \circ \gamma_t)\} \times \{t\}$$

and

$$\mathbb{K} = \bigcup_{t \in [0,1]} \mathbb{K}_t = \bigcup_{t \in [0,1]} \{(s, t) : s \in \text{Argmax}(\bar{p}_2 \circ \gamma_t)\} \subset \mathbb{T} \times [0,1].$$

Clearly the set  $\mathbb{K}$  is bounded. Let us show that  $\mathbb{K}$  is closed. Let  $(s_n, t_n)_{n \in \mathbb{N}} \subset \mathbb{K}$  be a sequence converging to  $(s, t)$ .

The sequence  $(t_n)_{n \in \mathbb{N}} \subset [0,1]$  converges to  $t \in [0,1]$ .

**Claim 2.2.2.**  $s \in \text{Argmax}(\bar{p}_2 \circ \gamma_t)$ .

*Proof.* Assume by contradiction that  $s$  does not belong to  $\text{Argmax}(\bar{p}_2 \circ \gamma_t)$ . That is, there exists  $\bar{s} \in \mathbb{T}$  such that  $\bar{p}_2 \circ \gamma_t(\bar{s}) > \bar{p}_2 \circ \gamma_t(s)$ . Denote

$$\rho = \bar{p}_2 \circ \gamma_t(\bar{s}) - \bar{p}_2 \circ \gamma_t(s) > 0.$$

Since  $(t_n)_{n \in \mathbb{N}}$  converges to  $t$ , since  $\tau \mapsto f_\tau$  is continuous for the  $\mathcal{C}^1$  compact-open topology and since both  $\bar{p}_2$  and  $\gamma$  are continuous, there exists  $\bar{n} \in \mathbb{N}$  such that for any  $n \in \mathbb{N}, n \geq \bar{n}$

$$\max_{x \in \mathbb{T}} |\bar{p}_2 \circ \gamma_t(x) - \bar{p}_2 \circ \gamma_{t_n}(x)| = \max_{x \in \mathbb{T}} |\bar{p}_2 \circ f_t \circ \gamma(x) - \bar{p}_2 \circ f_{t_n} \circ \gamma(x)| < \frac{\rho}{4}.$$

By the continuity of  $\bar{p}_2 \circ \gamma_t$  and since the sequence  $(s_n)_{n \in \mathbb{N}}$  converges to  $s$ , there exists  $\tilde{n} \geq \bar{n}, \tilde{n} \in \mathbb{N}$  such that for any  $n \in \mathbb{N}, n \geq \tilde{n}$

$$|\bar{p}_2 \circ \gamma_t(s) - \bar{p}_2 \circ \gamma_t(s_n)| < \frac{\rho}{4}.$$

Consequently for  $n \in \mathbb{N}, n \geq \tilde{n}$  we have

$$\bar{p}_2 \circ \gamma_{t_n}(\bar{s}) > \bar{p}_2 \circ \gamma_t(\bar{s}) - \frac{\rho}{4} = \bar{p}_2 \circ \gamma_t(s) + \frac{3}{4}\rho =$$



$$\begin{aligned}
&= (\bar{p}_2 \circ \gamma_t(s) - \bar{p}_2 \circ \gamma_t(s_n)) + (\bar{p}_2 \circ \gamma_t(s_n) - \bar{p}_2 \circ \gamma_{t_n}(s_n)) + \bar{p}_2 \circ \gamma_{t_n}(s_n) + \frac{3}{4}\rho > \\
&> -\frac{\rho}{4} - \frac{\rho}{4} + \bar{p}_2 \circ \gamma_{t_n}(s_n) + \frac{3}{4}\rho = \bar{p}_2 \circ \gamma_{t_n}(s_n) + \frac{\rho}{4} > \bar{p}_2 \circ \gamma_{t_n}(s_n).
\end{aligned}$$

This contradicts the fact that  $s_n$  belongs to  $\text{Argmax}(\bar{p}_2 \circ \gamma_{t_n})$  and we conclude.  $\square$

We deduce so that  $\mathbb{K}$  is closed and bounded, i.e. it is compact. Consider now the function

$$\mathbb{K} \ni (s, t) \mapsto (t, t \text{ Torsion}_t(f, \gamma(s), \gamma'(s)) + \text{Var}_\gamma(\gamma(s_0), \gamma(s))) \in [0, 1] \times \mathbb{R}.$$

It is continuous and, since  $\mathbb{K}$  is compact, its image is compact too. Observe that its image is actually the graph of the function  $\Phi_{|[0,1]}$ . From what remarked before, since the graph of  $\Phi_{|[0,1]}$  is compact, we have that  $\Phi_{|[0,1]}$  is continuous.

Using the same argument, we deduce that the function  $\Phi$  is continuous on every compact  $[0, n]$  for  $n \in \mathbb{N}^*$ . Consequently, the function  $\Phi : \mathbb{R}_+ \rightarrow \mathbb{Z}/2$  is continuous. This implies (as remarked above) that  $\Phi$  is the constant null function, concluding the proof.  $\square$

We finally prove Proposition 2.2.3.

*Proof of Proposition 2.2.3.* Fix  $n \in \mathbb{N}^*$  and let  $s_n \in \text{Argmax}(\bar{p}_2 \circ \gamma_n)$ . By Lemma 2.2.9 the value  $\Phi(n)$  does not depend on the element of  $\text{Argmax}(p_2 \circ \gamma_n)$  and by Lemma 2.2.10 the function  $\Phi$  is the constant zero function. Therefore

$$\Phi(n) = n \text{ Torsion}_n(f, \gamma(s_n), \gamma'(s_n)) + \text{Var}_\gamma(\gamma(s_0), \gamma(s_n)) = 0.$$

That is

$$|n \text{ Torsion}_n(f, \gamma(s_n), \gamma'(s_n))| = |\text{Var}_\gamma(\gamma(s_0), \gamma(s_n))| \leq C(\gamma),$$

i.e.  $\bar{z}(n) := \gamma(s_n) \in \gamma(\mathbb{T})$  is the required point.  $\square$

## 2.3 Torsion for tilt maps

The main reference for the definition of tilt maps is [Hu98] (see also [GR13]). We remark that the notion of tilt map is linked to the notion of positive (negative) paths presented in [Her83] and in [LC88].

### 2.3.1 Tilt maps on the bounded annulus

Let  $\gamma$  be a  $\mathcal{C}^1$  embedded curve  $\gamma : [0, 1] \rightarrow \mathbb{T} \times [0, 1]$  such that

$$\bar{p}_2 \circ \gamma(0) = 0 \quad \bar{p}_2 \circ \gamma(1) = 1,$$

where  $\bar{p}_2 : \mathbb{T} \times [0, 1] \rightarrow [0, 1]$  denotes the projection over the second coordinate and such that

$$\gamma((0, 1)) \subset \mathbb{T} \times (0, 1).$$

Denote as  $\chi$  the vertical vector  $(0, 1)$ . We define the angle function  $\text{tilt}(\gamma)$  as follows

$$\begin{aligned} \text{tilt}(\gamma) : [0, 1] &\rightarrow \mathbb{T} \\ t &\mapsto \theta(\chi, \gamma'(t)), \end{aligned} \quad (2.24)$$

where  $\theta(v, u)$  denotes the oriented angle between  $v$  and  $u$  with respect to the standard Riemannian metric and the counterclockwise orientation.

Let  $\widetilde{\text{tilt}}(\gamma) : [0, 1] \rightarrow \mathbb{R}$  be the continuous determination of the angle function  $\text{tilt}(\gamma)$  such that

$$\widetilde{\text{tilt}}(\gamma)(0) \in \left[-\frac{1}{4}, \frac{1}{4}\right].$$

From the definition of  $\widetilde{\text{tilt}}$ , we deduce the following property.

**Lemma 2.3.1.** *If  $t \in (0, 1]$  is such that*

$$\bar{p}_2 \circ \gamma(t) > \bar{p}_2 \circ \gamma(s)$$

*for any  $s < t$ , then*

$$\widetilde{\text{tilt}}(\gamma)(t) \in \left[-\frac{1}{4}, \frac{1}{4}\right].$$

*Proof.* Consider the lifted framework  $\mathbb{R} \times [0, 1]$  and denote as  $\Gamma : [0, 1] \rightarrow \mathbb{R} \times [0, 1]$  a lift of the curve  $\gamma$ . Observe that also in the lifted framework we have that  $p_2 \circ \Gamma(t) > p_2 \circ \Gamma(s)$  for any  $s < t$ .

Denote  $M = \max_{s \in [0, t]} p_1 \circ \Gamma(s)$ . Consider then the curve  $\psi$  obtained by concatenating the following ones (see Figure 2.3)

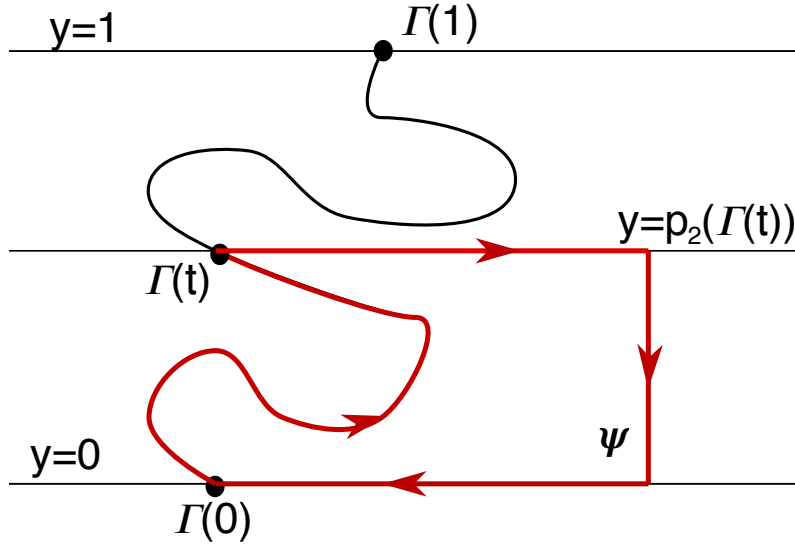


Figure 2.3 – The curve  $\psi$  built in the proof of Lemma 2.3.1.

- (i)  $\{\Gamma(s) : s \in [0, t]\}$ ;
- (ii) the horizontal segment  $\{(\tau, p_2 \circ \Gamma(t)) : \tau \in [p_1 \circ \Gamma(t), M + 1]\}$ ;
- (iii) the vertical segment  $\{(M + 1, p_2 \circ \Gamma(t) - \tau) : \tau \in [0, p_2 \circ \Gamma(t)]\}$ ;

(iv) the horizontal segment  $\{(M + 1 + p_1 \circ \Gamma(0) - \tau, 0) : \tau \in [p_1 \circ \Gamma(0), M + 1]\}$ .

Let us orient the curve  $\psi$  clockwise. Thanks to the choice of  $M$  and since  $\Gamma((0, 1)) \subset \mathbb{R} \times (0, 1)$ , the curve  $\psi$  is a piecewise  $\mathcal{C}^1$  closed curve without self-intersections. We are going to apply the Turning Tangent Theorem at  $\psi$  to calculate the angle variation of the vector tangent to  $\Gamma$  between  $\Gamma(0)$  and  $\Gamma(t)$ : such angle variation is  $\widetilde{tilt}(\gamma)(t) - \widetilde{tilt}(\gamma)(0)$ . Denote as

$$\alpha = \widetilde{tilt}(\gamma)(0) \in \left[-\frac{1}{4}, \frac{1}{4}\right]$$

the measure of the angle between  $\chi$  and  $\Gamma'(0)$  contained in  $[-\frac{1}{4}, \frac{1}{4}]$ . Denote as  $\beta$  the measure of the angle between  $\chi$  and  $\Gamma'(t)$  contained in  $[-\frac{1}{2}, \frac{1}{2}]$ . Observe in particular that  $\beta \in [-\frac{1}{4}, \frac{1}{4}]$  because the curve  $\Gamma$  crosses the horizontal line  $\mathbb{R} \times \{p_2 \circ \Gamma(t)\}$  at  $\Gamma(t)$  from the bottom up since we are assuming that  $p_2 \circ \Gamma(t) > p_2 \circ \Gamma(s)$  for any  $s < t$ .

In particular then  $\widetilde{tilt}(\gamma)(t) = \beta + k$  for some  $k \in \mathbb{Z}$ . Applying so the Turning Tangent Theorem we obtain

$$(\beta + k - \alpha) + (-\frac{1}{4} - \beta) - \frac{1}{4} - \frac{1}{4} + (\alpha - \frac{1}{4}) = -1,$$

that is  $k = 0$ . We conclude that  $\widetilde{tilt}(\gamma)(t) = \beta \in [-\frac{1}{4}, \frac{1}{4}]$  as desired.  $\square$

For any  $x \in \mathbb{T}$  let us denote as  $V_{(x,0)}$  the vertical line passing through the point  $(x, 0)$ , i.e.  $\{x\} \times [0, 1]$ . In the framework of the bounded annulus, the vertical  $V_{(x,0)}$  is parametrized as

$$[0, 1] \ni y \mapsto V_{(x,0)}(y) = (x, y) \in \mathbb{T} \times [0, 1].$$

**Definition 2.3.1** (Tilt map of the bounded annulus). A  $\mathcal{C}^1$  diffeomorphism  $f : \mathbb{T} \times [0, 1] \rightarrow \mathbb{T} \times [0, 1]$  isotopic to the identity is a positive (respectively negative) tilt map of the bounded annulus if for any  $x_0 \in \mathbb{T}$  it holds

$$\widetilde{tilt}(f \circ V_{(x_0,0)})(y) < 0 \quad (\text{respectively } > 0), \quad (2.25)$$

for any  $y \in [0, 1]$ .

**Remark 2.3.1.** Observe that in [Her83] and in [LC88] a path  $\gamma$  verifying  $\widetilde{tilt}(\gamma)(t) < 0$  (respectively  $> 0$ ) for any  $t$  is called a negative (respectively positive) path. Therefore every  $f \circ V_{(x_0,0)}$  is a negative path.

**Proposition 2.3.1.** Let  $f : \mathbb{T} \times [0, 1] \rightarrow \mathbb{T} \times [0, 1]$  be a  $\mathcal{C}^1$  diffeomorphism isotopic to the identity. Then for any  $z = (x, y) \in \mathbb{T} \times \{0\}$  it holds

$$Torsion_1(f, z, \chi) = \widetilde{tilt}(f \circ V_z)(y).$$

*Proof.* Referring to the notation introduced in Definition 1.1.2 in Chapter 1, the time-one torsion at  $z \in \mathbb{T} \times [0, 1]$  with respect to  $\chi$  is

$$\tilde{v}(f, z, \chi)(1) - \tilde{v}(f, z, \chi)(0),$$

where  $t \mapsto \tilde{v}(f, z, \chi)(t)$  is a lift of the oriented angle function  $t \mapsto \theta(\chi, Df_t(z)\chi)$ .

Consider the vertical  $V_z$  passing through  $z = (x, y)$ : we are interested in  $\widetilde{tilt}(f \circ V_z)$ . In

particular, observe that both  $\tilde{v}(f, z, \chi)(1)$  and  $\widetilde{tilt}(f \circ V_x)(y)$  are measures of the same angle  $\theta(\chi, Df(z)\chi)$ .

Consequently, the function

$$[0, 1] \ni y \mapsto \Psi(y) := \text{Torsion}_1(f, (x, y), \chi) - \widetilde{tilt}(f \circ V_{(x,0)})(y) \in \mathbb{Z}$$

is a continuous function which takes value in  $\mathbb{Z}$ . Therefore, it is constant.

Let us calculate  $\Psi(0)$ . On one hand, by definition of  $\widetilde{tilt}(f \circ V_{(x,0)})$ , it holds

$$\widetilde{tilt}(f \circ V_{(x,0)})(0) \in \left[-\frac{1}{4}, \frac{1}{4}\right].$$

On the other hand, since each  $f_t^2$  preserves the boundaries, it holds that for any  $t \in [0, 1]$  the angle

$$\theta(\chi, Df_t(x, 0)\chi) \in \left[-\frac{1}{4}, \frac{1}{4}\right]. \quad (2.26)$$

Since the time-one torsion does not depend on the chosen lift, select the lift

$$t \mapsto \tilde{v}(f, (x, 0), \chi)(t)$$

such that  $\tilde{v}(f, (x, 0), \chi)(0) = 0$ . By the continuity of the lift and because of (2.26), we deduce that

$$\tilde{v}(f, (x, 0), \chi)(1) \in \left[-\frac{1}{4}, \frac{1}{4}\right].$$

That is

$$\begin{aligned} \Psi(0) &= \text{Torsion}_1(f, (x, 0), \chi) - \widetilde{tilt}(f \circ V_{(x,0)})(0) = \\ &= \tilde{v}(f, (x, 0), \chi)(1) - \tilde{v}(f, (x, 0), \chi)(0) - \widetilde{tilt}(f \circ V_{(x,0)})(0) = \\ &= \tilde{v}(f, (x, 0), \chi)(1) - \widetilde{tilt}(f \circ V_z)(0) = 0. \end{aligned}$$

Since  $\Psi$  takes values in  $\mathbb{Z}$  and it is continuous, we conclude that  $\Psi$  is the zero constant function. That is, for any  $z \in \mathbb{T} \times [0, 1]$  it holds

$$\text{Torsion}_1(f, z, \chi) = \widetilde{tilt}(f \circ V_z)(y). \quad (2.27)$$

□

By the definition of positive tilt map, of negative-torsion map and from Proposition 2.3.1, we immediately deduce the following

**Corollary 2.3.1.** *Let  $f : \mathbb{T} \times [0, 1] \rightarrow \mathbb{T} \times [0, 1]$  be a  $\mathcal{C}^1$  diffeomorphism isotopic to the identity. Then,  $f$  is a negative-torsion map if and only if  $f$  is a positive tilt map.*

Another outcome is the following

**Corollary 2.3.2.** *Let  $f : \mathbb{T} \times [0, 1] \rightarrow \mathbb{T} \times [0, 1]$  be a positive (respectively negative) tilt map of the bounded annulus. Then for any  $n \in \mathbb{N}^*$  it holds that for any  $z \in \mathbb{T} \times [0, 1]$*

$$\text{Torsion}_n(f, z, \chi) < 0 \quad (\text{respectively } > 0).$$

*In particular for any  $z \in \mathbb{T} \times [0, 1]$ , whenever the limit exists, it holds*

$$\text{Torsion}(f, z) \leq 0 \quad (\text{respectively } \geq 0).$$

---

2. Where  $(f_t)_{t \in [0,1]}$  is an isotopy in  $\text{Diff}^1(\mathbb{T} \times [0, 1])$  joining the identity to  $f$ .

*Proof.* We proceed by induction. The statement for  $n = 1$  is an immediate consequence of Corollary 2.3.2 and of the definition of negative-torsion map.

Assume now that the result holds true for  $n$  and let us show it for  $n + 1$ . Fix  $z \in \mathbb{T} \times [0, 1]$ . By inductive hypothesis it holds that

$$n \text{Torsion}_n(f, z, \chi) < 0.$$

By Corollary 2.3.2 and by Definition 2.0.1 we know that  $\text{Torsion}_1(f, f^n(z), \chi) < 0$ . Let us apply Lemma 2.2.3 for  $f$  at  $z$  with respect to  $\mathcal{N} = 2, l_1 = n, l_2 = 1, \mathcal{K}_1 = \mathcal{K}_2 = 0$ . We conclude that

$$(n + 1) \text{Torsion}_{n+1}(f, z, \chi) < 0,$$

so in particular  $\text{Torsion}_{n+1}(f, z, \chi) < 0$ , as desired.

Whenever the limit exists, it clearly holds that  $\text{Torsion}(f, z) \leq 0$ .

□

### 2.3.2 Tilt maps on the unbounded annulus

Let  $\gamma$  be a  $\mathcal{C}^1$  embedded curve  $\gamma : \mathbb{R} \rightarrow \mathbb{A}$  such that

$$\lim_{t \rightarrow +\infty} \bar{p}_2 \circ \gamma(t) = +\infty \quad \text{and} \quad \lim_{t \rightarrow -\infty} \bar{p}_2 \circ \gamma(t) = -\infty, \quad (2.28)$$

where  $\bar{p}_2$  denotes the projection over the second coordinate on  $\mathbb{A}$ . The angle function  $\text{tilt}(\gamma)$  is defined by

$$\begin{aligned} \text{tilt}(\gamma) : \mathbb{R} &\rightarrow \mathbb{T} \\ t &\mapsto \theta(\chi, \gamma'(t)), \end{aligned} \quad (2.29)$$

where  $\theta(v, u)$  denotes the oriented angle between  $v$  and  $u$  with respect to the standard Riemannian metric and the counterclockwise orientation.

**Lemma 2.3.2.** *Let  $t \in \mathbb{R}$  be such that*

$$\bar{p}_2 \circ \gamma(t) > \bar{p}_2 \circ \gamma(s) \quad \forall s < t.$$

*Let  $\widetilde{\text{tilt}}(\gamma) : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous determination of the angle function in (2.29) such that  $\widetilde{\text{tilt}}(\gamma)(t) \in [-\frac{1}{4}, \frac{1}{4}]$ .*

*Let  $\bar{t} \in \mathbb{R}, \bar{t} \neq t$  be such that*

$$\bar{p}_2 \circ \gamma(\bar{t}) > \bar{p}_2 \circ \gamma(s) \quad \forall s < \bar{t}.$$

*Then  $\widetilde{\text{tilt}}(\gamma)(\bar{t})$  is in  $[-\frac{1}{4}, \frac{1}{4}]$ .*

*Proof.* Let us consider the lifted framework and denote as  $\Gamma : \mathbb{R} \rightarrow \mathbb{R}^2$  a lift of  $\gamma$ . For any  $s$  we have that  $\widetilde{\text{tilt}}(\gamma)(s) = \widetilde{\text{tilt}}(\Gamma)(s)$ . Thus, we will conclude by showing that  $\widetilde{\text{tilt}}(\Gamma)(\bar{t}) \in [-\frac{1}{4}, \frac{1}{4}]$ .

Assume without loss of generality that  $\bar{t} > t$  (if  $t > \bar{t}$  we argument similarly). Denote

$$H = \min_{s \in [t, \bar{t}]} p_2 \circ \Gamma(s).$$

Define

$$\tilde{t} = \max\{s \leq t : p_2 \circ \Gamma(s) = H - 1\}.$$

Such  $\tilde{t}$  is well-defined because  $\lim_{s \rightarrow -\infty} p_2 \circ \Gamma(s) = -\infty$  and by the continuity of  $p_2 \circ \Gamma$ . Moreover  $\tilde{t} < t$  and for any  $s \in (\tilde{t}, \bar{t}]$  it holds  $p_2 \circ \Gamma(s) > p_2 \circ \Gamma(\tilde{t}) = H - 1$ .

Denote

$$M = \max_{s \in [\tilde{t}, \bar{t}]} p_1 \circ \Gamma(s).$$

Consider now two closed simple curve  $\psi_1, \psi_2$  built as follows.

Let  $\psi_1$  be the piecewise  $\mathcal{C}^1$  closed simple curve obtained by concatenating the following curves (see Figure 2.4)

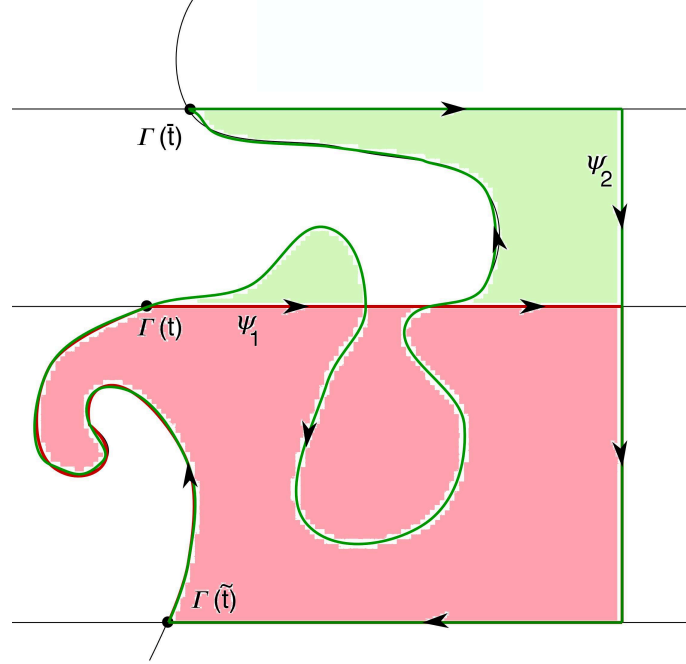


Figure 2.4 – The curve  $\psi_1$  is the boundary of the red region and the curve  $\psi_2$  is the boundary of the green region.

- (i)  $\{\Gamma(s) : s \in [\tilde{t}, t]\}$ ;
- (ii) the horizontal segment  $\{(\tau, p_2 \circ \Gamma(t)) : \tau \in [p_1 \circ \Gamma(t), M + 1]\}$ ;
- (iii) the vertical segment  $\{(M + 1, p_2 \circ \Gamma(t) + p_2 \circ \Gamma(\tilde{t}) - \tau) : \tau \in [p_2 \circ \Gamma(\tilde{t}), p_2 \circ \Gamma(t)]\}$ ;
- (iv) the horizontal segment  $\{(M + 1 + p_1 \circ \Gamma(\tilde{t}) - \tau, p_2 \circ \Gamma(\tilde{t})) : \tau \in [p_1 \circ \Gamma(\tilde{t}), M + 1]\}$ .

Let  $\psi_2$  be the piecewise  $\mathcal{C}^1$  closed simple curve obtained by concatenating the following curves (see Figure 2.4)

- (i)  $\{\Gamma(s) : s \in [\tilde{t}, \bar{t}]\}$ ;
- (ii) the horizontal segment  $\{(\tau, p_2 \circ \Gamma(\bar{t})) : \tau \in [p_1 \circ \Gamma(\bar{t}), M + 1]\}$ ;
- (iii) the vertical segment  $\{(M + 1, p_2 \circ \Gamma(\bar{t}) + p_2 \circ \Gamma(\tilde{t}) - \tau) : \tau \in [p_2 \circ \Gamma(\tilde{t}), p_2 \circ \Gamma(\bar{t})]\}$ ;
- (iv) the horizontal segment  $\{(M + 1 + p_1 \circ \Gamma(\tilde{t}) - \tau, p_2 \circ \Gamma(\tilde{t})) : \tau \in [p_1 \circ \Gamma(\tilde{t}), M + 1]\}$ .

The curves  $\psi_1, \psi_2$  do not have self-intersections thanks to the definition of  $\tilde{t}$  and of  $M$ . We orient these curves  $\psi_1, \psi_2$  clockwise. Denote now

- (a)  $\alpha \in [-\frac{1}{2}, \frac{1}{2}]$  the measure of the angle  $\theta(\chi, \Gamma'(\tilde{t}))$  contained in  $[-\frac{1}{2}, \frac{1}{2}]$ . In particular, the curve  $\Gamma$  is crossing at  $\Gamma(\tilde{t})$  the horizontal line  $\mathbb{R} \times \{H-1\}$  from the bottom up. Therefore,  $\alpha \in [-\frac{1}{4}, \frac{1}{4}]$ .
- (b)  $\beta \in [-\frac{1}{2}, \frac{1}{2}]$  the measure of the angle  $\theta(\chi, \Gamma'(t))$  contained in  $[-\frac{1}{2}, \frac{1}{2}]$ . Observe that  $\beta = \widetilde{tilt}(\Gamma)(t) \in [-\frac{1}{4}, \frac{1}{4}]$ .
- (c)  $\nu \in [-\frac{1}{2}, \frac{1}{2}]$  the measure of the angle  $\theta(\chi, \Gamma'(\bar{t}))$  contained in  $[-\frac{1}{2}, \frac{1}{2}]$ . We remark that also  $\nu$  belongs to  $[-\frac{1}{4}, \frac{1}{4}]$ . Indeed, since for any  $s < \bar{t}$  it holds  $p_2 \circ \Gamma(s) < p_2 \circ \Gamma(\bar{t})$ , the curve  $\Gamma$  is crossing at  $\Gamma(\bar{t})$  the horizontal line  $\mathbb{R} \times \{p_2 \circ \Gamma(\bar{t})\}$  from the bottom up.

Apply the Turning Tangent Theorem to the first curve  $\psi_1$ . In particular, since the angle variation along  $\Gamma_{[\tilde{t}, t]}$  does not depend on the chosen lift, we are going to consider  $\widetilde{tilt}(\Gamma)$  as lift and we obtain

$$\left(\widetilde{tilt}(\Gamma)(t) - \widetilde{tilt}(\Gamma)(\tilde{t})\right) + \left(-\frac{1}{4} - \beta\right) - \frac{1}{4} - \frac{1}{4} + \left(\alpha - \frac{1}{4}\right) = -1.$$

There exists  $j \in \mathbb{Z}$  such that  $\widetilde{tilt}(\Gamma)(\tilde{t}) = \alpha + j$ . Thus

$$(\beta - \alpha - j) - \beta + \alpha - 1 = -1,$$

i.e.  $j = 0$ .

Apply the Turning Tangent Theorem to the second curve  $\psi_2$ , choosing  $\widetilde{tilt}(\Gamma)$  as lift to calculate the angle variation along  $\Gamma_{[\bar{t}, t]}$ . There exists  $k \in \mathbb{Z}$  such that  $\widetilde{tilt}(\Gamma)(\bar{t}) = \nu + k$ . We so obtain

$$\left(\widetilde{tilt}(\Gamma)(\bar{t}) - \widetilde{tilt}(\Gamma)(\tilde{t})\right) + \left(-\frac{1}{4} - \nu\right) - \frac{1}{4} - \frac{1}{4} + \left(\alpha - \frac{1}{4}\right) = -1,$$

that is

$$(\nu + k - \alpha) - \nu + \alpha - 1 = -1.$$

We conclude that  $k = 0$ . Equivalently

$$\widetilde{tilt}(\Gamma)(\bar{t}) = \widetilde{tilt}(\gamma)(\bar{t}) = \nu \in \left[-\frac{1}{4}, \frac{1}{4}\right].$$

□

**Notation 2.3.1.** Thanks to conditions(2.28) and to Lemma 2.3.2, the continuous determination  $\widetilde{tilt}(\gamma)(\cdot)$  of the angle function in (2.29) such that for any  $t \in \mathbb{R}$  so that

$$\bar{p}_2 \circ \gamma(t) > \bar{p}_2 \circ \gamma(s) \quad \forall s < t$$

it holds  $\widetilde{tilt}(\gamma)(t)$  in  $[-\frac{1}{4}, \frac{1}{4}]$ , exists and it is unique. From now on,  $\widetilde{tilt}(\gamma)(\cdot)$  denotes such a continuous determination.

For any  $x \in \mathbb{T}$  let us denote as  $V_{(x,0)}$  the vertical line passing through the point  $(x, 0)$ , i.e.  $\{x\} \times \mathbb{R}$ . In the framework of the unbounded annulus, the vertical  $V_{(x,0)}$  is parametrized as

$$\mathbb{R} \ni y \mapsto V_{(x,0)}(y) = (x, y) \in \mathbb{A}.$$

**Definition 2.3.2** (Tilt maps of the unbounded annulus). A  $\mathcal{C}^1$  diffeomorphism  $f : \mathbb{A} \rightarrow \mathbb{A}$  isotopic to the identity is a positive (respectively negative) tilt map of the unbounded annulus if for any  $x_0 \in \mathbb{T}$  it holds

$$\widetilde{tilt}(f \circ V_{(x_0,0)})(y) < 0 \quad (\text{respectively } > 0), \quad (2.30)$$

for any  $y \in \mathbb{R}$ .

**Example 2.3.1.** An example of positive (respectively negative) tilt map (which a priori is not a twist map) is the composition of positive (respectively negative) twist maps. Indeed, any composition of positive twist maps is a negative-torsion map (actually it can be shown that any composition of negative-torsion maps is a negative-torsion map). By Corollary 2.3.3 we deduce that such a composition is a positive tilt map.

**Proposition 2.3.2.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a  $\mathcal{C}^1$  diffeomorphism isotopic to the identity. Then for any  $z \in \mathbb{A}$  it holds*

$$\text{Torsion}_1(f, z, \chi) = \widetilde{tilt}(f \circ V_z)(y).$$

*Proof.* Always referring to the notation introduced in Definition 1.1.2 in Chapter 1, the time-one torsion at  $z \in \mathbb{A}$  with respect to the vertical vector  $\chi$  is

$$\tilde{v}(f, z, \chi)(1) - \tilde{v}(f, z, \chi)(0),$$

where  $t \mapsto \tilde{v}(f, z, \chi)(t)$  is a lift of the oriented angle function  $t \mapsto \theta(\chi, Df_t(z)\chi)$ . Consider the vertical  $V_z$  passing through  $z = (x, y)$ , i.e.  $V_z = V_{(x,0)} = \{x\} \times \mathbb{R}$ . We consider  $\widetilde{tilt}(f \circ V_z)$ .

Observe that both  $\tilde{v}(f, z, \chi)(1)$  and  $\widetilde{tilt}(f \circ V_z)(y)$  are measures of the same angle  $\theta(\chi, Df(z)\chi)$ . Consider then the function

$$\mathbb{A} \ni z = (x, y) \mapsto \Psi(z) := \text{Torsion}_1(f, z, \chi) - \widetilde{tilt}(f \circ V_z)(y) \in \mathbb{R}.$$

It is a continuous function taking values in  $\mathbb{Z}$  and so it is constant. We are going to exhibit a point  $z \in \mathbb{A}$  such that  $\Psi(z) = 0$ : thus we can conclude that  $\Psi$  is the constant zero function. In particular, this will imply that, for any  $(x, y) \in \mathbb{A}$ ,

$$\text{Torsion}_1(f, (x, y), \chi) = \widetilde{tilt}(f \circ V_{(x,0)})(y). \quad (2.31)$$

Consider the  $\mathcal{C}^1$  essential curve  $\mathbb{T} \times \{0\}$  and its image  $f(\mathbb{T} \times \{0\})$ . Observe that the angle variation along  $\mathbb{T} \times \{0\}$  between any two points  $z_1, z_2 \in \mathbb{T} \times \{0\}$  is null, i.e.  $\text{Var}_{\mathbb{T} \times \{0\}}(z_1, z_2) = 0$ . Moreover, any point  $z_1 \in \mathbb{T} \times \{0\}$  is a point of maximal height of  $\mathbb{T} \times \{0\}$ .

Referring to (2.20), let  $\bar{z} = (x, 0) \in \mathbb{T} \times \{0\}$  be such that

$$(x, 0) \in \text{Argmax}(\bar{p}_2 \circ f|_{\mathbb{T} \times \{0\}}) = \{(x, 0) \in \mathbb{T} \times \{0\} : \bar{p}_2 \circ f(x, 0) = \max_{\xi \in \mathbb{T} \times \{0\}} \bar{p}_2 \circ f(\xi)\}.$$

As an outcome of Lemma 2.2.10 and of the choice of  $\bar{z}$ , it holds that

$$\text{Torsion}_1(f, \bar{z}, \mathcal{H}) + \text{Var}_{\mathbb{T} \times \{0\}}(z_1, \bar{z}) = \text{Torsion}_1(f, \bar{z}, \mathcal{H}) = 0,$$

where  $z_1 \in \mathbb{T} \times \{0\}$  is no matter which point of (maximal height of)  $\mathbb{T} \times \{0\}$ .

By Lemma 1.1.3 in Chapter 1, we deduce that

$$|\text{Torsion}_1(f, \bar{z}, \chi)| = |\text{Torsion}_1(f, \bar{z}, \chi) - \text{Torsion}_1(f, \bar{z}, \mathcal{H})| < \frac{1}{2}. \quad (2.32)$$



**Claim 2.3.1.** The point  $\bar{z} = (x, 0) \in \mathbb{T} \times \{0\}$  is such that for any  $s < 0$  it holds

$$\bar{p}_2 \circ f(x, 0) > \bar{p}_2 \circ f(x, s).$$

*Proof.* Consider the vertical  $V_{(x,0)}$  and its image  $f \circ V_{(x,0)}$ . Assume by contradiction that there exists  $s < 0$  such that

$$p_2 \circ f(x, s) \geq p_2 \circ f(x, 0).$$

On one hand, since  $f(x, 0)$  is a point of maximal height of  $f(\mathbb{T} \times \{0\})$ , we deduce that

$$\bar{p}_2 \circ f(x, s) \geq \bar{p}_2 \circ f(\xi, 0) \quad \forall \xi \in \mathbb{T}.$$

Consequently, since  $f(\mathbb{T} \times \{0\})$  separates the annulus into an upper and a lower unbounded regions, it holds that the point  $f(x, s)$  lies above or on the curve  $f(\mathbb{T} \times \{0\})$ .

On the other hand, since  $f$  preserves the boundaries, it holds

$$\lim_{y \rightarrow -\infty} \bar{p}_2 \circ f(x, y) = -\infty.$$

We so deduce that the curve  $f(\mathbb{T} \times \{0\})$  intersects the curve  $\{f(x, \xi) : \xi \leq s\}$ . This contradicts the fact that

$$(\mathbb{T} \times \{0\}) \cap \{(x, \xi) : \xi < 0\} = \emptyset$$

and that  $f$  is injective. □

Therefore, for any  $s < 0$  we have that  $\bar{p}_2 \circ f(x, s) < \bar{p}_2 \circ f(x, 0)$ . From the definition of  $\widetilde{tilt}$ , this implies that

$$\widetilde{tilt}(f \circ V_x)(0) \in \left[-\frac{1}{4}, \frac{1}{4}\right]. \quad (2.33)$$

Look now at  $\text{Torsion}_1(f, (x, 0), \chi)$ : since it does not depend on the chosen lift, assume that  $\tilde{v}(f, (x, 0), \chi)(0) = 0$  (see Definition 1.1.2 in Chapter 1). Both  $\tilde{v}(f, (x, 0), \chi)(1)$  and  $\widetilde{tilt}(f \circ V_x)(0)$  are measure of the same angle. So, by the choice of the lift, it holds

$$\tilde{v}(f, (x, 0), \chi)(1) - \widetilde{tilt}(f \circ V_x)(0) = \text{Torsion}_1(f, (x, 0), \chi) - \widetilde{tilt}(f \circ V_x)(0) \in \mathbb{Z}.$$

Since from (2.32) we have  $\text{Torsion}_1(f, (x, 0), \chi) \in (-\frac{1}{2}, \frac{1}{2})$  and from (2.33) we have  $\widetilde{tilt}(f \circ V_x)(0) \in [-\frac{1}{4}, \frac{1}{4}]$ , we conclude that

$$\text{Torsion}_1(f, (x, 0), \chi) = \widetilde{tilt}(f \circ V_x)(0).$$

Since the function  $\Psi$  is constant and  $\Psi(x, 0) = 0$ , for any  $z = (x, y) \in \mathbb{A}$  it holds  $\Psi(z) = 0$ , that is

$$\text{Torsion}_1(f, z, \chi) = \widetilde{tilt}(f \circ V_z)(y). \quad \square$$

From the definition of positive tilt map and of negative-torsion map and from Proposition 2.3.2, we deduce the following

**Corollary 2.3.3.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a  $C^1$  diffeomorphism isotopic to the identity. Then,  $f$  is a negative-torsion map if and only if  $f$  is a positive tilt map.*

As for the bounded case, we obtain the following outcome.

**Corollary 2.3.4.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a positive tilt map of the unbounded annulus. Then for any  $z \in \mathbb{A}$  and for any  $n \in \mathbb{N}^*$  it holds*

$$\text{Torsion}_n(f, z, \chi) < 0.$$

*In particular for any  $z \in \mathbb{A}$ , whenever the limit exists, it holds*

$$\text{Torsion}(f, z) \leq 0.$$

The proof of Corollary 2.3.4 is exactly that of Corollary 2.3.2 and so we omit it.

## 2.4 Birkhoff Theorem through torsion

Using the tool of torsion, we can prove Birkhoff's-theorem-like result (see [Bir22] and [Her83]) in a different hypothesis framework. The idea of using the torsion (i.e. the Maslov index) in order to prove a Birkhoff's-theorem-like result was already present in the works of M. Bialy and L. Polterovich (see [BP89], [Pol91] and [BP92]). This section arises from a question by V. Humilière.

**Theorem 2.4.1.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a negative-torsion (positive-torsion) map. Let  $\gamma : \mathbb{T} \rightarrow \mathbb{A}$  be a  $\mathcal{C}^1$   $f$ -invariant essential curve such that  $f|_\gamma$  is non wandering. Then  $\gamma$  is the graph of a  $\mathcal{C}^1$  function.*

We remark that on one hand we do not require that  $f$  is either a twist map or a conservative map. On the other hand  $f$  has to be a negative-torsion (positive-torsion) map and we require that the dynamics restricted to the  $\mathcal{C}^1$  curve is non-wandering.

Our proof is done by contradiction. We start by stating the main results that will be used in the proof. Let us first recall the definition of isotopy.

**Definition 2.4.1.** Let  $M, N$  be two smooth manifolds. An isotopy of  $M$  in  $N$  is an homotopy  $H : M \times [0, 1] \rightarrow N$  such that for any  $t \in [0, 1]$  the map  $H_t : M \rightarrow N, x \mapsto H(x, t)$  is a  $\mathcal{C}^0$  embedding.

**Proposition 2.4.1.** *An essential curve  $\gamma : \mathbb{T} \rightarrow \mathbb{A}$  is isotopic to either*

$$\mathbb{T} \ni \tau \mapsto \mathbf{c}_1(\tau) = (\tau, 0) \in \mathbb{A} \quad \text{or} \quad \mathbb{T} \ni \tau \mapsto \mathbf{c}_{-1}(\tau) = (-\tau, 0) \in \mathbb{A}.$$

Actually, we will show that any  $\mathcal{C}^r$  essential curve ( $r \geq 0$ ) is isotopic to either  $\mathbf{c}_1$  or to  $\mathbf{c}_{-1}$  through an isotopy  $H : \mathbb{T} \times [0, 1] \rightarrow \mathbb{A}$  such that for any  $s \in [0, 1]$  the map  $t \mapsto H(t, s)$  is a  $\mathcal{C}^r$  embedding of  $\mathbb{T}$  in  $\mathbb{A}$ .

We introduce some notations that will be used in the proof of Proposition 2.4.1.

**Notation 2.4.1.** Let  $\Gamma : \mathbb{T} \rightarrow \Gamma(\mathbb{T}) \subset \mathbb{R}^2$  be a simple closed curve in the plane. Thanks to Jordan's Theorem,  $\mathbb{R}^2 \setminus \Gamma(\mathbb{T})$  has two connected components, a bounded one and an unbounded one, such that  $\Gamma(\mathbb{T})$  is the common boundary. The bounded connected component is denoted  $\text{int}(\Gamma)$ . The unbounded one is denoted  $\text{ext}(\Gamma)$ . We denote as  $\mathbb{D}^2$  the closed unit disk. With an abuse of notation, in the sequel we identify  $\mathbb{R}^2$  with the complex plane.

*Proof.* The annulus is diffeomorphic to  $\mathbb{R}^2 \setminus \{0\}$  through the diffeomorphism

$$\mathbb{A} \ni (x, y) \mapsto g(x, y) = e^y e^{2\pi i x} \in \mathbb{R}^2 \setminus \{0\}.$$

Let  $\gamma : \mathbb{T} \rightarrow \mathbb{A}$  be a  $\mathcal{C}^r$  essential curve ( $r \geq 0$ ) and let  $\Gamma : \mathbb{T} \rightarrow \mathbb{R}^2$  be equal to  $g \circ \gamma$ . The curve  $\Gamma$  is so a  $\mathcal{C}^r$  simple closed curve in  $\mathbb{R}^2$ . The point  $0 \in \mathbb{R}^2$  belongs to  $\text{int}(\Gamma)$  because  $\gamma$  is an essential curve in  $\mathbb{A}$ .

We discuss separately the case  $r = 0$  and  $r \geq 1$ .

Case  $r = 0$ . By Schoenflies Theorem (see Theorem III.6.C in [Bin83]) there exists a homeomorphism  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  with compact support such that  $f(\overline{\text{int}(\Gamma)}) = \mathbb{D}^2$ , where  $\overline{\text{int}(\Gamma)}$  is the closure of  $\text{int}(\Gamma)$ .

Without loss of generality, we assume that  $f(0) = 0$ . Indeed, if this is not the case, denote as  $[0, f(0)]$  the segment joining the origin to  $f(0)$  and let  $\eta : \mathbb{R}^2 \rightarrow [0, 1]$  be a  $\mathcal{C}^\infty$  bump function such that  $\eta|_{\mathbb{R}^2 \setminus \mathbb{D}^2} = 0$  and  $\eta|_{[0, f(0)]} = 1$ .

Let  $\phi_1$  be the time-one flow of the vector field  $X = -\eta f(0)$ . Observe that  $\phi_1(f(0)) = 0$  and  $\phi_1|_{\mathbb{R}^2 \setminus \mathbb{D}^2} = \text{Id}$ . Then, replacing  $f$  with  $\phi_1 \circ f$ , we obtain the required homeomorphism.

The homeomorphism  $G = g^{-1} \circ f|_{\mathbb{R}^2 \setminus \{0\}} \circ g : \mathbb{A} \rightarrow \mathbb{A}$  is such that  $G(\gamma(\mathbb{T})) = \mathbb{T} \times \{0\}$ . Moreover, since  $f$  has compact support and  $\gamma(\mathbb{T})$  is compact, there exists  $M > 0$  such that  $G(\gamma(t) + (0, M)) = \gamma(t) + (0, M)$  for any  $t \in \mathbb{T}$ .

Consider the isotopy  $H : \mathbb{T} \times [0, 1] \rightarrow \mathbb{A}$

$$(t, s) \mapsto G(\gamma(t) + s(0, M)) - s(0, M).$$

It joins the curve  $t \mapsto H(t, 0) = G \circ \gamma(t)$  to the curve  $\gamma$ . The image of  $G \circ \gamma(\mathbb{T})$  is  $\mathbb{T} \times \{0\}$  and so the map  $t \mapsto p_1 \circ G \circ \gamma(t)$  is a homeomorphism of  $\mathbb{T}$ .

If  $t \mapsto p_1 \circ G \circ \gamma(t)$  preserves the orientation, then it is isotopic to  $\text{Id}_{\mathbb{T}}$  and therefore  $\gamma$  is isotopic to  $\mathbf{c}_1$ . Otherwise,  $t \mapsto p_1 \circ G \circ \gamma(t)$  is isotopic to  $-\text{Id}_{\mathbb{T}}$  and so  $\gamma$  is isotopic to  $\mathbf{c}_{-1}$ .

Case  $r \geq 1$ . By Theorem 8.3.7 in [Hir76] the closed unit disk  $\mathbb{D}^2$  is  $\mathcal{C}^r$  diffeomorphic to  $\overline{\text{int}(\Gamma)}$ .

That is, there exists a  $\mathcal{C}^r$  diffeomorphism  $f : \mathbb{D}^2 \rightarrow \overline{\text{int}(\Gamma)}$ . We can assume that  $f$  preserves the orientation, up to replace  $f$  with  $f \circ R$  where  $R(re^{2\pi i t}) = re^{-2\pi i t}$ . As done in the  $r = 0$  case, we can suppose that  $f(0) = 0$ .

Denote as  $\mathbb{S}^1$  the unit circle in  $\mathbb{R}^2$ . Then  $\varphi = f \circ \Gamma : \mathbb{T} \rightarrow \mathbb{R}^2 \setminus \{0\}$  is a parametrization of the unit circle. We construct now an isotopy  $H : \mathbb{T} \times [0, 1] \rightarrow \mathbb{R}^2 \setminus \{0\}$  which joins  $\varphi$  to  $\Gamma$ . Thus,  $g^{-1} \circ H$  is an isotopy on the annulus joining  $g^{-1} \circ \varphi$  to  $\gamma$ .

Since  $g^{-1} \circ \varphi(\mathbb{T}) = \mathbb{T} \times \{0\}$ , the map  $t \mapsto p_1 \circ g^{-1} \circ \varphi(t)$  is a  $\mathcal{C}^r$  diffeomorphism of  $\mathbb{T}$ . If it preserves the orientation, then  $\gamma$  is isotopic to  $\mathbf{c}_1$ . Otherwise,  $\gamma$  is isotopic to  $\mathbf{c}_{-1}$ .

The isotopy  $H$  is obtained by concatenating four isotopies. Since  $f$  is at least  $\mathcal{C}^1$ , the differential  $Df(0)$  is in  $\text{GL}(2, \mathbb{R})$  and so  $\mathbb{T} \ni t \mapsto Df(0)\varphi(t) \in \mathbb{R}^2$  is a  $\mathcal{C}^r$  embedding whose image is contained in  $\mathbb{R}^2 \setminus \{0\}$ .

Since the set of  $\mathcal{C}^1$  embeddings of  $\mathbb{T}$  in  $\mathbb{R}^2 \setminus \{0\}$  is open in the strong topology (see Theorem 1.1.4 in [Hir76]), there exists  $\varepsilon > 0$  such that every  $\mathcal{C}^r$  function of  $\mathbb{T}$  in  $\mathbb{R}^2 \setminus \{0\}$  which is  $\varepsilon$ -close to  $t \mapsto Df(0)\varphi(t)$  in the  $\mathcal{C}^1$ -distance is a  $\mathcal{C}^r$  embedding of  $\mathbb{T}$  in  $\mathbb{R}^2 \setminus \{0\}$ .

Let  $\bar{r} \in (0, 1]$  be such that  $t \mapsto \frac{f(\bar{r}\varphi(t))}{\bar{r}}$  is  $\varepsilon$ -close to  $t \mapsto Df(0)\varphi(t)$  in the  $\mathcal{C}^1$ -distance.

The final isotopy is obtained by concatenating the following ones.

- First we move the  $\mathcal{C}^r$  embedding  $\varphi$  up to the rescaled  $\mathcal{C}^r$  embedding  $\bar{r}\varphi$  using the isotopy  $(t, s) \mapsto ((1 - (1 - s)\bar{r}))\varphi(t)$ .
- Since  $f$  preserves the orientation, then  $Df(0)$  is in the same component of  $\text{GL}(2, \mathbb{R})$  as  $\mathbb{I}_2$ . A path in  $\text{GL}(2, \mathbb{R})$  provides an isotopy from  $\bar{r}\varphi$  to  $\bar{r}Df(0)\varphi$ .
- Consider the isotopy  $\hat{H} : \mathbb{T} \times [0, 1] \rightarrow \mathbb{R}^2 \setminus \{0\}$

$$(t, s) \mapsto sf(\bar{r}\varphi(t)) + (1 - s)\bar{r}Df(0)\varphi(t).$$

By the choice of  $\varepsilon$  and of  $\bar{r}$ , each  $\hat{H}(\cdot, s)$  is a  $\mathcal{C}^r$  embedding and the isotopy joins  $\bar{r}Df(0)\varphi$  to  $f(\bar{r}\varphi)$ .

- Finally, consider the isotopy  $\tilde{H} : \mathbb{T} \times [0, 1] \rightarrow \mathbb{R}^2 \setminus \{0\}$

$$(t, s) \mapsto f(((1 - s)\bar{r} + s)\varphi(t)).$$

It joins  $f(\bar{r}\varphi)$  to  $f \circ \varphi = \Gamma$  and for any  $s$  the map  $H(\cdot, s)$  is a  $\mathcal{C}^r$  embedding of  $\mathbb{T}$  in  $\mathbb{R}^2 \setminus \{0\}$ , because  $f$  fixes the origin.

□

The following lemmas give us a sufficient condition to deduce that a  $\mathcal{C}^1$  essential curve is actually the graph of a  $\mathcal{C}^1$  function.

**Lemma 2.4.1.** *Let  $\gamma : \mathbb{T} \rightarrow \mathbb{A}$  be an essential curve. Then every lift  $\Gamma : \mathbb{R} \rightarrow \mathbb{R}^2$  of  $\gamma$  is such that  $\Gamma(\tau + 1) = \Gamma(\tau) + (\delta, 0)$ , where  $\delta = 1$  if  $\gamma$  is homotopic to  $t \mapsto \mathbf{c}_1(t) = (t, 0)$  and  $\delta = -1$  if  $\gamma$  is homotopic to  $t \mapsto \mathbf{c}_{-1}(t) = (-t, 0)$ .*

*Proof.* Let  $\Gamma$  be a lift of  $\gamma$ . For all  $\tau \in \mathbb{R}$  we have that  $\Gamma(\tau + 1) = \Gamma(\tau) + (k, 0)$ , for some  $k \in \mathbb{Z}$ . Such an integer does not depend on  $\tau \in \mathbb{R}$ . From Proposition 2.4.1 and from the homotopy lifting property (see [Hat02]), there exists a lift of the homotopy joining  $\gamma$  to  $\mathbf{c}_1$  (or  $\mathbf{c}_{-1}$ ) which joins  $\Gamma$  to a lift of  $\mathbf{c}_1$  (or  $\mathbf{c}_{-1}$ ). In particular, denoting as  $\mathbf{C}_1$  the involved lift of  $\mathbf{c}_1$  (or  $\mathbf{C}_{-1}$  the involved lift of  $\mathbf{c}_{-1}$ ), we have that

$$\mathbf{C}_1(\tau + 1) - \mathbf{C}_1(\tau) = (k, 0) = \Gamma(\tau + 1) - \Gamma(\tau)$$

$$(\text{ or } \mathbf{C}_{-1}(\tau + 1) - \mathbf{C}_{-1}(\tau) = (k, 0) = \Gamma(\tau + 1) - \Gamma(\tau)).$$

Since  $\mathbf{C}_1$  (respectively  $\mathbf{C}_{-1}$ ) is a lift of  $\mathbf{c}_1$  (respectively of  $\mathbf{c}_{-1}$ ), we have that  $k = 1$  (respectively  $k = -1$ ).

□

**Lemma 2.4.2.** *Let  $\gamma$  be a  $\mathcal{C}^1$  essential curve. If  $\gamma$  is transversal to the vertical at every point, then  $\gamma$  is the graph of a function.*

*Proof.* Let  $\Gamma$  be a lift of  $\gamma$ . We show now that  $\Gamma$  is the graph of a  $\mathcal{C}^1$  function  $\phi : \mathbb{R} \rightarrow \mathbb{R}$ . Consider the  $\mathcal{C}^1$  function  $p_1 \circ \Gamma : \mathbb{R} \rightarrow \mathbb{R}$ . Since  $\Gamma$  is transversal to the vertical (because  $\Gamma$  is a lift of  $\gamma$  and  $\gamma$  is transversal to the vertical by hypothesis), it holds that  $D(p_1 \circ \Gamma)(\tau) \neq 0$  for any  $\tau \in \mathbb{R}$ . Assume, without loss of generality (the other case can be discussed similarly), that  $D(p_1 \circ \Gamma)(\tau) > 0$  for any  $\tau$ . Consequently,  $p_1 \circ \Gamma$  is an increasing diffeomorphism to its image.

By Lemma 2.4.1 and since  $p_1 \circ \Gamma$  is increasing, we have that for any  $\tau \in \mathbb{R}$

$$p_1 \circ \Gamma(\tau + 1) = p_1 \circ \Gamma(\tau) + 1. \tag{2.34}$$

From (2.34) and from the continuity of  $p_1 \circ \Gamma$ , we deduce that  $p_1 \circ \Gamma(\mathbb{R}) = \mathbb{R}$ . Therefore  $p_1 \circ \Gamma : \mathbb{R} \rightarrow \mathbb{R}$  is a  $\mathcal{C}^1$  diffeomorphism. Denote  $\phi = (p_1 \circ \Gamma)^{-1}$ . Thus, the  $\mathcal{C}^1$  function

$$\mathbb{R} \ni s \mapsto p_2 \circ \Gamma \circ \phi(s) \in \mathbb{R}$$

is such that  $\text{Graph}(p_2 \circ \Gamma \circ \phi) = \Gamma$ .

Let us show now that  $p_2 \circ \Gamma \circ \phi$  is 1-periodic. From (2.34) it holds for any  $\tau \in \mathbb{R}$

$$\phi(p_1 \circ \Gamma(\tau)) + 1 = \tau + 1 = \phi(p_1 \circ \Gamma(\tau + 1)) = \phi(p_1 \circ \Gamma(\tau) + 1),$$

that is  $\phi$  commutes with the translation by 1.

Thus for any  $s \in \mathbb{R}$  we have that

$$p_2 \circ \Gamma \circ \phi(s + 1) = p_2 \circ \Gamma(\phi(s) + 1) = p_2(\Gamma(\phi(s)) + (1, 0)) = p_2 \circ \Gamma \circ \phi(s)$$

and we deduce that  $p_2 \circ \Gamma \circ \phi$  is 1-periodic. Consequently, its projection on the annulus is well-defined and  $\gamma$  is the graph of the  $\mathcal{C}^1$  function  $\varphi : \mathbb{T} \rightarrow \mathbb{R}$  satisfying  $\varphi \circ p = p_2 \circ \Gamma \circ \phi$ , where  $p : \mathbb{R} \rightarrow \mathbb{T}$  is the covering map of  $\mathbb{T}$ . □

The following lemma provides an upper bound of the  $N$  finite-time torsion along the curve  $\gamma$ . The bound is independent of  $N$ .

**Notation 2.4.2.** Let  $x \in \mathbb{A}$  and let  $\delta \in (0, \frac{1}{4})$ . Denote

$$C(x, \chi, \delta) = \left\{ v \in T_x \mathbb{A} : \theta(\chi, v) \text{ or } \theta(-\chi, v) \text{ admits a measure in } (-\delta, \delta) \right\}.$$

**Lemma 2.4.3.** Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a negative-torsion map and let  $K$  be a compact  $f$ -invariant set. There exist  $\varepsilon \in (0, \frac{1}{2})$  and  $0 < \delta < \frac{\varepsilon}{8}$  such that for any  $x \in K$ , for any  $v \in C(x, \chi, \delta)$  (see Notation 2.4.2) and for any  $N \in \mathbb{N}^*$  it holds

$$N \text{ Torsion}_N(f, x, v) < -\frac{\varepsilon}{4} < 0.$$

We postpone the proof of Lemma 2.4.3 to Subsection 2.4.1.

The third result that we will use to prove Theorem 2.4.1 guarantees us that, in the framework of negative-torsion (positive-torsion) maps, we can use the angle variation along the curve  $\gamma$  to calculate the finite-time torsion.

**Lemma 2.4.4.** Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a negative-torsion (positive-torsion) map. Let  $\gamma : \mathbb{T} \rightarrow \mathbb{A}$  be a  $\mathcal{C}^1$   $f$ -invariant essential curve. Then for any  $s \in \mathbb{T}$  and for any  $N \in \mathbb{N}$  it holds

$$N \text{ Torsion}_N(f, \gamma(s), \gamma'(s)) = \text{Var}_\gamma(\gamma(s), \gamma(\bar{s}_N)),$$

where  $f^N \circ \gamma(s) = \gamma(\bar{s}_N)$ .

We postpone the proof of Lemma 2.4.4 to Subsection 2.4.2. We finally prove Theorem 2.4.1.

*Proof of Theorem 2.4.1.* Argue by contradiction and assume that  $\gamma$  is not a graph. Then from Lemma 2.4.2 there exists a point  $z = \gamma(t)$  such that  $\gamma'(t) \in \mathbb{R}\chi$ . Denote

$$\chi' = \begin{cases} \chi & \text{if } \gamma'(t) \in \mathbb{R}_+\chi \\ -\chi & \text{if } \gamma'(t) \in \mathbb{R}_-\chi. \end{cases}$$

In particular  $\theta(\chi', \gamma'(t))$  is zero.

Let  $\varepsilon \in (0, \frac{1}{2})$  and  $\delta \in (0, \frac{\varepsilon}{8})$  be the parameters given by Lemma 2.4.3 applied at the  $f$ -invariant compact set  $\gamma(\mathbb{T})$ .

The curve  $\gamma$  is  $\mathcal{C}^1$  and it is an embedding. There exists a neighborhood  $U \subset \mathbb{T}$  of  $t$  such that for any  $s \in U$  the oriented angle  $\theta(\chi', \gamma'(s))$  admits a measure in  $(-\delta, \delta)$ .

Since the dynamics  $f|_\gamma$  is non wandering, there exists  $N \in \mathbb{N}$  and  $\bar{s} \in U$  such that  $\bar{s}_N \in U$ , where  $\bar{s}_N$  is such that  $f^N \circ \gamma(\bar{s}) = \gamma(\bar{s}_N)$ .

Let us calculate  $N \text{Torsion}_N(f, \gamma(\bar{s}), \gamma'(\bar{s}))$ . From Lemma 2.4.4 we have

$$N \text{Torsion}_N(f, \gamma(\bar{s}), \gamma'(\bar{s})) = \text{Var}_\gamma(\gamma(\bar{s}), \gamma(\bar{s}_N)).$$

**Claim 2.4.1.**  $\text{Var}_\gamma(\gamma(\bar{s}), \gamma(\bar{s}_N)) \in (-2\delta, 2\delta)$ .

*Proof.* Since both  $\bar{s}$  and  $\bar{s}_N$  belong to  $U$ , we have that both the oriented angles  $\theta(\chi', \gamma'(\bar{s}))$  and  $\theta(\chi', \gamma'(\bar{s}_N))$  admit a measure in  $(-\delta, \delta)$ . Thus, the oriented angle  $\theta(\gamma'(\bar{s}), \gamma'(\bar{s}_N))$  admits a measure in  $(-2\delta, 2\delta)$ .

Since for any  $s \in U$  the oriented angle  $\theta(\chi', \gamma'(s))$  admits a measure in  $(-\delta, \delta)$ , since either  $[\bar{s}, \bar{s}_N]$  or  $[\bar{s}_N, \bar{s}]$  is contained in  $U$  and since  $\delta < \frac{1}{2}$ , we have that either  $\text{Var}_\gamma(\gamma(\bar{s}), \gamma(\bar{s}_N))$  or  $\text{Var}_\gamma(\gamma(\bar{s}_N), \gamma(\bar{s}))$  is in the interval  $(-2\delta, 2\delta)$ .

By properties (2) and (3) of Proposition 2.2.1 it holds

$$\begin{aligned} \text{Var}_\gamma(\gamma(\bar{s}_N), \gamma(\bar{s})) &= -\text{Var}_\gamma(\gamma(\bar{s}), \gamma(\bar{s}_N)) + \text{Var}_\gamma(\gamma(\bar{s}_N), \gamma(\bar{s}_N)) = \\ &= -\text{Var}_\gamma(\gamma(\bar{s}), \gamma(\bar{s}_N)). \end{aligned}$$

In both cases, we have that  $\text{Var}_\gamma(\gamma(\bar{s}), \gamma(\bar{s}_N)) \in (-2\delta, 2\delta)$ . □

Consequently, since  $\delta \in (0, \frac{\varepsilon}{8})$ ,

$$N \text{Torsion}_N(f, \gamma(\bar{s}), \gamma'(\bar{s})) = \text{Var}_\gamma(\gamma(\bar{s}), \gamma(\bar{s}_N)) \in \left(-\frac{\varepsilon}{4}, \frac{\varepsilon}{4}\right).$$

From Lemma 2.4.3, we have that  $N \text{Torsion}_N(f, \gamma'(\bar{s}), \gamma'(\bar{s})) < -\frac{\varepsilon}{4}$ . This is the required contradiction and we conclude. □

We highlight the fact that, in order to obtain the result of Theorem 2.4.1, we need to have information over the dynamics on the curve. Indeed, there exist non conservative positive twist maps that admit  $\mathcal{C}^1$  essential  $f$ -invariant curves which are not graphs of function. See for example Proposition 15.3 in [LC88].

**Remark 2.4.1.** We have actually shown that the curve  $\gamma$  is the graph of a function and it is always transverse to the vertical. Thus, since  $\gamma$  is  $\mathcal{C}^1$ , we deduce that  $\gamma$  is the graph of a Lipschitz function.

## 2.4.1 An upper bound of $N$ -finite time torsion: proof of Lemma 2.4.3

In order to show Lemma 2.4.3 we first need to prove the following

**Lemma 2.4.5.** *Let  $K$  be a compact  $f$ -invariant set. There exist  $\varepsilon \in (0, \frac{1}{2})$  and  $0 < \delta < \frac{\varepsilon}{8}$  such that for any  $x \in K$  and for any  $v \in C(x, \chi, \delta)$  (see Notation 2.4.2) it holds*

$$\text{Torsion}_1(f, x, v) < -\frac{\varepsilon}{2} < 0.$$

*Proof.* Since  $f$  is a negative-torsion map, since  $K$  is compact and since the function

$$T_K^1 \mathbb{A} \ni (x, v) \mapsto \Phi(x, v) := \text{Torsion}_1(f, x, v) \in \mathbb{R}$$

is continuous, there exists  $\varepsilon \in (0, \frac{1}{2})$  such that for any  $x \in K$

$$\Phi(x, \chi) = \text{Torsion}_1(f, x, \chi) \leq -\varepsilon < 0.$$

Remark that  $\Phi(x, \chi) = \text{Torsion}_1(f, x, \chi) = \text{Torsion}_1(f, x, -\chi) = \Phi(x, -\chi)$ .

Observe that  $\Phi^{-1}((-\infty, -\frac{\varepsilon}{2}))$  is a neighborhood of  $\{(x, \chi) : x \in K\} \cup \{(x, -\chi) : x \in K\}$ . Thus there exists  $\delta \in (0, \frac{\varepsilon}{8})$  such that

$$\{(x, v) : x \in K, v \in C(x, \chi, \delta)\}$$

is contained in  $\Phi^{-1}((-\infty, -\frac{\varepsilon}{2}])$ . □

*Proof of Lemma 2.4.3.* Let us proceed by induction. Let  $\delta \in (0, \frac{\varepsilon}{8})$  be given by Lemma 2.4.5. The case for  $N = 1$  is given by Lemma 2.4.5. Assume now that the result holds for  $N - 1$ , i.e. for any  $y \in K$  and for any  $w \in C(y, \chi, \delta)$  (see Notation 2.4.2) it holds  $(N - 1) \text{Torsion}_{N-1}(f, y, w) < -\frac{\varepsilon}{4}$ . Equivalently

$$\tilde{v}(f, y, w)(N - 1) - \tilde{v}(f, y, w)(0) < -\frac{\varepsilon}{4}.$$

Let  $x \in K$  and let  $v \in C(x, \chi, \delta)$  (see Notation 2.4.2). Since  $N \text{Torsion}_N(f, x, v) = N \text{Torsion}_N(f, x, -v)$ , assume without loss of generality that the angle  $\theta(\chi, v)$  admits a measure in  $(-\delta, \delta)$ .

Choose a lift so that  $\tilde{v}(f, x, v)(0) \in (-\delta, \delta)$ . Then,  $\tilde{v}(f, x, v)(N - 1) < -\frac{\varepsilon}{4} + \delta < -\frac{\varepsilon}{8}$ . Consider now the lift such that  $\tilde{v}(f, f^{N-1}(x), \chi)(0) = 0^3$ . In particular

$$\tilde{v}(f, x, v)(N - 1) < \tilde{v}(f, f^{N-1}(x), \chi)(0).$$

By Lemma 1.1.1 in Chapter 1, from the choice of the lift and from Lemma 2.4.5 (since  $\theta(\chi, \chi)$  clearly admits a measure in  $(-\delta, \delta)$ ) we have that

$$\tilde{v}(f, x, v)(N) < \tilde{v}(f, f^{N-1}(x), \chi)(1) = \tilde{v}(f, f^{N-1}(x), \chi)(1) - \tilde{v}(f, f^{N-1}(x), \chi)(0) < -\frac{\varepsilon}{2}.$$

Thanks to the choice of the lift  $\tilde{v}(f, x, v)(\cdot)$  we deduce that

$$\tilde{v}(f, x, v)(N) - \tilde{v}(f, x, v)(0) < -\frac{\varepsilon}{2} + \delta < -\frac{\varepsilon}{2} + \frac{\varepsilon}{8} < -\frac{\varepsilon}{4}$$

and we conclude. □

---

3. Careful! We are considering a lift with respect to a different point in  $T_K \mathbb{A}$ .

## 2.4.2 Finite-time torsion as angle variation along $\gamma$ : proof of Lemma 2.4.4

A first step to prove Lemma 2.4.4 is the following result.

**Lemma 2.4.6.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a negative-torsion map. Let  $\gamma : \mathbb{T} \rightarrow \mathbb{A}$  be a  $\mathcal{C}^1$   $f$ -invariant essential curve. Then for any  $s \in \mathbb{T}$  it holds*

$$\text{Torsion}_1(f, \gamma(s), \gamma'(s)) = \text{Var}_\gamma(\gamma(s), \gamma(\bar{s})),$$

where  $f \circ \gamma(s) = \gamma(\bar{s})$ .

*Proof.* Observe that both  $\text{Torsion}_1(f, \gamma(s), \gamma'(s))$  and  $\text{Var}_\gamma(\gamma(s), \gamma(\bar{s}))$  are measures of the same oriented angle  $\theta(\gamma'(s), Df(\gamma(s))\gamma'(s)) = \theta(\gamma'(s), \gamma'(\bar{s}))$ . Therefore, by continuity of  $s \mapsto \theta(\gamma'(s), Df(\gamma(s))\gamma'(s))$  and of  $s \mapsto \theta(\gamma'(s), \gamma'(\bar{s}))$ , there exists  $k \in \mathbb{Z}$  such that for any  $s \in \mathbb{T}$

$$\text{Torsion}_1(f, \gamma(s), \gamma'(s)) = \text{Var}_\gamma(\gamma(s), \gamma(\bar{s})) + k. \quad (2.35)$$

Let  $z(\infty) = \gamma(s(\infty)) \in \gamma(\mathbb{T})$  be a point given by Lemma 2.2.8. In particular there exists  $K \in \mathbb{N}$  such that for any  $N \in \mathbb{N}^*$  it holds

$$N \text{Torsion}_N(f, z(\infty), \chi) \in \left[ -\frac{K}{2}, 0 \right]. \quad (2.36)$$

From Lemma 1.1.3 in Chapter 1 we have that for any  $N \in \mathbb{N}^*$  it holds

$$N \text{Torsion}_N(f, \gamma(s(\infty)), \gamma'(s(\infty))) \in \left( -\frac{K+1}{2}, \frac{1}{2} \right), \quad (2.37)$$

At the same time, from (2.35) and since  $\gamma$  is  $f$ -invariant, we have that

$$\begin{aligned} N \text{Torsion}_N(f, \gamma(s(\infty)), \gamma'(s(\infty))) &= \sum_{i=0}^{N-1} \text{Torsion}_1(f, f^i \circ \gamma(s(\infty)), Df^i(\gamma(s(\infty)))\gamma'(s(\infty))) = \\ &= Nk + \sum_{i=0}^{N-1} \text{Var}_\gamma(\gamma(\bar{s}_i), \gamma(\bar{s}_{i+1})), \end{aligned}$$

where for any  $i \in \llbracket 0, N \rrbracket$  the point  $\bar{s}_i \in \mathbb{T}$  is such that  $\gamma(\bar{s}_i) = f^i(\gamma(s(\infty)))$ .

**Claim 2.4.2.** For any  $N \in \mathbb{N}^*$  it holds

$$\sum_{i=0}^{N-1} \text{Var}_\gamma(\gamma(\bar{s}_i), \gamma(\bar{s}_{i+1})) = \text{Var}_\gamma(\gamma(\bar{s}_0), \gamma(\bar{s}_N)).$$

*Proof.* Let us show the claim by induction. For  $N = 1$  there is nothing to prove. Assume that the claim holds for  $N - 1, N > 1$ . Consequently

$$\begin{aligned} \sum_{i=0}^{N-1} \text{Var}_\gamma(\gamma(\bar{s}_i), \gamma(\bar{s}_{i+1})) &= \sum_{i=0}^{N-2} \text{Var}_\gamma(\gamma(\bar{s}_i), \gamma(\bar{s}_{i+1})) + \text{Var}_\gamma(\gamma(\bar{s}_{N-1}), \gamma(\bar{s}_N)) = \\ &= \text{Var}_\gamma(\gamma(\bar{s}_0), \gamma(\bar{s}_{N-1})) + \text{Var}_\gamma(\gamma(\bar{s}_{N-1}), \gamma(\bar{s}_N)). \end{aligned}$$

By property (3) of Proposition 2.2.1, this last quantity is equal to  $\text{Var}_\gamma(\gamma(\bar{s}_0), \gamma(\bar{s}_N))$  and we conclude the proof of the claim.  $\square$



Consequently, for any  $N \in \mathbb{N}^*$

$$Nk = N \text{Torsion}_N(f, \gamma(s(\infty)), \gamma'(s(\infty))) - \text{Var}_\gamma(\gamma(s(\infty)), \gamma(\bar{s}_N)),$$

where  $\bar{s}_N \in \mathbb{T}$  is such that  $\gamma(\bar{s}_N) = f^N \circ \gamma(s(\infty))$ . Referring to Definition 2.2.3, we have that

$$|\text{Var}_\gamma(\gamma(s(\infty)), \gamma(\bar{s}_N))| \leq C(\gamma) < +\infty.$$

This observation, together with (2.37), implies that the application

$$\mathbb{N}^* \ni N \mapsto N \text{Torsion}_N(f, \gamma(s(\infty)), \gamma'(s(\infty))) - \text{Var}_\gamma(\gamma(s(\infty)), \gamma(\bar{s}_N)) \in \mathbb{Z}$$

is bounded. Thus, the only possible case is that  $k = 0$ . □

The proof of Lemma 2.4.4 is now an immediate corollary of Lemma 2.4.6.

*Proof of Lemma 2.4.4.* Let  $s \in \mathbb{T}$  and let  $N \in \mathbb{N}$ . Then, from Lemma 2.4.6,

$$\begin{aligned} N \text{Torsion}_N(f, \gamma(s), \gamma'(s)) &= \\ &= \sum_{i=0}^{N-1} \text{Torsion}_1(f, f^i \circ \gamma(s), Df^i(\gamma(s))\gamma'(s)) = \sum_{i=0}^{N-1} \text{Var}_\gamma(\gamma(\bar{s}_i), \gamma(\bar{s}_{i+1})), \end{aligned}$$

where for any  $i \in \llbracket 0, N-1 \rrbracket$  we denote  $\gamma(\bar{s}_i) = f^i(\gamma(s))$ .

From Claim 2.4.2 it holds

$$\sum_{i=0}^{N-1} \text{Var}_\gamma(\gamma(\bar{s}_i), \gamma(\bar{s}_{i+1})) = \text{Var}_\gamma(\gamma(s), \gamma(\bar{s}_N))$$

and so we conclude that  $N \text{Torsion}_N(f, \gamma(s), \gamma'(s)) = \text{Var}_\gamma(\gamma(s), \gamma(\bar{s}_N))$ , where  $\bar{s}_N \in \mathbb{T}$  is such that  $\gamma(\bar{s}_N) = f^N \circ \gamma(s)$ . □

An outcome of Lemma 2.4.4 (already proved by S. Crovisier for twist maps in [Cro03]) is the following

**Corollary 2.4.1.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a negative-torsion (positive-torsion) map. Let  $\gamma : \mathbb{T} \rightarrow \mathbb{A}$  be a  $\mathcal{C}^1$  essential  $f$ -invariant curve on  $\mathbb{A}$ . Then, for any  $s \in \mathbb{T}$  it holds*

$$\text{Torsion}(f, \gamma(s)) = 0.$$

*Proof.* Let  $s \in \mathbb{T}$ . Fix  $N \in \mathbb{N}$ . From Lemma 2.4.4 it holds

$$\text{Torsion}_N(f, \gamma(s), \gamma'(s)) = \frac{\text{Var}_\gamma(\gamma(s), \gamma(\bar{s}_N))}{N},$$

where  $\bar{s}_N \in \mathbb{T}$  is such that  $\gamma(\bar{s}_N) = f^N \circ \gamma(s)$ . Consequently, since the complexity of the curve  $\gamma$  (see Definition 2.2.3) is bounded, we deduce that

$$|\text{Torsion}_N(f, \gamma(s), \gamma'(s))| \leq \frac{C(\gamma)}{N}.$$

Consider then the limit as  $N$  goes to  $+\infty$  and conclude that  $\text{Torsion}(f, \gamma(s)) = 0$  for any  $s \in \mathbb{T}$ . □

## 2.5 Appendix of Chapter 2

In this Appendix we prove Lemma 2.2.3.

We recall the hypothesis of the statement. For  $a \in \mathbb{A}$ ,  $\mathcal{N} \in \mathbb{N}^*$ ,  $(\mathcal{K}_i)_{i \in \llbracket 0, \mathcal{N}-1 \rrbracket} \in \mathbb{N}^{\mathcal{N}}$  and  $l_0 = 0 < l_1 < \dots < l_{\mathcal{N}}$  with  $l_i \in \mathbb{N}$  we have for any  $i \in \llbracket 0, \mathcal{N}-1 \rrbracket$

$$\tilde{v}(I)(f^{l_i}(a), \chi, l_{i+1} - l_i) - \tilde{v}(I)(f^{l_i}(a), \chi, 0) < -\frac{\mathcal{K}_i}{2}. \quad (2.38)$$

**Remark 2.5.1.** Observe that (2.38) remains true also with respect to the vector  $-\chi$  instead of  $\chi$  and does not depend on the chosen continuous determination.

*Proof of Lemma 2.2.3.* Let  $\xi \in T_a \mathbb{A} \setminus \{0\}$ . Assume that  $\xi$  either has strictly positive first coordinate or  $\xi$  is  $\chi$ : we can choose the continuous determination  $\tilde{v}(I)(a, \xi, \cdot)$  so that

$$-\frac{1}{2} < \tilde{v}(I)(a, \xi, 0) \leq \tilde{v}(I)(a, \chi, 0) = 0. \quad (2.39)$$

Lemma 1.1.1 in Chapter 1 tells us that for any  $t \in \mathbb{R}$  we have

$$\tilde{v}(I)(a, \xi, t) \leq \tilde{v}(I)(a, \chi, t). \quad (2.40)$$

Then we are going to show that for any  $j \in \llbracket 1, \mathcal{N} \rrbracket$  it holds<sup>4</sup>

$$\tilde{v}(I)(a, \xi, l_j) < -\frac{\sum_{i=0}^{j-1} \mathcal{K}_i}{2}. \quad (2.41)$$

Let us show inequality (2.41) by induction.

(i) Consider  $j = 1$ . By (2.38) for  $i = 0$  and by (2.39), we have

$$\tilde{v}(I)(a, \chi, l_1) - \tilde{v}(I)(a, \chi, 0) = \tilde{v}(I)(a, \chi, l_1) < -\frac{\mathcal{K}_0}{2}. \quad (2.42)$$

By inequality (2.40) for  $t = l_1$  and by inequality (2.42) it holds

$$\tilde{v}(I)(a, \xi, l_1) < -\frac{\mathcal{K}_0}{2}.$$

Thus, (2.41) holds for  $j = 1$ .

(ii) Let consider now  $j > 1$  and assume that  $\tilde{v}(I)(a, \xi, l_{j-1}) < -\frac{\sum_{i=0}^{j-2} \mathcal{K}_i}{2}$ . We are going to show that

$$\tilde{v}(I)(a, \xi, l_j) < -\frac{\sum_{i=0}^{j-1} \mathcal{K}_i}{2}.$$

We start by choosing the continuous determination  $\tilde{v}(I)(f^{l_{j-1}}(a), Df^{l_{j-1}}(a)\xi, \cdot)$  so that

$$\tilde{v}(I)(a, \xi, l_{j-1}) = \tilde{v}(I)(f^{l_{j-1}}(a), Df^{l_{j-1}}(a)\xi, 0). \quad (2.43)$$

---

4. Observe that this inequality depends on the choice of the continuous determination.

This is possible since  $v(I)(a, \xi, l_{j-1})$  and  $v(I)(f^{l_{j-1}}(a), Df^{l_{j-1}}(a)\xi, 0)$  are the same angle. By the inductive hypothesis we have

$$\tilde{v}(I)(f^{l_{j-1}}(a), Df^{l_{j-1}}(a)\xi, 0) < -\frac{\sum_{i=0}^{j-2} \mathcal{K}_i}{2}. \quad (2.44)$$

If  $\sum_{i=0}^{j-2} \mathcal{K}_i \in \mathbb{N}$  is odd, choose the continuous determination  $\tilde{v}(I)(f^{l_{j-1}}(a), -\chi, \cdot)$  so that

$$\tilde{v}(I)(f^{l_{j-1}}(a), -\chi, 0) = -\frac{\sum_{i=0}^{j-2} \mathcal{K}_i}{2}. \quad (2.45)$$

Hence inequality (2.44) becomes

$$\tilde{v}(I)(f^{l_{j-1}}(a), Df^{l_{j-1}}(a)\xi, 0) < \tilde{v}(I)(f^{l_{j-1}}(a), -\chi, 0).$$

By Lemma 1.1.1 for  $t = l_j - l_{j-1}$

$$\tilde{v}(I)(f^{l_{j-1}}(a), Df^{l_{j-1}}(a)\xi, l_j - l_{j-1}) < \tilde{v}(I)(f^{l_{j-1}}(a), -\chi, l_j - l_{j-1}). \quad (2.46)$$

Inequality (2.38) with respect to  $-\chi$  instead of  $\chi$  for  $i = j - 1$  (see Remark 2.5.1) gives us

$$\tilde{v}(I)(f^{l_{j-1}}(a), -\chi, l_j - l_{j-1}) < -\frac{\mathcal{K}_{j-1}}{2} + \tilde{v}(I)(f^{l_{j-1}}(a), -\chi, 0).$$

This, together with (2.45), implies that

$$\tilde{v}(I)(f^{l_{j-1}}(a), -\chi, l_j - l_{j-1}) < -\frac{\sum_{i=0}^{j-1} \mathcal{K}_i}{2}. \quad (2.47)$$

By (2.46) and (2.47)

$$\tilde{v}(I)(f^{l_{j-1}}(a), Df^{l_{j-1}}(a)\xi, l_j - l_{j-1}) < -\frac{\sum_{i=0}^{j-1} \mathcal{K}_i}{2}. \quad (2.48)$$

Consider now the two continuous functions

$$t \mapsto \tilde{v}(I)(a, \xi, l_{j-1} + t)$$

and

$$t \mapsto \tilde{v}(I)(f^{l_{j-1}}(a), Df^{l_{j-1}}(a)\xi, t).$$

They are continuous determinations of the same angle function and by (2.43) they coincide for  $t = 0$ . Consequently they are equal at any  $t$ , in particular for  $t = l_j - l_{j-1}$ . So by (2.48)

$$\tilde{v}(I)(a, \xi, l_j) < -\frac{\sum_{i=0}^{j-1} \mathcal{K}_i}{2}.$$

If  $\sum_{i=0}^{j-2} \mathcal{K}_i \in \mathbb{N}$  is even, we choose the continuous determination  $\tilde{v}(I)(f^{l_{j-1}}(a), \chi, \cdot)$  so that

$$\tilde{v}(I)(f^{l_{j-1}}(a), \chi, 0) = -\frac{\sum_{i=0}^{j-2} \mathcal{K}_i}{2}$$

and we repeat the same argument.

We have so proved inequality (2.41) and we are going to conclude the proof of Lemma 2.2.3.

By inequality (2.41) for  $j = \mathcal{N}$  and by (2.39) we have

$$\tilde{v}(I)(a, \xi, l_{\mathcal{N}}) - \tilde{v}(I)(a, \xi, 0) < -\frac{\sum_{i=0}^{\mathcal{N}-1} \mathcal{K}_i}{2} + \frac{1}{2} \quad (2.49)$$

as desired. If  $\xi = \chi$ , then again inequality (2.41) for  $j = \mathcal{N}$  and the equality in (2.39) imply in particular that

$$\tilde{v}(I)(a, \chi, l_{\mathcal{N}}) - \tilde{v}(I)(a, \chi, 0) < -\frac{\sum_{i=0}^{\mathcal{N}-1} \mathcal{K}_i}{2}. \quad (2.50)$$

Assume now that  $\xi$  either has strictly negative first coordinate or  $\xi$  is  $-\chi$ . Then  $-\xi$  either has strictly positive first coordinate or  $-\xi$  is  $\chi$ . From (2.49) we have

$$\tilde{v}(I)(a, -\xi, l_{\mathcal{N}}) - \tilde{v}(I)(a, -\xi, 0) < -\frac{\sum_{i=0}^{\mathcal{N}-1} \mathcal{K}_i}{2} + \frac{1}{2}.$$

Since

$$\tilde{v}(I)(a, \xi, l_{\mathcal{N}}) - \tilde{v}(I)(a, \xi, 0) = \tilde{v}(I)(a, -\xi, l_{\mathcal{N}}) - \tilde{v}(I)(a, -\xi, 0), \quad (2.51)$$

we conclude that

$$\tilde{v}(I)(a, \xi, l_{\mathcal{N}}) - \tilde{v}(I)(a, \xi, 0) < -\frac{\sum_{i=0}^{\mathcal{N}-1} \mathcal{K}_i}{2} + \frac{1}{2}.$$

In particular, if  $\xi = -\chi$ , then by (2.50) and (2.51) it holds

$$\tilde{v}(I)(a, -\chi, l_{\mathcal{N}}) - \tilde{v}(I)(a, -\chi, 0) < -\frac{\sum_{i=0}^{\mathcal{N}-1} \mathcal{K}_i}{2}.$$

This concludes our proof. □



# Chapter 3

## Points of zero torsion for conservative twist maps

In this chapter we consider conservative twist maps of the annulus. In particular, we are interested in the torsion of bounded instability regions of  $\mathbb{A}$  (both bounded subannuli and periodic discs). We will show that any bounded instability region contains a subset of positive measure where the torsion is not null.

### 3.1 Conservative twist maps and instability zones

The manifold  $\mathbb{A} = \mathbb{T} \times \mathbb{R}$  is endowed with its standard Riemannian metric and trivialization. Denote as  $\omega = dx \wedge dy$  the area form on  $\mathbb{A}$ . With an abuse of notation, we denote as  $\omega$  also the measure associated to the area form on  $\mathbb{A}$ , i.e. the Lebesgue measure. Fix the counterclockwise orientation and consider the constant vector field  $\chi(z) = \chi = (0, 1)$ .

We recall that the function  $p : \mathbb{R} \rightarrow \mathbb{T}$  denotes the universal covering of the 1-dimensional torus, while  $p \times \text{Id}_{\mathbb{R}} : \mathbb{R}^2 \rightarrow \mathbb{A}$  denotes the universal covering of  $\mathbb{A}$ . Denote as  $p_1, p_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$  the projections over the first and second coordinates, respectively. With an abuse of notation, denote as  $p_1, p_2$  also the projections over the first and second coordinate, respectively, for the annulus  $\mathbb{A}$ .

Our main references for this section are [Arn16], [Ban88] and [Gol01].

**Definition 3.1.1** (Symplectic map). A  $\mathcal{C}^1$  diffeomorphism  $f : \mathbb{A} \rightarrow \mathbb{A}$  is symplectic if  $f^*\omega = \omega$ .

**Definition 3.1.2** (Conservative map). A  $\mathcal{C}^1$  diffeomorphism  $f : \mathbb{A} \rightarrow \mathbb{A}$  is conservative (or exact symplectic) if  $f^*\omega - \omega$  is an exact 1-form.

**Notation 3.1.1.** All along the section,  $f : \mathbb{A} \rightarrow \mathbb{A}$  is a **conservative twist map**.

We briefly recall the notation used for lifts of oriented angle functions. Let  $I = (f_t)_t$  in  $\text{Diff}^1(\mathbb{A})$  be an isotopy joining the identity to  $f$ . Define the function

$$v(I) : T\mathbb{A}_* \times \mathbb{R} \rightarrow \mathbb{T}$$

$$(z, \xi, t) \mapsto \theta(\chi, Df_t(x)\xi),$$

where  $T\mathbb{A}_* = \{(z, \xi) \in T\mathbb{A} : \xi \neq 0\}$ . Fix  $(z, \xi) \in T\mathbb{A}_*$  and denote as  $\tilde{v}(I)(z, \xi, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$  a continuous determination of the angle function  $v(I)(z, \xi, \cdot)$ .

**Remark 3.1.1.** Be careful! Although we have chosen a continuous determination  $\tilde{v}(f) : T\mathbb{A}_* \times \mathbb{R} \rightarrow \mathbb{R}$  to introduce the torsion, in the sequel sometimes we will be interested also in considering different determinations  $\tilde{v}(f)(z, \xi, \cdot)$  that are independently defined for different points  $(z, \xi) \in T\mathbb{A}_*$ .

We refer to Definitions 1.1.2 and 1.1.3 in Chapter 1 for the notion of (finite-time) torsion. We recall that the (finite-time) torsion on  $\mathbb{A}$  does not depend on the choice of the isotopy (see Proposition 1.3.2 in Chapter 1). Therefore, we will use the notations  $\text{Torsion}(f, z)$  and  $\text{Torsion}_n(f, z, \xi)$  for the torsion and the torsion at finite time.

We refer to [Arn16] for the following notions.

**Definition 3.1.3.** An *essential curve* is a  $\mathcal{C}^0$ -embedded circle in  $\mathbb{A}$  not homotopic to a point.

**Definition 3.1.4.** An *essential subannulus* of  $\mathbb{A}$  is a subset of the annulus that is homeomorphic to  $\mathbb{A}$  and contains an essential curve of  $\mathbb{A}$ .

**Notation 3.1.2.** Denote as  $\mathcal{I}(f)$  the union of all the invariant continuous graphs of  $f$  and as  $\mathcal{N}(f)$  its complement.

We are then interested in the dynamics on  $\mathcal{N}(f)$  and in particular in the torsion at points of  $\mathcal{N}(f)$ . We start by stating the following

**Proposition 3.1.1** (Proposition 2.17 in [Arn16]). *Let  $f$  be a conservative twist map. Every connected components of  $\mathcal{N}(f)$  is either a bounded disc or an essential subannulus of  $\mathbb{A}$ .*

- *When such a component is a disc  $D$ , then this disc is periodic i.e. there exists  $N \geq 1$  such that  $f^N(D) = D$ . Moreover, the boundary of  $D$  is the union of parts of two invariant continuous graphs that have the same rational rotation number.*
- *When such a component is an essential subannulus, then it is invariant by  $f$ , and each of the two components of its boundary is either  $\mathbb{T} \times \{\pm\infty\}$ <sup>1</sup> or an invariant continuous graph.*

**Definition 3.1.5.** An *instability zone* is a connected component of  $\mathcal{N}(f)$  which is an essential subannulus. An *instability disc* is a connected component of  $\mathcal{N}(f)$  which is a  $f$ -periodic disc.

Recall that, with an abuse of notation, we denote as  $\omega$  both the area form on  $\mathbb{A}$  and the measure associated, i.e. the Lebesgue measure. The main result of the chapter is then the following

**Theorem 3.1.1.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a conservative twist map. Let  $U \subset \mathbb{A}$  be a bounded connected component of  $\mathcal{N}(f)$ . Then*

$$\omega(\{z \in U : \text{Torsion}(f, z) \neq 0\}) > 0.$$

Theorem 3.1.1 is an outcome of Proposition 2.17 in [Arn16] (here Proposition 3.1.1) and of the following propositions.

**Proposition 3.1.2.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a conservative twist map. Let  $U \subset \mathbb{A}$  be a bounded instability zone. Then  $\omega(\{z \in U : \text{Torsion}(f, z) \neq 0\}) > 0$ .*

---

1. The boundary is considered in the compactification of  $\mathbb{A}$ , i.e. in  $\mathbb{T} \times \bar{\mathbb{R}}$ , where  $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ .

**Proposition 3.1.3.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a conservative twist map. Let  $U \subset \mathbb{A}$  be an instability disc. Then  $\omega(\{z \in U : \text{Torsion}(f, z) \neq 0\}) > 0$ .*

Proposition 3.1.2 will be an immediate outcome of Theorem 3.2.1 (see Section 3.2), while the proof of Proposition 3.1.3 is presented in Section 3.3.

**Remark 3.1.2.** Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a conservative positive twist map. Let us assume that, whenever it exists,  $\text{Torsion}(f, z)$  is zero for  $\omega$ -almost every  $z \in \mathbb{A}$ . As an outcome of Theorem 3.1.1, we deduce that there are neither bounded instability zones nor instability discs.

An outcome of Theorem 3.1.1 concerning the torsion of measures is then the following

**Corollary 3.1.1.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a conservative positive twist map and let  $\omega = dx \wedge dy$  be the area form on  $\mathbb{A}$ . With an abuse of notation, denote as  $\omega$  also the Lebesgue measure on  $\mathbb{A}$ . Let  $U$  be either a bounded instability zone or the orbit of a  $N$ -periodic instability disc  $D$  (i.e.  $\bigcup_{i=0}^{N-1} f^i(D)$ ). Then it holds*

$$-\frac{1}{2} \leq \text{Torsion}(f, \tilde{\omega}) < 0$$

where  $\tilde{\omega}$  is the normalized Lebesgue measure with respect to  $U$ , i.e.  $\tilde{\omega}(\cdot) = \frac{\omega(\cdot \cap U)}{\omega(U)}$ .

*Proof.* Since for any  $z \in \mathbb{A}$  it holds  $-\frac{1}{2} \leq \text{Torsion}(f, z) \leq 0$  (see Corollary 2.1.1) and since, from the definition of the torsion of a measure (see Definition 1.1.4), we have

$$\text{Torsion}(f, \tilde{\omega}) = \int_{\mathbb{A}} \text{Torsion}(f, z) d\tilde{\omega}(z) = \int_U \text{Torsion}(f, z) d\tilde{\omega}(z),$$

we already know that  $-\frac{1}{2} \leq \text{Torsion}(f, \tilde{\omega}) \leq 0$ . By Theorem 3.1.1 applied at  $U$ , the set  $V = \{z \in U : \text{Torsion}(f, z) \neq 0\}$  has positive Lebesgue measure. Actually, because of Corollary 2.1.1, it holds  $V = \{z \in U : \text{Torsion}(f, z) < 0\}$ .

Recall that by Ruelle's theorem in [Rue85] the torsion exists at  $\tilde{\omega}$ -almost every  $z \in U$ . In particular, the torsion exists and it is null for  $\tilde{\omega}$ -almost every  $z \in U \setminus V$ . Consequently

$$\begin{aligned} \text{Torsion}(f, \tilde{\omega}) &= \int_{\mathbb{A}} \text{Torsion}(f, z) d\tilde{\omega}(z) = \int_U \text{Torsion}(f, z) d\tilde{\omega}(z) = \\ &= \int_V \text{Torsion}(f, z) d\tilde{\omega}(z) + \int_{U \setminus V} \text{Torsion}(f, z) d\tilde{\omega}(z) = \int_V \text{Torsion}(f, z) d\tilde{\omega}(z) < 0. \end{aligned}$$

That is, it holds  $-\frac{1}{2} \leq \text{Torsion}(f, \tilde{\omega}) < 0$ . □

## 3.2 $\mathcal{C}^0$ -integrability of bounded sub-annuli

We start by introducing some notions and definitions and by presenting the main result (see Theorem 3.2.1) whose proof will take almost all the section.

**Definition 3.2.1.** A conservative twist map  $f$  is  $\mathcal{C}^0$  integrable if there exists a partition of  $\mathbb{A}$  into continuous closed invariant curves not homotopic to a point, any one of which is a continuous embedding of  $\mathbb{S}^1$  in  $\mathbb{A}$ .



Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a conservative positive twist map. Let  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a lift of  $f$ . Referring to [Ban88][Sections 1 and 7] (see also [Arn16][Proposition 1.8] and [Gol01][Chapter 1, Section 5A]), there exists a function

$$h : \mathbb{R}^2 \rightarrow \mathbb{R}$$

such that

$$F(x, y) = (X, Y) \quad \Leftrightarrow \quad y = -\frac{\partial}{\partial x} h(x, X), \quad Y = \frac{\partial}{\partial X} h(x, X). \quad (3.1)$$

The function  $h$  is called a *generating function* of  $F$  and

- (i)  $h$  is a  $\mathcal{C}^2$  function;
- (ii)  $h$  is invariant under translation of  $(1, 1)$ , that is

$$h(x + 1, X + 1) = h(x, X) \quad \forall (x, X) \in \mathbb{R}^2;$$

- (iii) for any  $(x, X) \in \mathbb{R}^2$  it holds  $\frac{\partial^2}{\partial x \partial X} h(x, X) < 0$ .

**Definition 3.2.2.** A sequence of real numbers  $(x_n)_{n \in \mathbb{Z}} \subset \mathbb{R}^{\mathbb{Z}}$  is a configuration for  $F$  if there exists a sequence  $(y_n)_{n \in \mathbb{Z}} \subset \mathbb{R}^{\mathbb{Z}}$  such that  $(x_n, y_n)_{n \in \mathbb{Z}}$  is an orbit for  $F$ , that is  $(x_n, y_n) = F^n(x_0, y_0)$  for any  $n \in \mathbb{Z}$ .

Observe that  $(x_n)_{n \in \mathbb{Z}}$  is a configuration if and only if

$$\frac{\partial}{\partial X} h(x_{n-1}, x_n) + \frac{\partial}{\partial x} h(x_n, x_{n+1}) = 0 \quad \forall n \in \mathbb{Z}. \quad (3.2)$$

**Definition 3.2.3** (Definition 3.6 in [Arn16]). Let  $(x_n)_{n \in \mathbb{Z}} \subset \mathbb{R}^{\mathbb{Z}}$  be a sequence of real numbers. The union over all  $n \in \mathbb{Z}$  of segments in the plane  $\mathbb{R}^2$  joining  $(n, x_n)$  to  $(n + 1, x_{n+1})$  is called the Aubry diagram of  $(x_n)_{n \in \mathbb{Z}}$ .

We say that the Aubry diagrams of two sequences  $(x_n)_{n \in \mathbb{Z}}, (\tilde{x}_n)_{n \in \mathbb{Z}}$  cross

- (i) between  $n$  and  $n + 1$  if  $(x_n - \tilde{x}_n)(x_{n+1} - \tilde{x}_{n+1}) < 0$ ;
- (ii) at  $n \in \mathbb{Z}$  if  $x_n = \tilde{x}_n$  and  $(x_{n-1} - \tilde{x}_{n-1})(x_{n+1} - \tilde{x}_{n+1}) < 0$ .

**Notation 3.2.1.** Let  $(x_n)_{n \in \mathbb{Z}}$  be a sequence of real numbers. We denote as  $\mathcal{L}((x_n)_{n \in \mathbb{Z}})$  the Aubry diagram of  $(x_n)_{n \in \mathbb{Z}}$ .

**Remark 3.2.1.** Observe that two configurations  $(x_n)_{n \in \mathbb{Z}}, (\tilde{x}_n)_{n \in \mathbb{Z}}$  such that  $x_{n_1} < \tilde{x}_{n_1}, x_{n_1+1} = \tilde{x}_{n_1+1}$  for some  $n_1 \in \mathbb{Z}$  actually cross at  $n_1 + 1$  according to point (ii) of Definition 3.2.3. Indeed we are now going to show that  $x_{n_1+2} > \tilde{x}_{n_1+2}$ . Equivalently we have that  $x_{n_1+1} = \tilde{x}_{n_1+1}$  and

$$(x_{n_1} - \tilde{x}_{n_1})(x_{n_1+2} - \tilde{x}_{n_1+2}) < 0.$$

Denote as  $(x_n, y_n)_{n \in \mathbb{Z}}, (\tilde{x}_n, \tilde{y}_n)_{n \in \mathbb{Z}}$  the orbits for  $F$  corresponding to  $(x_n)_{n \in \mathbb{Z}}, (\tilde{x}_n)_{n \in \mathbb{Z}}$ , respectively. Recall that the function  $\mathbb{R} \ni y \mapsto p_1 \circ F^{-1}(x_{n_1+1}, y) \in \mathbb{R}$  is a strictly decreasing diffeomorphism of  $\mathbb{R}$  and so its inverse function is strictly decreasing as well. From this and since  $y_{n_1+1} = (p_1 \circ F^{-1}(x_{n_1+1}, \cdot))^{-1}(x_{n_1}), \tilde{y}_{n_1+1} = (p_1 \circ F^{-1}(x_{n_1+1}, \cdot))^{-1}(\tilde{x}_{n_1}), x_{n_1} < \tilde{x}_{n_1}$ , it holds  $y_{n_1+1} > \tilde{y}_{n_1+1}$ .

Now the function  $\mathbb{R} \ni y \mapsto p_1 \circ F(x_{n_1+1}, y)$  is a strictly increasing diffeomorphism and, since  $y_{n_1+1} > \tilde{y}_{n_1+1}$ , we conclude that

$$x_{n_1+2} = p_1 \circ F(x_{n_1+1}, y_{n_1+1}) > p_1 \circ F(x_{n_1+1}, \tilde{y}_{n_1+1}) = \tilde{x}_{n_1+2}.$$

**Definition 3.2.4.** Let  $(x_n)_{n \in \mathbb{Z}}$  be a configuration for  $F$ . The sequence  $(\xi_n)_{n \in \mathbb{Z}} \subset \mathbb{R}^{\mathbb{Z}}$ ,  $\xi_n \in T_{x_n} \mathbb{R}$ , is a Jacobi field along the configuration  $(x_n)_{n \in \mathbb{Z}}$  if for all  $n \in \mathbb{Z}$

$$\frac{\partial^2}{\partial x \partial X} h(x_{n-1}, x_n) \xi_{n-1} + \left[ \frac{\partial^2}{\partial x^2} h(x_n, x_{n+1}) + \frac{\partial^2}{\partial X^2} h(x_{n-1}, x_n) \right] \xi_n + \frac{\partial^2}{\partial x \partial X} h(x_n, x_{n+1}) \xi_{n+1} = 0. \quad (3.3)$$

**Definition 3.2.5.** Two points  $x_M, x_N$  of a configuration  $(x_n)_{n \in \mathbb{Z}}$ ,  $M \neq N$ , are called *conjugate points* if there exists a non zero Jacobi field  $(\xi_n)_{n \in \mathbb{Z}}$  along  $(x_n)_{n \in \mathbb{Z}}$  such that  $\xi_M = \xi_N = 0$ .

**Notation 3.2.2.** Let  $z = (x, y) \in \mathbb{R}^2$  and denote as  $(x_n, y_n)_{n \in \mathbb{Z}}$  the orbit of  $z$  with respect to  $F$ . We say that  $z$  has conjugate points along its orbit if there exist  $M < N$ ,  $M, N \in \mathbb{Z}$  such that  $x_M, x_N$  are conjugate points.

**Remark 3.2.2.** For every  $z \in \mathbb{R}^2$ , the vertical space at  $z$  is

$$\mathcal{V}(z) := \ker(Dp_{1|T_z \mathbb{R}^2}).$$

Let  $M < N$ ,  $M, N \in \mathbb{Z}$ , and let  $(x_n)_{n \in \mathbb{Z}}$  be a configuration for  $F$ . Denote as  $(x_n, y_n)_{n \in \mathbb{Z}}$  the orbit associated to the configuration. We remark that, from a geometrical point of view, two points  $x_M, x_N$  of the configuration are conjugate if

$$\mathcal{V}(x_N, y_N) \cap DF^{N-M}(x_M, y_M)(\mathcal{V}(x_M, y_M)) \neq \{0\}.$$

**Notation 3.2.3.** Denote as a BIES an open subset  $U \subset \mathbb{A}$  which is a bounded,  $f$ -invariant essential subannulus.

**Remark 3.2.3.** Any instability zone is  $f$ -invariant. Any bounded instability zone is a BIES. The boundary of any BIES is the union of two invariant disjoint curves. By Birkhoff's theorem, the boundary is actually the union of two disjoint Lipschitz continuous graphs in  $\mathbb{A}$  (see [Bir22] and Chapter 1 in [Her83]).

The main result of the section concerns the relation between properties of torsion and  $\mathcal{C}^0$  integrability for a conservative positive twist map.

**Theorem 3.2.1.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a conservative positive twist map. Let  $U$  be a BIES for  $f$ . Then the following statements are equivalent:*

- (i)  $f|_U$  is  $\mathcal{C}^0$  integrable;
- (ii) the torsion exists and it is null for every  $z \in U$ ;
- (iii) the torsion is null for  $\omega$ -almost every  $z \in U$ .

The implication (ii)  $\Rightarrow$  (iii) of Theorem 3.2.1 is trivial. Let us begin with the proof of the implication (i)  $\Rightarrow$  (ii) of Theorem 3.2.1.

*Proof of (i)  $\Rightarrow$  (ii) of Theorem 3.2.1.* Assume  $f|_U$  is  $\mathcal{C}^0$  integrable and consider  $(x, y) \in U$ . By the  $\mathcal{C}^0$ -integrability condition, there exists a continuous closed  $f$ -invariant curve  $\Gamma$  not homotopic to a point such that  $(x, y) \in \Gamma$ . By Birkhoff's theorem (see [Bir22]), the curve  $\Gamma$  is the graph of a Lipschitz continuous function  $\gamma : \mathbb{T} \rightarrow \mathbb{R}$ . The graph of  $\gamma$ , i.e.  $\Gamma$ , is a closed well-ordered set and every point of it is an accumulation point of  $\Gamma$ . By Corollary 2.4 in [Cro03], every point of  $\Gamma$ , so in particular  $(x, y)$ , has zero torsion, according to Crovisier's definition. By Proposition 2.1.4 in Chapter 1, Crovisier's torsion is equivalent to Definition 1.1.3. Hence,  $(x, y)$  has zero torsion with respect to our definition.

By the arbitrariness of  $(x, y) \in U$ , we conclude that every point in  $U$  has zero torsion.  $\square$

*Sketch of the proof of (iii)  $\Rightarrow$  (i) of Theorem 3.2.1.* The proof of this last implication relies on the following two main propositions.

**Proposition 3.2.1.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a conservative positive twist map and let  $U$  be a BIES. If the torsion is zero for  $\omega$ -almost every point of  $U$ , then  $f$  has no conjugate points in  $U$ .*

**Proposition 3.2.2.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a conservative positive twist map and let  $U$  be a BIES. If  $f$  has no conjugate points in  $U$ , then  $f|_U$  is  $\mathcal{C}^0$  integrable.*

Mostly all the section concerns the proofs of these two main propositions (see Subsection 3.2.1 for the proof of Proposition 3.2.1 and Subsection 3.2.2 for that of Proposition 3.2.2). Admitting for the moment Propositions 3.2.1 and 3.2.2, the necessary implication follows immediately. □

From Theorem 3.2.1 we deduce the following

**Corollary 3.2.1.** *Let  $f : \mathbb{T} \times [0, 1] \rightarrow \mathbb{T} \times [0, 1]$  be a conservative positive twist map on the bounded annulus. Then,  $f$  is  $\mathcal{C}^0$  integrable if and only if the torsion is null at  $\omega$ -almost every point.*

*Proof.* The function  $f$  can be extended to a conservative positive twist map  $\tilde{f} : \mathbb{A} \rightarrow \mathbb{A}$  on the unbounded annulus such that  $\tilde{f}|_{\mathbb{T} \times [0, 1]} = f$  (see [LC91][Chapter 1, Section 2] and [MF94][Theorem 8.1]). The interior of the bounded annulus, that is  $\mathbb{T} \times (0, 1)$ , is a BIES for  $\tilde{f}$ . Applying Theorem 3.2.1 we conclude. □

An immediate outcome of Theorem 3.2.1 is also the proof of Proposition 3.1.2.

*Proof of Proposition 3.1.2.* Let  $U \subset \mathbb{A}$  be a bounded instability zone. From Remark 3.2.3 we have that  $U$  is a BIES. Since  $f|_U$  is not  $\mathcal{C}^0$  integrable (from the definition of the instability zone), we deduce that the torsion is not  $\omega$ -almost everywhere null. That is  $\omega(\{z \in U : \text{Torsion}(f, z) \neq 0\}) > 0$ , as claimed. □

### 3.2.1 Proof of Proposition 3.2.1

This section is devoted to the proof of Proposition 3.2.1. Actually, we are going to show a more general result. That is, the following proposition holds true.

**Proposition 3.2.3.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a conservative positive twist map. Let  $U \subset \mathbb{A}$  be an open invariant set such that  $\omega(U) < +\infty$ . If  $\omega$ -almost every  $z \in U$  has zero torsion, then  $f|_U$  has no conjugate points.*

The proof of Proposition 3.2.3 is an outcome of the following result.

**Proposition 3.2.4.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a conservative positive twist map. Let  $z \in \mathbb{A}$  be such that:*

- (i)  $z$  has a conjugate point;

(ii)  $z$  has a neighborhood  $U$  such that  $\omega(\cup_{n \in \mathbb{Z}} f^n(U)) < +\infty$ .

Then there exists an open neighborhood  $W_z \subset U$  of  $z$  such that for  $\omega$ -almost every  $z' \in W_z$  it holds  $\text{Torsion}(f, z') < 0$ .

Let us first show Proposition 3.2.3 by assuming Proposition 3.2.4.

*Proof of Proposition 3.2.3.* Let us argue by contradiction and suppose there exists  $z \in U$  which has a conjugate point. Since  $U$  is  $f$ -invariant and  $\omega(U) < +\infty$ , by Proposition 3.2.4 there exists a set of positive measure where the torsion is strictly negative. This contradicts the hypothesis and we conclude.  $\square$

The rest of the section concerns the proof of Proposition 3.2.4. Such proof relies on three main arguments:

- (1) the presence of a neighborhood of  $z$  of positive measure such that each of its points has strictly negative finite-time torsion;
- (2) a link between the returning time of a point  $z'$  to this neighborhood and the torsion at finite time at  $z'$ ;
- (3) the use of Birkhoff's Ergodic Theorem for evaluating returning times of points to the highlighted neighborhood.

Let us prove now Proposition 3.2.4.

*Proof.* By hypothesis, the point  $z \in \mathbb{A}$  has a conjugate point. That is (see Remark 3.2.2) there exists  $n \in \mathbb{N}^*$  and  $k \in \mathbb{Z}$  such that

$$n \text{Torsion}_n(I, z, \chi) = \tilde{v}(I)(z, \chi, n) - \tilde{v}(I)(z, \chi, 0) = -\frac{k}{2}. \quad (3.4)$$

Since  $f$  is a positive twist map, by Theorem 2.1.1 in Chapter 2, any finite-time torsion with respect to the vertical vector is strictly negative. Hence  $k \geq 1$ .

Since

$$t \mapsto v(I)(z, \chi, n+t) \quad \text{and} \quad t \mapsto v(I)(f^n(z), Df^n(z)\chi, t)$$

are the same angle function, the functions

$$t \mapsto \tilde{v}(I)(z, \chi, n+t) \quad \text{and} \quad t \mapsto \tilde{v}(I)(f^n(z), Df^n(z)\chi, t)$$

differ by an integer. Since  $Df^n(z)\chi$  is a vertical vector from (3.4) and by Theorem 2.1.1, for any  $m \in \mathbb{N}, m > n$  it holds

$$\tilde{v}(I)(f^n(z), Df^n(z)\chi, m-n) - \tilde{v}(I)(f^n(z), Df^n(z)\chi, 0) < 0. \quad (3.5)$$

Since the following differences do not depend on the choice of the continuous determination, we have that for any  $m \in \mathbb{N}, m > n$

$$\begin{aligned} & \tilde{v}(I)(z, \chi, m) - \tilde{v}(I)(z, \chi, n) = \\ & = \tilde{v}(I)(f^n(z), Df^n(z)\chi, m-n) - \tilde{v}(I)(f^n(z), Df^n(z)\chi, 0). \end{aligned}$$

So, by (3.5)

$$\tilde{v}(I)(z, \chi, m) - \tilde{v}(I)(z, \chi, n) < 0.$$

Consequently for any  $m \in \mathbb{N}, m > n$  we have

$$\tilde{v}(I)(z, \chi, m) - \tilde{v}(I)(z, \chi, 0) < -\frac{k}{2}. \quad (3.6)$$

Fix now  $m \in \mathbb{N}, m > n$ . By continuity of torsion at finite time with respect to the point, inequality (3.6) is an open condition and there exists a neighborhood  $W_z \subset U$  of  $z$  such that for any  $y \in W_z$

$$\tilde{v}(I)(y, \chi, m) - \tilde{v}(I)(y, \chi, 0) < -\frac{k}{2}. \quad (3.7)$$

The open set  $W_z$  has positive measure, denoted as  $\omega(W_z) =: \varepsilon > 0$ . We need now to prove that  $\omega$ -almost every point in  $W_z$  has negative torsion.

Observe that, by Ruelle's theorem in [Rue85], at  $\omega$ -almost every point of  $\cup_{n \in \mathbb{Z}} f^n(U)$  the torsion exists. Consequently, also at  $\omega$ -almost every point of  $W_z$  it exists. Recall that (by Corollary 2.1.1) the torsion is non positive whenever it exists.

We start now discussing the second main argument of the proof, i.e. the relation between returning times to  $W_z$  and finite time torsion.

Let  $N \in \mathbb{N}$  and let us introduce the notation

$$\begin{aligned} \square_N : \bigcup_{n \in \mathbb{Z}} f^n(U) &\longrightarrow \mathbb{N} \\ x &\mapsto \square_N(x) := \sum_{i=0}^{N-1} \mathbb{I}_{W_z}(f^{mi}(x)), \end{aligned} \quad (3.8)$$

where  $\mathbb{I}_{W_z}(\cdot)$  denotes the characteristic function of the set  $W_z$ . The function  $\square_N$  evaluated at  $x$  counts how many points of the segment of the orbit  $(f^{mi}(x))_{i \in [0, N-1]}$  of  $x$  under  $f^m$  are in  $W_z$ .

The following lemma provides us the required link between torsion and returning times (counted by  $\square_N(x)$ ).

**Lemma 3.2.1.** *For  $x \in W_z$  such that  $\square_N(x) \geq 2$  we have*

$$\tilde{v}(I)(x, \chi, mN) - \tilde{v}(I)(x, \chi, 0) < -\frac{\square_N(x)k}{2} + \frac{k}{2}. \quad (3.9)$$

The proof of Lemma 3.2.1 relies on the following rough idea. From inequality (3.7), every time that the orbit of a point  $x$  comes back to  $W_z$ , the torsion gains a negative contribution (less than  $-\frac{k}{2}$ ) over the successive  $m$ -lengthed time interval. Hence, when looking at the variation of a continuous determination of the angle function over a given time interval  $N$ , we consider the  $f^m$ -orbit of the point  $x$ . Each contribution of the torsion between consecutive points of the  $f^m$ -orbit with respect to the vertical vector is strictly negative (see Theorem 2.1.1), but contributions of the torsion corresponding to points coming back to  $W_z$  are strictly less than  $-\frac{k}{2}$ . Adding all the contributions, the variation of the continuous determination over the considered time interval  $N$  is strictly less than  $\square_N(x)$  times  $-\frac{k}{2}$ .

Actually, when coming back to  $W_z$ , we cannot directly use inequality (3.7) and Theorem 2.1.1 since we are not looking at torsion at finite time with respect to the vertical vector. We need to prove a more accurate estimation and we will show that, up to add a constant, the previous rough idea holds.

Lemma 2.2.3 will be the main tool in providing such an estimation and in proving Lemma 3.2.1. Therefore we recall Lemma 2.2.3 here (see Appendix 2.5 for the detailed proof).

**Lemma 2.2.3.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a  $C^1$  diffeomorphism isotopic to the identity. Let  $a \in \mathbb{A}$ . Let  $\mathcal{N} \in \mathbb{N}^*$ ,  $(\mathcal{K}_i)_{i \in \llbracket 0, \mathcal{N}-1 \rrbracket} \in \mathbb{N}^{\mathcal{N}}$  and  $l_0 = 0 < l_1 < \dots < l_{\mathcal{N}}$  with  $l_i \in \mathbb{N}$  for any  $i$ . Assume that for all  $i \in \llbracket 0, \mathcal{N}-1 \rrbracket$  it holds*

$$\tilde{v}(I)(f^{l_i}(a), \chi, l_{i+1} - l_i) - \tilde{v}(I)(f^{l_i}(a), \chi, 0) < -\frac{\mathcal{K}_i}{2}. \quad (3.10)$$

*Then for any vector  $\xi \in T_a \mathbb{A} \setminus \{0\}$  we have*

$$\tilde{v}(I)(a, \xi, l_{\mathcal{N}}) - \tilde{v}(I)(a, \xi, 0) < -\frac{\sum_{i=0}^{\mathcal{N}-1} \mathcal{K}_i}{2} + \frac{1}{2}. \quad (3.11)$$

*Moreover when  $\xi = \chi$  we have*

$$\tilde{v}(I)(a, \chi, l_{\mathcal{N}}) - \tilde{v}(I)(a, \chi, 0) < -\frac{\sum_{i=0}^{\mathcal{N}-1} \mathcal{K}_i}{2}. \quad (3.12)$$

An outcome of Lemma 2.2.3 is Lemma 2.2.2, that we restate here in the particular framework of positive twist maps (we refer to Lemma 2.2.2 for the proof).

**Lemma 3.2.2.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a positive twist map. Let  $a \in \mathbb{A}, n \in \mathbb{N}^*, k \in \mathbb{N}$  be such that*

$$\tilde{v}(I)(a, \chi, n) - \tilde{v}(I)(a, \chi, 0) < -\frac{k}{2}.$$

*Then for any  $l \in \mathbb{N}, l \geq n$  it holds*

$$\tilde{v}(I)(a, \chi, l) - \tilde{v}(I)(a, \chi, 0) < -\frac{k}{2}.$$

We now prove Lemma 3.2.1.

*Proof of Lemma 3.2.1.* We will show that we can apply Lemma 2.2.3 with respect to  $f^m$  in order to bound the quantity

$$\tilde{v}(I)(x, \chi, mN) - \tilde{v}(I)(x, \chi, 0)$$

for  $x \in W_z$ .

We are assuming that  $x \in W_z$  is such that  $\square_N(x) \geq 2$ , that is the  $f^m$ -orbit of the point  $x$  comes back to  $W_z$  at least another time within the time interval  $\llbracket 0, N-1 \rrbracket$ .

Denote as  $l_i$  the integer in  $\llbracket 0, N-1 \rrbracket$  such that  $f^{ml_i}(x)$  is the  $i$ -th point of the orbit of  $x$  that comes back to  $W_z$ . Let us apply Lemma 2.2.3 (used with respect to  $f^m$ ) with  $a = x$ ,  $\mathcal{N} = \square_N(x) - 1 \geq 1$  and  $\mathcal{K}_i = k$  for any  $i$ , where  $k \in \mathbb{N}$  is the positive integer of (3.7). For any  $i$ , since  $f^{ml_i}(x) \in W_z$ , by (3.7)

$$\tilde{v}(I)(f^{ml_i}(x), \chi, m) - \tilde{v}(I)(f^{ml_i}(x), \chi, 0) < -\frac{k}{2}$$

and, since  $l_{i+1} - l_i \geq 1$ , from Lemma 3.2.2 we deduce that

$$\tilde{v}(I)(f^{ml_i}(x), \chi, m(l_{i+1} - l_i)) - \tilde{v}(I)(f^{ml_i}(x), \chi, 0) < -\frac{k}{2}.$$

From Lemma 2.2.3, recalling that  $\mathcal{N} = \square_N(x) - 1$ , we have (see (3.12))

$$\tilde{v}(I)(x, \chi, ml_{\mathcal{N}}) - \tilde{v}(I)(x, \chi, 0) < -\frac{\square_N(x)k}{2} + \frac{k}{2}.$$

Since  $\square_N(x)k \in \mathbb{N}^*$  and  $mN \geq ml_{\mathcal{N}}$ , applying Lemma 3.2.2, we conclude

$$\tilde{v}(I)(x, \chi, mN) - \tilde{v}(I)(x, \chi, 0) < -\frac{\square_N(x)k}{2} + \frac{k}{2}.$$

□

The following Lemma concerns the last main argument of the proof and enables us to conclude.

**Lemma 3.2.3.** *For  $\omega$ -almost every  $x \in W_z$  it holds  $\text{Torsion}(f, x) < 0$ .*

*Proof of Lemma 3.2.3.* We start by remarking that for  $\omega$ -almost every  $x \in W_z$  the torsion exists (see [Rue85]). Since  $\frac{\square_N(x)}{N}$  is a Birkhoff's sum, its limit for  $N$  going to infinity exists for  $\omega$ -almost every  $x$ . Denote

$$A_1 := \{x \in W_z : \text{Torsion}(f, x) = 0\}.$$

We are going to prove that  $\omega(A_1) = 0$ . Let  $A$  be the set  $\cup_{n \in \mathbb{Z}} f^{mn}(A_1)$ . Then  $A$  is clearly  $f^m$ -invariant. The set  $A$  is contained in  $\cup_{n \in \mathbb{Z}} f^n(U)$ : since by hypothesis it holds  $\omega(\cup_{n \in \mathbb{Z}} f^n(U)) < +\infty$ , we have that  $\omega(A) < +\infty$ . Observe that  $f^m$  preserves the measure  $\omega|_A$ , defined as  $\omega|_A(\cdot) := \omega(\cdot \cap A)$ .

Apply then Birkhoff's Ergodic Theorem at  $(A, \omega|_A)$  with respect to  $f^m$  and  $\mathbb{I}_{W_z}(\cdot) \in L^1(A, \omega|_A)$ . Then it holds

$$\int_A \lim_{N \rightarrow +\infty} \frac{\square_N(x)}{N} d\omega(x) = \int_A \mathbb{I}_{W_z}(x) d\omega(x) = \omega(W_z \cap A). \quad (3.13)$$

Look now at  $\int_A \text{Torsion}(f, x) d\omega(x)$ .

On one hand, this integral is null by definition of the sets  $A_1$  and  $A$  and by the invariance of the torsion along the orbit of a point. On the other hand, we are going to show that for  $\omega$ -almost every  $x \in A$  it holds

$$\text{Torsion}(f, x) \leq -\frac{k}{2m} \lim_{N \rightarrow +\infty} \frac{\square_N(x)}{N}, \quad (3.14)$$

where  $k \in \mathbb{N}^*, m \in \mathbb{N}^*$  are the positive integers of (3.7).

By definition of  $A$ , for any  $x \in A$  there exists  $\tilde{x} \in A_1$  and  $n = n(x) \in \mathbb{Z}$  such that  $f^{nm}(\tilde{x}) = x$ .

Observe that for any  $N > |n|$  it holds

$$\left| \frac{\square_N(x)}{N} - \frac{\square_N(\tilde{x})}{N} \right| = \left| \frac{\sum_{i=n}^{N-1+n} \mathbb{I}_{W_z}(f^{im}(\tilde{x}))}{N} - \frac{\sum_{i=0}^{N-1} \mathbb{I}_{W_z}(f^{im}(\tilde{x}))}{N} \right| \leq \frac{2n}{N}$$

and so

$$\lim_{N \rightarrow +\infty} \frac{\square_N(x)}{N} = \lim_{N \rightarrow +\infty} \frac{\square_N(\tilde{x})}{N}. \quad (3.15)$$

By Poincaré Recurrence Theorem (see Theorem 4.1.19 in [KH95]) applied at  $A$  with respect to  $f$  and  $\omega|_A$ ,  $\omega|_A$ -almost every  $x \in A$  is recurrent. Moreover, by definition of  $A_1$  and  $A$ , the torsion exists at every  $x \in A$ .

In particular, because the torsion is invariant along the  $f$ -orbit of a point, we have

$$\text{Torsion}(f, x) = \text{Torsion}(f, \tilde{x}). \quad (3.16)$$

Let  $x \in A$  be a recurrent point. Note that the point  $\tilde{x}$  is recurrent. In particular, there exists  $\bar{N}(\tilde{x}) \in \mathbb{N}^*$  such that for any  $N \geq \bar{N}(\tilde{x})$  it holds  $\square_N(\tilde{x}) \geq 2$ .

Then, applying Lemma 3.2.1 at the point  $\tilde{x}$ , it holds for any  $N \geq \bar{N}(\tilde{x})$

$$\text{Torsion}_{mN}(f, \tilde{x}, \chi) < -\frac{\square_N(\tilde{x})k}{2mN} + \frac{k}{2mN}. \quad (3.17)$$

Consequently, from (3.15), (3.16) and (3.17)

$$\begin{aligned} \text{Torsion}(f, x) &= \text{Torsion}(f, \tilde{x}) = \lim_{N \rightarrow +\infty} \text{Torsion}_{mN}(f, \tilde{x}, \chi) \leq \\ &\leq \lim_{N \rightarrow +\infty} \left( -\frac{\square_N(\tilde{x})k}{2mN} + \frac{k}{2mN} \right) = -\frac{k}{2m} \lim_{N \rightarrow +\infty} \frac{\square_N(\tilde{x})}{N} = -\frac{k}{2m} \lim_{N \rightarrow +\infty} \frac{\square_N(x)}{N}. \end{aligned}$$

That is, for  $\omega|_A$ -almost every  $x \in A$  (i.e. for any  $x \in A$  at which the torsion exists and that is recurrent) inequality (3.14) is satisfied.

Look now back at  $\int_A \text{Torsion}(f, x) d\omega(x) = 0$ . Since (3.14) holds for  $\omega|_A$ -almost every  $x \in A$  and from (3.13), we have

$$0 = \int_A \text{Torsion}(f, x) d\omega(x) \leq -\frac{k}{2m} \int_A \lim_{N \rightarrow +\infty} \frac{\square_N(x)}{N} d\omega(x) = -\frac{k}{2m} \omega(W_z \cap A) \leq 0.$$

Consequently we have  $\omega(W_z \cap A) = 0$ . We then conclude that  $0 \leq \omega(A_1) = \omega(W_z \cap A_1) \leq \omega(W_z \cap A) = 0$ , that is  $\omega(A_1) = 0$  as desired.  $\square$

We have so exhibited a neighborhood  $W_z$  of  $z$  where  $\omega$ -almost every point has non zero torsion. Since  $f$  is a positive twist map, by Corollary 2.1.1, whenever it exists, the torsion is always non positive. We deduce so that  $\omega$ -almost every point in  $W_z$  has negative torsion.  $\square$

**Remark 3.2.4.** In Section 3.3 we will need the notion of over-conjugate points (see Definition 3.3.2). A point  $z \in \mathbb{A}$  has over-conjugate points if there exists  $m \in \mathbb{N}^*$  such that

$$\tilde{v}(I)(z, \chi, m) - \tilde{v}(I)(z, \chi, 0) \leq -\frac{1}{2}.$$

Observe that Proposition 3.2.4 holds true also if we assume as condition (i) that the point  $z$  has over-conjugate points.

We wonder if a similar result could hold from a topological point of view. The following question is due to J.P. Marco:

**Question 3.2.1.** Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a conservative positive twist map and let  $U$  be an invariant set of finite measure. If the set  $\{x \in U : \text{Torsion}(f, x) = 0\}$  contains a  $G_\delta$  dense set, then is it true that there are no conjugate points?



### 3.2.2 Proof of Proposition 3.2.2

In this Section we show that if there are no conjugate points in a BIES  $U$ , then  $f|_U$  is  $\mathcal{C}^0$ -integrable. The proof is an adaptation of the results in [CS96]. In [CS96], Cheng and Sun work with configurations on the whole unbounded annulus: here our framework is that of Proposition 3.2.1, that is a bounded domain.

We remark that the same result holds for conservative negative twist map, by changing  $f$  into  $f^{-1}$ .

The proof largely uses the theory of Aubry-Mather sets as sets of minimizing configurations for an appropriate action functional, as presented in [Ban88]. The hypothesis of not having conjugate points in  $U$  implies that any configuration in the bounded domain is in some Aubry-Mather set. This implies that the Aubry-Mather set of any given rotation number (in a suitable interval) is the graph of a continuous 1-periodic function. The projection over  $\mathbb{A}$  of such graphs is the desired partition into  $\mathcal{C}^0$  closed  $f$ -invariant essential curves.

The Subsection is organized as follows. First we introduce the framework of configurations on a bounded domain. Among such configurations, we focus on those minimizing the action functional  $H$  (see Definition 3.2.6). Then we show that minimizing configurations on the bounded domain are actually minimizers on the whole annulus. We need to adapt to the bounded framework the arguments of [CS96]. In particular, we prove that if there are no conjugate points in  $U$ , then any configuration in the bounded domain is also minimizing. We use then the properties of the rotation number of minimizing configurations as main tool to show that any Aubry-Mather set is a graph of a continuous 1-periodic function, concluding so our proof.

#### Framework and notations

Consistently with (3.1), let  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  denote a lift of the conservative positive twist map  $f$  and let  $h : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a generating function for  $F$ , that is

$$F(x, y) = (X, Y) \quad \Leftrightarrow \quad y = -\frac{\partial h(x, X)}{\partial x}, \quad Y = \frac{\partial h(x, X)}{\partial X}.$$

**Notation 3.2.4.** Let  $U \subset \mathbb{A}$  be a BIES (see Notation 3.2.3). Each component of the boundary of  $U$  is  $f$ -invariant and is the graph of a Lipschitz map. This is a theorem due to G. D. Birkhoff (see [Her83] for a complete proof).

Denote as  $\gamma_1, \gamma_2 : \mathbb{T} \rightarrow \mathbb{R}$  the continuous functions whose graphs are the lower and upper component of  $\partial U$  respectively. We write then

$$U = \{(x, y) \in \mathbb{A} : \gamma_1(x) < y < \gamma_2(x)\}. \quad (3.18)$$

Denote

$$\mathcal{U} = (p \times \text{Id}_{\mathbb{R}})^{-1}(U), \quad (3.19)$$

where  $p \times \text{Id}$  is the universal covering of  $\mathbb{A}$ .

Observe that  $\mathcal{U}$  is the intersection of

$$\mathcal{U}_1 = \{(x, y) \in \mathbb{R}^2 : \Gamma_1(x) < y\} \quad \text{and} \quad \mathcal{U}_2 = \{(x, y) \in \mathbb{R}^2 : y < \Gamma_2(x)\}, \quad (3.20)$$

where  $\Gamma_i = \gamma_i \circ p$  for  $i = 1, 2$ .

For  $(x, y) \in \mathcal{U}$ , the notation  $(x_n, y_n)_{n \in \mathbb{Z}} = (F^n(x, y))_{n \in \mathbb{Z}}$  refers to the orbit of  $(x, y)$  with

respect to  $F$ .

Let  $\mathcal{D} \subset \mathbb{R}^2$  be the set

$$\mathcal{D} = \{(x, X) \in \mathbb{R}^2 : p_1 \circ F(x, \Gamma_1(x)) < X < p_1 \circ F(x, \Gamma_2(x))\}, \quad (3.21)$$

which is the intersection of

$$\mathcal{D}_1 = \{(x, X) : p_1 \circ F(x, \Gamma_1(x)) < X\} \quad \text{and} \quad \mathcal{D}_2 = \{(x, X) : X < p_1 \circ F(x, \Gamma_2(x))\}.$$

**Remark 3.2.5.** A point  $(x, y) \in \mathbb{R}^2$  belongs to the subset  $\mathcal{U}$  if and only if the point  $(x, X) = (x, p_1 \circ F(x, y)) \in \mathbb{R}^2$  is in  $\mathcal{D}$ .

On one hand, let  $(x, y) \in \mathcal{U}$ , that is  $\Gamma_1(x) < y < \Gamma_2(x)$ . By the twist condition (see Definition 2.1) it holds

$$p_1 \circ F(x, \Gamma_1(x)) < p_1 \circ F(x, y) = X < p_1 \circ F(x, \Gamma_2(x)),$$

i.e.  $(x, X) \in \mathcal{D}$ .

On the other hand, let  $(x, X) \in \mathcal{D}$ , that is  $p_1 \circ F(x, \Gamma_1(x)) < X < p_1 \circ F(x, \Gamma_2(x))$ . Again by the twist condition, the inverse function of  $y \mapsto p_1 \circ F(x, y)$  remains an increasing homeomorphism. Therefore

$$\Gamma_1(x) < (p_1 \circ F(x, \cdot))^{-1}(X) = y < \Gamma_2(x),$$

i.e.  $(x, y) \in \mathcal{U}$ .

**Notation 3.2.5.** Denote as  $C(\mathcal{D})$  the set of configurations  $(x_n)_{n \in \mathbb{Z}}$  such that  $(x_n, x_{n+1}) \in \mathcal{D}$  for any  $n \in \mathbb{Z}$ .

**Remark 3.2.6.** If  $(x, y) \in \mathcal{U}$ , then, by the invariance of  $U$ , every point of its orbit  $(x_n, y_n) = F^n(x, y)$  is in  $\mathcal{U}$ . This observation and Remark 3.2.5 tell us that for any  $n \in \mathbb{Z}$  the point  $(x_n, x_{n+1})$  is in  $\mathcal{D}$ . Therefore the configuration  $(x_n)_n = (p_1 \circ F^n(x, y))_n$  associated to the point  $(x, y)$  is in  $C(\mathcal{D})$ .

## Minimizing configurations

**Definition 3.2.6.** For any  $M, N \in \mathbb{Z}, M \leq N$ , define the action functional  $H_{M,N}$  as

$$H_{M,N} : \mathbb{R}^{N-M+3} \longrightarrow \mathbb{R}$$

$$(x_{M-1}, x_M, \dots, x_N, x_{N+1}) \mapsto H_{M,N}(x_{M-1}, \dots, x_{N+1}) := \sum_{i=M-1}^{i=N} h(x_i, x_{i+1}),$$

where  $h : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a generating function for  $F$ .

**Definition 3.2.7.** Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a conservative positive twist map and let  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a lift of  $f$ . Let  $V \subset \mathbb{R}^2$ . The minimizing set  $\mathcal{M}(V)$  is the set of sequences  $(x_n)_{n \in \mathbb{Z}}$  such that  $(x_n, x_{n+1}) \in V$  for any  $n \in \mathbb{Z}$  and such that for any  $M, N \in \mathbb{Z}$  the segment  $(x_{M-1}, \dots, x_{N+1})$  minimizes the action functional  $H_{M,N}$  among all the segments  $(\tilde{x}_{M-1}, \dots, \tilde{x}_{N+1}) \in \mathbb{R}^{N-M+3}$  such that  $\tilde{x}_{M-1} = x_{M-1}, \tilde{x}_{N+1} = x_{N+1}$  and  $(\tilde{x}_n, \tilde{x}_{n+1}) \in V$  for any  $n \in \llbracket M-1, N+1 \rrbracket$ .

Denote  $\mathcal{M}(\mathbb{R}^2)$  as  $\mathcal{M}$  and call it the *minimizing set of  $f$* .

**Remark 3.2.7.** Let  $U \subset \mathbb{A}$  be a BIES. We refer to the notation introduced in **Framework and notations**. Let  $(x_n)_{n \in \mathbb{Z}}$  be a sequence in  $\mathcal{M}(\mathcal{D})$ . By definition,  $(x_n, x_{n+1}) \in \mathcal{D}$  for any  $n \in \mathbb{Z}$  and for any  $M, N \in \mathbb{Z}, M \leq N$ , the segment  $(x_{M-1}, \dots, x_{N+1})$  is a local minimum of the action functional  $H_{M,N}$  among segments  $(\tilde{x}_{M-1}, \dots, \tilde{x}_{N+1}) \in \mathbb{R}^{N-M+3}$  so that  $\tilde{x}_{M-1} = x_{M-1}, \tilde{x}_{N+1} = x_{N+1}$  and  $(\tilde{x}_n, \tilde{x}_{n+1}) \in \mathcal{D}$  for any  $n \in \llbracket M-1, N \rrbracket$ . Therefore, for any  $i \in \mathbb{Z}$  it holds

$$\frac{\partial h(x_{i-1}, x_i)}{\partial X} + \frac{\partial h(x_i, x_{i+1})}{\partial x} = 0.$$

The sequence  $(x_n)_{n \in \mathbb{Z}}$  is actually a configuration for  $F$  (see (3.2)). That is,  $(x_n)_{n \in \mathbb{Z}} \in C(\mathcal{D})$ .

The rest of the paragraph is devoted to the proof that minimizing the action functional among configurations corresponding to orbits of points in  $U$  is equivalent to minimizing the action functional among configurations corresponding to orbits of points in  $\mathbb{A}$ .

The next proposition is the core of the argument: the aim of the subsection will descend as a corollary of it. Proposition 3.2.5 compares configurations of two points, the first one lying under (or above) a  $f$ -invariant essential curve, the other one lying on such a curve, and having the same first coordinate projection. We will show that for any  $n > 0$  the images through  $F^n$  of the two points cannot have same first coordinate anymore.

**Proposition 3.2.5.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a conservative positive twist map and let  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a lift of  $f$ . Let  $\gamma : \mathbb{T} \rightarrow \mathbb{R}$  be a continuous function whose graph is  $f$ -invariant and let  $\Gamma = \gamma \circ p$ . For any  $x \in \mathbb{R}$  if  $y < \Gamma(x)$  (respectively  $y > \Gamma(x)$ ) then*

$$p_1 \circ F^n(x, y) < p_1 \circ F^n(x, \Gamma(x)) \quad (\text{respectively } p_1 \circ F^n(x, y) > p_1 \circ F^n(x, \Gamma(x)))$$

for any  $n \in \mathbb{N}^*$ .

*Proof.* The region  $U$  lying below the graph of  $\gamma$  is  $f$ -invariant. Denote  $\mathcal{U} = (p \times \text{Id}_{\mathbb{R}})^{-1}(U)$  and consider

$$\mathcal{G} := \{(x, X) \in \mathbb{R}^2 : X = p_1 \circ F(x, \Gamma(x))\},$$

$$\mathcal{D} := \{(x, X) \in \mathbb{R}^2 : X < p_1 \circ F(x, \Gamma(x))\}.$$

Denote as  $(x_n, y_n)_{n \in \mathbb{Z}}, (\xi_n, \Gamma(\xi_n))_{n \in \mathbb{Z}}$  the  $F$ -orbits of  $(x, y)$  and  $(x, \Gamma(x))$  respectively, while

$$(x_n)_{n \in \mathbb{Z}} = (p_1 \circ F^n(x, y))_{n \in \mathbb{Z}} \quad \text{and} \quad (\xi_n)_{n \in \mathbb{Z}} = (p_1 \circ F^n(x, \Gamma(x)))_{n \in \mathbb{Z}}$$

are their associated configurations. By the invariance of  $U$  and of the graph of  $\gamma$  and by adapting Remark 3.2.6, it holds that  $(x_n)_{n \in \mathbb{Z}} \in C(\mathcal{D})$  and  $(\xi_n)_{n \in \mathbb{Z}} \in C(\mathcal{G})$ . Observe that  $x_0 = \xi_0 = x$ . We are going to prove by induction that  $x_n < \xi_n$  for any  $n > 0$ , concluding so the proof (the case  $y > \Gamma(x)$  can be treated similarly).

(i) Since  $x_0 = \xi_0 = x$  and  $y < \Gamma(x)$ , by the twist condition we have

$$x_1 = p_1 \circ F(x, y) < p_1 \circ F(x, \Gamma(x)) = \xi_1.$$

(ii) Fix now  $n > 0$  and assume that  $x_n < \xi_n$ . The function  $x \mapsto p_1 \circ F(x, \Gamma(x))$  is the lift of an orientation preserving homeomorphism of the circle and so it is an increasing homeomorphism of  $\mathbb{R}$ . Consequently, by the inductive hypothesis,

$$p_1 \circ F(x_n, \Gamma(x_n)) < p_1 \circ F(\xi_n, \Gamma(\xi_n)) = \xi_{n+1}.$$

By the invariance of  $\mathcal{U}$ , we have that  $(x_n, y_n) \in \mathcal{U}$ , that is  $y_n < \Gamma(x_n)$ . Consequently, by the twist condition, it holds

$$x_{n+1} = p_1 \circ F(x_n, y_n) < p_1 \circ F(x_n, \Gamma(x_n)) < p_1 \circ F(\xi_n, \Gamma(\xi_n)) = \xi_{n+1}.$$

□

In particular, we deduce that if there exists a continuous function on  $\mathbb{T}$  whose graph is  $f$ -invariant and, consequently, which bounds an upper and a lower unbounded annuli  $U_+, U_-$ , then there cannot exist two segments of orbits  $(x_n, y_n)_{n \in \llbracket 0, N \rrbracket}$  in  $U_+$  and  $(\tilde{x}_n, \tilde{y}_n)_{n \in \llbracket 0, N \rrbracket}$  in  $U_-$  such that  $x_0 = \tilde{x}_0, x_N = \tilde{x}_N, N \geq 1$ .

**Reminder 3.2.1.** We recall here some previous notations. Let  $U \subset \mathbb{A}$  be a BIES. Let  $\gamma_1, \gamma_2 : \mathbb{T} \rightarrow \mathbb{A}$  be continuous functions such that

$$U = \{(x, y) \in \mathbb{A} : \gamma_1(x) < y < \gamma_2(x)\}.$$

Let  $\Gamma_i = \gamma_i \circ p$  for  $i = 1, 2$  and let  $\mathcal{U} = \{(x, y) \in \mathbb{R}^2 : \Gamma_1(x) < y < \Gamma_2(x)\}$ . Denote

$$\mathcal{D} = \{(x, X) \in \mathbb{R}^2 : p_1 \circ F(x, \Gamma_1(x)) < X < p_1 \circ F(x, \Gamma_2(x))\}.$$

The minimizing set  $\mathcal{M}$  is the set of sequences  $(x_n)_{n \in \mathbb{Z}}$  such that for any  $M, N \in \mathbb{Z}$  the segment  $(x_{M-1}, \dots, x_{N+1})$  minimizes the action functional  $H_{M,N}$  among all the segments  $(\tilde{x}_{M-1}, \dots, \tilde{x}_{N+1}) \in \mathbb{R}^{N-M+3}$  such that  $\tilde{x}_{M-1} = x_{M-1}$  and  $\tilde{x}_{N+1} = x_{N+1}$ .

The minimizing set  $\mathcal{M}(\mathcal{D})$  is the set of sequences  $(x_n)_{n \in \mathbb{Z}}$  such that  $(x_n, x_{n+1}) \in \mathcal{D}$  for any  $n \in \mathbb{Z}$  and such that for any  $M, N \in \mathbb{Z}$  the segment  $(x_{M-1}, \dots, x_{N+1})$  minimizes the action functional  $H_{M,N}$  among all the segments  $(\tilde{x}_{M-1}, \dots, \tilde{x}_{N+1}) \in \mathbb{R}^{N-M+3}$  such that  $\tilde{x}_{M-1} = x_{M-1}, \tilde{x}_{N+1} = x_{N+1}$  and  $(\tilde{x}_n, \tilde{x}_{n+1}) \in \mathcal{D}$  for any  $n \in \llbracket M-1, N+1 \rrbracket$ .

Denote as  $C(\mathcal{D})$  the set of configurations  $(x_n)_{n \in \mathbb{Z}}$  such that  $(x_n, x_{n+1}) \in \mathcal{D}$  for any  $n \in \mathbb{Z}$ .

The following result descends so as a corollary.

**Corollary 3.2.2.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a conservative positive twist map and let  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a lift of  $f$ . Let  $U \subset \mathbb{A}$  be a BIES. Then  $\mathcal{M}(\mathcal{D}) \subset \mathcal{M}$ .*

*Proof.* Denote as  $\gamma_1, \gamma_2 : \mathbb{T} \rightarrow \mathbb{R}$  the continuous functions whose graphs are the boundary components of  $U$ . Referring to (3.19), (3.20) and (3.21), we denote

$$\mathcal{U} = \{(x, y) \in \mathbb{R}^2 : \Gamma_1(x) < y < \Gamma_2(x)\}$$

and

$$\mathcal{D} = \{(x, X) \in \mathbb{R}^2 : p_1 \circ F(x, \Gamma_1(x)) < X < p_1 \circ F(x, \Gamma_2(x))\},$$

where  $\Gamma_i = \gamma_i \circ p$  for  $i = 1, 2$ .

We want to show that  $\mathcal{M}(\mathcal{D}) \subset \mathcal{M}$ . Argue by contradiction and suppose there exists  $(x_n)_{n \in \mathbb{Z}} \in \mathcal{M}(\mathcal{D})$  not in  $\mathcal{M}$ . Remark that  $(x_n)_{n \in \mathbb{Z}}$  is a configuration in  $C(\mathcal{D})$ . Since  $(x_n)_{n \in \mathbb{Z}}$  does not belong to  $\mathcal{M}$ , there exist  $M, N \in \mathbb{Z}, M \leq N$  and a segment  $(\tilde{x}_{M-1}, \dots, \tilde{x}_{N+1}) \in \mathbb{R}^{N-M+3}$  such that  $\tilde{x}_{M-1} = x_{M-1}, \tilde{x}_{N+1} = x_{N+1}, (\tilde{x}_n, \tilde{x}_{n+1})$  does not belong to  $\mathcal{D}$  for at least an integer  $n \in \llbracket M-1, N \rrbracket$  and

$$H_{M,N}(\tilde{x}_{M-1}, \tilde{x}_M, \dots, \tilde{x}_N, \tilde{x}_{N+1}) < H_{M,N}(x_{M-1}, x_M, \dots, x_N, x_{N+1}).$$

Choose  $(\tilde{x}_{M-1}, \dots, \tilde{x}_{N+1})$  that minimizes  $H_{M,N}$ . It satisfies (3.2). Therefore, we extend it to a configuration  $(\tilde{x}_n)_{n \in \mathbb{Z}}$  in  $C(\mathbb{R}^2)$  (in particular, it is in  $\mathcal{M}$ ). Observe that by the

invariance of  $\mathcal{D}$  for every  $n \in \mathbb{Z}$  it holds  $(\tilde{x}_n, \tilde{x}_{n+1}) \notin \mathcal{D}$ .

Let  $(x_n, y_n)_{n \in \mathbb{Z}}$  and  $(\tilde{x}_n, \tilde{y}_n)_{n \in \mathbb{Z}}$  be the orbits corresponding, respectively, to  $(x_n)_{n \in \mathbb{Z}}$  and  $(\tilde{x}_n)_{n \in \mathbb{Z}}$ . Without loss of generality, assume that

$$y_{M-1} < \Gamma_2(x_{M-1}) = \Gamma_2(\tilde{x}_{M-1}) \leq \tilde{y}_{M-1}.$$

Applying Proposition 3.2.5 we have that  $x_{N+1} < \tilde{x}_{N+1}$ , which is the desired contradiction.  $\square$

**Remark 3.2.8.** The minimizing set of  $f$  is non empty (see Theorem 3.17 in [Ban88]). Moreover, Mather proved that any configuration obtained from a point  $(x, y) \in \mathbb{A}$  lying on a  $f$ -invariant essential curve is in the minimizing set of  $f$  (see Proposition 2.8 in [Mat91]). We will show that, for any BIES  $U$ , its correspondent minimizing set  $\mathcal{M}(\mathcal{D})$  is non empty (see Remark 3.2.11).

**Proof of  $C(\mathcal{D}) = \mathcal{M}(\mathcal{D})$**

In this paragraph we show that if there are no conjugate points for  $f$  in  $U$ , then any configuration is minimizing. We are going to adapt to the bounded framework the arguments in [CS96].

In particular, the absence of conjugate points will imply that two Jacobi fields along a configuration cross at most once (see Definition 3.2.3). From this we deduce that any two configurations in  $C(\mathcal{D})$  can cross at most once. This will imply that all the configurations are also minimizers (see Definition 3.2.7).

All along the paragraph we refer to Notation 3.2.4 (see also Reminder 3.2.1).

**Notation 3.2.6.** Fix a configuration  $\mathbf{x} = (x_n)_{n \in \mathbb{Z}} \in C(\mathcal{D})$  and integers  $M \leq N$ . Consider then the action functional  $H_{M,N}$  (see Definition 3.2.6) on the set of  $(N - M + 3)$ -uples  $(\tilde{x}_{M-1}, \tilde{x}_M, \dots, \tilde{x}_N, \tilde{x}_{N+1})$  such that  $\tilde{x}_{M-1} = x_{M-1}, \tilde{x}_{N+1} = x_{N+1}$ .

We then consider

$$\begin{aligned} \mathbb{H}_{M,N}^{\mathbf{x}} : \mathbb{R}^{N-M+1} &\rightarrow \mathbb{R} \\ (\tilde{x}_M, \dots, \tilde{x}_N) &\mapsto \mathbb{H}_{M,N}^{\mathbf{x}}(\tilde{x}_M, \dots, \tilde{x}_N) := H_{M,N}(x_{M-1}, \tilde{x}_M, \dots, \tilde{x}_N, x_{N+1}) = \\ &= h(x_{M-1}, \tilde{x}_M) + \sum_{i=M}^{N-1} h(\tilde{x}_i, \tilde{x}_{i+1}) + h(\tilde{x}_N, x_{N+1}). \end{aligned} \quad (3.22)$$

The  $(N - M)$ -uple  $(x_M, \dots, x_N)$  is a critical point of  $\mathbb{H}_{M,N}^{\mathbf{x}}$ . Denote as  $\mathcal{H}_{M,N}^{\mathbf{x}}(\tilde{x}_M, \dots, \tilde{x}_N)$  the Hessian matrix of  $\mathbb{H}_{M,N}^{\mathbf{x}}$  evaluated at the  $(N - M)$ -uple  $(\tilde{x}_M, \dots, \tilde{x}_N)$ . When it will be clear by the context, we will omit the superscript  $\mathbf{x}$  on  $\mathbb{H}_{M,N}^{\mathbf{x}}, \mathcal{H}_{M,N}^{\mathbf{x}}$ .

**Lemma 3.2.4.** *Assume that  $f$  has no conjugate points in  $U$ . Then for any configuration  $\mathbf{x} = (x_n)_{n \in \mathbb{Z}} \in C(\mathcal{D})$  and for any integers  $M \leq N$  the hessian  $\mathcal{H}_{M,N}^{\mathbf{x}}$  is positive definite.*

*Proof.* Since  $f$  does not have conjugate points in  $U$ , we deduce, for any configuration  $\tilde{\mathbf{x}} = (\tilde{x}_n)_{n \in \mathbb{Z}}$  such that  $\tilde{x}_{M-1} = x_{M-1}$ , that  $\mathcal{H}_{M,N}^{\tilde{\mathbf{x}}}(\tilde{x}_M, \dots, \tilde{x}_N)$  does not have zero eigenvalue. Indeed, if  $(\xi_M, \dots, \xi_N)$  is an eigenvector of the zero eigenvalue for  $\mathcal{H}_{M,N}^{\tilde{\mathbf{x}}}(\tilde{x}_M, \dots, \tilde{x}_N)$ , then  $(0, \xi_M, \dots, \xi_N, 0)$  is a segment of a Jacobi field along  $\tilde{\mathbf{x}} = (\tilde{x}_n)_{n \in \mathbb{Z}}$  (see (3.3)). Since  $\tilde{\mathbf{x}} = (\tilde{x}_n)_{n \in \mathbb{Z}}$  is a configuration, by hypothesis it does not have conjugate points. So we deduce that  $(\xi_M, \dots, \xi_N)$  is the zero vector and we contradict the fact that  $(\xi_M, \dots, \xi_N)$

is an eigenvector.

As we will see in Remark 3.2.8 there always exist minimizing configurations in  $C(\mathcal{D})$ . By Corollary 3.2.2, they are also minimizing configurations in  $\mathcal{M}$ . By Lemma 5.3 in [Arn16], the Hessian  $\mathcal{H}_{M,N}^{\tilde{x}}$  evaluated at a minimizing configuration<sup>2</sup>  $\tilde{x} = (\tilde{x}_n)_{n \in \mathbb{Z}}$  such that  $\tilde{x}_{M-1} = x_{M-1}$  is positive definite. By the continuity of the eigenvalues and by the connectedness of  $\mathcal{D}$  (actually of the set of configurations  $\tilde{x} = (\tilde{x}_n)_{n \in \mathbb{Z}}$  such that  $\tilde{x}_{M-1} = x_{M-1}$ ), we deduce that  $\mathcal{H}_{M,N}^x(x_M, \dots, x_N)$  is positive definite too.  $\square$

In Definition 3.2.3 we have introduced the Aubry diagram  $\mathcal{L}((t_n)_{n \in \mathbb{Z}})$  for a sequence of real numbers  $(t_n)_{n \in \mathbb{Z}}$  and we have explained what it means that two sequences  $(t_n)_{n \in \mathbb{Z}}, (s_n)_{n \in \mathbb{Z}}$  cross. That is,  $(t_n)_{n \in \mathbb{Z}}$  and  $(s_n)_{n \in \mathbb{Z}}$  cross

(i) between  $n$  and  $n+1$  if  $(t_n - s_n)(t_{n+1} - s_{n+1}) < 0$ ;

(ii) at  $n \in \mathbb{Z}$  if  $t_n = s_n$  and  $(t_{n-1} - s_{n-1})(t_{n+1} - s_{n+1}) < 0$ .

**Lemma 3.2.5.** *Let  $f$  be with no conjugate points in  $U$ . Let  $(\xi_n)_{n \in \mathbb{Z}}$  and  $(\eta_n)_{n \in \mathbb{Z}}$  be two different Jacobi fields along the same configuration  $(x_n)_{n \in \mathbb{Z}} \in C(\mathcal{D})$ . Then their Aubry diagrams  $\mathcal{L}((\xi_n)_{n \in \mathbb{Z}})$  and  $\mathcal{L}((\eta_n)_{n \in \mathbb{Z}})$  cross at most once.*

*Proof.* The set of Jacobi vector fields along  $(x_n)_{n \in \mathbb{Z}}$  is a vector space. Denote as  $(\zeta_n)_{n \in \mathbb{Z}}$  the Jacobi vector field along  $(x_n)_{n \in \mathbb{Z}}$  obtained as the difference of  $(\xi_n)_{n \in \mathbb{Z}}$  and  $(\eta_n)_{n \in \mathbb{Z}}$ . That is, for any  $n \in \mathbb{Z}$  we have  $\zeta_n = \xi_n - \eta_n$ .

We are going to show that, given  $(\zeta_{M-1}, \zeta_M, \dots, \zeta_N, \zeta_{N+1})$  a segment of the Jacobi field  $(\zeta_n)_{n \in \mathbb{Z}}$  along  $(x_n)_{n \in \mathbb{Z}}$  such that

$$\zeta_{M-1} > 0 \quad \text{and} \quad \zeta_{N+1} \geq 0, \quad (3.23)$$

then  $\zeta_i > 0$  for  $i = M, \dots, N$ . This will imply our lemma: indeed, if by contradiction the Jacobi vector fields  $(\xi_n)_{n \in \mathbb{Z}}$  and  $(\eta_n)_{n \in \mathbb{Z}}$  intersect twice, then we have that (up to exchange the roles of the fields) there exists  $M < N$  such that  $\eta_{M-1} < \xi_{M-1}, \eta_{N+1} \leq \xi_{N+1}$  and  $\eta_i \geq \xi_i$  for some  $i \in \llbracket M, N \rrbracket$ . This would contradict our result.

Let us show that, assuming (3.23), it holds  $\zeta_i > 0$  for any  $i \in \llbracket M, N \rrbracket$ .

Denote as  $h_{i,i+1}$  the generating function at the point  $(x_i, x_{i+1})$ , i.e.  $h_{i,i+1} = h(x_i, x_{i+1})$ .

Use the definition of Jacobi field (see (3.3)) and write

$$\mathcal{H}_{M,N}^x(x_M, \dots, x_N) \begin{pmatrix} \zeta_M \\ \vdots \\ \zeta_N \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 h_{M,M+1}}{\partial x^2} + \frac{\partial^2 h_{M-1,M}}{\partial X^2} & \frac{\partial^2 h_{M,M+1}}{\partial x \partial X} & 0 & \dots & 0 \\ \frac{\partial^2 h_{M,M+1}}{\partial x \partial X} & \frac{\partial^2 h_{M+1,M+2}}{\partial x^2} + \frac{\partial^2 h_{M,M+1}}{\partial X^2} & \frac{\partial^2 h_{M+1,M+2}}{\partial x \partial X} & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \frac{\partial^2 h_{N-1,N}}{\partial x \partial X} & \frac{\partial^2 h_{N,N+1}}{\partial x^2} + \frac{\partial^2 h_{N-1,N}}{\partial X^2} \end{pmatrix} \begin{pmatrix} \zeta_M \\ \vdots \\ \zeta_N \end{pmatrix}$$

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2. Such minimizing configuration is in  $C(\mathcal{D})$ .

$$= \begin{pmatrix} -\frac{\partial^2}{\partial x \partial X} h(x_{M-1}, x_M) \zeta_{M-1} \\ 0 \\ \vdots \\ 0 \\ -\frac{\partial^2}{\partial x \partial X} h(x_N, x_{N+1}) \zeta_{N+1} \end{pmatrix}.$$

By (3.23), the first component of this vector is positive and the last one is non negative. Concerning the matrix  $\mathcal{H}_{M,N}^x$ , it is a tridiagonal symmetric matrix that is positive definite by Lemma 3.2.4: so the diagonal terms are strictly positive. The off-diagonal entries are strictly negative (since  $\partial^2 h / \partial x \partial X < 0$ ). Its inverse  $(\mathcal{H}_{M,N}^x(x_M, \dots, x_N))^{-1}$  is a symmetric, positive definite matrix whose entries are all positive (see Section 2.1 in [Meu92] or Section 3.4.1 in Appendix 3.4).

Since

$$(\mathcal{H}_{M,N}^x(x_M, \dots, x_N))^{-1} \begin{pmatrix} -\frac{\partial^2}{\partial x \partial X} h(x_{M-1}, x_M) \zeta_{M-1} \\ 0 \\ \vdots \\ 0 \\ -\frac{\partial^2}{\partial x \partial X} h(x_N, x_{N+1}) \zeta_{N+1} \end{pmatrix} = \begin{pmatrix} \zeta_M \\ \vdots \\ \zeta_N \end{pmatrix},$$

since all the entries involved are positive, we conclude that  $\zeta_i > 0$  for  $i = M, \dots, N$ .  $\square$

**Lemma 3.2.6.** *Assume that  $f$  has no conjugate points in  $U$ . Let  $(x_n)_{n \in \mathbb{Z}}, (\tilde{x}_n)_{n \in \mathbb{Z}}$  be two configurations in  $C(\mathcal{D})$ . Then, their Aubry diagrams  $\mathcal{L}((x_n)_{n \in \mathbb{Z}})$  and  $\mathcal{L}((\tilde{x}_n)_{n \in \mathbb{Z}})$  cross at most once.*

*Proof.* Argue by contradiction and suppose that  $\mathcal{L}((x_n)_{n \in \mathbb{Z}})$  and  $\mathcal{L}((\tilde{x}_n)_{n \in \mathbb{Z}})$  cross twice. Without loss of generality, we can assume that there exists  $N_0 \in \mathbb{Z}, N_0 > 1$  such that

$$x_0 < \tilde{x}_0 \quad \tilde{x}_1 \leq x_1 \quad x_{N_0} \leq \tilde{x}_{N_0}. \quad (3.24)$$

Let us define the affine functions

$$\begin{aligned} f_0 : [0, 1] &\rightarrow \mathbb{R} \\ t &\mapsto f_0(t) := x_0 + t(\tilde{x}_0 - x_0) \end{aligned}$$

and

$$\begin{aligned} f_1 : [0, 1] &\rightarrow \mathbb{R} \\ t &\mapsto f_1(t) := x_1 + t(\tilde{x}_1 - x_1). \end{aligned}$$

The following claim proves that the segment connecting  $(x_0, x_1)$  to  $(\tilde{x}_0, \tilde{x}_1)$  in  $\mathbb{R}^2$  is fully contained in  $\mathcal{D}$ . That is, any point on such a segment corresponds to a configuration in  $C(\mathcal{D})$ .

**Claim 3.2.1.** For all  $t \in [0, 1]$  we have

$$(f_0(t), f_1(t)) \in \mathcal{D}.$$

We postpone the proof of Claim 3.2.1 and we finish now proving Lemma 3.2.6. We can associate to any  $(f_0(t), f_1(t)) \in \mathcal{D}$  the corresponding point

$$y(t) := (p_1 \circ F(f_0(t), \cdot))^{-1}(f_1(t))$$

and, by Remark 3.2.5, the point  $(f_0(t), y(t))$  belongs to  $\mathcal{U}$ . By the invariance of  $\mathcal{U}$ , every point of the orbit for  $F$  of  $(f_0(t), y(t))$  is in  $\mathcal{U}$  and, by Remark 3.2.6, the configuration corresponding to the orbit is in  $C(\mathcal{D})$ . For any  $t \in [0, 1]$ , denote the configuration associated to  $(f_0(t), y(t))$  as  $(f_n(t))_{n \in \mathbb{Z}} \in C(\mathcal{D})$ .

In the following we exhibit a Jacobi field along a configuration which contradicts Lemma 3.2.5 and this will end our proof.

For any  $t \in [0, 1]$ , being  $(f_n(t))_{n \in \mathbb{Z}}$  a configuration in  $C(\mathcal{D})$ , we have that, for any  $n \in \mathbb{Z}$ ,

$$\frac{\partial}{\partial X} h(f_{n-1}(t), f_n(t)) + \frac{\partial}{\partial x} h(f_n(t), f_{n+1}(t)) = 0.$$

Differentiate now with respect to  $t$  and obtain

$$\begin{aligned} \frac{\partial^2}{\partial X \partial x} h(f_{n-1}(t), f_n(t)) f'_{n-1}(t) + \left[ \frac{\partial^2}{\partial X^2} h(f_{n-1}(t), f_n(t)) + \frac{\partial^2}{\partial x^2} h(f_n(t), f_{n+1}(t)) \right] f'_n(t) + \\ + \frac{\partial^2}{\partial x \partial X} h(f_n(t), f_{n+1}(t)) f'_{n+1}(t) = 0. \end{aligned}$$

For any  $t \in [0, 1]$ , the sequence  $(f'_n(t))_{n \in \mathbb{Z}}$  is then a Jacobi field along the configuration  $(f_n(t))_{n \in \mathbb{Z}}$ .

Observe in particular (by (3.24)) that

$$f'_0(t) = \tilde{x}_0 - x_0 > 0 \quad \text{and} \quad f'_1(t) = \tilde{x}_1 - x_1 \leq 0.$$

Since, again by (3.24),

$$x_{N_0} = f_{N_0}(0) \leq f_{N_0}(1) = \tilde{x}_{N_0}$$

there exists  $\bar{t} \in [0, 1]$  such that  $f'_{N_0}(\bar{t}) \geq 0$ .

Look then at the Jacobi field  $(f'_n(\bar{t}))_{n \in \mathbb{Z}}$  along the configuration  $(f_n(\bar{t}))_{n \in \mathbb{Z}} \in C(\mathcal{D})$ : it holds  $f'_0(\bar{t}) > 0$ ,  $f'_{N_0}(\bar{t}) \geq 0$  and  $f'_1(\bar{t}) \leq 0$ . Let us discuss the two possible cases.

- If  $f'_1(\bar{t}) < 0$ , the Jacobi field changes sign between 0 and 1 and then, since  $f'_{N_0}(\bar{t}) \geq 0$ , there is another change of sign somewhere between 1 and  $N_0 + 1$ .
- If  $f'_1(\bar{t}) = 0$ , then by the definition of the Jacobi field we have

$$\begin{aligned} \frac{\partial^2}{\partial x \partial X} h(f_0(\bar{t}), f_1(\bar{t})) f'_0(\bar{t}) + \left( \frac{\partial^2}{\partial x^2} h(f_1(\bar{t}), f_2(\bar{t})) + \frac{\partial^2}{\partial X^2} h(f_0(\bar{t}), f_1(\bar{t})) \right) f'_1(\bar{t}) + \\ + \frac{\partial^2}{\partial x \partial X} h(f_1(\bar{t}), f_2(\bar{t})) f'_2(\bar{t}) = \\ = \frac{\partial^2}{\partial x \partial X} h(f_0(\bar{t}), f_1(\bar{t})) f'_0(\bar{t}) + \frac{\partial^2}{\partial x \partial X} h(f_1(\bar{t}), f_2(\bar{t})) f'_2(\bar{t}) = 0. \end{aligned}$$

Since  $\frac{\partial^2}{\partial x \partial X} h < 0$ , it holds  $f'_2(\bar{t}) < 0$ . We are so changing sign at 1. Since  $f'_{N_0}(\bar{t}) \geq 0$ , we have another change of sign between 2 and  $N_0 + 1$  and we contradict again Lemma 3.2.5.



□

*Proof of Claim 3.2.1.* The set  $\mathcal{D}$  is

$$\{(x, X) \in \mathbb{R}^2 : p_1 \circ F(x, \Gamma_1(x)) < X < p_1 \circ F(x, \Gamma_2(x))\}.$$

Clearly,  $(f_0(0), f_1(0)) = (x_0, x_1)$  and  $(f_0(1), f_1(1)) = (\tilde{x}_0, \tilde{x}_1)$  belong to  $\mathcal{D}$ . That is

$$p_1 \circ F(x_0, \Gamma_1(x_0)) < x_1 < p_1 \circ F(x_0, \Gamma_2(x_0)), \quad (3.25)$$

$$p_1 \circ F(\tilde{x}_0, \Gamma_1(\tilde{x}_0)) < \tilde{x}_1 < p_1 \circ F(\tilde{x}_0, \Gamma_2(\tilde{x}_0)). \quad (3.26)$$

Observe that

$$\mathbb{R} \ni x \mapsto p_1 \circ F(x, \Gamma_1(x)) \in \mathbb{R} \quad \text{and} \quad \mathbb{R} \ni x \mapsto p_1 \circ F(x, \Gamma_2(x)) \in \mathbb{R} \quad (3.27)$$

are lifts of orientation preserving homeomorphisms of  $\mathbb{T}$ . Consequently, they are strictly increasing and commute with the translation of the quantity 1.

By definition of  $f_0$  and  $f_1$ , for any  $t \in (0, 1)$

$$x_0 < f_0(t) < \tilde{x}_0 \quad \text{and} \quad \tilde{x}_1 \leq f_1(t) \leq x_1. \quad (3.28)$$

By the strict monotonicity of the functions (3.27), thanks to (3.28) and (3.26) for any  $t \in (0, 1)$  we have

$$p_1 \circ F(f_0(t), \Gamma_1(f_0(t))) < p_1 \circ F(\tilde{x}_0, \Gamma_1(\tilde{x}_0)) < \tilde{x}_1 \leq f_1(t).$$

In particular

$$p_1 \circ F(f_0(t), \Gamma_1(f_0(t))) < f_1(t). \quad (3.29)$$

In a similar way, by (3.28) and (3.25), for any  $t \in (0, 1)$

$$f_1(t) \leq x_1 < p_1 \circ F(x_0, \Gamma_2(x_0)) < p_1 \circ F(f_0(t), \Gamma_2(f_0(t))).$$

In particular

$$f_1(t) < p_1 \circ F(f_0(t), \Gamma_2(f_0(t))). \quad (3.30)$$

Hence, by (3.29) and (3.30), for any  $t \in [0, 1]$  the point  $(f_0(t), f_1(t))_{t \in [0, 1]}$  belongs to the set  $\mathcal{D}$ .

□

The main result of this paragraph follows immediately from Lemma 3.2.6.

**Proposition 3.2.6.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a conservative positive twist map and let  $U$  be a BIES. If  $f$  has no conjugate points in  $U$  then all the configurations in  $C(\mathcal{D})$  are minimizing, i.e.  $C(\mathcal{D}) = \mathcal{M}(\mathcal{D}) \subset \mathcal{M}$ .*

*Proof.* Let  $\mathbf{x} = (x_n)_{n \in \mathbb{Z}} \in C(\mathcal{D})$ . Argue by contradiction and suppose that  $(x_n)_{n \in \mathbb{Z}} \notin \mathcal{M}(\mathcal{D})$ . That is, there exist  $M, N \in \mathbb{Z}, M \leq N$  and a segment  $(\tilde{x}_M, \dots, \tilde{x}_N)$  of another configuration  $(\tilde{x}_n)_{n \in \mathbb{Z}} \in C(\mathcal{D})$  (see Remark 3.2.7) with  $\tilde{x}_{M-1} = x_{M-1}, \tilde{x}_{N+1} = x_{N+1}$  such that (see (3.22))

$$\mathbb{H}_{M,N}^{\mathbf{x}}(\tilde{x}_M, \dots, \tilde{x}_N) < \mathbb{H}_{M,N}^{\mathbf{x}}(x_M, \dots, x_N).$$

The Aubry diagrams  $\mathcal{L}((x_n)_{n \in \mathbb{Z}})$  and  $\mathcal{L}((\tilde{x}_n)_{n \in \mathbb{Z}})$  cross twice at  $M - 1$  and  $N + 1$ , contradicting then Lemma 3.2.6. □

## Rotation number of minimizing configurations

This “independent” paragraph concerns properties of rotation number of minimizing configurations. They will be used in the next paragraph to conclude the proof of the  $\mathcal{C}^0$ -integrability. The main reference of the subsection is [Ban88].

**Notation 3.2.7.** The function  $f : \mathbb{A} \rightarrow \mathbb{A}$  is a conservative positive twist map and we choose a lift  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  of  $f$ .

We start by recalling the following result stated by Bangert (see Corollary 3.16 in [Ban88]).

**Proposition 3.2.7.** *There exists a map  $\rho : \mathcal{M} \rightarrow \mathbb{R}$ , continuous with respect to the induced product topology on  $\mathcal{M} \subset \mathbb{R}^{\mathbb{Z}}$ , such that*

(i) *for any  $(x_n)_{n \in \mathbb{Z}}$  for any  $i \in \mathbb{Z}$  it holds*

$$|x_i - x_0 - i\rho((x_n)_{n \in \mathbb{Z}})| < 1;$$

(ii) *if  $(x_n)_{n \in \mathbb{Z}}$  is periodic, that is there exist  $p, q \in \mathbb{N}$  such that  $x_q = x_0 + p$ , then  $\rho((x_n)_{n \in \mathbb{Z}}) = \frac{p}{q}$ ;*

(iii) *the function  $\rho$  is invariant by translations, i.e. for any  $(a, b) \in \mathbb{Z}^2$  we have*

$$\rho((x_{n-a} + b)_{n \in \mathbb{Z}}) = \rho((x_n)_{n \in \mathbb{Z}}).$$

The function  $\rho$  is called the *rotation number function* and  $\rho((x_n)_{n \in \mathbb{Z}})$  is called the *rotation number* of  $(x_n)_{n \in \mathbb{Z}} \in \mathcal{M}$ .

From condition (i) we immediately deduce that for any  $(x_n)_{n \in \mathbb{Z}} \in \mathcal{M}$

$$\rho((x_n)_{n \in \mathbb{Z}}) = \lim_{|n| \rightarrow +\infty} \frac{x_n}{n}. \quad (3.31)$$

**Remark 3.2.9.** Let  $\gamma : \mathbb{T} \rightarrow \mathbb{R}$  be a continuous map whose graph is  $f$ -invariant and denote  $\Gamma = \gamma \circ p$ . Then, we can always define the rotation number  $\rho$  for  $F$  of  $\Gamma$  as the rotation number of a configuration associated to a point  $(x, \Gamma(x))$ .

Indeed, by Remark 3.2.8, for any point of  $\text{Graph}(\Gamma)$  the corresponding configuration is minimizing and its rotation number is so well-defined. Moreover, all the points in  $\text{Graph}(\Gamma)$  provide the same rotation number.

Indeed, let  $x^1, x^2 \in \mathbb{R}$  be such that  $x^1 \leq x^2 < x^1 + 1$ . Consider the associated configurations  $(p_1 \circ F^n(x^1, \Gamma(x^1)))_{n \in \mathbb{Z}}$  and  $(p_1 \circ F^n(x^2, \Gamma(x^2)))_{n \in \mathbb{Z}}$ . For any  $n \in \mathbb{N}$  the application

$$\mathbb{R} \ni x \mapsto p_1 \circ F^n(x, \Gamma(x)) \in \mathbb{R}$$

is a homeomorphism since it is a lift of a homeomorphism of the circle. It is increasing since  $p_1 \circ F^n(x, \Gamma(x)) < p_1 \circ F^n(x, \Gamma(x)) + 1 = p_1 \circ F^n(x + 1, \Gamma(x + 1))$ .

In particular for any  $n \in \mathbb{N}$  it holds

$$p_1 \circ F^n(x^1, \Gamma(x^1)) \leq p_1 \circ F^n(x^2, \Gamma(x^2)) < p_1 \circ F^n(x^1 + 1, \Gamma(x^1 + 1)) = p_1 \circ F^n(x^1, \Gamma(x^1)) + 1.$$

Then

$$\rho((p_1 \circ F^n(x^1, \Gamma(x^1)))_{n \in \mathbb{Z}}) = \lim_{n \rightarrow +\infty} \frac{p_1 \circ F^n(x^1, \Gamma(x^1))}{n} \leq$$

$$\leq \lim_{n \rightarrow +\infty} \frac{p_1 \circ F^n(x^2, \Gamma(x^2))}{n} = \rho((p_1 \circ F^n(x^2, \Gamma(x^2)))_{n \in \mathbb{Z}})$$

and

$$\begin{aligned} \rho((p_1 \circ F^n(x^2, \Gamma(x^2)))_{n \in \mathbb{Z}}) &= \lim_{n \rightarrow +\infty} \frac{p_1 \circ F^n(x^2, \Gamma(x^2))}{n} \leq \\ &\leq \lim_{n \rightarrow +\infty} \frac{p_1 \circ F^n(x^1, \Gamma(x^1)) + 1}{n} = \rho((p_1 \circ F^n(x^1, \Gamma(x^1)))_{n \in \mathbb{Z}}). \end{aligned}$$

That is

$$\rho((p_1 \circ F^n(x^1, \Gamma(x^1)))_{n \in \mathbb{Z}}) = \rho((p_1 \circ F^n(x^2, \Gamma(x^2)))_{n \in \mathbb{Z}}).$$

**Notation 3.2.8.** For  $\rho \in \mathbb{R}$ , let  $\mathcal{M}_\rho$  denote the set of minimizing configurations whose rotation number is  $\rho$ .

We recall a fundamental result in [Ban88] concerning configurations in  $\mathcal{M}_\rho$ . See Theorems 4.1, 5.1, 5.3, 5.8 and Page 26 in [Ban88].

**Theorem 3.2.2.** *Let  $\rho \in \mathbb{R} \setminus \mathbb{Q}$ . Then configurations in  $\mathcal{M}_\rho$  cannot cross.*

*Let  $\rho \in \mathbb{Q}$ . Then  $\mathcal{M}_\rho$  is the disjoint union of three sets  $\mathcal{M}_\rho^{\text{per}} \sqcup \mathcal{M}_\rho^+ \sqcup \mathcal{M}_\rho^-$  and configurations in  $\mathcal{M}_\rho^{\text{per}} \sqcup \mathcal{M}_\rho^+$  (respectively  $\mathcal{M}_\rho^{\text{per}} \sqcup \mathcal{M}_\rho^-$ ) cannot cross.*

**Remark 3.2.10.** In the case of  $\rho \in \mathbb{Q}$ , Theorem 3.2.2 states that, given  $x_0 \in \mathbb{R}$ , there are at most two configurations  $(\tilde{x}_i)_{i \in \mathbb{Z}}, (\bar{x}_i)_{i \in \mathbb{Z}}$  in  $\mathcal{M}_\rho$  such that  $\tilde{x}_0 = \bar{x}_0 = x_0$ .

We are now going to consider a BIES  $U$ . Both its boundary components are bounded. We will be interested in the rotation numbers realized by orbits of points of  $U$  and we will introduce the definition of twist interval (see Definition 3.2.8). The following proposition goes back to Birkhoff (see [Bir32][Section 4]) and will assure us the well-definition and the “non-degeneracy” of the twist interval.

**Proposition 3.2.8.** *Let  $\psi_1, \psi_2 : \mathbb{T} \rightarrow \mathbb{R}$  be continuous maps whose graphs are  $f$ -invariant and such that for any  $x \in \mathbb{T}$  it holds  $\psi_1(x) < \psi_2(x)$ . Denote  $\Psi_1 = \psi_1 \circ p, \Psi_2 = \psi_2 \circ p$  and let  $\varphi_1, \varphi_2$  be the rotation numbers for  $F$  of  $\Psi_1, \Psi_2$ , respectively. Then  $\varphi_1 < \varphi_2$ .*

We refer to [Her83] for the proof of Proposition 3.2.8 (see Complement 2.4.4, page 10). From now until the end of the paragraph we refer to Notation 3.2.4 and to the following

**Notation 3.2.9.** Let  $\rho_1, \rho_2$  be the rotation numbers for  $F$  of  $\Gamma_1, \Gamma_2$ , respectively, where  $\Gamma_i = \gamma_i \circ p$  for  $i = 1, 2$  and  $\gamma_1, \gamma_2$  are the components of  $\partial U$ .

Remark that by Proposition 3.2.8 it holds  $\rho_1 < \rho_2$ . We can give the following

**Definition 3.2.8.** The real interval  $[\rho_1, \rho_2]$  is the twist interval for  $F$  of  $U$ .

Proposition 3.2.8 assures us that the twist interval of a BIES does not degenerate into a point. An outcome of Proposition 3.2.5 is the following

**Corollary 3.2.3.** *Let  $U$  be the BIES introduced in Notation 3.2.9. Denote as  $[\rho_1, \rho_2]$  the twist interval for  $F$  of  $U$ . Then*

$$\mathcal{M}(\mathcal{D}) \subseteq \bigcup_{\rho \in [\rho_1, \rho_2]} \mathcal{M}_\rho \subset \mathcal{M}.$$

*Proof.* Let  $(x_n)_{n \in \mathbb{Z}} \in \mathcal{M}(\mathcal{D})$ . By Corollary 3.2.2 it holds  $(x_n)_{n \in \mathbb{Z}} \in \mathcal{M}$ . We are going to show that  $\rho((x_n)_{n \in \mathbb{Z}})$  is in the twist interval  $[\rho_1, \rho_2]$ . Let  $(x_n, y_n)_{n \in \mathbb{Z}}$  be the corresponding orbit of the configuration  $(x_n)_{n \in \mathbb{Z}}$ . Apply then Proposition 3.2.5 to the points  $(x_0, \Gamma_1(x_0))$ ,  $(x_0, y_0)$  and  $(x_0, \Gamma_2(x_0))$ . For any  $n > 0$  we have

$$p_1 \circ F^n(x_0, \Gamma_1(x_0)) < x_n < p_1 \circ F^n(x_0, \Gamma_2(x_0)).$$

Consequently

$$\rho_1 = \lim_{n \rightarrow +\infty} \frac{p_1 \circ F^n(x_0, \Gamma_1(x_0))}{n} \leq \rho((x_n)_{n \in \mathbb{Z}}) = \lim_{n \rightarrow +\infty} \frac{x_n}{n} \leq \lim_{n \rightarrow +\infty} \frac{p_1 \circ F^n(x_0, \Gamma_2(x_0))}{n} = \rho_2.$$

Hence,  $(x_n)_{n \in \mathbb{Z}} \in \bigcup_{\rho \in [\rho_1, \rho_2]} \mathcal{M}_\rho$ .  $\square$

Actually, we can say something more precise concerning  $\mathcal{M}(\mathcal{D})$ .

**Lemma 3.2.7.** *Let  $U$  be the BIES introduced in Notation 3.2.4. Let  $[\rho_1, \rho_2]$  be the twist interval for  $F$  of  $U$ . Let  $(x_n)_{n \in \mathbb{Z}}$  be a minimizing configuration of rotation number for  $F$  equal to  $\rho((x_n)_{n \in \mathbb{Z}}) \in (\rho_1, \rho_2)$ . Then the orbit  $(x_n, y_n)_{n \in \mathbb{Z}}$  corresponding to the configuration  $(x_n)_{n \in \mathbb{Z}}$  is in  $U$ . Moreover*

$$\bigcup_{\rho \in (\rho_1, \rho_2)} \mathcal{M}_\rho \subset \mathcal{M}(\mathcal{D}).$$

*Proof.* The set  $U$  is the bounded open domain delimited by the graphs of  $\gamma_1$  and  $\gamma_2$ . Recall that we are assuming that  $\gamma_1 < \gamma_2$ . Let us show that  $(x_0, y_0)$  is above  $\gamma_1$ , that is  $y_0 > \gamma_1(x_0)$ .

Consider the configurations  $(x_n)_{n \in \mathbb{Z}}$  and  $(p_1 \circ F^n(x_0, \Gamma_1(x_0)))_{n \in \mathbb{Z}}$ . They are both minimizing configurations (see Proposition 2.8 in [Mat91]). By Lemma 3.1 in [Ban88], they cross at most once. Actually they cross at  $n = 0$ . Since

$$\lim_{n \rightarrow +\infty} \frac{x_n}{n} = \rho((x_n)_{n \in \mathbb{Z}}) > \rho = \rho((p_1 \circ F^n(x_0, \Gamma_1(x_0)))_{n \in \mathbb{Z}}) = \lim_{n \rightarrow +\infty} \frac{p_1 \circ F^n(x_0, \Gamma_1(x_0))}{n},$$

for any  $n > 0$  it holds  $x_n > p_1 \circ F^n(x_0, \Gamma_1(x_0))$ . In particular  $x_1 > p_1 \circ F(x_0, \Gamma_1(x_0))$  and, by the twist condition, we have  $y_0 > \Gamma_1(x_0)$ , where  $(x_0, y_0)$  is the point corresponding to the configuration  $(x_n)_{n \in \mathbb{Z}}$  in the lifted framework. Projecting on the annulus, it holds so  $y_0 > \gamma_1(x_0)$ .

An adapted argument shows that  $(x_0, y_0)$  is below  $\gamma_2$ , that is  $y_0 < \gamma_2(x_0)$ . Thus  $(x_0, y_0) \in U$ . By the invariance of  $U$  we conclude that the orbit  $(x_n, y_n)_{n \in \mathbb{Z}}$  is in  $U$ .

In particular, if  $(x_n)_{n \in \mathbb{Z}}$  is a configuration in  $\mathcal{M}_\rho$  for some  $\rho \in (\rho_1, \rho_2)$ , then  $(x_n)_{n \in \mathbb{Z}} \in C(\mathcal{D})$  and since it is minimizing we deduce that  $(x_n)_{n \in \mathbb{Z}} \in \mathcal{M}(\mathcal{D})$ .  $\square$

**Remark 3.2.11.** Since for any  $\rho \in \mathbb{R}$  the set  $\mathcal{M}_\rho$  is non empty (see Theorem 3.17 in [Ban88]), since  $\rho_1 < \rho_2$  by Proposition 3.2.8 and from Lemma 3.2.7, we immediately deduce that  $\mathcal{M}(\mathcal{D})$  is non empty.

The following lemma gives information about the order of rotation number of configurations having the same zero entry. We will use this property in the following paragraph.

**Lemma 3.2.8.** *Let  $(x, y^1), (x, y^2) \in \mathbb{A}$  be such that  $y^1 \leq y^2$ . Assume that their corresponding configurations  $(p_1 \circ F^n(x, y^1))_{n \in \mathbb{Z}}, (p_1 \circ F^n(x, y^2))_{n \in \mathbb{Z}}$  are minimizing. Then*

$$\rho((p_1 \circ F^n(x, y^1))_{n \in \mathbb{Z}}) \leq \rho((p_1 \circ F^n(x, y^2))_{n \in \mathbb{Z}}).$$

*Proof.* Denote as  $(x_n^1)_{n \in \mathbb{Z}}, (x_n^2)_{n \in \mathbb{Z}}$  the configurations  $(p_1 \circ F^n(x, y^1))_{n \in \mathbb{Z}}, (p_1 \circ F^n(x, y^2))_{n \in \mathbb{Z}}$ , respectively. Observe that  $x_0^1 = x_0^2 = x$  and, since  $y^1 \leq y^2$ , by the twist condition,  $x_1^1 \leq x_1^2$ . By Lemma 3.1 in [Ban88], minimizing configurations cross at most once and so for any  $n > 0$  we have  $x_n^1 \leq x_n^2$ . We now conclude, since

$$\rho((x_n^1)_{n \in \mathbb{Z}}) = \lim_{n \rightarrow +\infty} \frac{x_n^1}{n} \leq \lim_{n \rightarrow +\infty} \frac{x_n^2}{n} = \rho((x_n^2)_{n \in \mathbb{Z}}).$$

□

### End of the proof of Proposition 3.2.2

In this paragraph we show that if any configurations in  $C(\mathcal{D})$  is minimizing, then  $U$  admits a partition into continuous  $f$ -invariant essential curves.

Let  $U$  be a BIES and we refer to Notation 3.2.4. Denote as  $[\rho_1, \rho_2]$  the twist interval for  $F$  of  $U$  (see Definition 3.2.8).

Since any configuration in  $C(\mathcal{D})$  is minimizing (see Proposition 3.2.6), we associate a rotation number to every  $(x_n)_{n \in \mathbb{Z}} \in C(\mathcal{D})$ , that is to every  $(x, y) \in \mathcal{U}$ .

From Remark 3.2.9, also any configuration  $(x_n)_{n \in \mathbb{Z}}$  whose corresponding point is in  $\partial \mathcal{U}$  has a well-defined rotation number.

**Definition 3.2.9.** The function<sup>3</sup>

$$\begin{aligned} \mathcal{R} : cl(\mathcal{U}) &\subseteq \mathbb{R}^2 \rightarrow \mathbb{R} \\ (x, y) &\mapsto \mathcal{R}(x, y) := \rho((p_1 \circ F^n(x, y))_{n \in \mathbb{Z}}) \end{aligned}$$

is the rotation number function.

In other words,  $\mathcal{R}(x, y)$  is the rotation number of the configuration associated to the point  $(x, y) \in cl(\mathcal{U})$  (see Remarks 3.2.5, 3.2.6 and 3.2.9).

**Proposition 3.2.9.** *The function  $\mathcal{R} : cl(\mathcal{U}) \rightarrow \mathbb{R}$  is continuous.*

*Proof.* Let  $(x, y) \in cl(\mathcal{U})$  and fix  $\varepsilon > 0$ . We are going to show that there exists  $\delta > 0$  such that for any  $(x', y') \in cl(\mathcal{U})$  for which

$$\|(x, y) - (x', y')\| < \delta$$

it holds

$$|\mathcal{R}(x, y) - \mathcal{R}(x', y')| < \varepsilon.$$

Fix  $i \in \mathbb{N}, i \geq \lfloor \frac{4}{\varepsilon} \rfloor + 1$ . By the continuity of the function  $p_1 \circ F^i$ , there exists  $\delta \in (0, \frac{\varepsilon}{4})$  so that for any  $(x', y') \in cl(\mathcal{U})$  for which

$$\|(x, y) - (x', y')\| < \delta$$

it holds

$$|p_1 \circ F^i(x, y) - p_1 \circ F^i(x', y')| < \frac{\varepsilon}{4}.$$

Choose then  $(x', y') \in cl(\mathcal{U})$  so that  $\|(x, y) - (x', y')\| < \delta$ . Denote as  $(x_n)_{n \in \mathbb{Z}}, (x'_n)_{n \in \mathbb{Z}}$  the corresponding configurations  $(p_1 \circ F^n(x, y))_{n \in \mathbb{Z}}, (p_1 \circ F^n(x', y'))_{n \in \mathbb{Z}}$  which are in

$$C(\mathcal{D}) \cup \{(p_1 \circ F^n(x, \Gamma_1(x)))_{n \in \mathbb{Z}} : x \in \mathbb{R}\} \cup \{(p_1 \circ F^n(x, \Gamma_2(x)))_{n \in \mathbb{Z}} : x \in \mathbb{R}\}.$$

---

3.  $cl(\mathcal{U})$  denotes the closure of  $\mathcal{U}$ .

By Proposition 3.2.6, by Corollaries 3.2.2 and 3.2.3 and by Remark 3.2.9, their rotation numbers are well-defined and contained in  $[\rho_1, \rho_2]$ .

By property (i) of Proposition 3.2.7, it holds

$$\begin{aligned} |\mathcal{R}(x, y) - \mathcal{R}(x', y')| &= |\rho((x_n)_{n \in \mathbb{Z}}) - \rho((x'_n)_{n \in \mathbb{Z}})| \leq \\ &\leq \left| \frac{p_1 \circ F^i(x, y) - x}{i} - \frac{p_1 \circ F^i(x', y') - x'}{i} \right| + \frac{2}{i}. \end{aligned}$$

By the choice of  $\delta > 0$  and  $i \in \mathbb{N}$  made above, we have

$$|\mathcal{R}(x, y) - \mathcal{R}(x', y')| \leq \frac{|x - x'|}{i} + \frac{|p_1 \circ F^i(x, y) - p_1 \circ F^i(x', y')|}{i} + \frac{2}{i} < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon.$$

□

**Notation 3.2.10.** For any  $\alpha \in [\rho_1, \rho_2]$  denote as  $\mathcal{M}_\alpha(\mathcal{D})$  the set of (minimizing) configurations in  $C(\mathcal{D})$  with rotation number equal to  $\alpha$ <sup>4</sup>. Denote as  $\mathcal{U}_\alpha \subset \mathcal{U}$  the set of points  $(x, y) \in \mathcal{U}$  whose corresponding configuration is in  $\mathcal{M}_\alpha(\mathcal{D})$ . Denote as  $U_\alpha \subset U$  the set of points  $(x, y) \in U$  such that  $(p \times \text{Id}_{\mathbb{R}})^{-1}(x, y) \in \mathcal{U}_\alpha$ .

The following lemma assures us that above any point  $x_0 \in \mathbb{R}$  there is at most one point in  $U$  with prescribed rotation number.

**Lemma 3.2.9.** *Assume that  $C(\mathcal{D}) = \mathcal{M}(\mathcal{D})$ . Fix  $\rho \in [\rho_1, \rho_2]$  and  $x_0 \in \mathbb{R}$ . Then there cannot exist two different points  $(x_0, y_1), (x_0, y_2) \in \text{cl}(\mathcal{U})$  both with rotation number  $\rho$ .*

*Proof.* Argue by contradiction and assume that both  $(x_0, y), (x_0, \tilde{y}) \in \text{cl}(\mathcal{U})$  have rotation number  $\rho$ . Let us say (to fix the ideas) that  $y < \tilde{y}$ . Denote  $(x_n^1)_{n \in \mathbb{Z}}, (x_n^2)_{n \in \mathbb{Z}}$  their corresponding configurations. Both the configurations are minimizing. Indeed, if the corresponding point is in  $\mathcal{U}$ , then by the hypothesis  $C(\mathcal{D}) = \mathcal{M}(\mathcal{D})$  and since  $\mathcal{M}(\mathcal{D}) \subset \mathcal{M}$  (see Corollary 3.2.2), the configuration is minimizing. If the corresponding point is in  $\partial\mathcal{U}$ , the configuration is minimizing by Proposition 2.8 in [Mat91].

The two configurations cross at  $n = 0$ . Let us discuss the value of  $\rho$ .

- (i) If  $\rho \in \mathbb{R} \setminus \mathbb{Q}$ , then, by Theorem 4.1 in [Ban88] (here Theorem 3.2.2), different configurations in  $\mathcal{M}_\rho$  cannot cross and we have the desired contradiction.
- (ii) If  $\rho \in \mathbb{Q}$ , choose  $\hat{y} \in (y, \tilde{y})$  so that  $(x_0, \hat{y}) \in \mathcal{U}$ . Its corresponding configuration is minimizing since  $C(\mathcal{D}) = \mathcal{M}(\mathcal{D}) \subset \mathcal{M}$  and by Lemma 3.2.8 we deduce that its rotation number is  $\rho$ .

In [Ban88] (see Section 5, Page 26, here Theorem 3.2.2 and Remark 3.2.10), Bangert shows that above any  $x_0 \in \mathbb{R}$  there are at most two points whose configurations are minimizing and have the same rational rotation number. This gives us the required contradiction.

□

The property that any configuration in  $C(\mathcal{D})$  is minimizing and Lemma 3.2.9 enable us to characterize the whole set  $\mathcal{M}(\mathcal{D})$  in terms of rotation numbers.

---

4. By Corollary 3.2.3, any minimizing configuration in  $\mathcal{M}(\mathcal{D})$  has rotation number in  $[\rho_1, \rho_2]$ .

**Proposition 3.2.10.** *It holds*

$$\mathcal{M}(\mathcal{D}) = \bigcup_{\rho \in (\rho_1, \rho_2)} \mathcal{M}_\rho.$$

*Proof.* By Corollary 3.2.3 and by Lemma 3.2.7 we know that

$$\bigcup_{\rho \in (\rho_1, \rho_2)} \mathcal{M}_\rho \subset \mathcal{M}(\mathcal{D}) \subset \bigcup_{\rho \in [\rho_1, \rho_2]} \mathcal{M}_\rho.$$

Argue by contradiction and assume there exists a configuration  $(x_n)_{n \in \mathbb{Z}}$  in  $\mathcal{M}(\mathcal{D})$  which does not belong to  $\bigcup_{\rho \in (\rho_1, \rho_2)} \mathcal{M}_\rho$ . Consequently,  $\rho((x_n)_{n \in \mathbb{Z}}) \in \{\rho_1, \rho_2\}$ . Without loss of generality we assume that  $\rho((x_n)_{n \in \mathbb{Z}}) = \rho_1$  (the case  $\rho_2$  is treated similarly).

Denote  $(x_0, y_0) \in \mathcal{U}$  the point that corresponds to the configuration  $(x_n)_{n \in \mathbb{Z}}$ . Consider now the points  $(x_0, y_0), (x_0, \Gamma_1(x_0))$ : they belong to the closure of  $\mathcal{U}$  and have the same rotation number. This contradicts Lemma 3.2.9 and we conclude.  $\square$

The next Lemma shows that for any  $\alpha \in (\rho_1, \rho_2)$  the projection over the zero entry  $p_0 : \mathcal{M}_\alpha(\mathcal{D}) \rightarrow \mathbb{R}$  is surjective.

**Lemma 3.2.10.** *Let  $p_0 : C(\mathcal{D}) \rightarrow \mathbb{R}$  be the projection over the 0-th entry of a configuration in  $C(\mathcal{D})$ . For any  $\alpha \in (\rho_1, \rho_2)$  the projection  $p_0(\mathcal{M}_\alpha(\mathcal{D}))$  is  $\mathbb{R}$ .*

*Proof.* Fix  $\alpha \in (\rho_1, \rho_2)$  and fix  $x_0 \in \mathbb{R}$ . By Remark 3.2.9, we have that  $\mathcal{R}(x_0, \Gamma_1(x_0)) = \rho_1$  and  $\mathcal{R}(x_0, \Gamma_2(x_0)) = \rho_2$ . By the continuity of the function  $\mathcal{R}$  (see Proposition 3.2.9), there exists  $y_0 \in (\Gamma_1(x_0), \Gamma_2(x_0))$  such that  $(x_0, y_0) \in \mathcal{U}$  and  $\mathcal{R}(x_0, y_0) = \alpha$ . Thus, the configuration associated to the point  $(x_0, y_0)$  is minimizing with rotation number  $\alpha$ .  $\square$

By Proposition 3.2.10, for any  $\alpha \in (\rho_1, \rho_2)$  we have that  $\mathcal{M}_\alpha(\mathcal{D}) = \mathcal{M}_\alpha$ .

In order to show that  $f|_U$  is  $\mathcal{C}^0$  integrable, we are going to exhibit a partition of  $U$  into continuous invariant essential curves. In particular, this partition will be given by graphs of functions  $\gamma_\alpha$ ,  $\alpha \in (\rho_1, \rho_2)$ , each of which corresponds to  $U_\alpha$  (see Notation 3.2.10).

**Proposition 3.2.11.** *Assume that  $C(\mathcal{D}) = \mathcal{M}(\mathcal{D})$ . Then, for any  $\alpha \in (\rho_1, \rho_2)$  the set  $U_\alpha$  is the graph of a continuous function  $\gamma_\alpha : \mathbb{T} \rightarrow \mathbb{R}$ .*

*Proof.* Fix  $\alpha \in (\rho_1, \rho_2)$ . By Lemmas 3.2.9 and 3.2.10, for any  $x \in \mathbb{R}$  there exists a unique  $y = y(x, \alpha) \in \mathbb{R}$  such that:

- $(x, y) \in \mathcal{U}$ ;
- $\mathcal{R}(x, y) = \alpha$ .

Let us define the function

$$\begin{aligned} \Gamma_\alpha : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto y(x, \alpha). \end{aligned}$$

**Claim 3.2.2.** The function  $\Gamma_\alpha$  is 1-periodic.

Consider  $x, x+1 \in \mathbb{R}$ . Since both  $\mathcal{U}$  and the rotation number function  $\mathcal{R}$  are invariant by  $(1,0)$ -translations, it holds  $(x+1, y(x, \alpha)) \in \mathcal{U}$  and

$$\begin{aligned}\mathcal{R}(x+1, y(x, \alpha)) &= \rho((p_1 \circ F^n(x+1, y(x, \alpha)))_{n \in \mathbb{Z}}) = \rho((p_1 \circ F^n(x, y(x, \alpha)))_{n \in \mathbb{Z}} + 1) = \\ &= \rho((p_1 \circ F^n(x, y(x, \alpha)))_{n \in \mathbb{Z}}) = \alpha.\end{aligned}$$

Then, thanks to the unicity assured by Lemma 3.2.9, we conclude that  $y(x, \alpha) = y(x+1, \alpha)$  as desired. Equivalently, for any  $x \in \mathbb{R}$  it holds

$$\Gamma_\alpha(x+1) = \Gamma_\alpha(x).$$

Define now  $\gamma_\alpha : \mathbb{T} \rightarrow \mathbb{R}$  as the unique function so that

$$\Gamma_\alpha = \gamma_\alpha \circ p.$$

In particular, the graph of  $\gamma_\alpha$  is the projection over the annulus of the graph of  $\Gamma_\alpha$ .

**Claim 3.2.3.** The function  $\gamma_\alpha$  is continuous.

Let us start by showing that  $\text{Graph}(\Gamma_\alpha)$  is closed. This will imply that also  $\text{Graph}(\gamma_\alpha)$  is closed.

Let  $(x_n, \Gamma_\alpha(x_n))_{n \in \mathbb{N}}$  be a sequence in  $\text{Graph}(\Gamma_\alpha)$  converging to  $(x, y)$ . By definition of  $\Gamma_\alpha$ , each  $(x_n, \Gamma_\alpha(x_n))$  has rotation number  $\alpha$  and so by the continuity of  $\mathcal{R}$  on  $cl(\mathcal{U})$  we deduce that  $\mathcal{R}(x, y) = \alpha$ . In particular, as an outcome of Proposition 3.2.10 we deduce that  $(x, y) \in \mathcal{U}$ . Since there exists a unique point  $\Gamma_\alpha(x)$  so that  $\mathcal{R}(x, \Gamma_\alpha(x)) = \alpha$ , we conclude that  $y = \Gamma_\alpha(x)$ , i.e.  $(x, y) \in \text{Graph}(\Gamma_\alpha)$  and so the graph of  $\Gamma_\alpha$  is closed.

Since  $\text{Graph}(\gamma_\alpha)$  is contained in  $U$  and  $U$  is bounded, we deduce that  $\text{Graph}(\gamma_\alpha)$  is also bounded. Thus,  $\text{Graph}(\gamma_\alpha)$  is compact. By Theorem 5.6.34 in [Soh03], we then conclude that  $\gamma_\alpha$  is continuous, as desired.

Finally, by the definition of  $\gamma_\alpha$ , its graph is  $U_\alpha$ .

□

We can now prove Proposition 3.2.2.

*Proof of Proposition 3.2.2.* Let  $(x, y) \in U$  and let  $(x_n)_{n \in \mathbb{Z}}$  be its corresponding configuration. By Proposition 3.2.6,  $(x_n)_{n \in \mathbb{Z}} \in C(\mathcal{D}) = \mathcal{M}(\mathcal{D}) \subset \mathcal{M}$  and so it has rotation number  $\alpha \in (\rho_1, \rho_2)$  by Proposition 3.2.10. By definition of  $U_\alpha$ , the point  $(x, y)$  belongs to  $U_\alpha$  and so by Proposition 3.2.11 it lies on the graph of  $\gamma_\alpha$ .

By the invariance of  $U$  and that of the rotation number, the graph of  $\gamma_\alpha$  is  $f$ -invariant.

Finally, let  $\alpha_1, \alpha_2 \in (\rho_1, \rho_2), \alpha_1 \neq \alpha_2$ . Then the graphs of  $\gamma_{\alpha_1}$  and of  $\gamma_{\alpha_2}$  cannot intersect. The graphs of  $\gamma_\alpha, \alpha \in (\rho_1, \rho_2)$  provide then the required partition of  $U$  into continuous  $f$ -invariant essential curves, showing that  $f|_U$  is  $C^0$ -integrable.

□

### 3.3 Torsion of instability discs

The main result of this section is Proposition 3.1.3, that we recall here.

**Proposition 3.1.3.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a conservative twist map. Let  $U \subset \mathbb{A}$  be an instability disc. Then  $\omega(\{z \in U : \text{Torsion}(f, z) \neq 0\}) > 0$ .*



### 3.3.1 Zero-torsion set and over-conjugate points

A fundamental tool for our proof is the notion of over-conjugate points. We start by presenting different definitions of conjugate points, according to what appear in the literature.

**Definition 3.3.1** (conjugate points). The point  $z \in \mathbb{A}$  has a conjugate point if there exists  $n \in \mathbb{N}^*$  and  $k \in \mathbb{Z}$  such that

$$\tilde{v}(I)(z, \chi, n) - \tilde{v}(I)(z, \chi, 0) = -\frac{k}{2}.$$

Definition 3.3.1 coincides with Definition 3.2.5 and it is used in [CS96] and [Arc16].

**Remark 3.3.1.** Since  $f$  is a positive twist map, by Theorem 2.1.1, any finite-time torsion with respect to the vertical vector is negative. Hence, in Definition 3.3.1, it holds  $k \geq 1$ .

**Definition 3.3.2** (over-conjugate points). The point  $z \in \mathbb{A}$  has a over-conjugate point if there exists  $n \in \mathbb{N}^*$  such that

$$\tilde{v}(I)(z, \chi, n) - \tilde{v}(I)(z, \chi, 0) \leq -\frac{1}{2}.$$

We will see that the fundamental notion of conjugate points for our purposes is actually Definition 3.3.2.

**Definition 3.3.3** ( $I$ -conjugate points). Let  $I = (f_t)_t$  be an isotopy in  $\text{Diff}^1(\mathbb{A})$  joining the identity to  $f$ . The point  $z \in \mathbb{A}$  has a  $I$ -conjugate point with respect to  $I$  if there exists  $t \in \mathbb{R}_+$  such that

$$\tilde{v}(I)(z, \chi, t) - \tilde{v}(I)(z, \chi, 0) = -\frac{1}{2}.$$

Definition 3.3.3 is equivalent to the one adopted for example in [AABZ15] within the framework of Tonelli Hamiltonian flows.

**Remark 3.3.2.** Since the finite-time torsion on  $\mathbb{A}$  does not depend on the choice of the isotopy, Definitions 3.3.1 and 3.3.2 do not depend on  $I = (f_t)_t$ . Nevertheless, Definition 3.3.3 depends on the choice of the isotopy.

**Remark 3.3.3.** Observe that if  $z \in \mathbb{A}$  has a conjugate point, then it has also a over-conjugate point.

If  $z \in \mathbb{A}$  has a over-conjugate point, then it has also a  $I$ -conjugate point (with respect to any isotopy).

**Definition 3.3.4** (Local twist map). A  $\mathcal{C}^1$  diffeomorphism isotopic to the identity  $f : \mathbb{A} \rightarrow \mathbb{A}$  is a positive (negative) local twist map at  $(x, y) \in \mathbb{A}$  if for any lift  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  of  $f$  and any lift  $X \in \mathbb{R}$  of  $x$  there exists an open interval  $I \subset \mathbb{R}$  of  $y$  such that

$$I \ni \xi \mapsto p_1 \circ F(X, \xi) \in \mathbb{R}$$

is an increasing (decreasing) diffeomorphism to its image.

A  $\mathcal{C}^1$  diffeomorphism isotopic to the identity  $f : \mathbb{A} \rightarrow \mathbb{A}$  is a positive (negative) local twist map on  $U \subset \mathbb{A}$  if it is a positive (negative) local twist map at every  $(x, y) \in U$ .

The following result links the notions of over-conjugate points and of local twist maps.

**Proposition 3.3.1.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a positive twist map (not necessarily conservative) and let  $U \subset \mathbb{A}$ . The following conditions are equivalent:*

- (i) *no points of  $U$  have over-conjugate points;*
- (ii) *for any  $n \in \mathbb{N}^*$  the map  $f^n$  is a positive local twist map on  $U$ .*

Observe that  $U \subset \mathbb{A}$  can be no matter which subset of  $\mathbb{A}$ !

*Proof.* Let us show the two implications.

- (i)  $\Rightarrow$  (ii) Since  $f$  is a positive twist map, by Theorem 2.1.1 in Chapter 2, and since by hypothesis there are no points with over-conjugate points, it holds that for any  $z \in U$  and for any  $n \in \mathbb{N}^*$

$$\tilde{v}(I)(z, \chi, n) - \tilde{v}(I)(z, \chi, 0) \in \left(-\frac{1}{2}, 0\right).$$

This implies that for any  $n \in \mathbb{N}^*$  we have

$$D(p_1 \circ f^n)(z)\chi > 0.$$

Consequently, considering  $z = (x, y) \in U$ , for any lift  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  of  $f$  and for any lift  $X \in \mathbb{R}$  of  $x$ , the function

$$\mathbb{R} \ni \xi \mapsto p_1 \circ F^n(X, \xi) \in \mathbb{R}$$

has positive derivative at  $y$ . Since the function is  $\mathcal{C}^1$ , we deduce from the Inverse Function Theorem that there exists a neighborhood  $I \subset \mathbb{R}$  of  $y$  so that

$$I \ni \xi \mapsto p_1 \circ F^n(X, \xi) \in \mathbb{R}$$

is an increasing  $\mathcal{C}^1$  diffeomorphism to its image. Hence, by the arbitrariness of  $z \in U$ , we conclude that for any  $n \in \mathbb{N}^*$  the map  $f^n$  is a positive local twist map on  $U$ .

- (ii)  $\Rightarrow$  (i) Argue by contradiction and assume there exists  $z \in U$  which has over-conjugate points, i.e. there exists  $m \in \mathbb{N}^*$  such that

$$\tilde{v}(I)(z, \chi, m) - \tilde{v}(I)(z, \chi, 0) \leq -\frac{1}{2}.$$

We are now going to show the existence of  $N \in \mathbb{N}^*$  so that for some (and so any) lift  $F^N$  of  $f^N$  it holds

$$D(p_1 \circ F^N)(z)\chi \leq 0.$$

This will allow us to conclude since it is a contradiction with the fact that  $f^N$  is a positive local twist map at  $z \in U$ .

**Claim 3.3.1.** There exists  $N \in \mathbb{N}^*$  such that  $D(p_1 \circ F^N)(z)\chi \leq 0$ .

If

$$\tilde{v}(I)(z, \chi, m) - \tilde{v}(I)(z, \chi, 0) \in \left[ -k, -k + \frac{1}{2} \right]$$

for some  $k \in \mathbb{Z}$ , then there is nothing to prove since  $D(p_1 \circ F^m)(z)\chi \leq 0$  and so the required  $N$  is  $m$ .

Assume so that

$$\tilde{v}(I)(z, \chi, m) - \tilde{v}(I)(z, \chi, 0) \in \left( -k - \frac{1}{2}, -k \right)$$

for some  $k \in \mathbb{Z}$ . That is, the angle  $\theta(\chi, Df^m(z)\chi)$  admits a measure in  $(-\frac{1}{2}, 0)$ . Since by hypothesis  $z$  has a over-conjugate point at  $m$ , it holds  $k \geq 1$ .

Moreover, by Theorem 2.1.1 in Chapter 2, we have that

$$\text{Torsion}_1(f, z, \chi) \in \left( -\frac{1}{2}, 0 \right)$$

and consequently  $m > 1$ . Define now  $\bar{n} \in \mathbb{N}^*$  as the maximum integer in  $\llbracket 1, m \rrbracket$  so that

- for any  $1 \leq n \leq \bar{n}$  it holds  $\tilde{v}(I)(z, \chi, n) - \tilde{v}(I)(z, \chi, 0) \in (-\frac{1}{2}, 0)$ ;
- $\tilde{v}(I)(z, \chi, \bar{n} + 1) - \tilde{v}(I)(z, \chi, 0) \leq -\frac{1}{2}$ .

We will now show that

$$\tilde{v}(I)(z, \chi, \bar{n} + 1) - \tilde{v}(I)(z, \chi, 0) \in \left[ -1, -\frac{1}{2} \right]$$

and so  $D(p_1 \circ f^{\bar{n}+1})(z)\chi \leq 0$ , concluding our proof.

By definition of  $\bar{n}$ , we have that

$$\tilde{v}(I)(z, \chi, \bar{n}) - \tilde{v}(I)(z, \chi, 0) > -\frac{1}{2}. \quad (3.32)$$

Choose continuous determinations such that

$$\tilde{v}(I)(z, \chi, 0) = 0 \quad \text{and} \quad \tilde{v}(I)(f^{\bar{n}}(z), -\chi, 0) = -\frac{1}{2}. \quad (3.33)$$

As observed in Remark 3.1.1, we highlight the fact that we are interested in considering different determinations  $\tilde{v}(I)(x, \xi, \cdot)$  independently defined for different  $(x, \xi) \in T\mathbb{A}_*$ .

Denote as  $w$  the vector  $Df^{\bar{n}}(z)\chi$  and choose a continuous determination such that

$$\tilde{v}(I)(f^{\bar{n}}(z), w, 0) = \tilde{v}(I)(z, \chi, \bar{n}). \quad (3.34)$$

Consequently, by (3.32), (3.33) and (3.34), we have that

$$\tilde{v}(I)(f^{\bar{n}}(z), w, 0) > \tilde{v}(I)(f^{\bar{n}}(z), -\chi, 0).$$

From Lemma 1.1.1 in Chapter 1 we deduce that

$$\tilde{v}(I)(f^{\bar{n}}(z), w, 1) > \tilde{v}(I)(f^{\bar{n}}(z), -\chi, 1).$$

Since  $t \mapsto \tilde{v}(I)(f^{\bar{n}}(z), w, t)$  and  $t \mapsto \tilde{v}(I)(z, \chi, \bar{n} + t)$  are lifts of the same angle function that coincide at  $t = 0$ , we deduce that they are equal. Hence

$$\tilde{v}(I)(z, \chi, \bar{n} + 1) > \tilde{v}(I)(f^{\bar{n}}(z), -\chi, 1).$$

By the choice of the lifts, since  $\tilde{v}(I)(z, \chi, 0) = 0$ , it holds

$$\tilde{v}(I)(z, \chi, \bar{n} + 1) - \tilde{v}(I)(z, \chi, 0) = \tilde{v}(I)(z, \chi, \bar{n} + 1) > \tilde{v}(I)(f^{\bar{n}}(z), -\chi, 1). \quad (3.35)$$

By Theorem 2.1.1 we have that

$$\tilde{v}(I)(f^{\bar{n}}(z), -\chi, 1) - \tilde{v}(I)(f^{\bar{n}}(z), -\chi, 0) \in \left(-\frac{1}{2}, 0\right),$$

so it holds, by (3.33),

$$\tilde{v}(I)(f^{\bar{n}}(z), -\chi, 1) > -\frac{1}{2} + \tilde{v}(I)(f^{\bar{n}}(z), -\chi, 0) = -1. \quad (3.36)$$

From (3.35) and (3.36) we conclude that

$$\tilde{v}(I)(z, \chi, \bar{n} + 1) - \tilde{v}(I)(z, \chi, 0) > -1.$$

From the definition of  $\bar{n}$ , we have that

$$-1 < \tilde{v}(I)(z, \chi, \bar{n} + 1) - \tilde{v}(I)(z, \chi, 0) \leq -\frac{1}{2}.$$

That is, the required integer  $N$  is  $\bar{n} + 1$ , concluding so the proof. □

The following result concerns the link between full-measure zero-torsion sets, conjugate points and local twist maps.

**Proposition 3.3.2.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a conservative positive twist map. Let  $U \subset \mathbb{A}$  be an open  $f$ -invariant set such that  $\omega(U) < +\infty$  (where  $\omega$  denotes the Lebesgue measure). Then the following conditions are equivalent.*

- (1) *The torsion at  $z$  is zero for every  $z \in U$ .*
- (2) *The torsion at  $z$  is zero for  $\omega$ -almost every  $z \in U$ .*
- (3) *No points  $z \in U$  have over-conjugate points.*
- (4) *For any  $n \in \mathbb{N}^*$  the map  $f^n$  is a positive local twist map on  $U$ .*

*Proof.* (1)  $\Rightarrow$  (2) This implication is trivial.

(2)  $\Rightarrow$  (3) This implication comes from Proposition 3.2.4 and Remark 3.2.4.

(3)  $\Rightarrow$  (1) Let  $z \in U$ . Since  $z$  has no over-conjugate points, by definition for any  $n \in \mathbb{N}^*$  it holds

$$\tilde{v}(I)(z, \chi, n) - \tilde{v}(I)(z, \chi, 0) > -\frac{1}{2}.$$

By Theorem 2.1.1, since  $f$  is a positive twist map, it holds also that for any  $n \in \mathbb{N}^*$

$$\tilde{v}(I)(z, \chi, n) - \tilde{v}(I)(z, \chi, 0) < 0.$$

Consequently

$$\text{Torsion}(f, z) = \lim_{n \rightarrow +\infty} \frac{\tilde{v}(I)(z, \chi, n) - \tilde{v}(I)(z, \chi, 0)}{n} = 0.$$

(3)  $\Leftrightarrow$  (4) This equivalence is exactly the content of Proposition 3.3.1.

□

Let  $U \subset \mathbb{A}$  be an open  $f$ -invariant set. As an outcome of Proposition 3.3.2 we deduce that the torsion at  $z \in U$  is zero for  $\omega$ -almost every  $z \in U$  if and only if it is zero for every  $z \in U$ .

**Remark 3.3.4.** Let  $U \subset \mathbb{A}$  be an open set such that  $\omega(U) < +\infty$  and such that there exists  $N \in \mathbb{N}^*$  so that  $f^N(U) = U$ . Denote as

$$\mathfrak{U} = \bigcup_{i=0}^{N-1} f^i(U).$$

Observe that  $\omega(\{z \in U : \text{Torsion}(f, z) \neq 0\}) = 0$  if and only if  $\omega(\{z \in \mathfrak{U} : \text{Torsion}(f, z) \neq 0\}) = 0$ .

Indeed, assume that  $\omega(\{z \in U : \text{Torsion}(f, z) \neq 0\}) = 0$ . Observe that

$$\{z \in \mathfrak{U} : \text{Torsion}(f, z) \neq 0\} = \bigcup_{i=0}^{N-1} \{x \in f^i(U) : \text{Torsion}(f, x) \neq 0\}.$$

Consequently

$$\omega(\{z \in \mathfrak{U} : \text{Torsion}(f, z) \neq 0\}) \leq \sum_{i=0}^{N-1} \omega(\{x \in f^i(U) : \text{Torsion}(f, x) \neq 0\}).$$

Since the torsion is invariant along the  $f$ -orbit of the point, for any  $i \in \llbracket 0, N-1 \rrbracket$  it holds

$$\begin{aligned} \{x \in f^i(U) : \text{Torsion}(f, x) \neq 0\} &= \\ &= \{y \in U : \text{Torsion}(f, f^{-i}(y)) \neq 0\} = \{y \in U : \text{Torsion}(f, y) \neq 0\}. \end{aligned}$$

Hence we conclude because

$$\omega(\{z \in \mathfrak{U} : \text{Torsion}(f, z) \neq 0\}) \leq \sum_{i=0}^{N-1} \omega(\{x \in U : \text{Torsion}(f, x) \neq 0\}) = 0.$$

Consequently, by Proposition 3.3.2 and by this observation, if the torsion is null at  $\omega$ -almost every point of  $U$  then there are no points with over-conjugate points in  $\mathfrak{U}$ .

### 3.3.2 About instability discs: proof of Proposition 3.1.3

In this Subsection we finally prove Proposition 3.1.3, that is

**Proposition 3.1.3.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a conservative twist map. Let  $U \subset \mathbb{A}$  be an instability disc. Then  $\omega(\{z \in U : \text{Torsion}(f, z) \neq 0\}) > 0$ .*

Remark that, by Proposition 3.1.1, the proof of Proposition 3.1.3, together with Proposition 3.1.2, concludes the proof of Theorem 3.1.1: every bounded connected component of  $\mathcal{N}(f)$  has a positive-measure set of points with non-zero torsion.

The proof of Proposition 3.1.3 is made by contradiction. Assuming that  $\omega$ -almost every  $z \in U$  has zero torsion, by Remark 3.3.4 every point in  $U$  is free of over-conjugate points. The absence of over-conjugate points allows us to build an invariant foliation in  $U$ . The leaf of the foliation to which  $z \in U$  belongs is obtained as limit (as  $n$  goes to  $+\infty$ ) of images through  $f^n$  of vertical lines passing through  $f^{-n}(z)$ .

The idea of the construction is inspired by the construction of Green bundles (see [Gre58] and [Arn10]), but we work on the surface instead of working on the tangent spaces.

**Notation 3.3.1.** In the sequel we are going to consider the standard Euclidean distance, denoted as  $d$ .

#### The construction of the foliation $(\mathcal{G}_z)_{z \in U}$

We are going to build our foliation in a more general framework.

**Hypothesis (H).** Let  $U \subset \mathbb{A}$  be a non empty open bounded set homeomorphic to an open disc so that there exists  $N \in \mathbb{N}^*$  such that  $f^N(U) = U$  and so that

$$\omega(\{z \in U : \text{Torsion}(f, z) \neq 0\}) = 0.$$

In order to prove Proposition 3.1.3, we will argue by contradiction and assume that  $U$  is an instability disc such that the torsion is  $\omega$ -almost everywhere null: in particular  $U$  will verify Hypothesis (H) (see Definition 3.1.5).

**Remark 3.3.5.** Let  $U \subset \mathbb{A}$  satisfy Hypothesis (H). Then from Remark 3.3.4 every points in  $U$  is free of over-conjugate points.

Let fix a lift  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  of  $f$ . Recall that  $p_1, p_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$  denote the projection over the first and the second coordinate, respectively. Denote as  $\mathcal{U} \subset \mathbb{R}^2$  a connected component of the lift of  $U$  on  $\mathbb{R}^2$ . In particular,  $\mathcal{U}$  is an open set, homeomorphic to an open disc and there exists  $k \in \mathbb{Z}$  such that

$$F^N(\mathcal{U}) = \mathcal{U} + (k, 0) = \{z + (k, 0) : z \in \mathcal{U}\}.$$

Let  $z \in \mathcal{U}$ . Denote as  $V_{F^{-n}(z)}^{CC, \mathcal{U}}$  the connected component of

$$V_{F^{-n}(z)} \cap F^{-n}(\mathcal{U}) = \{(p_1(F^{-n}(z)), y) : y \in \mathbb{R}\} \cap F^{-n}(\mathcal{U})$$

which contains  $F^{-n}(z)$ . Observe that  $\partial V_{F^{-n}(z)}^{CC, \mathcal{U}} \subset \partial F^{-n}(\mathcal{U})$  and, since  $V_{F^{-n}(z)}^{CC, \mathcal{U}}$  is an open segment in the real line, it holds that  $\partial V_{F^{-n}(z)}^{CC, \mathcal{U}}$  is made up of two distinct points.

Denote then as  $\mathcal{G}_{n,z}$  the image  $F^n(V_{F^{-n}(z)}^{CC,\mathcal{U}})$ : observe that for any  $n \in \mathbb{N}$  the point  $z$  belongs to  $\mathcal{G}_{n,z}$ . Moreover, the boundary  $\partial\mathcal{G}_{n,z}$  is contained in  $\partial\mathcal{U}$  by the invariance of the boundary.

We start now discussing the structure of these  $\mathcal{G}_{n,z}$ .

**Lemma 3.3.1.** *For any  $z \in \mathcal{U}$  and for any  $n \in \mathbb{N}^*$  the set  $\mathcal{G}_{n,z}$  is the graph of a function*

$$\Gamma_{n,z} : (a_{n,z}, b_{n,z}) \rightarrow \mathbb{R},$$

where  $(a_{n,z}, b_{n,z}) \subset \mathbb{R}$ .

*Proof.* By Remark 3.3.5, there are no points with over-conjugate points in  $U$ . By Proposition 3.3.1 for any  $n \in \mathbb{N}^*$  the map  $f^n$  is a local positive twist map on  $\bigcup_{i \in \mathbb{Z}} f^i(U) = \bigcup_{i=0}^{N-1} f^i(U)$ .

Fix now  $n \in \mathbb{N}^*$  and consider  $V_{F^{-n}(z)}^{CC,\mathcal{U}}$ . The function

$$p_2(V_{F^{-n}(z)}^{CC,\mathcal{U}}) \ni \xi \mapsto p_1 \circ F^n(p_1(F^{-n}(z)), \xi) \in \mathbb{R}$$

is then an increasing diffeomorphism to its image. Its inverse function is so defined on an open interval that we denote as  $(a_{n,z}, b_{n,z}) \subset \mathbb{R}$ . Denote such an inverse function as

$$(a_{n,z}, b_{n,z}) \ni x \mapsto \mathcal{F}_{n,z}(x) \in p_2(V_{F^{-n}(z)}^{CC,\mathcal{U}})$$

and remark that it remains an increasing diffeomorphism to its image.

Define now the function

$$\begin{aligned} \Gamma_{n,z} &: (a_{n,z}, b_{n,z}) \rightarrow \mathbb{R} \\ x &\mapsto \Gamma_{n,z}(x) := p_2 \circ F^n(p_1(F^{-n}(z)), \mathcal{F}_{n,z}(x)). \end{aligned}$$

Thus, by construction, the graph of  $\Gamma_{n,z}$  is exactly the set  $\mathcal{G}_{n,z}$  and we conclude the proof.  $\square$

For the construction of the foliation  $(\mathcal{G}_z)_{z \in U}$  we are going to use Green bundles. We recall here the main definitions.

**Notation 3.3.2.** Let  $z \in \mathbb{A}$  and let  $n \in \mathbb{Z} \setminus \{0\}$ . Let  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the fixed lift of  $f$ . Let  $\mathfrak{z} \in \mathbb{R}^2$  be a lift of  $z$ . Denote as

$$G_n(z) = Df^n(f^{-n}(z))\mathcal{V}(f^{-n}(z)) \in T_z\mathbb{A},$$

where  $\mathcal{V}(f^{-n}(z)) = \ker D\bar{p}_1(f^{-n}(z))$  is the vertical line in  $T_{f^{-n}(z)}\mathbb{A}$ . The slope of  $G_n(z)$ , when defined, is denoted as  $s_n(z)$  and so

$$G_n(z) = \{(\delta, s_n(z)\delta) : \delta \in \mathbb{R}\}.$$

We observe that, thanks to the trivialization of the tangent bundle, the subspace  $G_n(z)$  can be identified with the subspace

$$DF^n(F^{-n}(\mathfrak{z}))\mathcal{V}(F^{-n}(\mathfrak{z})) \subset T_{\mathfrak{z}}\mathbb{R}^2.$$

In the sequel, with an abuse of notation, we denote as  $s_n(z)$  both the slope of  $G_n(z)$  and the slope of  $DF^n(F^{-n}(\mathfrak{z}))\mathcal{V}(F^{-n}(\mathfrak{z}))$ .

**Remark 3.3.6.** We will see that the absence of over-conjugate pointsimplies that the slope  $s_n(z)$  is well-defined for every  $n \in \mathbb{N}^*$  at every  $z \in U$  (see Remark 3.3.7).

**Notation 3.3.3.** For  $z \in U$  and  $n \in \mathbb{N}^*$ , denote as  $\theta_n(z)$  the measure contained in  $\left(-\frac{1}{2}, \frac{1}{2}\right]$  of the oriented angle between the positive horizontal vector  $\mathcal{H} = (1, 0)$  and the vector

$$Df^n(f^{-n}(z))\chi \in T_{f^{-n}(z)}\mathbb{A},$$

where  $\chi = (0, 1)$ .

We observe (and clarify later, see Remark 3.3.7) that the absence of over-conjugate pointsimplies that  $\theta_n(z) \in \left(-\frac{1}{4}, \frac{1}{4}\right)$ .

We want now to make explicit the link between  $s_n(z)$ ,  $\theta_n(z)$  and  $\tilde{v}(f)(f^{-n}(z), \chi, n)$ <sup>5</sup>. It holds (see Figure 3.1)

$$s_n(z) = \tan \left( 2\pi \left( \tilde{v}(f)(f^{-n}(z), \chi, n) + \frac{1}{4} \right) \right) = \tan (2\pi\theta_n(z)). \quad (3.37)$$

Indeed,  $\theta_n(z)$  is a measure of the oriented angle  $\theta(\mathcal{H}, Df(f^{-n}(z))\chi)$ , while  $\tilde{v}(f)(f^{-n}(z), \chi, n)$

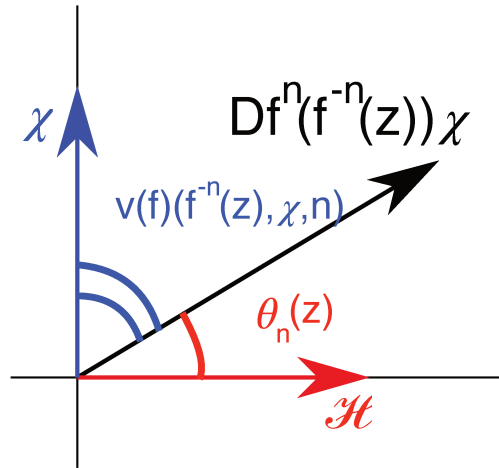


Figure 3.1 – The link between  $s_n(z)$ ,  $\theta_n(z)$  and  $\tilde{v}(f)(f^{-n}(z), \chi, n)$ .

is a measure of the oriented angle

$$\theta(\chi, Df^n(f^{-n}(z))\chi) = \theta(\chi, \mathcal{H}) + \theta(\mathcal{H}, Df^n(f^{-n}(z))\chi).$$

Consequently

$$\theta_n(z) - \tilde{v}(f)(f^{-n}(z), \chi, n) = \frac{1}{4} + k \quad \text{for some } k \in \mathbb{Z}. \quad (3.38)$$

By the periodicity of the tangent function, we deduce that  $\tan(2\pi(\tilde{v}(f)(f^{-n}(z), \chi, n) + \frac{1}{4})) = \tan(2\pi\theta_n(z))$ .

From the definition of  $s_n(z)$  and of  $\theta_n(z)$ , the equality  $s_n(z) = \tan(2\pi\theta_n(z))$  follows immediately, if we admit also  $\infty$  as possible values of  $s_n(z)$ .

---

5. Since the finite-time torsion does not depend on the choice of the isotopy on  $\mathbb{A}$  (see Proposition 1.3.2), we omit the dependance on  $I = (f_t)_t$ .



**Remark 3.3.7.** If there are no points with over-conjugate points, then for any  $z$  and any  $n \in \mathbb{N}^*$  the value  $s_n(z)$  is finite. Indeed, by the absence of over-conjugate points (see Definition 3.3.2), for any  $z \in U$  and for any  $n \in \mathbb{N}^*$  it holds

$$\tilde{v}(f)(f^{-n}(z), \chi, n) \in \left(-\frac{1}{2} + k, k\right) \quad \text{for some } k \in \mathbb{Z}.$$

Thus, from the definition of  $\theta_n(z)$  and from (3.38), we have that

$$\theta_n(z) \in \left(-\frac{1}{4}, \frac{1}{4}\right).$$

Finally, from (3.37), we conclude that  $|s_n(z)| = |\tan(2\pi\theta_n(z))| < +\infty$ .

We recall here the definitions of Green set and Green bundles of  $f$ , referring to [Arn16] and [Arn10].

The Green set of  $f$  is denoted as  $Green(f)$  and it is the set of points of  $\mathbb{A}$  such that along the whole orbit of these points it holds for any  $n \geq 1$

$$s_{-n}(x) < s_{-n-1}(x) < s_{n+1}(x) < s_n(x).$$

**Definition 3.3.5.** If  $x \in Green(f)$ , the two Green bundles at  $x$  are subspaces, denoted as  $G_+(x), G_-(x)$ , contained in the tangent space  $T_x\mathbb{A}$  with slopes  $s_+(x), s_-(x)$ , respectively, where

$$s_+(x) = \lim_{n \rightarrow +\infty} s_n(x) \quad \text{and} \quad s_-(x) = \lim_{n \rightarrow +\infty} s_{-n}(x).$$

Let us recall the characterization of the Green bundles presented in Theorem 7 in [Arn16].

**Theorem 3.3.1** (Theorem 7 in [Arn16]). *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a positive conservative twist map. The following conditions are equivalent:*

- (i)  $x \in Green(f)$ ;
- (ii) all along the  $f$ -orbit of  $x$  for any  $n \geq 1$  it holds  $s_n(x) > s_{-1}(x)$ .

Thanks to this last characterization we deduce the following

**Proposition 3.3.3.** *Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a conservative positive twist map and let  $U$  be an open bounded periodic set homeomorphic to an open disc such that  $\omega(\{z \in U : Torsion(f, z) \neq 0\}) = 0$ . Then*

$$U \subset Green(f).$$

*Proof.* Fix  $z \in U$ . From what observed in (3.37), it holds for any  $m \geq 1$

$$s_m(z) = \tan \left( 2\pi \left( \tilde{v}(f)(f^{-m}(z), \chi, m) + \frac{1}{4} \right) \right). \quad (3.39)$$

Consider now the slope  $s_{-1}(z)$ : since the tangent function is  $\pi$ -periodic, it can be defined also as the tangent of the angle between  $\mathcal{H}$  and  $Df^{-1}(f(z))(-\chi)$ . Thus

$$s_{-1}(z) = \tan \left( 2\pi \left( \tilde{v}(f^{-1})(f(z), -\chi, 1) + \frac{1}{4} \right) \right). \quad (3.40)$$

The function  $f^{-1}$  is a negative twist map and so, by Theorem 2.1.1 and Remark 2.1.2, it holds that

$$\tilde{v}(f^{-1})(f(z), -\chi, 1) \in \left(-\frac{1}{2} + k, k\right) \quad \text{for some } k \in \mathbb{Z}.$$

Consequently,  $|s_{-1}(z)| = \left| \tan \left( 2\pi \left( \tilde{v}(f^{-1})(f(z), -\chi, 1) + \frac{1}{4} \right) \right) \right| < +\infty$ .

By the periodicity of the tangent function, equalities (3.39) and (3.40) do not depend on the choice of the continuous determinations  $\tilde{v}(f)(f^{-m}(z), \chi, \cdot)$  and  $\tilde{v}(f^{-1})(f(z), -\chi, \cdot)$ .

Choose the continuous determination of  $t \mapsto v(f^{-1})(f(z), -\chi, t)$  such that  $\tilde{v}(f^{-1})(f(z), -\chi, 0) = -\frac{1}{2}$ . Therefore, by Theorem 2.1.1 and Remark 2.1.2,

$$\tilde{v}(f^{-1})(f(z), -\chi, 1) \in \left(-\frac{1}{2}, 0\right). \quad (3.41)$$

By hypothesis the set of points in  $U$  with not zero torsion has zero  $\omega$  measure. By Remark 3.3.4, there are no points with over-conjugate points in  $\bigcup_{i=0}^{N-1} f^i(U)$ , where  $N$  denotes the period of  $U$ . Hence, since for any  $m \geq 1$  the point  $f^{-m}(z)$  belongs to  $\bigcup_{i=0}^{N-1} f^i(U)$ , we have that for any  $l \in \mathbb{N}^*$

$$\tilde{v}(f)(f^{-m}(z), \chi, l) - \tilde{v}(f)(f^{-m}(z), \chi, 0) \in \left(-\frac{1}{2}, 0\right). \quad (3.42)$$

We are going now to use the characterization of the Green set of Theorem 7 in [Arn16] (here Theorem 3.3.1). That is,  $z \in \text{Green}(f)$  if and only if for any  $n \geq 1$  it holds

$$s_n(z) > s_{-1}(z). \quad (3.43)$$

Argue by contradiction and assume that there exists  $n \in \mathbb{N}^*$  such that  $s_n(z) \leq s_{-1}(z)$ .

Choose the continuous determination of  $t \mapsto v(f)(f^{-n}(z), \chi, t)$  such that  $\tilde{v}(f)(f^{-n}(z), \chi, 0) = 0$ . Thus, by (3.42),

$$\tilde{v}(f)(f^{-n}(z), \chi, n) \in \left(-\frac{1}{2}, 0\right). \quad (3.44)$$

By the contradiction hypothesis, by (3.39) and by (3.40) it holds

$$\begin{aligned} \tan \left( 2\pi \left( \tilde{v}(f)(f^{-n}(z), \chi, n) + \frac{1}{4} \right) \right) &= s_n(z) \leq \\ &\leq s_{-1}(z) = \tan \left( 2\pi \left( \tilde{v}(f^{-1})(f(z), -\chi, 1) + \frac{1}{4} \right) \right). \end{aligned} \quad (3.45)$$

By the choices of the lifts, i.e. by (3.41) and (3.44), we deduce from (3.45) that

$$\tilde{v}(f)(f^{-n}(z), \chi, n) \leq \tilde{v}(f^{-1})(f(z), -\chi, 1). \quad (3.46)$$

Choose a continuous determination of  $t \mapsto v(f)(z, Df^n(f^{-n}(z))\chi, t)$  such that

$$\tilde{v}(f)(z, Df^n(f^{-n}(z))\chi, 0) = \tilde{v}(f)(f^{-n}(z), \chi, n).$$

Choose a continuous determination of  $t \mapsto v(f)(z, Df^{-1}(f(z))(-\chi), t)$  such that

$$\tilde{v}(f)(z, Df^{-1}(f(z))(-\chi), 0) = \tilde{v}(f^{-1})(f(z), -\chi, 1).$$

From Lemma 1.1.1 in Chapter 1, since from (3.46)

$$\tilde{v}(f)(z, Df^n(f^{-n}(z))\chi, 0) \leq \tilde{v}(f)(z, Df^{-1}(f(z))(-\chi), 0),$$

we have that

$$\tilde{v}(f)(z, Df^n(f^{-n}(z))\chi, 1) \leq \tilde{v}(f)(z, Df^{-1}(f(z))(-\chi), 1). \quad (3.47)$$

Since  $t \mapsto \tilde{v}(f)(f^{-n}(z), \chi, n+t)$  and  $t \mapsto \tilde{v}(f)(z, Df^n(f^{-n}(z))\chi, t)$  are lifts of the same angle function that coincide at  $t = 0$ , we deduce that

$$\tilde{v}(f)(z, Df^n(f^{-n}(z))\chi, 1) = \tilde{v}(f)(f^{-n}(z), \chi, n+1). \quad (3.48)$$

From (3.42) and from the choice of the lift we have

$$\tilde{v}(f)(f^{-n}(z), \chi, n+1) \in \left(-\frac{1}{2}, 0\right). \quad (3.49)$$

Consider now  $\tilde{v}(f)(z, Df^{-1}(f(z))(-\chi), 1)$ . Since  $f$  is a positive twist map, by Proposition 2.1.2, we have

$$\tilde{v}(f)(z, Df^{-1}(f(z))(-\chi), 1) - \tilde{v}(f)(z, Df^{-1}(f(z))(-\chi), 0) \in \left(-1, \frac{1}{2}\right). \quad (3.50)$$

Since  $\tilde{v}(f)(z, Df^{-1}(f(z))(-\chi), 0) = \tilde{v}(f^{-1})(f(z), -\chi, 1)$  and from (3.41), it holds

$$\tilde{v}(f)(z, Df^{-1}(f(z))(-\chi), 0) \in \left(-\frac{1}{2}, 0\right).$$

Consequently, from (3.50), it holds

$$\tilde{v}(f)(z, Df^{-1}(f(z))(-\chi), 1) \in \left(-\frac{3}{2}, \frac{1}{2}\right). \quad (3.51)$$

Observe that  $\tilde{v}(f)(z, Df^{-1}(f(z))(-\chi), 1)$  is a measure of the oriented angle between  $\chi$  and  $Df(z)Df^{-1}(f(z))(-\chi) = -\chi$ . That is

$$\tilde{v}(f)(z, Df^{-1}(f(z))(-\chi), 1) = -\frac{1}{2} + k \quad \text{for some } k \in \mathbb{Z}.$$

From (3.51) we deduce that  $\tilde{v}(f)(z, Df^{-1}(f(z))(-\chi), 1) = -\frac{1}{2}$ .

This implies, together with (3.49), (3.47) and (3.48), that

$$-\frac{1}{2} < \tilde{v}(f)(f^{-n}(z), \chi, n+1) \leq \tilde{v}(f)(z, Df^{-1}(f(z))(-\chi), 1) = -\frac{1}{2},$$

which is the required contradiction.

We so conclude that for any  $n \geq 1$  it holds  $s_n(z) > s_{-1}(z)$ . Equivalently, from Theorem 7 in [Arn16] (here Theorem 3.3.1),  $z \in \text{Green}(f)$ .

□

It holds that every  $z \in \bigcup_{i=0}^{N-1} f^i(U)$  is in  $Green(f)$ . Consequently, for any  $z \in \bigcup_{i=0}^{N-1} f^i(U)$  and for any  $n \in \mathbb{N}, n > 1$  we have that

$$s_{-1}(z) < s_{-n}(z) < s_n(z) < s_1(z). \quad (3.52)$$

Denote as

$$K := \max \left( \max_{z \in cl(\bigcup_{i=0}^{N-1} f^i(U))} |s_{-1}(z)|, \max_{z \in cl(\bigcup_{i=0}^{N-1} f^i(U))} |s_1(z)| \right), \quad (3.53)$$

where  $cl(\bigcup_{i=0}^{N-1} f^i(U))$  denotes the closure of  $\bigcup_{i=0}^{N-1} f^i(U)$ .

**Remark 3.3.8.** Observe that

$$s_{-1}(z) = \tan \left( 2\pi \left( \tilde{v}(f^{-1})(f(z), \chi, 1) + \frac{1}{4} \right) \right)$$

and

$$s_1(z) = \tan \left( 2\pi \left( \tilde{v}(f)(f^{-1}(z), \chi, 1) + \frac{1}{4} \right) \right).$$

Since  $f$  is a positive twist map and  $f^{-1}$  is a negative twist map, from Theorem 2.1.1 and Remark 2.1.2, we deduce that for any  $z \in \mathbb{A}$  both  $s_{-1}(z)$  and  $s_1(z)$  are finite. Since  $s_{-1}(z), s_1(z)$  depend continuously on  $z$  and since the closure of  $\bigcup_{i=0}^{N-1} f^i(U)$  is bounded, we conclude that the constant  $K$ , defined in (3.53), is finite.

We are now going to show that every function  $\Gamma_{n,z}$  is a  $K$ -Lipschitz function.

**Lemma 3.3.2.** *For every  $z \in \mathcal{U}$  and for every  $n \geq 1$  the function  $\Gamma_{n,z} : (a_{n,z}, b_{n,z}) \rightarrow \mathbb{R}$  is  $K$ -Lipschitz.*

*Proof.* Fix  $z \in \mathcal{U}$  and  $n \in \mathbb{N}^*$ . Consider the function  $\Gamma_{n,z} : (a_{n,z}, b_{n,z}) \rightarrow \mathbb{R}$ . Let  $x, y \in (a_{n,z}, b_{n,z}), x < y$ . By definition of  $\Gamma_{n,z}$  it holds that  $F^{-n}(x, \Gamma_{n,z}(x)), F^{-n}(y, \Gamma_{n,z}(y))$  both belong to  $V_{F^{-n}(z)}^{CC, \mathcal{U}}$ , that is the connected component of

$$V_{F^{-n}(z)} \cap F^{-n}(\mathcal{U})$$

that contains  $F^{-n}(z)$ . Denote now  $X = p_2 \circ F^{-n}(x, \Gamma_{n,z}(x)), Y = p_2 \circ F^{-n}(y, \Gamma_{n,z}(y))$ . In particular

$$\left\{ (p_1(F^{-n}(z)), \delta) : \delta \in [X, Y] \right\} \subset V_{F^{-n}(z)}^{CC, \mathcal{U}} \subset F^{-n}(\mathcal{U}).$$

By Proposition 3.3.2 and by Remark 3.3.4, at every point of  $\bigcup_{i \in \mathbb{Z}} f^i(U)$  the map  $f^n$  is a positive local twist map. As a consequence, it holds that  $X < Y$ .

By Cauchy's mean value theorem, there exists  $w \in (X, Y)$  such that

$$\frac{p_2 \circ F^n(p_1 \circ F^{-n}(z), Y) - p_2 \circ F^n(p_1 \circ F^{-n}(z), X)}{p_1 \circ F^n(p_1 \circ F^{-n}(z), Y) - p_1 \circ F^n(p_1 \circ F^{-n}(z), X)} = \frac{D(p_2 \circ F^n)(p_1 \circ F^{-n}(z), w)\chi}{D(p_1 \circ F^n)(p_1 \circ F^{-n}(z), w)\chi}. \quad (3.54)$$

Observe that

$$\frac{D(p_2 \circ F^n)(p_1 \circ F^{-n}(z), w)\chi}{D(p_1 \circ F^n)(p_1 \circ F^{-n}(z), w)\chi} = s_n(p_1 \circ F^{-n}(z), w).$$

From (3.54), by the definition of  $K$  (see (3.53)) and from (3.52), it holds

$$|p_2 \circ F^n(p_1 \circ F^{-n}(z), Y) - p_2 \circ F^n(p_1 \circ F^{-n}(z), X)| \leq$$

$$\leq K|p_1 \circ F^n(p_1 \circ F^{-n}(z), Y) - p_1 \circ F^n(p_1 \circ F^{-n}(z), X)|.$$

We then conclude remarking that

$$p_2 \circ F^n(p_1 \circ F^{-n}(z), Y) - p_2 \circ F^n(p_1 \circ F^{-n}(z), X) = \Gamma_{n,z}(y) - \Gamma_{n,z}(x)$$

and

$$p_1 \circ F^n(p_1 \circ F^{-n}(z), Y) - p_1 \circ F^n(p_1 \circ F^{-n}(z), X) = y - x,$$

that gives us the desired inequality

$$|\Gamma_{n,z}(y) - \Gamma_{n,z}(x)| \leq K|y - x|.$$

□

**Remark 3.3.9.** For every  $z \in \bigcup_{i \in \mathbb{Z}} F^i(\mathcal{U})$  and every  $n \geq 1$ , each function  $\Gamma_{n,z}$  can be extended on the closed interval  $[a_{n,z}, b_{n,z}]$  so that  $\Gamma_{n,z}$  is still continuous and  $K$ -Lipschitz (see Section 3.4.2 in Appendix 3.4).

With an abuse of notation, we refer to every such extension still as  $\Gamma_{n,z}$ .

**Remark 3.3.10.** For any  $n \geq 1$  and any  $z \in \mathcal{U}$ , the points  $(a_{n,z}, \Gamma_{n,z}(a_{n,z}))$  and  $(b_{n,z}, \Gamma_{n,z}(b_{n,z}))$  are points on the boundary of  $\mathcal{U}$ .

For a fixed  $z \in \mathcal{U}$ , we focus now on the relations between the graphs  $\Gamma_{n,z}$  as  $n \geq 1$  varies.

**Proposition 3.3.4.** *Let  $z \in \mathcal{U}$ . The graphs  $(\text{Graph}(\Gamma_{n,z}))_{n \geq 1}$ , i.e.  $(\mathcal{G}_{n,z})_{n \geq 1}$  are well-ordered (see Figure 3.2). That is*

(i) *for any  $x \in (p_1(z), b_{n,z}) \cap (p_1(z), b_{n+1,z})$  it holds*

$$\Gamma_{n+1,z}(x) < \Gamma_{n,z}(x);$$

(ii) *for any  $x \in (a_{n,z}, p_1(z)) \cap (a_{n+1,z}, p_1(z))$  it holds*

$$\Gamma_{n,z}(x) < \Gamma_{n+1,z}(x).$$

*Proof.* Fix  $n \geq 1$  and consider the case (i) (the case (ii) can be treated similarly), i.e. let  $x \in (p_1(z), b_{n,z}) \cap (p_1(z), b_{n+1,z})$ .

Define the function

$$[p_1(z), \min(b_{n,z}, b_{n+1,z})) \ni s \mapsto \Psi(s) = \Gamma_{n,z}(s) - \Gamma_{n+1,z}(s) \in \mathbb{R}.$$

Observe that  $\Psi(p_1(z)) = 0$ . Since, by Proposition 3.3.3, it holds  $U \subset \text{Green}(f)$ , we have that  $s_{n+1}(z) < s_n(z)$  from the definition of  $\text{Green}(f)$ <sup>6</sup>.

That is, by (3.37),

$$\begin{aligned} s_{n+1}(z) &= \tan \left( 2\pi \left( \tilde{v}(F)(F^{-(n+1)}(z), \chi, n+1) + \frac{1}{4} \right) \right) < \\ &< \tan \left( 2\pi \left( \tilde{v}(F)(F^{-n}(z), \chi, n) + \frac{1}{4} \right) \right) = s_n(z). \end{aligned} \quad (3.55)$$

---

6. With an abuse of notation, in order to lighten the notation, we denote as  $s_n(z)$  both the slope of  $G_n((p \times \text{Id})(z))$  and the slope of  $DF^n(F^{-n}(z))\mathcal{V}(F^{-n}(z))$ .

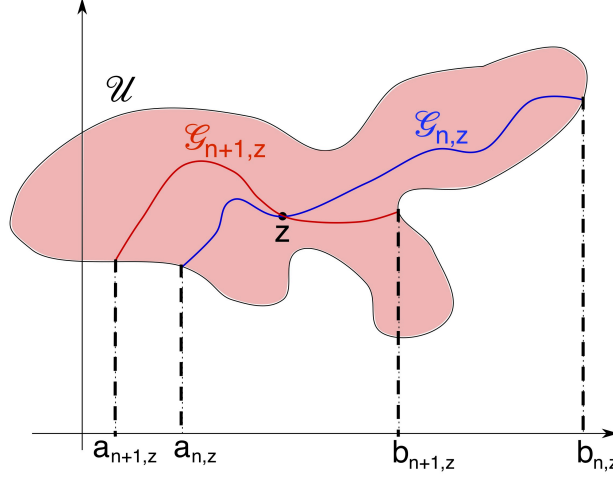


Figure 3.2 – The graphs of  $(\Gamma_{n,z})_{n \geq 1}$  are well-ordered.

Observe that  $s_n(z) = \Gamma'_{n,z}(p_1(z))$  and  $s_{n+1}(z) = \Gamma'_{n+1,z}(p_1(z))$  (see Notation 3.3.2). From the definition of  $\Psi$  and since  $s_{n+1}(z) < s_n(z)$  (see (3.55)), it holds that  $\Psi'(p_1(z)) > 0$ . So there exists  $\bar{s}$  such that  $[p_1(z), p_1(z) + \bar{s}] \subset [p_1(z), \min(b_{n,z}, b_{n+1,z})]$  and for any  $s \in (p_1(z), p_1(z) + \bar{s}]$  it holds  $\Psi(s) > \Psi(p_1(z)) = 0$ .

We claim now that for any  $s \in (p_1(z), \min(b_{n,z}, b_{n+1,z}))$  it holds  $\Psi(s) > 0$ . Argue by contradiction and assume there exists  $\xi \in (p_1(z), \min(b_{n,z}, b_{n+1,z}))$  such that  $\Psi(\xi) \leq 0$ . By the continuity of the function  $\Psi$  there exists  $X \in (p_1(z), \min(b_{n,z}, b_{n+1,z}))$ <sup>7</sup> such that  $\Psi(X) = 0$ . Equivalently,  $\Gamma_{n,z}(X) = \Gamma_{n+1,z}(X)$ .

Thus, we have that  $F^{-n}(X, \Gamma_{n,z}(X)) = F^{-n}(X, \Gamma_{n+1,z}(X))$ .

From the definition of  $\Gamma_{n,z}$  and  $\Gamma_{n+1,z}$  it holds that  $F^{-n}(X, \Gamma_{n,z}(X))$  belongs to  $V_{F^{-n}(z)}^{CC, \mathcal{U}}$ , while  $F^{-n}(X, \Gamma_{n+1,z}(X))$  belongs to  $F(V_{F^{-(n+1)}(z)}^{CC, \mathcal{U}})$ .

Since  $f$  is a positive twist map, the image  $F(V_{F^{-(n+1)}(z)}^{CC, \mathcal{U}})$  is a graph over its projection on the first coordinate and

$$F(V_{F^{-(n+1)}(z)}^{CC, \mathcal{U}}) \cap V_{F^{-n}(z)}^{CC, \mathcal{U}} = \{F^{-n}(z)\}.$$

Since  $F^{-n}(X, \Gamma_{n,z}(X)) = F^{-n}(X, \Gamma_{n+1,z}(X))$  is not  $F^{-n}(z)$  because  $X > p_1(z)$ , we obtain the desired contradiction and we conclude.  $\square$

The key idea for building our invariant foliation is considering the limit as  $n \rightarrow +\infty$  of well-ordered leaves  $(\mathcal{G}_{n,z})_{n \geq 1}$  at every  $z \in \mathcal{U}$ .

**Notation 3.3.4.** Let  $z \in \mathcal{U}$ . Denote

$$a_z := \limsup_{n \rightarrow +\infty} a_{n,z} \quad \text{and} \quad b_z := \liminf_{n \rightarrow +\infty} b_{n,z}.$$

Since  $\mathcal{U}$  is bounded, there exists  $C > 0$  such that for every  $z \in \mathcal{U}$  it holds  $|a_z| \leq C, |b_z| \leq C$ .

**Lemma 3.3.3.** *Let  $z \in \mathcal{U}$ . Then  $a_z < p_1(z) < b_z$ .*

<sup>7</sup>. Actually we have that  $X \in (p_1(z) + \bar{s}, \min(b_{n,z}, b_{n+1,z}))$ .

*Proof.* Since  $\mathcal{U}$  is open, there exists  $\varepsilon > 0$  such that the closed ball  $\overline{B_\varepsilon(z)} \subset \mathcal{U}$ . We are now going to build a “security rectangle” around  $z$  whose projection over the first coordinate is contained in  $(a_{n,z}, b_{n,z})$  for any  $n \geq 1$ .

Recall that, from Lemma 3.3.2, every function  $\Gamma_{n,z}$  is  $K$ -Lipschitz. Consider the horizontal  $K$ -cone at  $z$ , i.e.

$$C_z^{K,hor} = \{w \in \mathbb{R}^2 : |p_2(w - z)| \leq K|p_1(w - z)|\},$$

and intersect it with  $\partial B_\varepsilon(z)$ , see Figure 3.3. We so obtain the vertices of our “security

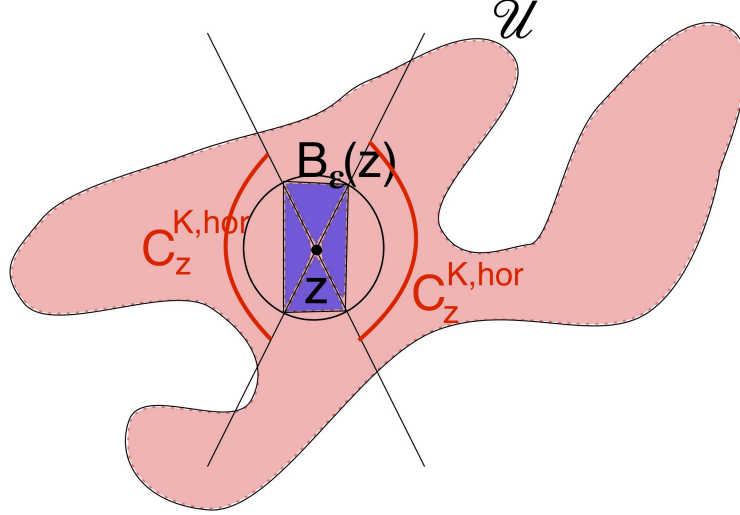


Figure 3.3 – How to build the “security rectangle” around  $z$ .

rectangle”  $R \subset \mathcal{U}$ , which is, for some  $0 < \delta < \varepsilon$ ,

$$R = [p_1(z) - \delta, p_1(z) + \delta] \times [p_2(z) - K\delta, p_2(z) + K\delta].$$

By Lemma 3.3.2, every  $\Gamma_{n,z}$  is  $K$ -Lipschitz. We claim now that, for any  $n \geq 1$ , the function  $\Gamma_{n,z}$  is defined on  $p_1(R) = [p_1(z) - \delta, p_1(z) + \delta]$ . Indeed, assume by contradiction that  $b_{n,z} < p_1(z) + \delta$ . Since  $\Gamma_{n,z}$  is  $K$ -Lipschitz, we deduce that  $(b_{n,z}, \Gamma_{n,z}(b_{n,z})) \in R \subset \overline{B_\varepsilon(z)} \subset \mathcal{U}$ . This contradicts the fact that the point  $(b_{n,z}, \Gamma_{n,z}(b_{n,z}))$  belongs to the boundary of  $\mathcal{U}$  (see Remark 3.3.10).

We so conclude that

$$a_z = \limsup_{n \rightarrow +\infty} a_{n,z} \leq p_1(z) - \delta < p_1(z) < p_1(z) + \delta \leq \liminf_{n \rightarrow +\infty} b_{n,z} = b_z.$$

□

We start now discussing the convergence of the functions  $(\Gamma_{n,z})_{n \geq 1}$ , in particular the pointwise convergence.

**Lemma 3.3.4.** *Let  $z \in \mathcal{U}$ . The sequence  $(\Gamma_{n,z})_{n \geq 1}$  converges pointwise on  $(a_z, b_z)$ .*

*Proof.* Let  $x \in (a_z, b_z)$ . If  $p_1(z) = x$ , then the sequence  $(\Gamma_{n,z}(x))_{n \geq 1}$  is constant and equal to  $p_2(z)$ .

Assume now that  $p_1(z) \neq x$  and denote  $\eta := \min((x - a_z), (b_z - x)) > 0$ . We discuss the two possible cases.

(i) Let  $p_1(z) < x$ . By definition of  $b_z$ , there exists  $\bar{n} \in \mathbb{N}$  such that

$$\inf_{m \geq \bar{n}} b_{m,z} > b_z - \eta \geq x.$$

So for any  $m \geq \bar{n}$  it holds that  $x \in (p_1(z), b_{m,z})$  and the function  $\Gamma_{m,z}$  is well-defined at  $x$ . By Proposition 3.3.4 the sequence  $(\Gamma_{m,z}(x))_{m \geq \bar{n}}$  is decreasing and bounded (because it is contained in  $\mathcal{U}$  which is bounded). Therefore,  $(\Gamma_{m,z}(x))_{m \geq 1}$  converges. Denote

$$\Gamma_z(x) := \lim_{n \rightarrow +\infty} \Gamma_{n,z}(x).$$

(ii) Similar arguments allow us to conclude in the case  $p_1(z) > x$ . □

So far, we have defined at  $z \in \mathcal{U}$  the function

$$\begin{aligned} \Gamma_z : (a_z, b_z) &\rightarrow \mathbb{R} \\ x &\mapsto \Gamma_z(x) := \lim_{n \rightarrow +\infty} \Gamma_{n,z}(x). \end{aligned} \tag{3.56}$$

**Lemma 3.3.5.** *Let  $z \in \mathcal{U}$ . The function  $\Gamma_z : (a_z, b_z) \rightarrow \mathbb{R}$  is  $K$ -Lipschitz (so in particular uniformly continuous).*

*Proof.* Let  $x, y \in (a_z, b_z)$ . There exist  $\tilde{n} \in \mathbb{N}$  such that for any  $n \geq \tilde{n}$  both  $x$  and  $y$  are in  $(a_{n,z}, b_{n,z})$ .

Fix  $\varepsilon > 0$ . Since the sequences  $(\Gamma_{n,z}(x))_{n \geq \tilde{n}}$ ,  $(\Gamma_{n,z}(y))_{n \geq \tilde{n}}$  converge to  $\Gamma_z(x), \Gamma_z(y)$  respectively, there exists  $\bar{n} \in \mathbb{N}$ ,  $\bar{n} \geq \tilde{n}$  such that for any  $n \geq \bar{n}$  it holds

$$|\Gamma_z(x) - \Gamma_{n,z}(x)| < \frac{\varepsilon}{2} \quad \text{and} \quad |\Gamma_z(y) - \Gamma_{n,z}(y)| < \frac{\varepsilon}{2}.$$

Consequently for any  $n \geq \bar{n}$  we have

$$\begin{aligned} &|\Gamma_z(x) - \Gamma_z(y)| \leq \\ &\leq |\Gamma_z(x) - \Gamma_{n,z}(x)| + |\Gamma_{n,z}(x) - \Gamma_{n,z}(y)| + |\Gamma_{n,z}(y) - \Gamma_z(y)| < |\Gamma_{n,z}(x) - \Gamma_{n,z}(y)| + \varepsilon. \end{aligned}$$

Since for all  $n \geq 1$  the function  $\Gamma_{n,z}$  is  $K$ -Lipschitz, we deduce that  $|\Gamma_z(x) - \Gamma_z(y)| < K|x - y| + \varepsilon$ .

By the arbitrariness of  $\varepsilon$ , we conclude that  $|\Gamma_z(x) - \Gamma_z(y)| \leq K|x - y|$ . That is,  $\Gamma_z$  is  $K$ -Lipschitz (so in particular uniform continuous). □

We can actually say something more about the convergence of  $(\Gamma_{n,z}|_{[p_1(z), b_z]})_{n \geq 1}$  and of  $(\Gamma_{n,z}|_{[a_z, p_1(z)]})_{n \geq 1}$ , which are decreasing and increasing sequences, respectively. Indeed, by Dini's Lemma (see for example Theorem 7.13 in [Rud76]) it holds

**Fact 3.3.1.** *On any compact subset of  $(a_z, b_z)$  the sequence  $(\Gamma_{n,z})_{n \geq 1}$  uniformly converges to the function  $\Gamma_z$ .*

**Remark 3.3.11.** For any  $z \in \mathcal{U}$ , we can extend the function  $\Gamma_z$  on the closed interval  $[a_z, b_z]$  (in an unique way) so that  $\Gamma_z : [a_z, b_z] \rightarrow \mathbb{R}$  is uniform continuous and  $K$ -Lipschitz (see Section 3.4.2 in Appendix 3.4). With an abuse of notation, we refer to such an extension still as  $\Gamma_z$ .



**Lemma 3.3.6.** *Let  $z \in \mathcal{U}$ . The points  $(a_z, \Gamma_z(a_z))$  and  $(b_z, \Gamma_z(b_z))$  belong to the boundary of  $\mathcal{U}$ .*

*Proof.* We are going to show the existence of a sequence  $(a_{n_k, z}, \Gamma_{n_k, z}(a_{n_k, z}))_{k \geq 1}$  which belongs to the boundary of  $\mathcal{U}$  and converges to  $(a_z, \Gamma_z(a_z))$ . Since the boundary of  $\mathcal{U}$  is closed, we will then deduce that  $(a_z, \Gamma_z(a_z)) \in \partial\mathcal{U}$ .

By definition of  $a_z$ , there exists a subsequence  $(a_{n_k, z})_{k \geq 1}$  converging to  $a_z$ . It is so sufficient showing that

$$\lim_{k \rightarrow +\infty} \Gamma_{n_k, z}(a_{n_k, z}) = \Gamma_z(a_z).$$

By Lemma 3.4.2 in Section 3.4.2, the value  $\Gamma_z(a_z)$  is the limit  $\lim_{m \rightarrow +\infty} \Gamma_z(x_m)$ , where  $(x_m)_{m \geq 1}$  is a sequence in  $(a_z, b_z)$  converging to  $a_z$ .

Recall that, by Lemma 3.4.2 in Section 3.4.2, for any  $n \geq 1$  the functions  $\Gamma_{n, z} : [a_{n, z}, b_{n, z}] \rightarrow \mathbb{R}$  and the function  $\Gamma_z : [a_z, b_z] \rightarrow \mathbb{R}$  are all  $K$ -Lipschitz.

Fix now  $\varepsilon > 0$ . Let  $\bar{m} \in \mathbb{N}$  be such that for any  $m \geq \bar{m}$  it holds

$$0 < \eta_m := x_m - a_z < \frac{\varepsilon}{4K}.$$

Let  $\bar{k} = \bar{k}(\bar{m}) \in \mathbb{N}$  be such that for any  $k \geq \bar{k}$  it holds

$$|a_{n_k, z} - a_z| < \frac{\eta_{\bar{m}}}{2} = \frac{x_{\bar{m}} - a_z}{2}$$

and so  $a_{n_k, z} < a_z + \frac{\eta_{\bar{m}}}{2} < x_{\bar{m}}$ .

In particular, for any  $k \geq \bar{k}$ , the function  $\Gamma_{n_k, z}$  is well-defined at  $x_{\bar{m}}$ .

Since  $(\Gamma_{n, z})_{n \in \mathbb{N}}$  converges pointwise to  $\Gamma_z$ , there exists  $\tilde{k} \in \mathbb{N}$ ,  $\tilde{k} \geq \bar{k}$  such that for any  $k \geq \tilde{k}$  it holds

$$|\Gamma_{n_k, z}(x_{\bar{m}}) - \Gamma_z(x_{\bar{m}})| < \frac{\varepsilon}{3}.$$

Hence, for  $k \geq \tilde{k}$  it holds

$$\begin{aligned} & |\Gamma_{n_k, z}(a_{n_k, z}) - \Gamma_z(a_z)| \leq \\ & \leq |\Gamma_{n_k, z}(a_{n_k, z}) - \Gamma_{n_k, z}(x_{\bar{m}})| + |\Gamma_{n_k, z}(x_{\bar{m}}) - \Gamma_z(x_{\bar{m}})| + |\Gamma_z(x_{\bar{m}}) - \Gamma_z(a_z)| \leq \\ & \leq K|a_{n_k, z} - x_{\bar{m}}| + \frac{\varepsilon}{3} + K|x_{\bar{m}} - a_z|, \end{aligned}$$

where in the last inequality we use the fact that both  $\Gamma_{n_k, z}$  and  $\Gamma_z$  are  $K$ -Lipschitz on the closure of their own domain of definition.

Now

$$|\Gamma_{n_k, z}(a_{n_k, z}) - \Gamma_z(a_z)| \leq K|a_{n_k, z} - a_z| + 2K|a_z - x_{\bar{m}}| + \frac{\varepsilon}{3} < \frac{\varepsilon}{8} + \frac{\varepsilon}{2} + \frac{\varepsilon}{3} = \frac{23}{24}\varepsilon < \varepsilon.$$

Consequently, the sequence  $(a_{n_k, z}, \Gamma_{n_k, z}(a_{n_k, z}))_k \subset \partial\mathcal{U}$  converges to  $(a_z, \Gamma_z(a_z))$ . Thus,  $(a_z, \Gamma_z(a_z))$  belongs to the boundary of  $\mathcal{U}$ .

A similar argument shows that also  $(b_z, \Gamma_z(b_z)) \in \partial\mathcal{U}$ .

□

**Remark 3.3.12.** In Lemma 3.3.6 we actually show that there exists a sequence  $(a_{n_k, z}, \Gamma_{n_k, z}(a_{n_k, z}))_{k \geq 1}$  (respectively  $(b_{n_k, z}, \Gamma_{n_k, z}(b_{n_k, z}))_{k \geq 1}$ ) which converges to  $(a_z, \Gamma_z(a_z))$  (respectively to  $(b_z, \Gamma_z(b_z))$ ).

We have so proved that the function

$$\Gamma_z : [a_z, b_z] \rightarrow \mathbb{R}$$

is uniformly continuous and  $K$ -Lipschitz. Its graph is contained in the closure of  $\mathcal{U}$  and  $(a_z, \Gamma_z(a_z)), (b_z, \Gamma_z(b_z)) \in \partial\mathcal{U}$ .

**Notation 3.3.5.** Denote as  $\widehat{\mathcal{G}}_z$  the graph of  $\Gamma_z$  over  $[a_z, b_z]$ . Since  $z \in \widehat{\mathcal{G}}_z \cap \mathcal{U}$ , we have that  $\widehat{\mathcal{G}}_z \cap \mathcal{U} \neq \emptyset$ . The leaf of our foliation  $\mathcal{G}_z$  is the connected component of  $\widehat{\mathcal{G}}_z \cap \mathcal{U}$  containing  $z$ .

We still denote  $\mathcal{G}_z$  as the graph of  $\Gamma_z : (a_z, b_z) \rightarrow \mathbb{R}$ , although if the domain of  $\Gamma_z$  whose graph is  $\mathcal{G}_z$  could be strictly contained in the initial definition of  $(a_z, b_z)$  (see Figure 3.4). Denote as  $cl(\mathcal{G}_z)$  the closure of  $\mathcal{G}_z$ : from the definition of  $\mathcal{G}_z$ , it holds that  $cl(\mathcal{G}_z)$  is  $\mathcal{G}_z \cup \{(a_z, \Gamma_z(a_z)), (b_z, \Gamma_z(b_z))\}$ , where the points  $(a_z, \Gamma_z(a_z)), (b_z, \Gamma_z(b_z))$  belong to the boundary  $\partial\mathcal{U}$ .

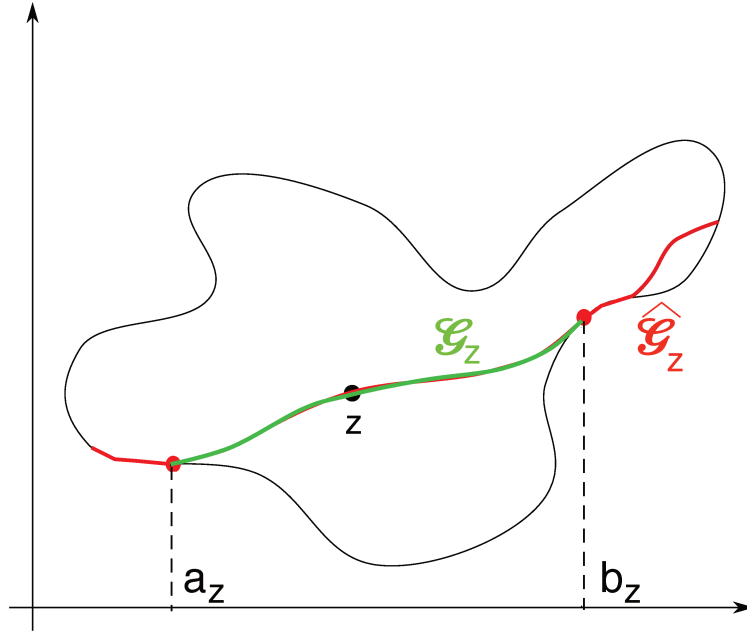


Figure 3.4 – The closed leaf  $\widehat{\mathcal{G}}_z$  and how to define the leaf  $\mathcal{G}_z$ .

### About intersection of leaves of $(\mathcal{G}_z)_{z \in U}$

In this paragraph we show that some “strict” intersections between different leaves are not possible.

**Definition 3.3.6** (Strict intersection of leaves). Let  $\mathcal{G}_z = \text{Graph}(\Gamma_z), \mathcal{G}_{z'} = \text{Graph}(\Gamma_{z'})$  be leaves of  $z, z' \in \mathcal{U}$ . The leaves  $\mathcal{G}_z, \mathcal{G}_{z'}$  intersect strictly if there exists  $x_1, x_2 \in (a_z, b_z) \cap (a_{z'}, b_{z'})$  such that

$$\Gamma_z(x_1) < \Gamma_{z'}(x_1) \quad \text{and} \quad \Gamma_z(x_2) > \Gamma_{z'}(x_2).$$

**Remark 3.3.13.** Let  $\mathcal{G}_z, \mathcal{G}_{z'}$  be leaves that intersect strictly. Assume that  $x_1 < x_2$  are such that  $\Gamma_z(x_1) < \Gamma_{z'}(x_1)$  and  $\Gamma_z(x_2) > \Gamma_{z'}(x_2)$ . Then there always exists  $x \in (x_1, x_2)$  such that  $\Gamma_z(x) - \Gamma_{z'}(x) = 0$ .

**Proposition 3.3.5.** *Let  $\mathcal{G}_z, \mathcal{G}_{z'}$  be different leaves. Then they cannot intersect strictly.*

*Proof.* Argue by contradiction and assume that  $\mathcal{G}_z, \mathcal{G}_{z'}$  are different leaves that intersect strictly. So, by Remark 3.3.13, there exists  $x_1, x_2, x \in (a_z, b_z) \cap (a_{z'}, b_{z'})$  so that (see Figure 3.5)

$$\Gamma_z(x_1) - \Gamma_{z'}(x_1) > 0 = \Gamma_z(x) - \Gamma_{z'}(x) > \Gamma_z(x_2) - \Gamma_{z'}(x_2).$$

Since  $x_1, x_2 \in (a_z, b_z) \cap (a_{z'}, b_{z'})$  there exists  $\bar{n} \in \mathbb{N}$  such that for any  $n \geq \bar{n}$  the functions

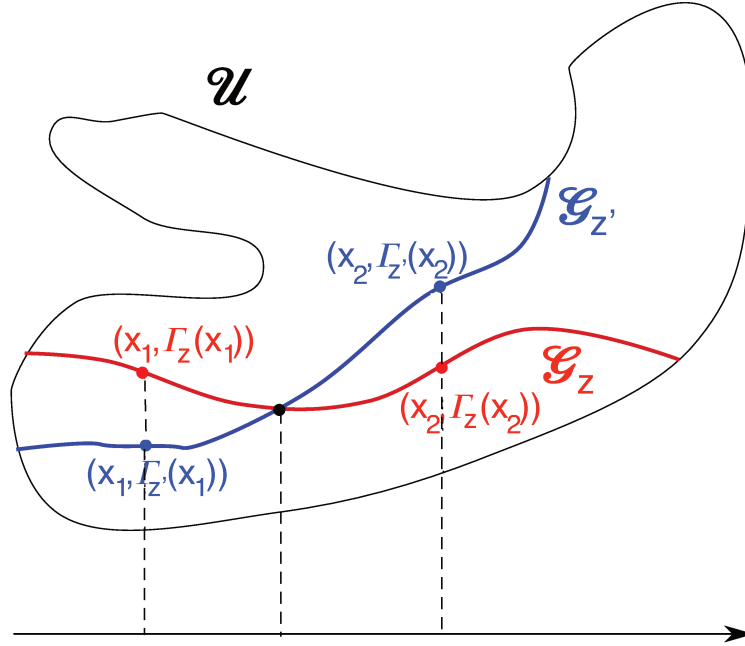


Figure 3.5 – The contradiction hypothesis of a strict intersection between leaves.

$\Gamma_{n,z}, \Gamma_{n,z'}$  are both defined at  $x_1, x_2$ . Moreover,  $(\Gamma_{n,z})_{n \geq \bar{n}}$  and  $(\Gamma_{n,z'})_{n \geq \bar{n}}$  converge, when defined, pointwise to  $\Gamma_z$  and  $\Gamma_{z'}$  respectively.

Denote as

$$\varepsilon := \Gamma_z(x_1) - \Gamma_{z'}(x_1) > 0 \quad \text{and} \quad \eta := \Gamma_{z'}(x_2) - \Gamma_z(x_2) > 0.$$

Let  $\Delta = \min(\varepsilon, \eta) > 0$ . There exists  $\tilde{n} \geq \bar{n}$  such that for any  $n \geq \tilde{n}$  it holds

- (i)  $|\Gamma_{n,z}(x_1) - \Gamma_z(x_1)| < \frac{\Delta}{3}$  and  $|\Gamma_{n,z}(x_2) - \Gamma_z(x_2)| < \frac{\Delta}{3}$ ;
- (ii)  $|\Gamma_{n,z'}(x_1) - \Gamma_{z'}(x_1)| < \frac{\Delta}{3}$  and  $|\Gamma_{n,z'}(x_2) - \Gamma_{z'}(x_2)| < \frac{\Delta}{3}$ .

Consequently for any  $n \geq \tilde{n}$  it holds

$$\begin{aligned} & \Gamma_{n,z}(x_1) - \Gamma_{n,z'}(x_1) = \\ & = (\Gamma_{n,z}(x_1) - \Gamma_z(x_1)) + (\Gamma_z(x_1) - \Gamma_{z'}(x_1)) + (\Gamma_{z'}(x_1) - \Gamma_{n,z'}(x_1)) > \\ & > -\frac{\Delta}{3} + \varepsilon - \frac{\Delta}{3} \geq \frac{\Delta}{3} > 0. \end{aligned}$$

Similarly, for any  $n \geq \tilde{n}$  we deduce that

$$\Gamma_{n,z'}(x_2) - \Gamma_{n,z}(x_2) > -\frac{\Delta}{3} + \eta - \frac{\Delta}{3} \geq \frac{\Delta}{3} > 0.$$

Fix now  $n \geq \tilde{n}$ . Because of the continuity of the function  $\xi \mapsto \Gamma_{n,z}(\xi) - \Gamma_{n,z'}(\xi)$  and since  $\Gamma_{n,z}(x_1) - \Gamma_{n,z'}(x_1) > 0 > \Gamma_{n,z}(x_2) - \Gamma_{n,z'}(x_2)$ , we deduce that there exists  $X \in (x_1, x_2)$  such that both  $\Gamma_{n,z}$  and  $\Gamma_{n,z'}$  are defined at  $X$  and  $\Gamma_{n,z}(X) - \Gamma_{n,z'}(X) = 0$ .

From the definitions of  $\Gamma_{n,z}, \Gamma_{n,z'}$ , we have exhibited a point of intersection between  $F^n(V_{F^{-n}(z)})$  and  $F^n(V_{F^{-n}(z')})$ .

That is, the vertical lines  $V_{F^{-n}(z)}$  and  $V_{F^{-n}(z')}$  intersect at some point and thus they must coincide. We deduce so that the leaves  $\text{Graph}(\Gamma_{n,z})$  and  $\text{Graph}(\Gamma_{n,z'})$  coincide, contradicting the fact that  $\Gamma_{n,z}(x_1) \neq \Gamma_{n,z'}(x_1)$  and  $\Gamma_{n,z}(x_2) \neq \Gamma_{n,z'}(x_2)$ .  $\square$

### About the invariance of leaves of $(\mathcal{G}_z)_{z \in U}$

We discuss now the dynamics over each leaf and we will show the invariance of the foliation. That is

**Proposition 3.3.6.** *Let  $z \in \mathcal{U}$ . Then  $F(\mathcal{G}_z) = \mathcal{G}_{F(z)}$ .*

**Remark 3.3.14.** In the previous sections we have built the leaves of the foliation  $\mathcal{G}_z$  for  $z \in \mathcal{U}$ . The construction can be done for any  $z \in F^i(\mathcal{U})$  for  $i \in \llbracket 0, N-1 \rrbracket$ , where  $N$  is the period of  $U$ .

*Proof.* We are going to prove that  $F(\mathcal{G}_z) \subset \mathcal{G}_{F(z)}$ . Once we will prove this inclusion, we can conclude. Indeed, if by contradiction there exists a point in  $\mathcal{G}_{F(z)} \setminus F(\mathcal{G}_z)$ , then, since we are assuming that  $F(\mathcal{G}_z) \subset \mathcal{G}_{F(z)}$ , it holds that  $F(a_z, \Gamma_z(a_z))$  or  $F(b_z, \Gamma_z(b_z))$  is in the open set  $F(\mathcal{U})$ . This contradicts the fact that  $\{(a_z, \Gamma_z(a_z)), (b_z, \Gamma_z(b_z))\} \subset \partial \mathcal{U}$  (see Lemma 3.3.6 and Notation 3.3.5) and that  $F(\partial \mathcal{U}) = \partial(F(\mathcal{U}))$ .

Argue by contradiction and assume that there exists  $w \in F(\mathcal{G}_z)$  but  $w \notin \mathcal{G}_{F(z)}$ . Actually, we can say that  $w \notin \widehat{\mathcal{G}_{F(z)}}$ . Indeed, if  $w$  would belong to  $\widehat{\mathcal{G}_{F(z)}} \setminus \mathcal{G}_{F(z)}$ , then it should belong to:

- either the boundary of  $F(\mathcal{U})$ , contradicting the hypothesis of being in  $F(\mathcal{G}_z) \subset F(\mathcal{U})$  and the fact that  $F(\mathcal{U})$  is open;
- or to a connected component of  $\widehat{\mathcal{G}_{F(z)}} \cap F(\mathcal{U})$  which does not contain  $F(z)$ . Since  $F$  is a homeomorphism and by the definition of  $\mathcal{G}_z$ , this contradicts the fact that  $w \in F(\mathcal{G}_z)$  and that  $F(\mathcal{G}_z)$  remains connected and contains  $F(z)$ .

In particular,  $w = F(x, \Gamma_z(x))$  for some  $x \in (a_z, b_z)$ . Without loss of generality we can assume that  $x \neq p_1(z)$ .

Denote

$$d(w, \mathcal{G}_{F(z)}) = \inf_{\zeta \in \mathcal{G}_{F(z)}} d(w, \zeta) \geq \min_{\zeta \in \widehat{\mathcal{G}_{F(z)}}} d(w, \zeta) =: \varepsilon > 0.$$

By the uniform continuity of  $F$  on  $\overline{\mathcal{U}}$ , there exists  $\delta > 0$  such that for any  $r, s \in \overline{\mathcal{U}}$  so that  $d(r, s) < \delta$  it holds that  $d(F(r), F(s)) < \frac{\varepsilon}{3}$ .

There exists  $\bar{n} \in \mathbb{N}$  such that for any  $n \geq \bar{n}$  we have

$$x \in (a_{n,z}, b_{n,z}) \quad \text{and} \quad |\Gamma_{n,z}(x) - \Gamma_z(x)| < \delta.$$

Consequently for any  $n \geq \bar{n}$  we have<sup>8</sup>

$$d(F(x, \Gamma_z(x)), F(x, \Gamma_{n,z}(x))) < \frac{\varepsilon}{3}.$$

Assume now that  $p_1(z) < x < b_z$ . For any  $n \in \mathbb{N}$ , when defined, it holds

$$\Gamma_z(x) \leq \Gamma_{n+1,z}(x) < \Gamma_{n,z}(x). \quad (3.57)$$

For any  $n \geq \bar{n}$  denote  $X(n) = p_1 \circ F(x, \Gamma_{n,z}(x))$ . From (3.57) and since  $F$  is the lift of a positive twist map, for any  $n \geq \bar{n}$  it holds  $X(n+1) < X(n)$ .

The sequence  $(X(n))_{n \geq \bar{n}}$  is decreasing and bounded (since  $F(\mathcal{U})$  is bounded). Therefore, it converges and we denote as

$$X := \lim_{m \rightarrow +\infty} X(m) \leq X(n) \quad \forall n \geq \bar{n}.$$

Observe that for any  $n \geq \bar{n}$  the point  $F(x, \Gamma_{n,z}(x))$  belongs to

$$F^{n+1}(V_{F^{-n}(z)}) = F^{n+1}(V_{F^{-(n+1)}(F(z))}) = \text{Graph}(\Gamma_{n+1,F(z)}).$$

Hence, for any  $n \geq \bar{n}$  it holds

$$X(n) \in (a_{n+1,F(z)}, b_{n+1,F(z)})$$

and

$$F(x, \Gamma_{n,z}(x)) = (X(n), \Gamma_{n+1,F(z)}(X(n))).$$

Because of the absence of over-conjugate points and because of Proposition 3.3.2, for any  $m \geq 1$  the function  $f^m$  is a positive local twist map at every point. Thus, since we are assuming that  $p_1(z) < x$  and since  $f^n$  is a local positive twist map, for any  $n \geq \bar{n}$  it holds that

$$p_2 \circ F^{-n}(z) < p_2 \circ F^{-n}(x, \Gamma_{n,z}(x)).$$

Since also  $f^{n+1}$  is a local positive twist map and since  $p_1 \circ F^{-n}(z) = p_1 \circ F^{-n}(x, \Gamma_{n,z}(x))$ , we deduce that

$$p_1(F(z)) = p_1 \circ F^{n+1}(F^{-n}(z)) < p_1 \circ F^{n+1}(F^{-n}(x, \Gamma_{n,z}(x))) = X(n).$$

Since for any  $n \geq \bar{n}$  it holds

$$X \leq X(n) < b_{n+1,F(z)}, \quad (3.58)$$

we conclude that

$$p_1(F(z)) \leq X \leq \liminf_{n \rightarrow +\infty} b_{n,F(z)} = b_{F(z)}.$$

The function  $\Gamma_{F(z)}$  is so defined at  $X$  and so  $(X, \Gamma_{F(z)}(X)) \in \widehat{\mathcal{G}_{F(z)}}$ .

Consider now

$$d(w, (X, \Gamma_{F(z)}(X))) \geq d(w, \widehat{\mathcal{G}_{F(z)}}) = \varepsilon > 0. \quad (3.59)$$

For any  $n \geq \bar{n}$  we have

$$\begin{aligned} d(w, (X, \Gamma_{F(z)}(X))) &= d(F(x, \Gamma_z(x)), (X, \Gamma_{F(z)}(X))) \leq \\ &\leq d(F(x, \Gamma_z(x)), F(x, \Gamma_{n,z}(x))) + d(F(x, \Gamma_{n,z}(x)), (X, \Gamma_{F(z)}(X))) < \end{aligned}$$

---

8. Recall that we are considering the standard Euclidean distance, see Notation 3.3.1.

$$\begin{aligned}
&< \frac{\varepsilon}{3} + d((X(n), \Gamma_{n+1, F(z)}(X(n))), (X, \Gamma_{F(z)}(X))) \leq \\
&\leq \frac{\varepsilon}{3} + d(X(n), X) + d(\Gamma_{n+1, F(z)}(X(n)), \Gamma_{F(z)}(X)) \leq \\
&\leq \frac{\varepsilon}{3} + d(X(n), X) + d(\Gamma_{n+1, F(z)}(X(n)), \Gamma_{n+1, F(z)}(X)) + d(\Gamma_{n+1, F(z)}(X), \Gamma_{F(z)}(X)).
\end{aligned}$$

Every  $\Gamma_{n+1, F(z)}$  is  $K$ -Lipschitz (see (3.53)) and, since  $(X(n))_{n \in \mathbb{N}}$  converges to  $X$ , there exists  $\tilde{n} \geq \bar{n}$  such that for any  $n \geq \tilde{n}$  it holds  $d(X(n), X) < \frac{\varepsilon}{3(1+K)}$ . Hence

$$d(X(n), X) + d(\Gamma_{n+1, F(z)}(X(n)), \Gamma_{n+1, F(z)}(X)) < \frac{\varepsilon}{3}.$$

Therefore for  $n \geq \tilde{n}$  we have that

$$d(w, (X, \Gamma_{F(z)}(X))) < \frac{2}{3}\varepsilon + d(\Gamma_{n+1, F(z)}(X), \Gamma_{F(z)}(X)). \quad (3.60)$$

If  $X < b_{F(z)}$ , then by the pointwise convergence of  $(\Gamma_{n+1, F(z)})_{n \in \mathbb{N}}$  towards  $\Gamma_{F(z)}$  there exists  $N \geq \tilde{n}$  such that for any  $n \geq N$  it holds

$$d(\Gamma_{n+1, F(z)}(X), \Gamma_{F(z)}(X)) < \frac{\varepsilon}{3}.$$

Consequently for  $n \geq N$ , from (3.59) and from (3.60), we conclude that

$$\varepsilon \leq d(w, (X, \Gamma_{F(z)}(X))) < \frac{2}{3}\varepsilon + d(\Gamma_{n+1, F(z)}(X), \Gamma_{F(z)}(X)) < \varepsilon,$$

obtaining so the required contradiction.

Suppose now that  $X = b_{F(z)}$ . Let  $(y_m)_{m \in \mathbb{N}}$  be a sequence in  $(a_{F(z)}, b_{F(z)})$  converging to  $b_{F(z)}$ . In particular it holds that  $\Gamma_{F(z)}(X) = \Gamma_{F(z)}(b_{F(z)}) = \lim_{m \rightarrow +\infty} \Gamma_{F(z)}(y_m)$ .

Fix now  $\bar{m} \in \mathbb{N}$  so that

$$d(y_{\bar{m}}, b_{F(z)}) < \frac{\varepsilon}{9K} \quad \text{and} \quad d(\Gamma_{F(z)}(y_{\bar{m}}), \Gamma_{F(z)}(b_{F(z)})) < \frac{\varepsilon}{9}. \quad (3.61)$$

Remark that for any  $m$  it holds  $y_m < X = b_{F(z)} = \liminf_{n \rightarrow +\infty} b_{n, F(z)}$ . Consequently, for any  $n \geq \tilde{n}$ , from (3.60) and (3.61), we have

$$\begin{aligned}
\varepsilon &\leq d(w, (X, \Gamma_{F(z)}(X))) < \frac{2}{3}\varepsilon + d(\Gamma_{n+1, F(z)}(X), \Gamma_{F(z)}(X)) \leq \\
&\leq \frac{2}{3}\varepsilon + d(\Gamma_{n+1, F(z)}(X), \Gamma_{n+1, F(z)}(y_{\bar{m}})) + d(\Gamma_{n+1, F(z)}(y_{\bar{m}}), \Gamma_{F(z)}(y_{\bar{m}})) + d(\Gamma_{F(z)}(y_{\bar{m}}), \Gamma_{F(z)}(X)) < \\
&< \frac{2}{3}\varepsilon + Kd(X, y_{\bar{m}}) + d(\Gamma_{n+1, F(z)}(y_{\bar{m}}), \Gamma_{F(z)}(y_{\bar{m}})) + \frac{\varepsilon}{9} < \\
&< \frac{2}{3}\varepsilon + \frac{2}{9}\varepsilon + d(\Gamma_{n+1, F(z)}(y_{\bar{m}}), \Gamma_{F(z)}(y_{\bar{m}})).
\end{aligned}$$

By the definition of  $a_z$  and  $b_z$  there exists  $N \in \mathbb{N}$ ,  $N \geq \tilde{n}$  such that for any  $n \geq N$  we have that  $y_{\bar{m}} \in (a_{n, z}, b_{n, z})$ .

By the pointwise convergence of  $(\Gamma_{n+1, F(z)})_{n \in \mathbb{N}}$  to  $\Gamma_{F(z)}$  at  $y_{\bar{m}}$ , there exists  $\bar{N} \geq N$  such that for any  $n \geq \bar{N}$  it holds

$$d(\Gamma_{n+1, F(z)}(y_{\bar{m}}), \Gamma_{F(z)}(y_{\bar{m}})) < \frac{\varepsilon}{9}.$$

Therefore, we conclude that for  $n \geq \bar{N}$

$$\varepsilon \leq d(w, (X, \Gamma_{F(z)}(X))) < \frac{2}{3}\varepsilon + \frac{2}{9}\varepsilon + \frac{\varepsilon}{9} = \varepsilon,$$

that is the required contradiction.

The case  $p_1(z) > x$  can be discussed similarly and so we conclude the proof.  $\square$

We are interested in the dynamics over the instability disc on the annulus.

**Notation 3.3.6.** Let  $z \in \mathcal{U}$ . The closed leaf  $cl(\mathcal{G}_z)$  divides  $\mathcal{U}$  into two (simply) connected components because  $\mathcal{U}$  is homeomorphic to an open disc and because  $\mathcal{G}_z \subset \mathcal{U}$  and  $(cl(\mathcal{G}_z) \setminus \mathcal{G}_z) \subset \partial\mathcal{U}$  (see Theorems V.11.7 and VI.5.1 in [New61]).

Denote as  $\mathcal{U}_{z,+}$  the connected component lying above  $\mathcal{G}_z$ , i.e. the connected component containing

$$V_z^{CC,+} := \{(p_1(z), y) \in V_z^{CC,\mathcal{U}} : y > p_2(z)\}.$$

Denote as  $\mathcal{U}_{z,-}$  the connected component lying below  $\mathcal{G}_z$ , i.e. the connected component containing

$$V_z^{cc,-} := \{(p_1(z), y) \in V_z^{CC,\mathcal{U}} : y < p_2(z)\}.$$

See Figure 3.6.

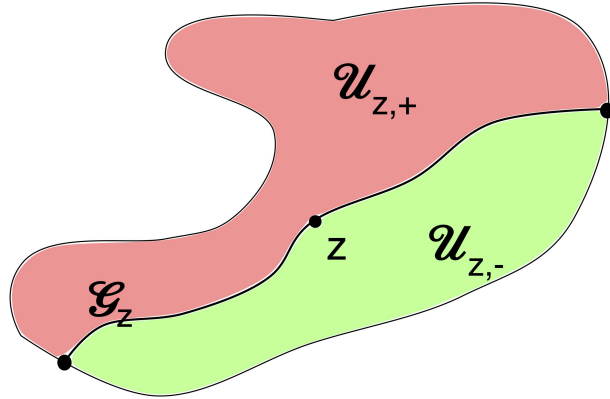


Figure 3.6 – The connected components  $\mathcal{U}_{z,+}$  and  $\mathcal{U}_{z,-}$ .

Similarly, the notations  $(F(\mathcal{U}))_{F(z),+}$  and  $(F(\mathcal{U}))_{F(z),-}$  refer to the connected components of  $F(\mathcal{U}) \setminus \mathcal{G}_{F(z)}$  lying, respectively, above and below the leaf  $\mathcal{G}_{F(z)}$ .

**Proposition 3.3.7.**  $F(\mathcal{U}_{z,+}) = (F(\mathcal{U}))_{F(z),+}$  and  $F(\mathcal{U}_{z,-}) = (F(\mathcal{U}))_{F(z),-}$ .

*Proof.* By Proposition 3.3.6 we have that  $F(\mathcal{G}_z) = \mathcal{G}_{F(z)}$ . Since  $F$  is a homeomorphism, two cases can occur. Either

$$F(\mathcal{U}_{z,+}) = (F(\mathcal{U}))_{F(z),+} \text{ and } F(\mathcal{U}_{z,-}) = (F(\mathcal{U}))_{F(z),-}$$

or

$$F(\mathcal{U}_{z,+}) = (F(\mathcal{U}))_{F(z),-} \text{ and } F(\mathcal{U}_{z,-}) = (F(\mathcal{U}))_{F(z),+}.$$

Let  $x \in (a_z, b_z)$  be such that  $x > p_1(z)$ . Let  $\bar{n} \in \mathbb{N}$  be such that  $x \in (a_{\bar{n},z}, b_{\bar{n},z})$  and  $p_1 \circ F(x, \Gamma_{\bar{n},z}(x)) = X(\bar{n}) \in (a_{F(z)}, b_{F(z)})$ .

By the definition of  $\Gamma_z(x)$  and of  $(\Gamma_{n,z}(x))_{n \in \mathbb{N}}$  we have that  $\Gamma_{\bar{n},z}(x) > \Gamma_z(x)$ . Thus the point  $(x, \Gamma_{\bar{n},z}(x))$  belongs to  $\mathcal{U}_+$ .

Since  $f^m$  is a positive local twist map for any  $m \in \mathbb{N}$  (in particular for  $\bar{n}, \bar{n}+1$ ) we deduce that

$$p_2 \circ F^{-\bar{n}}(x, \Gamma_{\bar{n},z}(x)) > p_2 \circ F^{-\bar{n}}(z)$$

and consequently

$$X(\bar{n}) := p_1 \circ F(x, \Gamma_{\bar{n},z}(x)) > p_1 \circ F(z). \quad (3.62)$$

From (3.62), since  $F(\text{Graph}(\Gamma_{\bar{n},z})) = \text{Graph}(\Gamma_{\bar{n}+1,F(z)})$  and since  $X(\bar{n}) \in (a_{F(z)}, b_{F(z)})$ , we deduce that

$$\Gamma_{\bar{n}+1,F(z)}(X(\bar{n})) \geq \Gamma_{F(z)}(X(\bar{n})).$$

Actually, since  $F(x, \Gamma_{\bar{n},z}(x))$  does not belong to  $\mathcal{G}_{F(z)}$ , it holds that  $\Gamma_{\bar{n}+1,F(z)}(X(\bar{n})) > \Gamma_{F(z)}(X(\bar{n}))$ . Equivalently

$$(X(\bar{n}), \Gamma_{\bar{n}+1,F(z)}(X(\bar{n}))) = F(x, \Gamma_{\bar{n},z}(x)) \in (F(\mathcal{U}))_{F(z),+}.$$

Since we have exhibited a point of  $\mathcal{U}_{z,+}$  whose image through  $F$  is contained in  $(F(\mathcal{U}))_{F(z),+}$ , we conclude that

$$F(\mathcal{U}_{z,+}) = (F(\mathcal{U}))_{F(z),+} \quad \text{and} \quad F(\mathcal{U}_{z,-}) = (F(\mathcal{U}))_{F(z),-}.$$

□

We are now going to show that the projection on the annulus of every leaf  $\mathcal{G}_z$  is  $f$ -periodic.

**Proposition 3.3.8.** *Let  $p \times \text{Id} : \mathbb{R}^2 \rightarrow \mathbb{A}$  be the projection on the annulus and let  $U = (p \times \text{Id})(\mathcal{U})$ . For any  $\bar{z} \in U \subset \mathbb{A}$  there exists  $M \in \mathbb{N}$  such that*

$$(p \times \text{Id})(\mathcal{G}_{F^{MN}(z)}) = (p \times \text{Id})(\mathcal{G}_z)$$

where  $N$  is the period of  $U$  and  $z \in \mathcal{U}$  is the lift of  $\bar{z}$ .

*Proof.* We work on the lifted framework. Let  $z', z'' \in \bigcup_{i \in \mathbb{Z}} F^{iN}(\mathcal{U})$ . In  $\mathbb{R}^2$  we say that two leaves  $\mathcal{G}_{z'}, \mathcal{G}_{z''}$  are  $t$ -equivalent if there exists  $k \in \mathbb{Z}$  so that

$$\mathcal{G}_{z'} + (k, 0) = \{x + (k, 0) : x \in \mathcal{G}_{z'}\} = \mathcal{G}_{z''}.$$

Clearly, two leaves are  $t$ -equivalent in  $\mathbb{R}^2$  if and only if their projections  $p \times \text{Id}$  on the annulus coincide.

Fix  $z \in \mathcal{U}$ . Argue by contradiction and assume that for any  $M \in \mathbb{N}$  the leaves  $\mathcal{G}_z$  and  $\mathcal{G}_{F^{MN}(z)}$  are not  $t$ -equivalent. In particular,  $\mathcal{G}_z$  and  $\mathcal{G}_{F^N(z)}$  are not  $t$ -equivalent, that is

$$\mathcal{G}_z + (k, 0) \neq \mathcal{G}_{F^N(z)} = F^N(\mathcal{G}_z),$$

where  $k \in \mathbb{Z}$  is such that  $F^N(z) \in \mathcal{U} + (k, 0)$  (so it depends on the lift  $F$ ). The last equality is Proposition 3.3.6.



**Claim 3.3.2.** If  $\mathcal{G}_z$  and  $\mathcal{G}_{F^N(z)}$  are not  $t$ -equivalent, then

$$(\mathcal{U}_{z,+} + (k, 0)) \cap (F^N(\mathcal{U}))_{F^N(z),+} = \emptyset.$$

*Proof.* From Proposition 3.3.5, since different leaves cannot intersect strictly, it holds that either  $(\mathcal{U}_{z,+} + (k, 0)) \subset (F^N(\mathcal{U}))_{F^N(z),+}$  or  $(F^N(\mathcal{U}))_{F^N(z),+} \subset (\mathcal{U}_{z,+} + (k, 0))$  or the two components are disjoint (see Figure 3.7).

If  $(F^N(\mathcal{U}))_{F^N(z),+}$  is (strictly) contained in  $\mathcal{U}_{z,+} + (k, 0)$  (see (a) in Figure 3.7), then

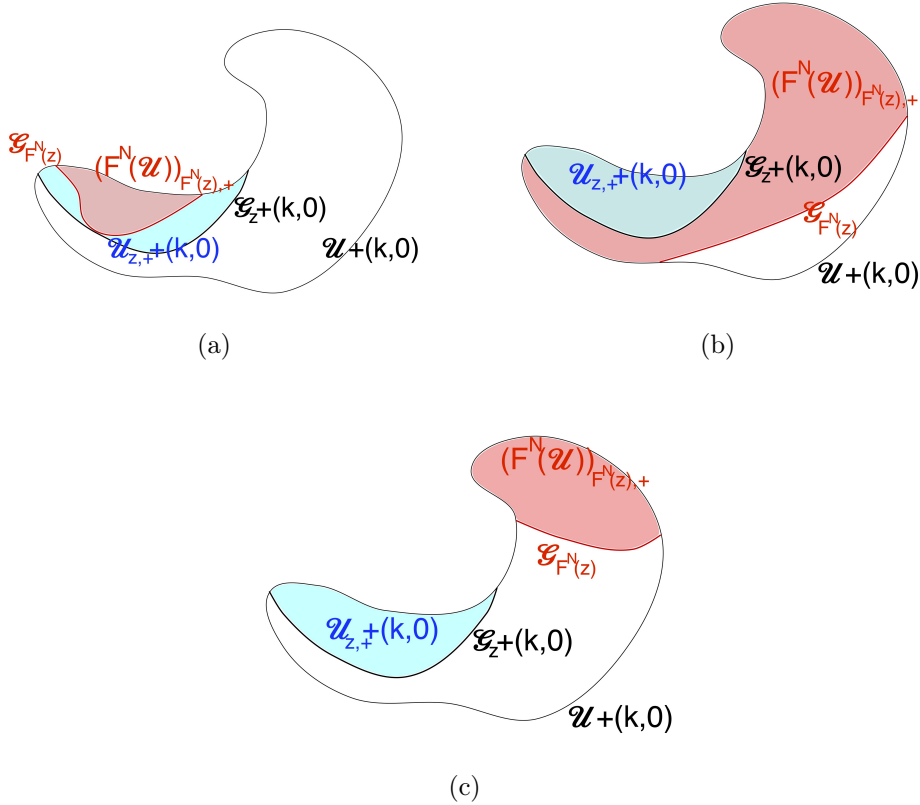


Figure 3.7 – The three possible cases discussed in Claim 3.3.2.

$$\omega((F^N(\mathcal{U}))_{F^N(z),+}) < \omega(\mathcal{U}_{z,+}). \quad (3.63)$$

Indeed, if there exists  $w \in \mathcal{U}_{z,+} + (k, 0)$  and  $w \notin \overline{(F^N(\mathcal{U}))_{F^N(z),+}}$  (i.e.  $w \notin \mathcal{G}_{F^N(z)}$ ), then we have that  $\omega((F^N(\mathcal{U}))_{F^N(z),+}) < \omega(\mathcal{U}_{z,+})$  because  $\omega$  is positive on open sets.

If every  $w \in \mathcal{U}_{z,+} + (k, 0) \setminus (F^N(\mathcal{U}))_{F^N(z),+}$  is contained in  $\overline{(F^N(\mathcal{U}))_{F^N(z),+}} \setminus (F^N(\mathcal{U}))_{F^N(z),+}$ , then every such  $w$  has to be in  $\mathcal{G}_{F^N(z)}$  because it cannot belong to  $\partial F(\mathcal{U})$ .

Consequently, it holds

$$(F^N(\mathcal{U}))_{F^N(z),+} \subset \mathcal{U}_{z,+} + (k, 0) \subset (F^N(\mathcal{U}))_{F^N(z),+} \cup \mathcal{G}_{F^N(z)} \subset \overline{(F^N(\mathcal{U}))_{F^N(z),+}}.$$

Since  $\mathcal{U}_{z,+} + (k, 0)$  is open, we have that  $\mathcal{U}_{z,+} + (k, 0) = (F^N(\mathcal{U}))_{F^N(z),+}$ .

Thus, we deduce that  $\mathcal{G}_z + (k, 0) = \mathcal{G}_{F^N(z)}$ , contradicting the hypothesis that the two leaves are not  $t$ -equivalent.

Assuming so (3.63), from Proposition 3.3.7, we conclude that

$$\omega(F^N(\mathcal{U}_{z,+})) = \omega((F^N(\mathcal{U}))_{F^N(z),+}) < \omega(\mathcal{U}_{z,+}),$$

contradicting the fact that  $F$  is conservative. The same argument holds if  $\mathcal{U}_{z,+} + (k, 0) \subset (F^N(\mathcal{U}))_{F^N(z),+}$  (see (b) in Figure 3.7).

We conclude that  $(\mathcal{U}_{z,+} + (k, 0)) \cap (F^N(\mathcal{U}))_{F^N(z),+} = \emptyset$  (see (c) in Figure 3.7).  $\square$

Arguing similarly, for any  $i, j \in \mathbb{N}, i \neq j$ , since by contradiction every  $\mathcal{G}_{F^{iN}(z)}$  and  $\mathcal{G}_{F^{jN}(z)}$  are not  $t$ -equivalent, from Claim 3.3.2 we have that

$$((F^{iN}(\mathcal{U}))_{F^{iN}(z),+} + (k(i-j), 0)) \cap (F^{jN}(\mathcal{U}))_{F^{jN}(z),+} = \emptyset.$$

Since  $F$  is conservative and since  $\omega$  is invariant by translations, we obtain that

$$\omega(\mathcal{U}) \geq \sum_{i \in \mathbb{N}} \omega((F^{iN}(\mathcal{U}))_{F^{iN}(z),+}) = +\infty,$$

which contradicts the fact that  $\omega(\mathcal{U})$  is finite.  $\square$

### Proof of Proposition 3.1.3

We finally prove Proposition 3.1.3. That is, any instability disc has a subset of positive measure whose points have non zero torsion.

*Proof of Proposition 3.1.3.* Argue by contradiction and assume that  $\omega$ -almost every  $z \in U$  has zero torsion. Equivalently,  $U \subset \mathbb{A}$  satisfies Hypothesis (H). We can so build the foliation  $(\mathcal{G}_z)_{z \in U}$  made up of periodic leaves.

Let  $\mathcal{U}$  be a connected component of  $(p \times \text{Id})^{-1}(U)$ . Fix  $z \in \mathcal{U}$ . Let  $M \in \mathbb{N}$  be such that

$$(p \times \text{Id})(\mathcal{G}_z) = (p \times \text{Id})(\mathcal{G}_{F^{MN}(z)}),$$

that is the leaves  $\mathcal{G}_z$  and  $\mathcal{G}_{F^{MN}(z)}$  are  $t$ -equivalent. Let  $k \in \mathbb{Z}$  be such that  $F^{MN}(\mathcal{U}) = \mathcal{U} + (k, 0)$ . Consequently

$$\mathcal{G}_z + (k, 0) = \mathcal{G}_{F^{MN}(z)}. \quad (3.64)$$

By Proposition 3.3.2 and by Remark 3.3.4, every  $f^n$  is a positive local twist map, in particular for  $n = MN$ .

Consider  $V_z^{CC, \mathcal{U}}$  (i.e. the connected component of  $V_z \cap \mathcal{U}$  containing  $z$ ) and  $F^{MN}(V_z^{CC, \mathcal{U}})$ . Since  $f^{MN}$  is a positive local twist map, the image  $F^{MN}(V_z^{CC, \mathcal{U}})$  is a “partial” graph and

$$p_2(V_z^{CC, \mathcal{U}}) \ni y \mapsto p_1 \circ F^{MN}(p_1(z), y) \in \mathbb{R}$$

is an increasing diffeomorphism to its image.

According to Notation 3.3.6,  $\mathcal{U}_{z,+}$  and  $\mathcal{U}_{z,-}$  denote the two connected components of  $\mathcal{U} \setminus \mathcal{G}_z$  lying respectively above and below  $\mathcal{G}_z$ .

From Proposition 3.3.7 we deduce that

$$F^{MN}(\mathcal{U}_{z,+}) = (F^{MN}(\mathcal{U}))_{F^{MN}(z),+}$$

and, since from (3.64) we have that  $\mathcal{G}_z + (k, 0) = \mathcal{G}_{F^{MN}(z)}$ , we conclude that

$$\mathcal{U}_{z,+} + (k, 0) = F^{MN}(\mathcal{U}_{z,+}).$$

In particular

$$F^{MN}(V_z^{CC,+}) \subset \mathcal{U}_{z,+} + (k, 0) \quad \text{and} \quad F^{MN}(V_z^{CC,-}) \subset \mathcal{U}_{z,-} + (k, 0).$$

Denote now as  $\mathcal{W}_{z,R}$  and  $\mathcal{W}_{z,L}$  the connected components of  $\mathcal{U} \setminus V_z^{CC,\mathcal{U}}$  that lie respectively locally on the right and locally on the left of the vertical segment  $V_z^{CC,\mathcal{U}}$ .

Let  $\mathcal{V} = \mathcal{U}_{z,+} \cap \mathcal{W}_{z,R}$ , see Figure 3.8. Since  $F$  is conservative, it holds that  $\omega(\mathcal{V}) = \omega(F^{MN}(\mathcal{V}))$ .

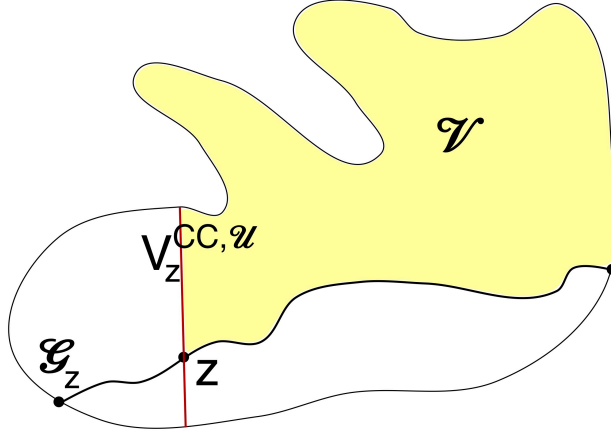


Figure 3.8 – The set  $\mathcal{V} = \mathcal{W}_{z,R} \cap \mathcal{U}_{z,+}$ .

**Claim 3.3.3.**  $p_1(F^{MN}(z)) < p_1(z) + k$ .

*Proof of Claim 3.3.3.* Argue by contradiction and assume that  $p_1(z) + k \leq p_1(F^{MN}(z))$ . Observe that  $F^{MN}(\mathcal{V})$  is one of the four connected components of

$$(\mathcal{U} + (k, 0)) \setminus ((\mathcal{G}_z + (k, 0)) \cup F^{MN}(V_z^{CC,\mathcal{U}})).$$

From Proposition 3.3.7 we know that  $F^{MN}(\mathcal{V}) \subset \mathcal{U}_{z,+} + (k, 0)$ .

Moreover, since  $f^{MN}$  is a positive local twist map and it preserves the orientation, the image  $F^{MN}(\mathcal{W}_{z,R})$  is the connected component of  $(\mathcal{U} + (k, 0)) \setminus F^{MN}(V_z^{CC,\mathcal{U}})$  that lies locally below  $F^{MN}(V_z^{CC,\mathcal{U}})$  (which is a partial graph).

Since  $F^{MN}$  is a positive local twist map and by contradiction hypothesis, for any  $\zeta \in V_z^{CC,+}$  it holds

$$p_1 \circ F^{MN}(\zeta) > p_1 \circ F^{MN}(z) \geq p_1(z) + k.$$

That is, the image  $F^{MN}(V_z^{CC,+})$  is contained in  $\mathcal{W}_{z,R} + (k, 0)$ .

Consequently we have that  $F^{MN}(\mathcal{V}) \subset \mathcal{V} + (k, 0)$ . Actually it is strictly contained since  $F^{MN}(V_z^{CC,+})$  is a partial graph.

Thus,  $\omega(F^{MN}(\mathcal{V})) < \omega(\mathcal{V})$  and this contradicts the fact that  $F$  is conservative. □

Consider now the set  $\mathcal{U}_{z,-} \cap \mathcal{W}_{z,L}$ . Arguing as in Claim 3.3.3 for  $\mathcal{V}$ , we deduce that

$$p_1(F^{MN}(z)) > p_1(z) + k.$$

This provide us a contradiction. The absurd was assuming that  $\omega$ -almost every  $z \in U$  has null torsion and we conclude that

$$\omega(\{z \in U : \text{Torsion}(f, z) \neq 0\}) > 0.$$

□

## 3.4 Appendix of Chapter 3

### 3.4.1 About tridiagonal symmetric positive definite matrices

This Section of the Appendix is devoted to prove a technical fact about tridiagonal, symmetric, positive definite matrices that is used in Lemma 3.2.5. Let us introduce the following notation.

**Notation 3.4.1.** The set  $TSP(n)$  is the set of matrices  $A \in M_n(\mathbb{R})$  which are tridiagonal, symmetric, positive definite and whose off-diagonal terms are all negative. That is, a matrix  $A \in TSP(n)$  is of the form

$$A = \begin{pmatrix} \alpha_1 & \beta_1 & 0 & \dots & 0 \\ \beta_1 & \alpha_2 & \beta_2 & \dots & 0 \\ 0 & \beta_2 & \alpha_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \beta_{n-1} \\ 0 & \dots & \dots & \beta_{n-1} & \alpha_n \end{pmatrix},$$

where  $\alpha_i > 0$  for all  $i \in \llbracket 1, n \rrbracket$  and  $\beta_i < 0$  for all  $i \in \llbracket 1, n-1 \rrbracket$ .

**Lemma 3.4.1.** *Let  $A \in TSP(n)$ . Then its inverse matrix  $A^{-1}$  has all positive entries.*

*Proof.* Let us prove the result by induction over  $n \in \mathbb{N}^*$ .

For  $n = 1$  we are considering a scalar  $\alpha_1$ . Clearly its inverse  $\frac{1}{\alpha_1}$  is positive since  $\alpha_1 > 0$ . Assume the statement holds true for matrices in  $TSP(n)$ . Let us show it for matrices in  $TSP(n+1)$ . Consider a matrix  $B$  in  $TSP(n+1)$ , i.e.

$$B = \left( \begin{array}{c|c} A & \begin{matrix} 0 \\ \vdots \\ 0 \\ \beta_n \end{matrix} \\ \hline \begin{matrix} 0 & \dots & 0 & \beta_n \end{matrix} & \alpha_{n+1} \end{array} \right),$$

where  $A \in TSP(n)$ ,  $\beta_n < 0$  and  $\alpha_{n+1} > 0$ . Its inverse matrix is also symmetric and positive definite: denote it as

$$\left( \begin{array}{c|c} C & v \\ \hline v^t & \gamma \end{array} \right),$$

where  $C \in M_n(\mathbb{R})$ ,  $v \in \mathbb{R}^n$  and  $\gamma \in \mathbb{R}$ . Then we have

$$\left( \begin{array}{c|c} A & \begin{matrix} 0 \\ \vdots \\ 0 \\ \beta_n \end{matrix} \\ \hline \begin{matrix} 0 & \dots & 0 & \beta_n \end{matrix} & \alpha_{n+1} \end{array} \right) \left( \begin{array}{c|c} C & v \\ \hline v^t & \gamma \end{array} \right) = \mathbb{I}_{n+1},$$

where  $\mathbb{I}_{n+1}$  is the identity matrix of size  $n + 1$ . Equivalently

$$\left( \begin{array}{c|c} AC + \begin{pmatrix} \mathbb{O}_{(n-1) \times n} \\ \beta_n v^t \end{pmatrix} & Av + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \beta_n \gamma \end{pmatrix} \\ \hline (0 \ \dots \ 0 \ \beta_n) C + \alpha_{n+1} v^t & \alpha_{n+1} \gamma \end{array} \right) = \mathbb{I}_{n+1}.$$

Consequently, since  $\alpha_{n+1} > 0$  and  $\alpha_{n+1} \gamma = 1$ , we deduce that  $\gamma > 0$ .

Since

$$Av + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \beta_n \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^n,$$

we have that

$$v = A^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -\beta_n \gamma \end{pmatrix}.$$

Since  $-\beta_n \gamma$  is positive and since all the entries of  $A^{-1}$  are positive by inductive hypothesis, we deduce that all entries of the vector  $v$  are positive.

We also have that

$$AC + \begin{pmatrix} \mathbb{O}_{(n-1) \times n} \\ \beta_n v^t \end{pmatrix} = \mathbb{I}_n.$$

Consequently

$$C = A^{-1} + A^{-1} \begin{pmatrix} \mathbb{O}_{(n-1) \times n} \\ -\beta_n v^t \end{pmatrix}.$$

Since all the entries of  $A^{-1}$  are positive by inductive hypothesis and since all the entries of the vector  $-\beta_n v^t$  are positive, we conclude that also all the entries of  $C$  are positive. That is, the inverse matrix of  $B$  has all positive entries. □

### 3.4.2 Extension theorem

In this Section of the Appendix we recall a useful classical result on extension theorems that is largely used in the construction of the foliation  $(\mathcal{G}_z)_{z \in U}$  in Section 3.3.2 to extend the functions  $\Gamma_{n,z}$  on boundary points.

**Lemma 3.4.2.** *Let  $X$  be a complete space and let  $E \subset X$ . Let  $f : E \rightarrow \mathbb{R}$  be a uniform continuous  $K$ -Lipschitz function. Then there exists a unique extension  $\bar{f} : \bar{E} \rightarrow \mathbb{R}$  of  $f$  which is uniform continuous and  $K$ -Lipschitz.*

*Proof.* Define the function  $\bar{f}$  as follows:

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in E \\ \lim_{\substack{y \rightarrow x \\ y \in E}} f(y) & \text{if } y \in \bar{E} \setminus E. \end{cases}$$

We start by showing the following

**Claim 3.4.1.** For  $x \in \bar{E} \setminus E$  the limit  $\lim_{\substack{y \rightarrow x \\ y \in E}} f(y) \in \mathbb{R}$  exists.

Let  $x \in \bar{E} \setminus E$  and let  $(x_n)_{n \in \mathbb{N}} \subset E$  be a sequence converging towards  $x$ . In particular,  $(x_n)_{n \in \mathbb{N}}$  is a Cauchy sequence and consequently  $(f(x_n))_{n \in \mathbb{N}}$  is a Cauchy sequence too. Denote as  $\bar{y} \in \mathbb{R}$  the limit of  $(f(x_n))_{n \in \mathbb{N}}$ . Let  $(x'_n)_{n \in \mathbb{N}}$  be another sequence in  $E$  converging to  $x$ . Fix  $\varepsilon > 0$ .

By the uniform continuity of  $f$ , there exists  $\delta > 0$  so that for any  $z, w \in E$ ,  $d(z, w) < \delta$  it holds that  $|f(z) - f(w)| < \varepsilon$ . Since both  $(x_n)_{n \in \mathbb{N}}$  and  $(x'_n)_{n \in \mathbb{N}}$  converge to  $x$ , the distance  $d(x_n, x'_n)$  is tending to 0 as  $n$  goes to  $+\infty$ . Let  $\bar{n} \in \mathbb{N}$  be such that for any  $n \geq \bar{n}$  we have  $d(x_n, x'_n) < \delta$ . Consequently for any  $n \geq \bar{n}$  it holds that  $|f(x_n) - f(x'_n)| < \varepsilon$ . By the arbitrariness of  $\varepsilon$ , we conclude that  $\lim_{n \rightarrow +\infty} f(x'_n) = \bar{y}$ . Equivalently, the limit  $\lim_{\substack{y \rightarrow x \\ y \in E}} f(y)$  exists.

The function  $\bar{f}$  is so well-defined and it is continuous. This implies its uniqueness. We can now prove the following

**Claim 3.4.2.** The function  $\bar{f}$  is uniform continuous.

Argue by contradiction and assume there exists  $\varepsilon_0 > 0$  such that for any  $\delta > 0$  there are  $x, y \in \bar{E}$  so that  $d(x, y) < \delta$  and  $|\bar{f}(x) - \bar{f}(y)| \geq \varepsilon_0$ .

Since  $f$  is uniform continuous, at least one among  $x$  and  $y$  has to be in  $\bar{E} \setminus E$ .

Let  $\delta_0 > 0$  be such that for any  $z, w \in E$ ,  $d(z, w) < \delta_0$  it holds  $|f(z) - f(w)| = |\bar{f}(z) - \bar{f}(w)| < \frac{\varepsilon_0}{4}$ .

Fix now  $\delta = \frac{\delta_0}{4}$ . Let  $x, y \in \bar{E}$  be the points such that, by contradiction hypothesis,  $d(x, y) < \frac{\delta_0}{4}$  and  $|\bar{f}(x) - \bar{f}(y)| \geq \varepsilon_0$ .

Let  $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}$  be sequences in  $E$  converging respectively to  $x$  and  $y$ . In particular, let  $n \in \mathbb{N}$  be large enough such that

$$d(x, x_n) < \frac{\delta_0}{4} \quad \text{and} \quad d(y, y_n) < \frac{\delta_0}{4}.$$

Since both  $x_n$  and  $y_n$  belong to  $E$ , by the uniform continuity of  $f$ , we have that  $|f(x_n) - f(y_n)| < \frac{\varepsilon_0}{4}$  because  $d(x_n, y_n) < \delta_0$ .

Up to choose a bigger  $n$ , by the continuity of  $\bar{f}$ , we can also assume that

$$|\bar{f}(x) - \bar{f}(x_n)| < \frac{\varepsilon_0}{4} \quad \text{and} \quad |\bar{f}(y) - \bar{f}(y_n)| < \frac{\varepsilon_0}{4}.$$

Consequently

$$\varepsilon_0 \leq |\bar{f}(x) - \bar{f}(y)| \leq |\bar{f}(x) - \bar{f}(x_n)| + |f(x_n) - f(y_n)| + |\bar{f}(y_n) - \bar{f}(y)| < \frac{3\varepsilon_0}{4},$$

which is the required contradiction.

Since  $f$  is  $K$ -Lipschitz, we can also deduce the following

**Claim 3.4.3.** The function  $\bar{f}$  is  $K$ -Lipschitz.

Indeed, let  $x, y \in \bar{E}$  and let  $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}$  be sequences in  $E$  converging to  $x, y$  respectively.

Fix  $\varepsilon > 0$ . There exists  $\bar{n} \in \mathbb{N}$  such that for any  $n \geq \bar{n}$  it holds

$$d(x, x_n) < \frac{\varepsilon}{4K} \quad \text{and} \quad d(y, y_n) < \frac{\varepsilon}{4K}, \tag{3.65}$$

where  $K$  is the Lipschitz constant of  $f$ . Moreover, by the continuity of  $\bar{f}$ , there exists  $\tilde{n} \in \mathbb{N}$  such that for any  $n \geq \tilde{n}$  we have

$$|\bar{f}(x) - \bar{f}(x_n)| < \frac{\varepsilon}{4} \quad \text{and} \quad |\bar{f}(y) - \bar{f}(y_n)| < \frac{\varepsilon}{4}.$$

Then for any  $n \geq \max(\bar{n}, \tilde{n})$  we have

$$|\bar{f}(x) - \bar{f}(y)| \leq |\bar{f}(x) - \bar{f}(x_n)| + |\bar{f}(x_n) - \bar{f}(y_n)| + |\bar{f}(y_n) - \bar{f}(y)| < \frac{\varepsilon}{2} + |\bar{f}(x_n) - \bar{f}(y_n)|.$$

Now, since  $x_n, y_n \in E$  and since  $f$  is  $K$ -Lipschitz, it holds that

$$|f(x_n) - f(y_n)| \leq Kd(x_n, y_n).$$

Consequently, from (3.65),

$$|\bar{f}(x) - \bar{f}(y)| < \frac{\varepsilon}{2} + Kd(x_n, y_n) \leq \frac{\varepsilon}{2} + k(d(x_n, x) + d(x, y) + d(y, y_n)) < Kd(x, y) + \varepsilon.$$

From the arbitrariness of  $\varepsilon$ , we conclude that  $|\bar{f}(x) - \bar{f}(y)| \leq Kd(x, y)$ , that is also  $\bar{f}$  is  $K$ -Lipschitz. □

# Chapter 4

## Torsion of horseshoes

Let  $S$  be a surface among  $\mathbb{R}^2, \mathbb{A}, \mathbb{T}^2$ : endow  $S$  with the standard Euclidean metric and the standard trivialization. Fix the counterclockwise orientation of the plane  $\mathbb{R}^2$  and the constant vector field  $\mathcal{H} = (1, 0)$ .

Let  $f : S \rightarrow S$  be a  $\mathcal{C}^1$  diffeomorphism isotopic to the identity. In particular,  $f$  preserves the orientation. For any  $x \in S$ , we denote as  $\mathcal{O}(x, f)$  the orbit of  $x$  with respect to  $f$ . Recall that, if  $N \subset S$  and  $x \in N$ , we denote as  $CC(N, x)$  the connected component of  $N$  containing  $x$ .

If  $S = \mathbb{R}^2$ , we make the further assumption that  $f$  has compact support.

We briefly recall the notions of (finite-time) torsion. Let  $I = (f_t)_{t \in \mathbb{R}}$  be an isotopy joining the identity to  $f$  in  $\text{Diff}^1(S)$ . For any  $(x, v) \in TS, v \neq 0$ , we recall the notation of the oriented angle function

$$\mathbb{R}_+ \ni t \mapsto v(I)(x, v, t) = \theta(\mathcal{H}, Df_t(x)v) \in \mathbb{T}.$$

Let  $\mathbb{R} \ni t \mapsto \tilde{v}(I)(x, v, t) \in \mathbb{R}$  be a continuous determination of the previous angle function. The torsion at finite time  $n \in \mathbb{N}, n \neq 0$  at  $(x, v)$  is then

$$\text{Torsion}_n(I, x, v) := \frac{1}{n} (\tilde{v}(I)(x, v, n) - \tilde{v}(I)(x, v, 0)).$$

The (asymptotic) torsion at  $x \in S$  is the limit, when it exists,

$$\text{Torsion}(I, x) := \lim_{n \rightarrow +\infty} \text{Torsion}_n(I, x, v).$$

**Remark 4.0.1.** The torsion, both at finite-time and asymptotic, does not depend on the choice of the continuous determination (see Proposition 1.1.1 in Chapter 1). The (asymptotic) torsion does not depend on the choice of the tangent vector  $v \in T_x S \setminus \{0\}$  (see Proposition 1.1.3 in Chapter 1). Moreover, the torsion does not depend on the choice of the isotopy if  $S = \mathbb{T}^2$  (see Section 2 in [BB13] and here Remark 1.3.1 in Chapter 1) or if  $S = \mathbb{A}$  (see Proposition 1.3.2). If  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  has compact support then the torsion does not depend on the choice of the isotopy as long as we consider isotopies with compact support<sup>1</sup> (see Section 2 in [BB13]). Therefore, in the sequel we omit the dependance on the isotopy.

Let  $q \in S$  be a periodic hyperbolic point for  $f$ . Let  $N > 0$  be the period of  $q$  for  $f$ . In

---

1. An isotopy  $I$  has compact support if for any  $t \in [0, 1]$  the support of  $f_t$  is in a compact set, independent of  $t$ .



the sequel, we will largely use classical results about hyperbolic sets that are recalled in Appendix A.

Assume  $p \in S$  is a point of transverse homoclinic intersection of  $\mathcal{O}(q, f)$  not belonging to the orbit of  $q$ . Without loss of generality (see Fact A.0.1), we assume that  $p \in (W^u(q) \cap W^s(q)) \setminus \{q\}$ .

Let  $\lambda_1, \lambda_2$  be the eigenvalues of  $Df^N(q) : T_q S \rightarrow T_q S$ . By the existence of  $p$  we have that  $\lambda_1, \lambda_2 \in \mathbb{R}$  and

$$0 < |\lambda_1| < 1 < |\lambda_2|.$$

Up to consider  $f^2$  instead of  $f$ , we can assume that  $\lambda_1$  and  $\lambda_2$  are both positive.

Concerning the torsion at finite time  $N$  at  $q$ , since the fields of half-lines of the stable and unstable directions are preserved (because the eigenvalues of  $Df^N(q)$  are positive), there exists  $k \in \mathbb{Z}$  such that

$$N \text{Torsion}_N(f, q, v) = N \text{Torsion}_N(f, q, w) = k \quad (4.1)$$

for  $v \in E_q^u, w \in E_q^s$ , where  $E_q^{u/s}$  denotes respectively the unstable and the stable subspace in  $T_q S$ .

**Notation 4.0.1.** To simplify the notation, from now on when not specified we consider the dynamics of  $f^N$  so that  $q$  is a hyperbolic fixed point for  $f^N$  and we denote as  $\mathfrak{f}$  the diffeomorphism  $f^N$  to lighten the notation.

In particular, concerning the  $N$ -time torsion for  $f$  at  $q$ , we have that

$$N \text{Torsion}_N(f, q, v) = \text{Torsion}_1(\mathfrak{f}, q, v) = k, \quad (4.2)$$

for  $v \in E_q^u$  or  $v \in E_q^s$ .

## 4.1 Statement of the main results

In this introductory section we state the main theorem of the chapter and present the main corollaries and outcomings.

The point  $q$  is a fixed hyperbolic point for  $\mathfrak{f}$  with transverse homoclinic intersections. To simplify the notation, assume that  $\text{Torsion}_1(\mathfrak{f}, q, v) = 0$ , where  $v \in E_q^u$ .

Let  $p \in W^u(q) \cap W^s(q), p \neq q$  be a suitable transverse homoclinic point, i.e. such that:

- the point  $p$  belongs to the local stable manifold of  $q$ ,
- the unstable manifold (respectively the stable manifold) at  $p$  has almost the same slope as  $E_q^u$  (respectively  $E_q^s$ ): to be precise, we compare slopes of the images of  $T_p W^u(q)$  and  $E_q^u$  through the differential of a chart.

See (a) of Figure 4.1.

We are going to construct a horseshoe, i.e. a uniformly hyperbolic set for some  $\mathfrak{f}^n$  such that  $\mathfrak{f}^n$  restricted to the horseshoe is conjugated to a shift dynamics, as follows. See Section 4.3. Consider a small rectangle  $R$  which contains the fixed point  $q$  and the point of homoclinic intersection  $p$  and which stretches along the local stable manifold. Then, there exists  $n \in \mathbb{N}$  such that  $\mathfrak{f}^n(R)$  stretches along the unstable manifold and contains the homoclinic point  $p$ . We focus on the two connected components of  $\mathfrak{f}^n(R) \cap R$  containing  $p$  and  $q$ . Denote them as  $V_1$  and  $V_0$  respectively. See (b) of Figure 4.1. We consider then

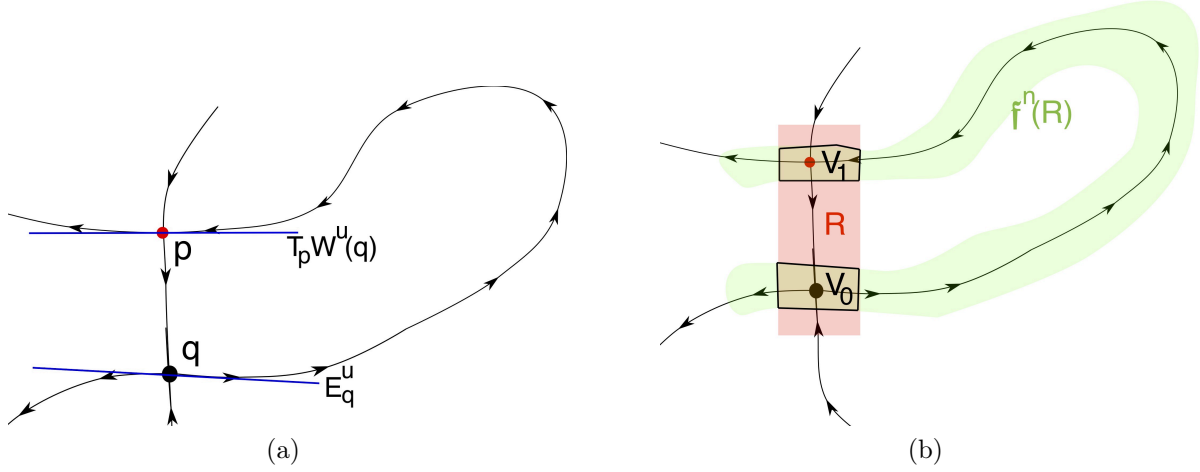


Figure 4.1

the  $f^n$ -invariant set  $H = \bigcap_{i \in \mathbb{Z}} f^{in}(V_0 \cup V_1)$ . The set  $H$  is a uniformly hyperbolic set for  $f^n$  such that  $(H, f^n)$  is conjugated to the shift dynamics  $(\{0, 1\}^{\mathbb{Z}}, S)$  (see Proposition 4.3.2). That is, we associate to every  $x \in H$  a sequence  $(\delta_i(x))_{i \in \mathbb{Z}} = (\delta_i)_{i \in \mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$  such that

$$f^{in}(x) \in V_{\delta_i}.$$

Our aim is so calculating the torsion at points of the horseshoe  $H$ . Since the asymptotic torsion does not depend on the tangent vector, we calculate the torsion for  $f$  at finite time  $n$  at points of  $H$  with respect to vectors belonging to the unstable subspace. The unstable subspace is well-defined because  $H$  is hyperbolic. Moreover the unstable subspace is invariant for  $Df^n$ .

Consider the angle variation of the vector tangent to the unstable manifold varying between  $q$  and  $p$ . Such angle variation admits a measure which is either almost null or almost equal to  $\frac{1}{2}$ . The unstable angle variation of  $(q, p)$  is the integer  $m \in \mathbb{Z}$  such that the angle variation of the vector tangent to the unstable manifold between  $q$  and  $p$  is almost equal to  $\frac{m}{2}$ .

The key point of our result is showing that for any  $x \in H$  we have that

$$n \text{ Torsion}_n(f, x, v) \quad \text{is almost equal to} \quad \frac{m}{2} \delta_1(x),$$

where the vector  $v$  belongs to the unstable subspace of  $x$ .

Using then a non trivial induction argument, we deduce the main theorem (see Theorem 4.4.1 and Corollary 4.4.1).

**Theorem.** *Let  $f : S \rightarrow S$  be a  $C^1$  diffeomorphism isotopic to the identity. Let  $q \in S$  be a fixed hyperbolic point with transverse homoclinic intersections. Assume that  $\text{Torsion}(f, q) = 0$ . Let  $p \in W^u(q) \cap W^s(q)$  be a suitable transverse homoclinic point. Let  $H$  be the associated horseshoe. Let  $m \in \mathbb{Z}$  be the unstable angle variation of  $(q, p)$ . For any  $x \in H$  the set of limit points of the sequences<sup>2</sup>*

$$( \text{Torsion}_l(f, x) )_{l \in \mathbb{N}} \quad \text{and} \quad \left( \frac{m \sum_{i=0}^l \delta_i(x)}{2n \frac{l}{l}} \right)_{l \in \mathbb{N}}$$

2. We do not make explicit the vector with respect to which we calculate the torsion to simplify the presentation of the main result.

are the same. The torsion of  $x$  exists if and only if the limit  $\lim_{l \rightarrow +\infty} \frac{\sum_{i=1}^l \delta_i(x)}{l}$  exists and, whenever it exists,

$$\text{Torsion}(\mathbf{f}, x) = \frac{m}{2n} \lim_{l \rightarrow +\infty} \frac{\sum_{i=1}^l \delta_i(x)}{l}.$$

We highlight the role played by the unstable angle variation  $m$ . Indeed, if  $m = 0$ , then the torsion exists and it is null (i.e. equal to  $\text{Torsion}(\mathbf{f}, q)$ ) at every point of  $H$ . In order to have non trivial torsion values, we are interested in the cases when  $m \neq 0$ .

Observe that if there are points  $x_1, x_2 \in H$  whose torsion values are different, then the unstable angle variation of  $(q, p)$  is non null. Moreover, if there are two periodic points of  $H$  with the same period, the first one with reflexion<sup>3</sup>, the second without, then again the unstable angle variation is non null.

We then state some interesting outcomings when the unstable angle variation  $m$  is non null. See Section 4.6.

**Corollary A.** *Let  $m \neq 0$ . For any  $[\alpha, \beta] \subset [0, m]$  there exists  $x \in H$  such that the set of limit points of  $(\text{Torsion}_l(\mathbf{f}, x))_{l \in \mathbb{N}}$  is  $[\alpha, \beta]$ .*

**Corollary B** *Let  $m \neq 0$ . For any  $\alpha \in [0, m]$  the set of points of  $H$  whose torsion equals  $\alpha$  is dense in  $H$ .*

In particular, if  $m$  is non null, then we have examples of points where the torsion does not exist.

**Corollary C.** *Let  $m \neq 0$ . The set of points of  $H$  at which the torsion does not exist contains a dense  $G_\delta$ -subset of  $H$ .*

Some consequences can be deduced also concerning torsion of measures.

**Corollary D.** *Let  $m \neq 0$ . For any  $\alpha \in [0, m]$  there exists an ergodic  $\mathbf{f}$ -invariant measure  $\mu$  with compact support such that  $\text{Torsion}(\mathbf{f}, \mu) = \alpha$ .*

If  $S = \mathbb{R}^2$  or  $S = \mathbb{A}$ , then there are conditions to obtain transverse homoclinic points with non zero unstable angle variation. See Lemma 4.5.3 and Remarks 4.5.1 and 4.5.2.

**Proposition A.** *Let  $S$  be either  $\mathbb{A}$  or  $\mathbb{R}^2$ . Let  $\mathbf{f} : S \rightarrow S$  be a  $C^1$  diffeomorphism isotopic to the identity. Assume that there is a periodic hyperbolic point  $q$  with transverse homoclinic intersections. Suppose that*

- *every homoclinic intersection is transverse,*
- *or  $\mathbf{f}$  is  $C^\infty$ .*

*Then there exists a point  $p$  of transverse homoclinic intersection such that the unstable angle variation of  $(q, p)$  is non null.*

---

3. That is, the differential of  $\mathbf{f}^N$  at the point, where  $N$  is the period of the point, has negative eigenvalues.

Through Proposition A and using Katok's result (see [Kat80]) which links topological entropy and transverse homoclinic intersections, we obtain sufficient conditions concerning the torsion to have dynamics with null topological entropy  $h_{top}$ .

**Corollary E.** *Let  $S$  be either  $\mathbb{T} \times [0, 1]$  or  $\mathbb{D}^2$ . Let  $f : S \rightarrow S$  be a  $\mathcal{C}^{1+\varepsilon}$  diffeomorphism ( $\varepsilon > 0$ ) such that:*

- *every homoclinic intersection of a hyperbolic periodic point is transverse,*
- *or  $f$  is  $\mathcal{C}^\infty$ .*

*If the torsion exists everywhere, then  $h_{top}(f) = 0$ .*

## 4.2 Choice of an adapted neighborhood for transverse homoclinic intersections

Fix

$$0 < \varepsilon < \frac{1}{12}.$$

This will be the maximum error that we will admit in calculating torsion at finite-time.

### 4.2.1 Choice of an adapted neighborhood of $q$

The main aim of this Subsection is selecting a suitable neighborhood of the fixed hyperbolic point.

**Lemma 4.2.1.** *There exists a neighborhood  $O$  of  $q$  and a chart  $\phi : O \rightarrow \mathbb{R}^2$  such that*

- (i)  $\phi(q) = (0, 0)$ ;
- (ii)  $\phi(O) \subset (-1, 1)^2$ ;
- (iii)  $\phi(CC(W^s(q) \cap O, q)) = \{0\} \times (-1, 1)$  and  $\phi(CC(W^u(q) \cap O, q)) = (-1, 1) \times \{0\}$ .

*Proof.* Let  $E_q^s$  and  $E_q^u$  be the stable and unstable subspaces of  $Df(q)$ , respectively. Let  $\psi : O \rightarrow (-1, 1)^2$  be a chart such that  $\psi(q) = (0, 0)$  and

$$D\psi(q)T_q W^u(q) = \mathbb{R} \times \{0\} \quad \text{and} \quad D\psi(q)T_q W^s(q) = \{0\} \times \mathbb{R}. \quad (4.3)$$

Recall that both  $W^u(q)$  and  $W^s(q)$  are  $\mathcal{C}^1$  1-dimensional immersed submanifolds. So, up to restrict the neighborhood  $O$ , there exist  $\mathcal{C}^1$  diffeomorphisms to their images

$$\gamma : (-1, 1) \rightarrow O, \quad \Gamma : (-1, 1) \rightarrow O$$

such that  $\gamma(0) = \Gamma(0) = q$ ,  $\gamma(-1, 1) = CC(W^u(q) \cap O, q)$  and  $\Gamma(-1, 1) = CC(W^s(q) \cap O, q)$ .

Let  $p_1, p_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$  be the projections over the first and the second coordinate respectively and consider then  $p_1 \circ \psi \circ \gamma : (-1, 1) \rightarrow \mathbb{R}$  and  $p_2 \circ \psi \circ \Gamma : (-1, 1) \rightarrow \mathbb{R}$ . Since both

$$Dp_1(\psi \circ \gamma(0)) \circ D\psi(\gamma(0))\gamma'(0) \quad \text{and} \quad Dp_2(\psi \circ \Gamma(0)) \circ D\psi(\Gamma(0))\Gamma'(0)$$

are not zero, by the inverse function theorem there exists  $\delta > 0$  such that

$$p_1 \circ \psi \circ \gamma : (-\delta, \delta) \rightarrow \mathbb{R}, \quad p_2 \circ \psi \circ \Gamma : (-\delta, \delta) \rightarrow \mathbb{R}$$

are  $\mathcal{C}^1$  diffeomorphisms to their images. Up to decrease  $\delta > 0$  (since  $p_1 \circ \psi \circ \gamma(0) = p_2 \circ \psi \circ \Gamma(0) = 0$ ), we can assume that  $p_1 \circ \psi \circ \gamma(-\delta, \delta) \subset (-1, 1)$  and  $p_2 \circ \psi \circ \Gamma(-\delta, \delta) \subset (-1, 1)$ . The functions  $\psi \circ \gamma : (-\delta, \delta) \rightarrow \mathbb{R}^2$  and  $\psi \circ \Gamma : (-\delta, \delta) \rightarrow \mathbb{R}^2$  are graphs of functions in the first and the second coordinates, respectively, of  $\mathbb{R}^2$ . That is, there exist  $\mathcal{C}^1$  functions

$$g_u : (-1, 1) \rightarrow (-1, 1), \quad g_s : (-1, 1) \rightarrow (-1, 1)$$

so that

$$\psi \circ \gamma|_{(-\delta, \delta)} \subseteq \{(x, g_u(x)) : x \in (-1, 1)\}$$

and

$$\psi \circ \Gamma|_{(-\delta, \delta)} \subseteq \{(g_s(y), y) : y \in (-1, 1)\}.$$

In particular it holds that  $g_u(0) = g_s(0) = 0$  and, by (4.3),  $g'_u(0) = g'_s(0) = 0$ .

Consider then the function  $\Phi : (-1, 1)^2 \rightarrow \mathbb{R}^2$  which deforms the standard vertical-horizontal foliation into the foliation made up of vertical translations of the graph of  $g_u$  (function in the first coordinate) and of horizontal translations of the graph of  $g_s$  (function in the second coordinate), see Figure 4.2.

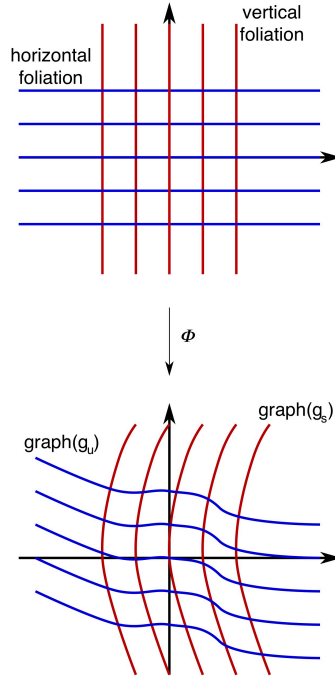


Figure 4.2 – Deformation of vertical-horizontal foliations into  $graph(g_s)$ – $graph(g_u)$  foliations.

That is

$$\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$(x, y) \mapsto \Phi(x, y) = (x + g_s(y), y + g_u(x)).$$

Observe that  $D\Phi(0, 0) = \mathbb{I}_2$ <sup>4</sup> and so by the inverse function theorem there exists a neighborhood  $W$  of  $0 \in \mathbb{R}^2$  such that  $W \subset (-1, 1)^2$  and  $\Phi|_W$  is a  $\mathcal{C}^1$  diffeomorphism to its image. By considering then

$$\phi := \Phi^{-1} \circ \psi : \psi^{-1}(\Phi(W)) \rightarrow \mathbb{R}^2,$$

---

4. Where  $\mathbb{I}_2$  denotes the 2-dimensional identity matrix.

we obtain a chart that verifies the required conditions.  $\square$

Let  $O$  be a neighborhood of  $q$  and  $\phi$  a chart given by Lemma 4.2.1. For any  $x, y \in O$  we identify the tangent spaces  $T_x S \cong T_y S \cong \mathbb{R}^2$  through the chart  $\phi$ .

**Notation 4.2.1.** Let  $O$  be a neighborhood of  $q$  given by Lemma 4.2.1 and let  $x, y \in O$ . Let  $E \subset T_x S$  and  $F \subset T_y S$  be 1-dimensional subspaces. The angle between  $E$  and  $F$  is denoted as  $\theta(E, F)$  and it is

$$\theta(E, F) = |\bar{\theta}(v', w')| = \min_{\substack{v \in E \cap \mathbb{S}^1 \\ w \in F \cap \mathbb{S}^1}} |\bar{\theta}(v, w)|,$$

where  $\theta(v, w) := \theta(\mathcal{H}, v) - \theta(\mathcal{H}, w)$ <sup>5</sup> and where  $\bar{\theta}$  is the measure of the oriented angle  $\theta$  contained in  $[-\frac{1}{2}, \frac{1}{2})$ .

**Condition 4.2.1.** Let  $O$  be an open neighborhood of  $q$  given by Lemma 4.2.1. Assume that

- (i) for any  $x \in CC(W^s(q) \cap O, q)$  the angle  $\theta(E_q^s, T_x W^s(q))$  admits a measure in  $(-\frac{\varepsilon}{4}, \frac{\varepsilon}{4})$ ;
- (ii) for any  $x \in CC(W^u(q) \cap O, q)$  the angle  $\theta(E_q^u, T_x W^u(q))$  admits a measure in  $(-\frac{\varepsilon}{4}, \frac{\varepsilon}{4})$ .

**Condition 4.2.2.** Let  $O$  be an open neighborhood of  $q$  given by Lemma 4.2.1. Assume that there exists  $0 < \delta < \frac{\varepsilon}{2}$  so that  $O \subseteq B_\delta(q) = \{x \in O : d(x, q) < \delta\}$ <sup>6</sup>,

$$CC(W^u(q) \cap O, q) = W_{loc, \delta}^u(q)$$

and

$$CC(W^s(q) \cap O, q) = W_{loc, \delta}^s(q),$$

where  $W_{loc, \delta}^s(q), W_{loc, \delta}^u(q)$  are the local stable and unstable manifolds of  $q$  respectively for  $\mathfrak{f}$  (see the Local (Un)Stable Manifold Theorem, here Theorem A.0.1).

**Remark 4.2.1.** Let  $O$  be a neighborhood of  $q$  satisfying Condition 4.2.2. Remark that  $\mathfrak{f}^n(x) \in O$  for any  $n \geq 0$  if and only if  $x \in W_{loc, \delta}^s(q)$  and  $\mathfrak{f}^{-n}(x) \in O$  for any  $n \geq 0$  if and only if  $x \in W_{loc, \delta}^u(q)$ .

**Definition 4.2.1.** Let  $O$  be a neighborhood of  $q$  satisfying Condition 4.2.2. A future-first-entry point for  $O$  is a point  $x \in \mathcal{O}(p, \mathfrak{f})$  such that

- (i)  $x \in W_{loc, \delta}^s(q) \subset O$ ;
- (ii)  $\mathfrak{f}^{-1}(x) \notin W_{loc, \delta}^s(q)$ .

A past-first-entry point for  $O$  is a point  $x \in \mathcal{O}(p, \mathfrak{f})$  such that

- (i)  $x \in W_{loc, \delta}^u(q) \subset O$ ;
- (ii)  $\mathfrak{f}(x) \notin W_{loc, \delta}^u(q)$ .

---

5. Recall that  $\theta(\mathcal{H}, v)$  is the oriented angle between the non zero vectors  $\mathcal{H}, v$  with respect to the standard Riemannian metric and orientation.

6. The distance  $d$  on  $O$  is the one inherited from the fixed metric.

**Remark 4.2.2.** Let  $O$  be a neighborhood of  $q$  satisfying Condition 4.2.2. Then the future-first-entry and past-first-entry points for  $O$  (with respect to  $\mathcal{O}(p, \mathfrak{f})$ ) are well-defined and unique.

Recall that the set  $\{q\} \cup \mathcal{O}(p, \mathfrak{f})$  is hyperbolic for  $\mathfrak{f}$  (see Fact A.0.2). Define now the functions

$$\begin{aligned} \{q\} \cup \mathcal{O}(p, \mathfrak{f}) \ni x &\mapsto E_x^u \cap \mathbb{S}^1 \in \bigcup_{x \in \{q\} \cup \mathcal{O}(p, \mathfrak{f})} \mathcal{G}_1(T_x S) \cap \mathbb{S}^1, \\ \{q\} \cup \mathcal{O}(p, \mathfrak{f}) \ni x &\mapsto E_x^s \cap \mathbb{S}^1 \in \bigcup_{x \in \{q\} \cup \mathcal{O}(p, \mathfrak{f})} \mathcal{G}_1(T_x S) \cap \mathbb{S}^1, \end{aligned}$$

where  $\mathcal{G}_1(T_x S)$  is the Grassmannian of the 1-dimensional subspaces in  $T_x S$ . They are continuous functions with respect to the Hausdorff distance.

Define for  $i = u, s$  the functions (see Notation 4.2.1)

$$\{q\} \cup \mathcal{O}(p, \mathfrak{f}) \ni x \mapsto \theta(\mathbb{R}\mathcal{H}, E_x^u) \in \mathbb{T},$$

$$\{q\} \cup \mathcal{O}(p, \mathfrak{f}) \ni x \mapsto \theta(\mathbb{R}\mathcal{H}, E_x^s) \in \mathbb{T}.$$

Observe that these functions are continuous.

**Condition 4.2.3.** Let  $O$  be an open neighborhood of  $q$  satisfying Condition 4.2.2. Since the angle function is continuous with respect to the vectors, there exists  $\delta > 0$  so that if  $v, w \in T^1 O$ ,  $\|v\| = \|w\| = 1$  and  $\|v - w\| < \eta^7$ , then the oriented angle

$$\theta(w, v) = \theta(\mathcal{H}, v) - \theta(\mathcal{H}, w)$$

admits a measure in  $(-\frac{\varepsilon}{4}, \frac{\varepsilon}{4})$ .

Assume that for every  $x \in \mathcal{O}(p, \mathfrak{f}) \cap W_{loc, \delta}^u(q)$  the connected component  $CC(W^s(q) \cap O, x)$  is  $\eta - \mathcal{C}^1$  close to  $W_{loc, \delta}^s(q)$  and for every  $x \in \mathcal{O}(p, \mathfrak{f}) \cap W_{loc, \delta}^s(q)$  the connected component  $CC(W^u(q) \cap O, x)$  is  $\eta - \mathcal{C}^1$  close to  $W_{loc, \delta}^u(q)$ .

We refer to Definition A.0.5 in Appendix A for the notion of  $\eta - \mathcal{C}^1$  close.

**Remark 4.2.3.** Observe in particular that Conditions 4.2.1 and 4.2.3 imply that

- for every  $x \in \mathcal{O}(p, \mathfrak{f}) \cap W_{loc, \delta}^u(q)$  the angle  $\theta(E_q^s, E_x^s)$  admits a measure in  $(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ ;
- for every  $x \in \mathcal{O}(p, \mathfrak{f}) \cap W_{loc, \delta}^s(q)$  the angle  $\theta(E_q^u, E_x^u)$  admits a measure in  $(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ .

**Lemma 4.2.2.** Let  $q$  be a fixed hyperbolic point for  $\mathfrak{f}$  and let  $p$  be a transverse homoclinic point for  $q$ . There exist two continuous vector fields  $e^u, e^s$

$$(\{q\} \cup \mathcal{O}(p, \mathfrak{f})) \ni x \mapsto e_x^u \in E_x^u \cap \mathbb{S}^1,$$

$$(\{q\} \cup \mathcal{O}(p, \mathfrak{f})) \ni x \mapsto e_x^s \in E_x^s \cap \mathbb{S}^1.$$

Let  $O$  be an open neighborhood of  $q$  satisfying Conditions 4.2.1, 4.2.2 and 4.2.3. Then for any  $x \in (\{q\} \cup \mathcal{O}(p, \mathfrak{f})) \cap (W_{loc, \delta}^s(q) \cup W_{loc, \delta}^u(q))$  the angles  $\theta(e_q^u, e_x^u)$  and  $\theta(e_q^s, e_x^s)$  admit a measure in  $(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ .

---

7. We are identifying tangent spaces at different points of  $O$  through the chart.

The ideas of the proof of Lemma 4.2.2 are the same of those of the proof of Lemma 4.2.4. Hence, we send to the proof of Lemma 4.2.4 in the following Subsection.

Let us introduce the cone field property (with respect to  $\mathfrak{f}$ ).

**Definition 4.2.2.** A set  $U \subset S$  satisfies the cone field property for  $\eta \in \mathbb{R}_+, \xi, \delta \in (0, 1) \cap \mathbb{R}, m \in \mathbb{N}^*$  if there exist a splitting  $E^1 \oplus E^2 = T_U S$  and a cone field  $(C_x^\eta)_{x \in U}$  where

$$C_x^\eta = \{v \in T_x S : v = v^1 + v^2, v^1 \in E_x^1, v^2 \in E_x^2, \|v^2\| \leq \eta \|v^1\|\}$$

such that

- (i) for any  $x \in U$  it holds  $\dim E_x^1 = \dim E_x^2 = 1$ ;
- (ii) for any  $x \in U \cap \mathfrak{f}^{-1}(U)$  it holds  $D\mathfrak{f}(x)C_x^\eta \subset C_{\mathfrak{f}(x)}^{\eta^\delta}$ ;
- (iii) for any  $x \in U$ 
  - for any  $v \in C_x^\eta$  it holds  $\|D\mathfrak{f}^m(x)v\| \geq \frac{1}{\xi}\|v\|$ ;
  - for any  $w \notin \text{int}(C_x^\eta)$  it holds  $\|D\mathfrak{f}^{-m}(x)w\| \geq \frac{1}{\xi}\|w\|$ .

**Condition 4.2.4.** Let  $O$  be a neighborhood of the fixed hyperbolic point  $q$  which satisfies the cone field property.

We refer to Appendix B for a detailed discussion of the open cone field property.

**Condition 4.2.5.** Let  $O$  be an open neighborhood of  $q$  satisfying Condition 4.2.2 such that

$$\mathcal{O}(p, \mathfrak{f}) \cap O \subset W_{loc, \delta}^s \cup W_{loc, \delta}^u.$$

That is the  $\mathfrak{f}$ -orbit of  $p$  intersects  $O$  only along the local stable manifold of  $q$  and the local unstable manifold of  $q$ .

**Condition 4.2.6.** Let  $O$  be an open neighborhood of  $q$  such that for any  $x \in (\{q\} \cup \mathcal{O}(p, \mathfrak{f})) \cap O$  it holds

$$|\text{Torsion}_1(\mathfrak{f}, x, e_x^u) - \text{Torsion}_1(\mathfrak{f}, q, e_q^u)| < \frac{\varepsilon}{2}$$

and

$$|\text{Torsion}_1(\mathfrak{f}, x, e_x^s) - \text{Torsion}_1(\mathfrak{f}, q, e_q^s)| < \frac{\varepsilon}{2}.$$

For the following condition we refer to [Con75] and [Mos73].

**Condition 4.2.7.** Let  $O$  be a neighborhood of  $q$  given by Lemma 4.2.1 and satisfying Conditions 4.2.2 and 4.2.3. Let  $p, \mathfrak{f}^{-M}(p) \in O$  be respectively future-first-entry and past-first-entry points for  $O$  (see Definition 4.2.1).

Referring to the chart of Lemma 4.2.1, assume that  $\phi(p) \in \{0\} \times \mathbb{R}_+, \phi(\mathfrak{f}^{-M}(p)) \in \mathbb{R}_+ \times \{0\}$ . Suppose that the set  $RV$  is contained in  $O$ , where  $RV$  is the image through  $\phi^{-1}$  of the intersection of the following sets (see Figure 4.3-(a))

- $\{x \geq 0, y \geq 0\}$ ;



- $Ipo-graph(u_0) := \{(x, y) : y \leq u_0(x)\}$ , i.e. the set of all points that lie on or below the graph of  $u_0$ , where the graph of  $u_0$  denotes  $\phi(CC(W^u(q) \cap O, p))$ ;
- $Left-graph(s_0) := \{(x, y) : x \leq s_0(y)\}$ , i.e. the set of all points that lie on or at the left of the graph of  $s_0$ , where the graph of  $s_0$  denotes  $\phi(CC(W^s(q) \cap O, f^{-M}(p)))$ .

From now on, for sake of simplicity we omit the chart  $\phi$  in our notation. Therefore, we speak of first and second coordinates also on the neighborhood of  $q$ .

**Notation 4.2.2.** Let  $O$  be a neighborhood of  $q$  satisfying Condition 4.2.7. Let  $p, f^{-M}(p) \in O$  be respectively future-first-entry and past-first-entry points for  $O$  (see Definition 4.2.1). For any  $i \geq 0$  we have that  $f^i(p) \in W_{loc, \delta}^s(q)$  and  $f^{-(M+i)}(p) \in W_{loc, \delta}^u(q)$ . Moreover, by Condition 4.2.3, for any  $i \geq 0$  it holds

- $CC(W^u(q) \cap O, f^i(p))$  is a graph with respect to the first coordinate and we denote it as the graph of  $u_i$ ;
- $CC(W^s(q) \cap O, f^{-(M+i)}(p))$  is a graph with respect to the second coordinate and we denote it as the graph of  $s_i$ .

We denote then for any  $i \geq 0$  (see Figure 4.3-(a))

$$RV(i) = \{x \geq 0, y \geq 0\} \cap Ipo-graph(u_0) \cap Left-graph(s_i),$$

where

- $Ipo-graph(u_0) = \{(x, y) : y \leq u(x)\}$  is the set of all points that lie on or below the graph of  $u_0$  (that is  $CC(W^u(q) \cap O, p)$ );
- $Left-graph(s_i) = \{(x, y) : x \leq s_i(y)\}$  is the set of all points that lie on or at the left of the graph of  $s_i$  (that is  $CC(W^s(q) \cap O, f^{-(M+i)N}(p))$ ).

We similarly denote for any  $i \geq 0$  (see Figure 4.3-(b))

$$RH(i) = \{x \geq 0, y \geq 0\} \cap Ipo-graph(u_i) \cap Left-graph(s_0),$$

where

- $Ipo-graph(u_i) = \{(x, y) : y \leq u(x)\}$  is the set of all points that lie on or below the graph of  $u_i$  (that is  $CC(W^u(q) \cap O, f^i(p))$ );
- $Left-graph(s_0) = \{(x, y) : x \leq s_0(y)\}$  is the set of all points that lie on or at the left of the graph of  $s_0$  (that is  $CC(W^s(q) \cap O, f^{-M}(p))$ ).

**Remark 4.2.4.** Let  $O$  be a neighborhood of  $q$  and let  $p, f^{-M}(p)$  be respectively future-first-entry and past-first-entry points for  $O$  with respect to which Condition 4.2.7 holds. Remark then, referring to Notation 4.2.2, that

$$(RV(i))_{i \geq 0} \quad \text{and} \quad (RH(i))_{i \geq 0}$$

are both decreasing sequences of sets contained in  $O$ .

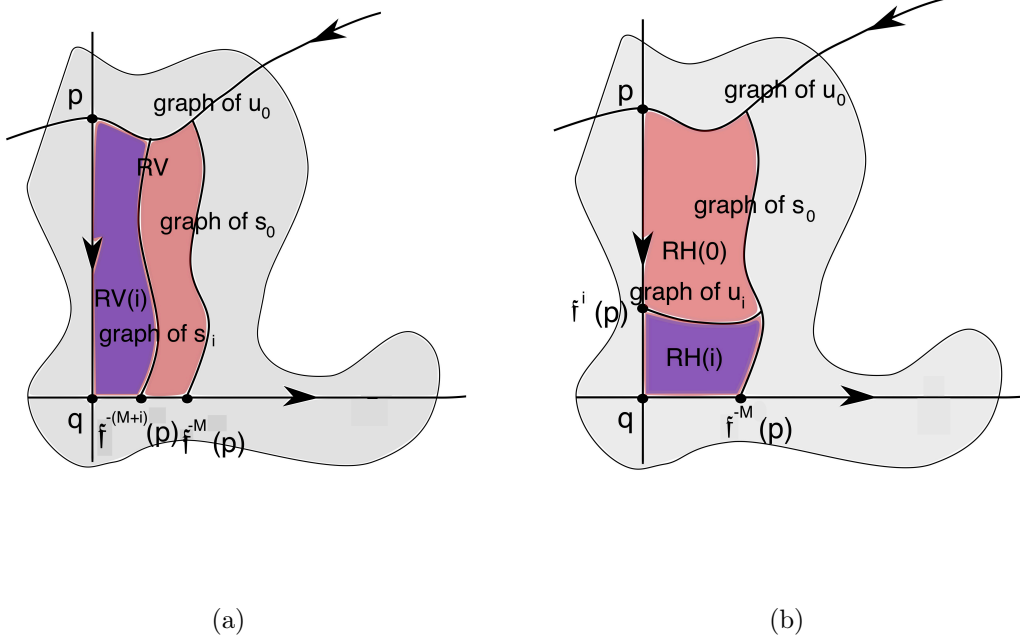


Figure 4.3 – Example of vertical rectangles  $RV(i)$  and of horizontal rectangles  $RH(i)$ .

**Definition 4.2.3.** Let  $q \in S$  be a fixed hyperbolic point for  $f$  and let  $p \in (W^u(q) \cap W^s(q)) \setminus \{q\}$ . Fix  $0 < \varepsilon < \frac{1}{12}$ .

A neighborhood  $O$  of  $q$  satisfying Conditions 4.2.1, 4.2.2, 4.2.3, 4.2.4, 4.2.5, 4.2.6 and 4.2.7, with respect to  $\varepsilon$  is called an **adapted neighborhood of  $q$  for  $\mathcal{O}(p, f)$  with respect to  $\varepsilon$** .

**Lemma 4.2.3.** Let  $q \in S$  be a fixed hyperbolic point for  $f$  and let  $p \in (W^u(q) \cap W^s(q)) \setminus \{q\}$ . Fix  $0 < \varepsilon < \frac{1}{12}$  and let  $W$  be an open neighborhood of  $q$ . Then there exists an adapted neighborhood  $O_\varepsilon$  of  $q$  for  $\mathcal{O}(p, f)$  with respect to  $\varepsilon$  contained in  $W$ .

*Proof.* Let  $O$  be a neighborhood of  $q$  given by Lemma 4.2.1: up to shrink  $O$ , we can assume it is contained in  $W$ . In particular,  $CC(W^s(q) \cap O, q)$  is sent by a chart  $\phi$  on  $\{0\} \times (-1, 1)$  and for any  $x \in CC(W^s(q) \cap O, q)$  it holds  $D\phi(x)T_x W^s(q) = \mathbb{R}(0, 1)$ . Because of the continuity of the angle function and because the stable manifold  $W^s(q)$  is a  $\mathcal{C}^1$  immersed submanifold, there exists a neighborhood  $O_1 \subset O$  of  $q$  such that for any  $x \in CC(W^s(q) \cap O_1, q)$  the angle

$$\theta(E_q^s, T_x W^s(q))$$

admits a measure in  $(-\frac{\varepsilon}{4}, \frac{\varepsilon}{4})$ . Similarly, we find a neighborhood  $O_2 \subset O_1$  such that for any  $x \in CC(W^u(q) \cap O_2, q)$  the angle

$$\theta(E_q^u, T_x W^u(q))$$

admits a measure in  $(-\frac{\varepsilon}{4}, \frac{\varepsilon}{4})$ . That is,  $O_2$  satisfies Condition 4.2.1. Remark that any neighborhood of  $q$  contained in  $O_2$  still satisfies Condition 4.2.1.

Apply the Local (Un)Stable Manifold Theorem at  $q$  for  $f$  (see Theorem A.0.1). Then,

there exists a neighborhood  $O_3 \subset O_2$  of  $q$  satisfying Condition 4.2.2. That is  $CC(W^u(q) \cap O_3, q) = W_{loc, \delta}^u(q)$  and  $CC(W^s(q) \cap O_3, q) = W_{loc, \delta}^s(q)$  for some  $0 < \delta < \frac{\varepsilon}{2}$ . This condition is verified for any neighborhood of  $q$  contained in  $O_3$ , up to decrease  $\delta$ .

Thanks to the  $\lambda$ -lemma (see Theorem A.0.3) we find a neighborhood  $O_4 \subset O_3$  that verifies also Condition 4.2.3. Again, we remark that such a condition still holds true if we shrink the neighborhood of the hyperbolic point.

The point  $q$  is hyperbolic and so in particular it satisfies the cone field property (see Proposition A.0.1). Such a criterion is an open condition in  $S$  (see Appendix B). Hence, there exists a neighborhood  $O_5$  of  $q$  contained in  $O_4$  verifying the cone field property, that is satisfying Condition 4.2.4. Observe that we can extend the cone field property by asking that for any

$$x \in \mathcal{O}(p, \mathbf{f}) \cap O_5 \cap [CC(W^s(q) \cap O_5, q) \cup CC(W^u(q) \cap O_5, q)]$$

the stable and unstable splitting is exactly

$$T_x W^s(q) \oplus T_x W^u(q).$$

The connected component  $CC(O_5, q)$  is a neighborhood of  $q$  which satisfies all the conditions 4.2.1, 4.2.2, 4.2.3 and 4.2.4.

Since  $\lim_{n \rightarrow \pm\infty} \mathbf{f}^n(p) = q$ , there is a finite number of points of  $\mathcal{O}(p, \mathbf{f})$  which do not belong to  $CC(W^u(q) \cap O_5, q) \cup CC(W^s(q) \cap O_5, q)$ . Therefore we find a neighborhood  $O_6$  of  $q$  contained in  $CC(O_5, q)$  such that all the previous conditions hold and such that

$$\mathcal{O}(p, f^N) \cap O_6 \subset CC(W^u(q) \cap O_6, q) \cup CC(W^s(q) \cap O_6, q).$$

That is,  $O_6$  satisfies also Condition 4.2.5.

Since the vector field  $e^u$  (respectively  $e^s$ ) is continuous (see Lemma 4.2.2) and since the torsion at finite time is continuous, also the function

$$x \mapsto \text{Torsion}_1(\mathbf{f}, x, e_x^u) \quad (\text{respectively } x \mapsto \text{Torsion}_1(\mathbf{f}, x, e_x^s))$$

is continuous. So, there exists a neighborhood  $O_7$  of  $q$  contained in  $O_6$  such that it satisfies Condition 4.2.6. Observe that  $O_7$  satisfies also Conditions from 4.2.1 to 4.2.4.

The intersection  $\mathcal{O}(p, \mathbf{f}) \cap O_7$  is contained in  $CC(W^s(q) \cap O_6, q) \cup CC(W^u(q) \cap O_6, q)$ . There is a finite number of points of  $\mathcal{O}(p, \mathbf{f}) \cap O_7$  which do not belong to  $CC(W^s(q) \cap O_7, q) \cup CC(W^u(q) \cap O_7, q)$ . We can so shrink  $O_7$  in order to exclude those points and verify also Condition 4.2.5.

Consider then  $p$  and  $\mathbf{f}^{-M}(p)$  (for some  $M > 0$ ) respectively future-first-entry and past-first-entry points for  $O_7$  (see Definition 4.2.1). By the  $\lambda$ -lemma (see Theorem A.0.3), there exists  $i \geq 0$  such that the “deformed” vertical rectangle  $RV(i)$  (see Notation 4.2.2) is contained in  $O_7$ . We then find a suitable small open neighborhood  $O_8$  of the rectangle  $RV(i)$  contained in  $O_7$ . That is, the neighborhood  $O_8$  satisfies all the conditions 4.2.1, 4.2.2, 4.2.3, 4.2.4, 4.2.5, 4.2.6 and 4.2.7.

We so conclude that  $O_8$  is an adapted neighborhood of  $q$  for  $\mathcal{O}(p, \mathbf{f})$  with respect to  $\varepsilon$  contained in  $W$ .

□

Fix an adapted neighborhood  $O_\varepsilon$  of  $q$  for  $\mathcal{O}(p, \mathbf{f})$ . Assume that the point  $p$  is a future-first-entry point for  $O_\varepsilon$  and let  $\mathbf{f}^{-J}(p)$  be a past-first-entry point for  $O_\varepsilon$ . Since  $p \in W_{loc, \delta}^s(q)$

and  $\mathfrak{f}^{-J}(p) \in W_{loc,\delta}^u(q)$  there exist  $t_s, t_u \in [0, 1]^8$  such that

$$p = \phi^{-1}(0, t_s) \quad \text{and} \quad \mathfrak{f}^{-J}(p) = \phi^{-1}(t_u, 0).$$

Define so the  $\mathcal{C}^1$  curves

$$\begin{aligned} \gamma_s : [0, t_s] &\rightarrow S \\ t &\mapsto \phi^{-1}(0, t_s - t), \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} \gamma_u : [0, t_u] &\rightarrow S \\ t &\mapsto \phi^{-1}(t, 0). \end{aligned} \tag{4.5}$$

They are respectively  $[p, q]^s$ , i.e. the subset of the local stable manifold of  $q$  connecting  $p$  to  $q$ , and  $[q, \mathfrak{f}^{-J}(p)]^u$ , i.e. the subset of the local unstable manifold connecting  $q$  to  $\mathfrak{f}^{-J}(p)$ .

Remark that  $\gamma_s(0) = p$ ,  $\gamma_s(t_s) = q$ ,  $\gamma_u(0) = q$  and  $\gamma_u(t_u) = \mathfrak{f}^{-J}(p)$ .

We fix an orientation of the stable and unstable subspaces at  $q$  by choosing as  $e_q^u \in E_q^u \cap \mathbb{S}^1$  and as  $e_q^s \in E_q^s \cap \mathbb{S}^1$  the vectors

$$\frac{\gamma_u'(0)}{\|\gamma_u'(0)\|} = e_q^u \quad \text{and} \quad \frac{\gamma_s'(t_s)}{\|\gamma_s'(t_s)\|} = e_q^s. \tag{4.6}$$

Denote the curve

$$\begin{aligned} \Gamma_u : [0, t_u] &\rightarrow S \\ t &\rightarrow \mathfrak{f}^J(\phi^{-1}(t, 0)), \end{aligned} \tag{4.7}$$

i.e. it is  $[q, p]^u$ , the subset of the unstable manifold connecting  $q$  to  $p$ . Observe that  $\Gamma_u(0) = q$  and  $\Gamma_u(t_u) = p$ . See Figure 4.4.

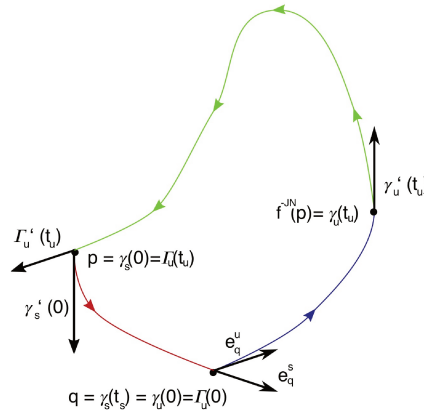


Figure 4.4

**Remark 4.2.5.** Concerning the choice of the orientation of  $E_q^u$  and  $E_q^s$ , it does not affect the choice of the adapted neighborhood  $O_\varepsilon$ . Indeed the only condition that could possibly depend on this choice is Condition 4.2.6 (concerning finite-time torsion). Since  $E_q^u \cap \mathbb{S}^1 = \{e_q^u, -e_q^u\}$  and since

$$\text{Torsion}_1(\mathfrak{f}, q, e_q^u) = \text{Torsion}_1(\mathfrak{f}, q, -e_q^u),$$

Condition 4.2.6 holds with respect to no matter which fixed  $e_q^u$ . The analogous condition holds also with respect to  $e_q^s \in E_q^s \cap \mathbb{S}^1$ .

---

8. Up to reparametrize, we can assume  $t_s, t_u$  non negative.

### 4.2.2 Choice of an adapted neighborhood of $\{q\} \cup \mathcal{O}(p)$

In this Subsection we are going to choose a neighborhood of the hyperbolic set  $\{q\} \cup \mathcal{O}(p, \mathfrak{f})$  with respect to the fixed  $0 < \varepsilon < \frac{1}{12}$ .

**Condition 4.2.8.** Let  $U$  be an open neighborhood of  $\{q\} \cup \mathcal{O}(p, \mathfrak{f})$  with a finite number  $n \in \mathbb{N}$  of connected components and such that

- (i) the connected component  $\tilde{U} := CC(U, q)$  containing  $q$  is an adapted neighborhood of  $q$  for  $\mathcal{O}(p, \mathfrak{f})$  with respect to  $\varepsilon$  (see Definition 4.2.3);
- (ii) every other connected component of  $U$  not containing  $q$  meets  $\{q\} \cup \mathcal{O}(p, \mathfrak{f})$  only at one point of  $\mathcal{O}(p, \mathfrak{f})$ .

**Remark 4.2.6.** Let  $U$  be an open neighborhood of  $\{q\} \cup \mathcal{O}(p, \mathfrak{f})$  satisfying Condition 4.2.8. In particular,  $CC(U, q)$  is an adapted neighborhood of  $q$  for  $\mathcal{O}(p, \mathfrak{f})$ . Denote then as  $p$  the future-first-entry point for  $CC(U, q)$  (see Definition 4.2.1). Then,  $U$  has  $n \in \mathbb{N}$  connected components if and only if  $\mathfrak{f}^{-n}(p)$  is the past-first-entry point for  $CC(U, q)$ .

**Condition 4.2.9.** Let  $U$  be an open neighborhood of the hyperbolic set  $\{q\} \cup \mathcal{O}(p, f^N)$  which satisfies the cone field property (for  $\mathfrak{f}$ )<sup>9</sup>.

Observe in particular that we extend the (un)stable splitting (and so the (un)stable cone field) on a neighborhood of  $\{q\} \cup \mathcal{O}(p)$  so that for any  $x \in \{q\} \cup \mathcal{O}(p, \mathfrak{f})$  the splitting is the hyperbolic one

$$T_x W^u(q) \oplus T_x W^s(q).$$

See Proposition B.0.2 in Appendix B.

**Notation 4.2.3.** Let  $U$  be an open neighborhood of  $\{q\} \cup \mathcal{O}(p, \mathfrak{f})$  satisfying Condition 4.2.9. Denote the maximal invariant set for  $\mathfrak{f}$  contained in  $U$  as

$$\Lambda(U) := \bigcap_{n \in \mathbb{Z}} \mathfrak{f}^n(U).$$

**Remark 4.2.7.** Thanks to the cone field criterion, the set  $\Lambda(U)$  defined in Notation 4.2.3 is hyperbolic for  $\mathfrak{f}$  and it strictly contains  $\{q\} \cup \mathcal{O}(p, \mathfrak{f})$ , since this last set is not isolated (see Fact A.0.3). Denote as  $E_{\Lambda(U)}^u \oplus E_{\Lambda(U)}^s$  the corresponding  $D\mathfrak{f}$ -invariant splitting of  $T_{\Lambda(U)}S$ .

**Condition 4.2.10.** Let  $U$  be an open neighborhood of  $\{q\} \cup \mathcal{O}(p, \mathfrak{f})$  satisfying Conditions 4.2.8 and 4.2.9. Let  $\Lambda(U)$  be the maximal  $\mathfrak{f}$ -invariant set contained in  $U$ . Recall that  $\Lambda(U)$  is hyperbolic (see Remark 4.2.7). Assume that (we refer to Notation 4.2.1)

- for any  $x \in \Lambda(U) \cap \tilde{U}$  both the angle  $\theta(E_q^u, E_x^u)$  between  $E_x^u$  and  $E_q^u$  and the angle  $\theta(E_q^s, E_x^s)$  between  $E_x^s$  and  $E_q^s$  are in  $(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ ;
- for every connected component  $V$  of  $U$  not containing  $q$ , for any  $x \in \Lambda(U) \cap V$  both the angle  $\theta(E_y^u, E_x^u)$  between  $E_x^u$  and  $E_y^u$  and the angle  $\theta(E_y^s, E_x^s)$  between  $E_x^s$  and  $E_y^s$  are in  $(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ , where  $y$  is the only point in  $V \cap \mathcal{O}(p, \mathfrak{f})$ .

---

9. We refer to Appendix B for a detailed discussion of the cone field property.

**Notation 4.2.4.** Let  $U$  be a neighborhood of  $\{q\} \cup \mathcal{O}(p, \mathfrak{f})$  which satisfies Condition 4.2.8. The number of connected components of  $U$  is denoted  $n$ . The connected component of  $U$  containing  $q$  is denoted  $\tilde{U}$ . The point  $p$  denotes the future-first-entry point for  $\tilde{U}$  and so  $\mathfrak{f}^{-n}(p)$  is the past-first-entry point for  $\tilde{U}$ . For any  $i \in \llbracket 1, n-1 \rrbracket$  denote as  $U_i$  the connected component of  $U$  which meets  $\mathcal{O}(p, \mathfrak{f})$  only at  $\mathfrak{f}^{-i}(p)$ .

**Lemma 4.2.4.** *Let  $q$  be a hyperbolic fixed point for  $\mathfrak{f}$  and let  $p \in (W^s(q) \cap W^u(q)) \setminus \{q\}$ . Let  $U$  be an open neighborhood of  $\{q\} \cup \mathcal{O}(p, \mathfrak{f})$  satisfying Conditions 4.2.8, 4.2.9 and 4.2.10.*

*There exist two continuous vector fields*

$$\Lambda(U) \ni x \mapsto e_x^u \in E_x^u \cap \mathbb{S}^1,$$

$$\Lambda(U) \ni x \mapsto e_x^s \in E_x^s \cap \mathbb{S}^1$$

*such that*

— *for any  $i \in \llbracket 1, n-1 \rrbracket$ , for any  $x \in \Lambda(U) \cap U_i$  the angles*

$$\theta(e_x^u, e_{\mathfrak{f}^{-i}(p)}^u) \quad \text{and} \quad \theta(e_x^s, e_{\mathfrak{f}^{-i}(p)}^s)$$

*both admit a measure in  $(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ ;*

— *for  $x \in \Lambda(U) \cap \tilde{U}$  the angles*

$$\theta(e_x^u, e_q^u) \quad \text{and} \quad \theta(e_x^s, e_q^s)$$

*both admit a measure in  $(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ .*

*Proof.* First of all, let us choose  $e_q^u \in E_q^u \cap \mathbb{S}^1$  and  $e_{\mathfrak{f}^{-i}(p)}^u \in E_{\mathfrak{f}^{-i}(p)}^u \cap \mathbb{S}^1$  for any  $i \in \llbracket 1, n-1 \rrbracket$ . We are going to show the existence and the continuity of the vector field  $e^u$ . Then the proof for  $e^s$  is analogous.

Consider  $\Lambda(U) \cap \tilde{U}$ . Observe that, since each  $E_x^u$  is 1-dimensional, every  $E_x^u \cap \mathbb{S}^1$  contains only two vectors  $\{v, -v\}$ . The vector field  $e^u$  is so defined, on  $\Lambda(U) \cap \tilde{U}$ , as

— at  $q$  it is the chosen  $e_q^u$ ;

— at  $x \in \Lambda(U) \cap \tilde{U}$ ,  $e_x^u$  is the vector such that the angle  $\theta(e_q^u, e_x^u)$  admits a measure in  $(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ .

The vector  $e_x^u$  is defined for any  $x \in \Lambda(U) \cap \tilde{U}$ . Indeed, the angle between  $E_q^u$  and  $E_x^u$  is (see Notation 4.2.1)

$$\min_{\substack{w \in E_q^u \cap \mathbb{S}^1 \\ v \in E_x^u \cap \mathbb{S}^1}} |\bar{\theta}(w, v)| = \min\{|\bar{\theta}(e_q^u, v)|, |\bar{\theta}(e_q^u, -v)|, |\bar{\theta}(-e_q^u, v)|, |\bar{\theta}(-e_q^u, -v)|\},$$

where  $v \in E_x^u \cap \mathbb{S}^1$ . Recall that the notation  $|\bar{\theta}(w, v)|$  refers to the absolute value of the measure of the angle contained in  $[-\frac{1}{2}, \frac{1}{2}]$ . In particular the angle  $\theta(E_x^u, E_q^u)$  is

$$\min\{|\bar{\theta}(e_q^u, v)|, |\bar{\theta}(e_q^u, -v)|\}, \quad \text{where } v \in E_x^u \cap \mathbb{S}^1,$$

since  $|\bar{\theta}(e_q^u, v)| = |\bar{\theta}(-e_q^u, -v)|$  and  $|\bar{\theta}(e_q^u, -v)| = |\bar{\theta}(-e_q^u, v)|$ . By Condition 4.2.10, we have that

$$\min\{|\bar{\theta}(e_q^u, v)|, |\bar{\theta}(e_q^u, -v)|\} < \frac{\varepsilon}{2}.$$

So we define  $e_x^u \in \{v, -v\}$  so that  $|\bar{\theta}(e_q^u, e_x^u)| < \frac{\varepsilon}{2}$ . That is such that  $\theta(e_q^u, e_x^u)$  admits a measure in  $(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ .

The vector  $e_x^u$  is uniquely defined. Indeed,  $\theta(e_q^u, -e_x^u)$  admits a measure in  $(\frac{1}{2} - \frac{\varepsilon}{2}, \frac{1}{2} + \frac{\varepsilon}{2})$  and, since  $0 < \varepsilon < \frac{1}{12}$ , we deduce that

$$|\bar{\theta}(e_q^u, e_x^u)| < |\bar{\theta}(e_q^u, -e_x^u)|.$$

Let us prove the continuity of  $e^u$  on  $\Lambda(U) \cap \tilde{U}$ . Fix  $x \in \Lambda(U) \cap \tilde{U}$  and fix  $\delta > 0$ . By the continuity of the splitting  $E_{\Lambda(U)}^u \oplus E_{\Lambda(U)}^s$ , there exists a neighborhood  $W$  of  $x$  such that for any  $y \in \Lambda(U) \cap W$  we have

$$d_H(E_x^u \cap \mathbb{S}^1, E_y^u \cap \mathbb{S}^1) < \delta.$$

In particular

$$\min_{v \in E_y^u \cap \mathbb{S}^1} \|e_x^u - v\| = \min\{\|e_x^u - e_y^u\|, \|e_x^u + e_y^u\|\} < \delta.$$

By showing that  $\|e_x^u - e_y^u\| < \|e_x^u + e_y^u\|$ , we immediately conclude.

**Claim 4.2.1.**  $\|e_x^u - e_y^u\| < \|e_x^u + e_y^u\|$ .

*Proof of Claim 4.2.1.* Observe that, thanks to the definition of  $e^u$ , the angle  $\theta(e_x^u, e_y^u)$  admits a measure in  $(-\varepsilon, \varepsilon)$  and the angle  $\theta(e_x^u, -e_y^u)$  admits a measure in  $(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon)$ . We look now at  $\|e_x^u + e_y^u\|^2$ . Recall that we are considering the Euclidean norm on  $\mathbb{R}^2 \cong T_x \tilde{U}$  for any  $x \in \Lambda(U) \cap \tilde{U}$  and we are identifying all the tangent spaces thanks to the trivialization. Denote as  $\langle \cdot, \cdot \rangle$  the standard scalar product. Hence

$$\|e_y^u + e_x^u\|^2 = \|e_y^u - e_x^u\|^2 + 4\langle e_x^u, e_y^u \rangle.$$

Since the angle  $\theta(e_x^u, e_y^u)$  admits a measure in  $(-\varepsilon, \varepsilon)$  and since  $0 < \varepsilon < \frac{1}{12}$ , we have that

$$\langle e_x^u, e_y^u \rangle = \cos(2\pi \theta(e_x^u, e_y^u)) > \frac{1}{2}.$$

Therefore, we conclude that

$$\|e_y^u + e_x^u\| = \|e_y^u - e_x^u\|^2 + 4\langle e_x^u, e_y^u \rangle > \|e_x^u - e_y^u\|^2 + 2 > \|e_x^u - e_y^u\|.$$

□

The same argument can be repeated on each  $\Lambda(U) \cap U_i$  for  $i \in \llbracket 1, n-1 \rrbracket$  and this shows the continuity of the vector field  $e^u$ .

□

**Condition 4.2.11.** Let  $U$  be an open neighborhood of  $\{q\} \cup \mathcal{O}(p, \mathfrak{f})$  satisfying Conditions 4.2.8, 4.2.9 and 4.2.10. Let  $p$  be the future-first-entry point for  $\tilde{U}$ . Then suppose that

— for  $x \in \Lambda(U) \cap \tilde{U}$  it holds

$$|\text{Torsion}_1(\mathbf{f}, x, e_x^u) - \text{Torsion}_1(\mathbf{f}, q, e_q^u)| < \frac{\varepsilon}{2}$$

and

$$|\text{Torsion}_1(\mathbf{f}, x, e_x^s) - \text{Torsion}_1(\mathbf{f}, q, e_q^s)| < \frac{\varepsilon}{2};$$

— for any  $i \in \llbracket 1, n-1 \rrbracket$ , for any  $x \in \Lambda(U) \cap U_i$  it holds

$$|\text{Torsion}_1(\mathbf{f}, x, e_x^u) - \text{Torsion}_1(\mathbf{f}, \mathbf{f}^{-i}(p), e_{\mathbf{f}^{-i}(p)}^u)| < \frac{\varepsilon}{2}$$

and

$$|\text{Torsion}_1(\mathbf{f}, x, e_x^s) - \text{Torsion}_1(\mathbf{f}, \mathbf{f}^{-i}(p), e_{\mathbf{f}^{-i}(p)}^s)| < \frac{\varepsilon}{2}.$$

**Definition 4.2.4.** Let  $q \in S$  be a fixed hyperbolic point for  $\mathbf{f}$  and let  $p \in (W^s(q) \pitchfork W^u(q)) \setminus \{q\}$ . Fix  $0 < \varepsilon < \frac{1}{12}$ .

Then an open neighborhood of  $\{q\} \cup \mathcal{O}(p, \mathbf{f})$  that satisfies Conditions 4.2.8, 4.2.9, 4.2.10 and 4.2.11 with respect to  $\varepsilon$  is an **adapted neighborhood of  $\{q\} \cup \mathcal{O}(p, \mathbf{f})$  with respect to  $\varepsilon$** .

**Fact 4.2.1.** Let  $q \in S$  be a fixed hyperbolic point for  $\mathbf{f}$  and let  $p \in (W^s(q) \pitchfork W^s(q)) \setminus \{q\}$ . Fix  $0 < \varepsilon < \frac{1}{12}$ .

Let  $W$  be an open neighborhood of  $\{q\} \cup \mathcal{O}(p, \mathbf{f})$ . Then there exists an adapted neighborhood of  $\{q\} \cup \mathcal{O}(p, \mathbf{f})$  with respect to  $\varepsilon$  contained in  $W$ .

*Proof.* From Lemma 4.2.3, there exists an adapted neighborhood  $O_\varepsilon$  of  $q$  for  $\mathcal{O}(p, \mathbf{f})$  with respect to  $\varepsilon$  contained in  $CC(W, q)$  (since  $CC(W, q)$  is a neighborhood of  $q$ ). Denote as  $\mathbf{f}^K(p), \mathbf{f}^{(K-n)}(p) \in \mathcal{O}(p, \mathbf{f})$  the future-first-entry and past-first-entry points, respectively, for  $O_\varepsilon$  (see Definition 4.2.1). Since  $O_\varepsilon$  is an adapted neighborhood of  $q$ , by Condition 4.2.5 for any  $i \in \llbracket 1, n-1 \rrbracket$  the point  $\mathbf{f}^{(K-i)}(p)$  does not belong to  $O_\varepsilon$ . Consider so for every  $i \in \llbracket 1, n-1 \rrbracket$  open connected neighborhoods  $U_i$  of  $\mathbf{f}^{(K-i)}(p)$ , contained in  $W$ , disjoint from  $O_\varepsilon$  and such that each  $U_i$  meets  $\mathcal{O}(p, \mathbf{f})$  just at  $\mathbf{f}^{(K-i)}(p)$ .

The neighborhood

$$W_1 := O_\varepsilon \cup \bigcup_{i=1}^{n-1} U_i$$

is so a neighborhood of  $\{q\} \cup \mathcal{O}(p, \mathbf{f})$  contained in  $W$  and satisfying Condition 4.2.8.

The set  $\{q\} \cup \mathcal{O}(p, \mathbf{f})$  is hyperbolic for  $\mathbf{f}$ , hence it satisfies the cone field property (see Proposition A.0.2). Since this property is open in  $S$  (see Appendix B), we find an open neighborhood  $W_2$  of  $\{q\} \cup \mathcal{O}(p, \mathbf{f})$  contained in  $W_1$  which satisfies the cone field property, that is Condition 4.2.9.

Up to restrict  $W_2$ , we can assume that  $\tilde{W}_2 := CC(W_2, q)$  is an adapted neighborhood of  $q$  for  $\mathcal{O}(p, \mathbf{f})$  with respect to  $\varepsilon$  and that  $W_2 \setminus \tilde{W}_2$  is made up of a finite number of connected components, each of which intersects  $\{q\} \cup \mathcal{O}(p, \mathbf{f})$  at just one point of  $\mathcal{O}(p, \mathbf{f})$ . That is,  $W_2$  satisfies Conditions 4.2.8 and 4.2.9.

Observe that if two neighborhoods of  $\{q\} \cup \mathcal{O}(p, \mathbf{f})$  are such that the first is contained in



the second, then the maximal  $\mathfrak{f}$ -invariant subset of the first neighborhood is contained in the maximal  $\mathfrak{f}$ -invariant subset of the second.

By Condition 4.2.9 and by the cone field criterion (see Proposition A.0.2), the  $\mathfrak{f}$ -invariant set  $\Lambda(W_2)$  is hyperbolic for  $\mathfrak{f}$ . Recall that the hyperbolic splitting of  $\Lambda(W_2)$  is continuous. Also the angle functions

$$\Lambda(W_2) \times \Lambda(W_2) \ni (x, y) \mapsto \theta(E_x^i, E_y^i) \in \mathbb{T} \quad \text{for } i = u, s$$

are continuous. So, we find a neighborhood  $W_3$  contained in  $W_2$  satisfying Condition 4.2.8, i.e. a neighborhood

$$W_3 := V \cup \bigcup_{i=1}^m V_i,$$

where  $V$  is an adapted neighborhood of  $q$ , such that

- for every  $x \in \Lambda(W_2) \cap V$  the angles  $\theta(E_q^u, E_x^u)$  and  $\theta(E_q^s, E_x^s)$  both are in  $(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$

and such that, denoting as  $\mathfrak{f}^L(p)$  the future-first-entry point for  $V$ ,

- for every  $x \in \Lambda(W_2) \cap V_i$  the angles  $\theta(E_{\mathfrak{f}^{(L-i)}(p)}^u, E_x^u)$  and  $\theta(E_{\mathfrak{f}^{(L-i)}(p)}^s, E_x^s)$  both are in  $(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ , where  $\mathfrak{f}^{(L-i)}(p)$  is the only point of  $\mathcal{O}(p, \mathfrak{f})$  belonging to  $V_i$ .

Then, the neighborhood  $W_3$  is a neighborhood of  $\{q\} \cup \mathcal{O}(p, \mathfrak{f})$  which satisfies Condition 4.2.9 (the cone field property) because it is contained in  $W_2$  and, since  $\Lambda(W_3) \subset \Lambda(W_2)$ , it satisfies also Condition 4.2.10 (concerning the angles of the (un)stable spaces).

By the continuity of the vector fields  $e^u, e^s$  (see Lemma 4.2.4) and since the torsion at finite time is continuous, there exists a neighborhood  $W_4$  contained in  $W_3$  (so in particular  $\Lambda(W_4) \subset \Lambda(W_3)$ ) which satisfies Conditions 4.2.8 and 4.2.9 and such that

- for any  $x \in \Lambda(W_3) \cap CC(W_4, q)$  it holds

$$|\text{Torsion}_1(\mathfrak{f}, x, e_x^u) - \text{Torsion}_1(\mathfrak{f}, q, e_q^u)| < \frac{\varepsilon}{2},$$

$$|\text{Torsion}_1(\mathfrak{f}, x, e_x^s) - \text{Torsion}_1(\mathfrak{f}, q, e_q^s)| < \frac{\varepsilon}{2}.$$

- for any  $i \in \mathbb{Z}$  such that  $\mathfrak{f}^i(p) \notin CC(W_4, q)$  at any  $x \in \Lambda(W_3) \cap CC(W_4, \mathfrak{f}^i(p))$  it holds

$$|\text{Torsion}_1(\mathfrak{f}, x, e_x^u) - \text{Torsion}_1(\mathfrak{f}, \mathfrak{f}^i(p), e_{\mathfrak{f}^i(p)}^u)| < \frac{\varepsilon}{2},$$

$$|\text{Torsion}_1(\mathfrak{f}, x, e_x^s) - \text{Torsion}_1(\mathfrak{f}, \mathfrak{f}^i(p), e_{\mathfrak{f}^i(p)}^s)| < \frac{\varepsilon}{2}.$$

Up to shrink the connected components other than  $CC(W_4, q)$ , the neighborhood  $W_4$  can be chosen so that it satisfies also Condition 4.2.10 (concerning angles of (un)stable subspaces). Consequently, we conclude that  $W_4$  is an adapted neighborhood of  $\{q\} \cup \mathcal{O}(p, \mathfrak{f})$  with respect to  $\varepsilon$  contained in  $W$ .

□

### 4.3 Construction of the horseshoe

In the sequel we are going to construct the so-called horseshoe dynamics.

Let us recall our framework. Let  $f^N = \mathbf{f} : S \rightarrow S$  be a  $\mathcal{C}^1$  diffeomorphism on  $S$  (where  $S$  is either  $\mathbb{R}^2$  or  $\mathbb{A}$  or  $\mathbb{T}^2$ ) isotopic to the identity. Let  $q \in S$  be a fixed hyperbolic point for  $\mathbf{f}$  (i.e. a periodic hyperbolic point for  $f$  of period  $N$ ) such that the eigenvalues of  $D\mathbf{f}(q)$  are  $\lambda_1, \lambda_2 \in \mathbb{R}$  so that  $0 < \lambda_1 < 1 < \lambda_2$ . Let  $p \in (W^u(q) \cap W^s(q)) \setminus \{q\}$ .

Let  $0 < \varepsilon < \frac{1}{12}$ . Fix an adapted neighborhood  $U_\varepsilon$  of  $\{q\} \cup \mathcal{O}(p, \mathbf{f})$  with respect to  $\varepsilon$  (see Definition 4.2.4).

Denote as  $\tilde{U}$  the connected component  $CC(U_\varepsilon, q)$ : by Condition 4.2.8, it is an adapted neighborhood of  $q$  for  $\mathcal{O}(p, \mathbf{f})$  with respect to  $\varepsilon$ . Up to change the point, let  $p$  be the future-first-entry point for  $\tilde{U}$  (see Definition 4.2.1). Denote as  $\mathbf{f}^{-n_u}(p)$  the past-first-entry point for  $\tilde{U}$ .

The neighborhood  $U_\varepsilon$  has  $n_u$  connected components (see Remark 4.2.6) and we denote as  $U_i$  the connected component which meets  $\mathcal{O}(p, \mathbf{f})$  only at  $\mathbf{f}^{-i}(p)$  (see Figure 4.5).

From now on, on  $\tilde{U}$  we use the coordinates given by the chart  $\phi$  of Lemma 4.2.1. That is, the local unstable and stable manifolds are contained respectively in  $\mathbb{R} \times \{0\}$  and  $\{0\} \times \mathbb{R}$  with respect to these coordinates. Up to contract or dilate the chart, we assume that

$$\phi(CC(W^s(q) \cap \tilde{U}, q)) = \{0\} \times B_1(0) \quad \text{and} \quad \phi(CC(W^u(q) \cap \tilde{U}, q)) = B_1(0) \times \{0\},$$

where  $CC(W^s(q) \cap \tilde{U}, q)$  (respectively  $CC(W^u(q) \cap \tilde{U}, q)$ ) is the connected component of  $W^s(q) \cap \tilde{U}$  (respectively  $W^u(q) \cap \tilde{U}$ ) that contains  $q$ .

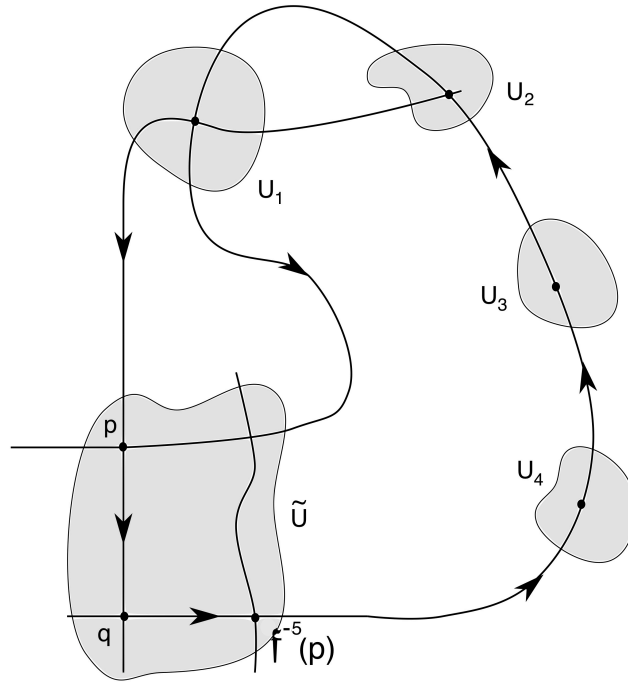


Figure 4.5 – An adapted neighborhood  $U_\varepsilon$  of  $\{q\} \cup \mathcal{O}(p, f^N)$  with  $n_u = 5$ .

Up to modify the chart  $\phi$  as  $z \mapsto (-p_1 \circ \phi(z), p_2 \circ \phi(z))$  or as  $z \mapsto (p_1 \circ \phi(z), -p_2 \circ \phi(z))$ , assume that  $\phi(p) \in \{0\} \times \mathbb{R}_+$  and  $\phi(\mathbf{f}^{-n_u}(p)) \in \mathbb{R}_+ \times \{0\}$ .

Define for any  $r \in (0, 1)$  the following sets

$$D_r^s := \{(0, y) \in \mathbb{R}^2 : y \in [0, r)\} \quad \text{and} \quad D_r^u := \{(x, 0) \in \mathbb{R}^2 : x \in [0, r)\}.$$

Since  $p, \mathfrak{f}^{-n_u}(p)$  are the future-first-entry and the past-first-entry points for  $\tilde{U}$  respectively, we have that

$$p \in \phi^{-1}(D_1^s) \setminus \mathfrak{f}(\phi^{-1}(D_1^s))$$

and

$$\mathfrak{f}^{-n_u}(p) \in \phi^{-1}(D_1^u) \setminus \mathfrak{f}^{-1}(\phi^{-1}(D_1^u)).$$

**Assumption 4.3.1.** Up to slightly modify the chart  $\phi$ , we assume that

$$p \in \text{int}(\phi^{-1}(D_1^s) \setminus \mathfrak{f}(\phi^{-1}(D_1^s))) \quad (4.8)$$

and

$$\mathfrak{f}^{-n_u}(p) \in \text{int}(\phi^{-1}(D_1^u) \setminus \mathfrak{f}^{-1}(\phi^{-1}(D_1^u))). \quad (4.9)$$

**Lemma 4.3.1.** *There exist  $\delta_u \in (0, 1]$  and  $\delta_s \in (0, 1]$  such that*

$$p \in \text{int}(\mathfrak{f}^{-n_u}(\phi^{-1}(D_{\delta_s}^s)) \setminus \mathfrak{f}^{-(n_u-1)}(\phi^{-1}(D_{\delta_s}^s)))$$

and

$$p \in \text{int}(\mathfrak{f}^{n_u}(\phi^{-1}(D_{\delta_u}^u)) \setminus \mathfrak{f}^{(n_u-1)}(\phi^{-1}(D_{\delta_u}^u))).$$

In order to prove Lemma 4.3.1, we introduce the following functions

$$\begin{aligned} (0, 1] \ni \delta &\mapsto l^s(\delta) \in \mathbb{N}, \\ l^s(\delta) &:= \min\{n \geq 0 : \mathfrak{f}^n(p) \in D_\delta^s\} \end{aligned} \quad (4.10)$$

and

$$\begin{aligned} (0, 1] \ni \delta &\mapsto l^u(\delta) \in \mathbb{N}, \\ l^u(\delta) &:= \min\{n \geq 0 : \mathfrak{f}^{-n}(p) \in D_\delta^u\}. \end{aligned} \quad (4.11)$$

**Remark 4.3.1.** Observe that for any  $n \geq l^s(\delta)$  it holds  $\mathfrak{f}^n(p) \in D_\delta^s$  and that for any  $n < l^s(\delta)$  it holds  $\mathfrak{f}^n(p) \notin D_\delta^s$ . Similarly for any  $n \geq l^u(\delta)$  it holds  $\mathfrak{f}^{-n}(p) \in D_\delta^u$  and for any  $n < l^u(\delta)$  it holds  $\mathfrak{f}^{-n}(p) \notin D_\delta^u$ .

**Remark 4.3.2.** We remark also that the functions  $\delta \mapsto l^s(\delta)$  and  $\delta \mapsto l^u(\delta)$  are both non increasing.

**Remark 4.3.3.** The integer  $l^s(\delta) \in \mathbb{N}$  is the unique integer such that

$$p \in \mathfrak{f}^{-l^s(\delta)}(\phi^{-1}(D_\delta^s)) \setminus \mathfrak{f}^{-(l^s(\delta)-1)}(\phi^{-1}(D_\delta^s)), \quad (4.12)$$

while  $l^u(\delta) \in \mathbb{N}$  is the unique integer such that

$$p \in \mathfrak{f}^{l^u(\delta)}(\phi^{-1}(D_\delta^u)) \setminus \mathfrak{f}^{(l^u(\delta)-1)}(\phi^{-1}(D_\delta^u)). \quad (4.13)$$

Lemma 4.3.1 follows then from the following claim.

**Claim 4.3.1.** For any  $J \in \mathbb{N}, J \geq l^s(1)$  there exists  $\delta_s \in (0, 1]$  such that  $l^s(\delta_s) = J$ . Similarly, for any  $J \in \mathbb{N}, J \geq l^u(1)$  there exists  $\delta_u \in (0, 1]$  such that  $l^u(\delta_u) = J$ .

*Proof of Claim 4.3.1.* We just prove the first assertion. The second one follows similarly. By Assumption 4.3.1, we have that  $p \in \text{int}(\phi^{-1}(D_1^s) \setminus \mathfrak{f}(\phi^{-1}(D_1^s)))$ . Consequently, for any  $n \geq 0$  it holds that

$$\mathfrak{f}^n(p) \in \text{int}(\mathfrak{f}^n(\phi^{-1}(D_1^s)) \setminus \mathfrak{f}^{(n+1)}(\phi^{-1}(D_1^s))).$$

Observe that, for any natural integer  $n$ , the set  $\mathfrak{f}^n(\phi^{-1}(D_1^s)) \setminus \mathfrak{f}^{(n+1)}(\phi^{-1}(D_1^s))$  is a fundamental domain of the connected component  $B^s$  of  $W^s(q) \setminus \{q\}$  that contains  $p$ . That is the orbit of any  $x \in B^s$  meets such a domain at just one point.

There exists  $\delta(n) \in (0, 1]$  such that

$$\mathfrak{f}^n(\phi^{-1}(D_1^s)) = \phi^{-1}(D_{\delta(n)}^s).$$

Therefore

$$\mathfrak{f}^n(p) \in \text{int}(\phi^{-1}(D_{\delta(n)}^s) \setminus \mathfrak{f}(\phi^{-1}(D_{\delta(n)}^s)))$$

and so

$$p \in \text{int}(\mathfrak{f}^{-n}(\phi^{-1}(D_{\delta(n)}^s)) \setminus \mathfrak{f}^{-(n-1)}(\phi^{-1}(D_{\delta(n)}^s))),$$

i.e.  $n = l^s(\delta(n))$ . By choosing  $n = J$ , we have that  $\delta(J) = \delta_s \in (0, 1]$  is so that  $l^s(\delta_s) = J$ .  $\square$

*Proof of Lemma 4.3.1.* By Claim 4.3.1, for any  $J \in \mathbb{N}, J \geq \max(l^s(1), l^u(1))$  there exist  $\delta_s, \delta_u \in (0, 1]$  such that  $l^s(\delta_s) = l^u(\delta_u) = J$ . Remark that  $l^s(1) = 1$  and  $l^u(1) = n_u$  from (4.8) and (4.9). By choosing then  $J = n_u$ , we immediately conclude.  $\square$

**Remark 4.3.4.** Up to slightly modify  $\delta_s, \delta_u$ , we can assume that  $l^s(\delta_s) = l^u(\delta_u) = n_u$  and (see Figure 4.6)

- (i)  $\mathfrak{f}^{-n_u}(\phi^{-1}(D_{\delta_s}^s)) \subset \tilde{U}$ ;
- (ii)  $\mathfrak{f}^{-n_u}(\phi^{-1}(0, \delta_s)) \notin CC(W^u(q) \cap \tilde{U}, p)$ ;
- (iii)  $\mathfrak{f}^{n_u}(\phi^{-1}(\delta_u, 0)) \in CC(W^u(q) \cap \tilde{U}, p) \setminus CC(W^s(q) \cap \tilde{U}, q)$ .

**Remark 4.3.5.** Since  $\tilde{U} = CC(U_\varepsilon, q)$  is an adapted neighborhood of  $q$  for  $\mathcal{O}(p, \mathfrak{f})$  with respect to  $\varepsilon$ , by Condition 4.2.3 we have that  $CC(W^u(q) \cap \tilde{U}, p)$  is a graph with respect to the first coordinate (through  $\phi$ ) and that  $CC(W^s(q) \cap \tilde{U}, \mathfrak{f}^{-n_u}(p))$  is a graph with respect to the second coordinate (through  $\phi$ ). See Figure 4.7.

**Notation 4.3.1.** The notations  $p_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $p_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$  refer to the projections over the first and the second coordinates respectively.

**Notation 4.3.2.** Let  $\mathcal{V}_1$  be a rectangle-shaped neighborhood of  $p$  contained in  $\tilde{U}$ . That is, there exist two closed intervals  $I_1, I_2$  contained respectively in  $\mathbb{R} \times \{0\}, \{0\} \times \mathbb{R}$  such that  $\phi(\mathcal{V}_1) = p_1(I_1) \times p_2(I_2)$ . Observe that  $0 \in p_1(I_1)$ .

By the continuity of  $\mathfrak{f}^{-i}$  for  $i \in \llbracket 1, n_u \rrbracket$  and since  $\mathfrak{f}^{-n_u}(p) \in \tilde{U}$  and  $\mathfrak{f}^{-i}(p) \in U_i$  for any  $i \in \llbracket 1, n_u - 1 \rrbracket$ , we can choose  $\mathcal{V}_1$  so that:

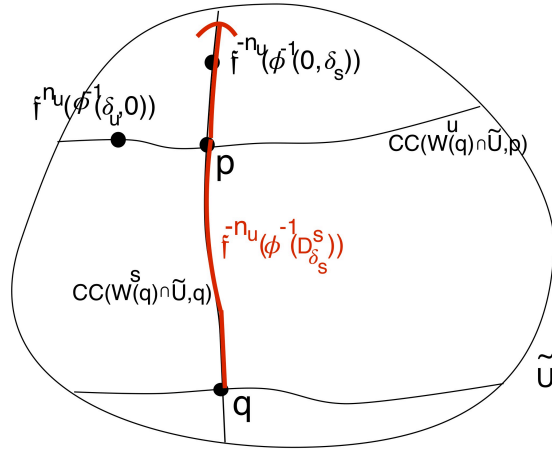


Figure 4.6

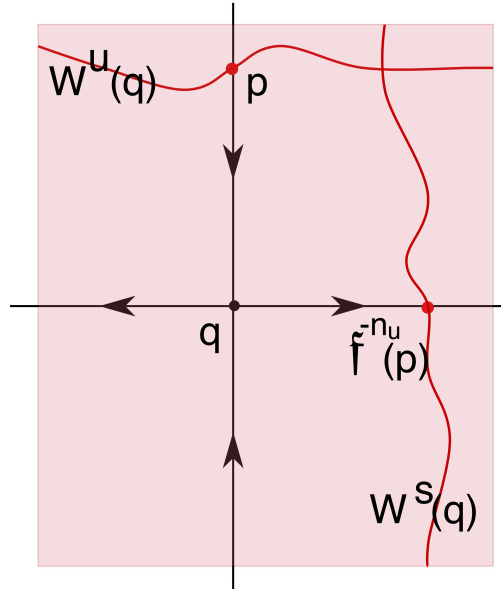


Figure 4.7 –  $CC(W^u(q) \cap \tilde{U}, p)$  (respectively  $CC(W^s(q) \cap \tilde{U}, f^{-n_u}(p))$ ) is a graph with respect to the first (respectively second) coordinate.

- $f^{-i}(\mathcal{V}_1)$  is contained in  $U_i$  for any  $i \in \llbracket 1, n_u - 1 \rrbracket$ ;
- $f^{-n_u}(\mathcal{V}_1)$  is contained in  $\tilde{U}$ .

Let  $\mathcal{V}_0$  be a rectangle-shaped neighborhood of  $q$  contained in  $\tilde{U}$ . Observe that  $0 \in p_1 \circ \phi(\mathcal{V}_0)$  and  $0 \in p_2 \circ \phi(\mathcal{V}_0)$ . By the continuity of  $f^i$  for any  $i \in \llbracket 1, n_u \rrbracket$  and since  $f^{-i}(q) \in \tilde{U}$  for any  $i \in \llbracket 1, n_u \rrbracket$ , we can choose  $\mathcal{V}_0$  so that:

- $f^{-i}(\mathcal{V}_0)$  is contained in  $\tilde{U}$  for any  $i \in \llbracket 1, n_u \rrbracket$ .

**Notation 4.3.3.** From now on, in order to lighten the notation, we omit to write  $\phi, \phi^{-1}$ .

By Condition 4.2.4 of  $\tilde{U}$  adapted neighborhood of  $q$  for  $\mathcal{O}(p)$  (the cone field property), for every  $x \in \tilde{U}$  we can define its stable and unstable cones. Denote as  $C_x^s$  and  $C_x^u$  the

stable and unstable cones at  $x \in \tilde{U}$  respectively<sup>10</sup>. For every  $x \in \tilde{U}$  we can assume that  $C_x^s \cap C_x^u = \{0\}$ .

Let us now introduce the notion of stable and unstable curves and of rectangles. We refer to Appendix C for a detailed discussion of these notions.

**Definition 4.3.1.** Let  $\gamma : [0, 1] \rightarrow \tilde{U}$  be a  $\mathcal{C}^1$  embedding. Then  $\gamma$  is an unstable curve if for any  $t \in [0, 1]$  it holds

$$\gamma'(t) \in C_{\gamma(t)}^u.$$

Similarly,  $\gamma$  is a stable curve if for any  $t \in [0, 1]$  it holds

$$\gamma'(t) \in C_{\gamma(t)}^s.$$

A rectangle  $R$  is a  $\mathcal{C}^1$  embedding

$$R : [0, 1]^2 \rightarrow \tilde{U}$$

such that for any  $t \in [0, 1]$  we have that  $R(\{t\} \times [0, 1])$  is a stable curve and  $R([0, 1] \times \{t\})$  is an unstable curve.

**Notation 4.3.4.** Let  $R$  be a rectangle. The stable boundary of  $R$  is

$$\partial^s R = \partial_0^s R \cup \partial_1^s R = R(\{0\} \times [0, 1]) \cup R(\{1\} \times [0, 1]),$$

while the unstable boundary of  $R$  is

$$\partial^u R = \partial_0^u R \cup \partial_1^u R = R([0, 1] \times \{0\}) \cup R([0, 1] \times \{1\}).$$

Let  $j \in \mathbb{N}$ . Let  $Q = Q(j)$  be a rectangle contained in  $\tilde{U}$  such that

- $\partial_0^s Q = cl(D_{\delta_s}^s)$ ;
- denoting as  $[q, \mathfrak{f}^{-(n_u+j)}(p)]^u$  the subset of the local unstable manifold of  $q$  connecting  $q$  to  $\mathfrak{f}^{-(n_u+j)}(p)$ , it holds that
$$[q, \mathfrak{f}^{-(n_u+j)}(p)]^u \subset \partial_0^u Q \subseteq \mathfrak{f}^{-j}(D_{\delta_u}^u);$$
- $\mathfrak{f}^{-n_u}(Q) \subset \tilde{U}$ .

See Figure 4.8. Denote as  $(\mathcal{V}_t)_{t \in [0, 1]}$  the foliation in  $Q$  made up of stable curves  $(Q(\{t\} \times [0, 1]))_{t \in [0, 1]}$ . Denote as  $(\mathcal{H}_s)_{s \in [0, 1]}$  the foliation of  $Q$  made up of unstable curves  $(Q([0, 1] \times \{s\}))_{s \in [0, 1]}$ .

Denote as

$$R = \mathfrak{f}^{-n_u}(Q).$$

**Condition 4.3.1.** Let  $j_1 \in \mathbb{N}$  be such that for any  $j \geq j_1$  the points  $q$  and  $p$  belong to different connected components of

$$\mathfrak{f}^{(2n_u+j)}(R) \cap R,$$

denoted respectively as  $V_0(j)$  and  $V_1(j)$ .

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<sup>10</sup>. In order to lighten the notation, we omit the parameter  $\eta$  referring to the size of the cones. See Appendixes B and C

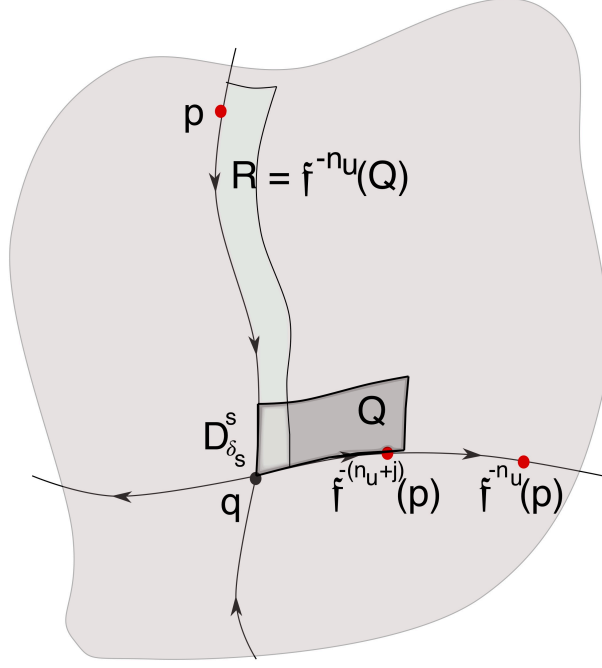


Figure 4.8 – The rectangle  $Q \subset \tilde{U}$ .

**Condition 4.3.2.** Let  $\mathcal{V}_1, \mathcal{V}_0$  be the neighborhoods of  $p$  and  $q$  respectively introduced in Notation 4.3.2. Let  $j_2 \in \mathbb{N}$  be such that for any  $j \geq j_2$  the connected components of  $\mathfrak{f}^{(2n_u+j)}(R) \cap R$  containing  $q$  and  $p$  (that is  $V_0(j)$  and  $V_1(j)$ , see Condition 4.3.1) are contained in  $\mathcal{V}_1$  and  $\mathcal{V}_0$  respectively.

**Condition 4.3.3.** Denote as  $\mathcal{F}$  the foliation whose leaves are the connected components of

$$\mathfrak{f}^{-n_u}(\mathcal{V}_t) \cap (V_0(j) \cup V_1(j))$$

as  $t \in [0, 1]$ .

Let  $j_3 \in \mathbb{N}$  be such that for any  $j \geq j_3$  for any  $s \in [0, 1]$  the image  $\mathfrak{f}^{(n_u+j)}(\mathcal{H}_s)$  intersects each leaf of  $\mathcal{F}$  only once and transversally.

**Proposition 4.3.1.** *There exists  $j \in \mathbb{N}$  such that the rectangle  $Q(j)$  is well-defined and Conditions 4.3.1, 4.3.2 and 4.3.3 hold.*

*Proof.* Recall that  $p \in \tilde{U}$  and  $\mathfrak{f}^{-n_u}(p) \in \tilde{U}$  denote respectively the future-first-entry and the past-first-entry points for  $\tilde{U}$  (see Definition 4.2.1). With an abuse of notation we identify  $\tilde{U}$  with its image in  $\mathbb{R}^2$  through the chart  $\phi$ .

The unstable and stable subspaces  $E^u$  and  $E^s$  extend continuously on  $\tilde{U}$  (see Proposition B.0.2 in Appendix B). Actually, the unstable subspace can be extended such that for  $x \in CC(W^u(q) \cap \tilde{U}, q) \cup CC(W^u(q) \cap \tilde{U}, p)$  it holds  $T_x W^u(q) = E_x^u$ .

There exists a continuous vector field  $e^u : \tilde{U} \rightarrow E_V^u$  such that for any  $z \in \tilde{U}$  the vector  $e_z^u$  belongs to  $E_z^u \cap \mathbb{S}^1$  (see Proposition B.0.3 in Appendix B). Observe that for  $z \in CC(W^u(q) \cap \tilde{U}, q)$  we have that  $D\phi(z)e_z^u \in \mathbb{R} \times \{0\}$ .

For any  $(0, y) \in D_{\delta_s}^s$  the vector  $(0, 1)$  belongs to  $\text{int } C_{(0,y)}^s$ . By the continuity of the stable cones, there exists a compact rectangle-shaped neighborhood  $\mathcal{W}$  of  $\phi^{-1}(D_{\delta_s}^s)$  contained in  $\tilde{U}$  such that for any  $z \in \mathcal{W}$  the vector  $D\phi^{-1}(\phi(z))(0, 1)$  belongs to  $\text{int } C_z^s$ . In particular any vertical segment contained in  $\mathcal{W}$  is a stable curve.

Since<sup>11</sup> (see Remark 4.3.4 and Notation 4.3.2):

- $\mathfrak{f}^{-n_u}(D_{\delta_s}^s)$  is contained in  $\tilde{U}$ ;
- $p_1(\phi \circ \mathfrak{f}^{-n_u} \circ \phi^{-1}(D_{\delta_s}^s)) = \{0\} \subset p_1(\mathcal{V}_0) \cap p_1(\mathcal{V}_1)$ ;
- $\phi \circ \mathfrak{f}^{n_u} \circ \phi^{-1}(\delta_u, 0) \notin \phi \circ \mathfrak{f}^{-n_u} \circ \phi^{-1}(D_{\delta_s}^s)$ ;

there exists a neighborhood  $\mathbb{W}$  of  $\mathfrak{f}^{-n_u} \circ \phi^{-1}(D_{\delta_s}^s)$  contained in  $\tilde{U}$  such that

$$p_1(\mathbb{W}) \subset p_1(\mathcal{V}_0) \cap p_1(\mathcal{V}_1) \quad \text{and} \quad \mathfrak{f}^{n_u} \circ \phi^{-1}(\delta_u, 0) \notin cl(\mathbb{W}).$$

In particular, there exists a ball  $B_{\zeta'}(\mathfrak{f}^{n_u}(\delta_u, 0)) \subset \tilde{U}$  centered at  $\mathfrak{f}^{n_u} \circ \phi^{-1}(\delta_u, 0)$  and disjoint from  $\mathbb{W}$ .

Observe that  $p_2(CC(W^u(q) \cap \tilde{U}, q) \cap \mathbb{W}) = \{0\}$  and so it is clearly contained in  $p_2(\mathcal{V}_0)$ .

In addition, up to restrict  $\mathbb{W}$ , we can assume that the projection over the second coordinate of  $CC(W^u(q) \cap \tilde{U}, p) \cap \mathbb{W}$  is strictly contained in  $p_2(\mathcal{V}_1)$ . See Figure 4.9.

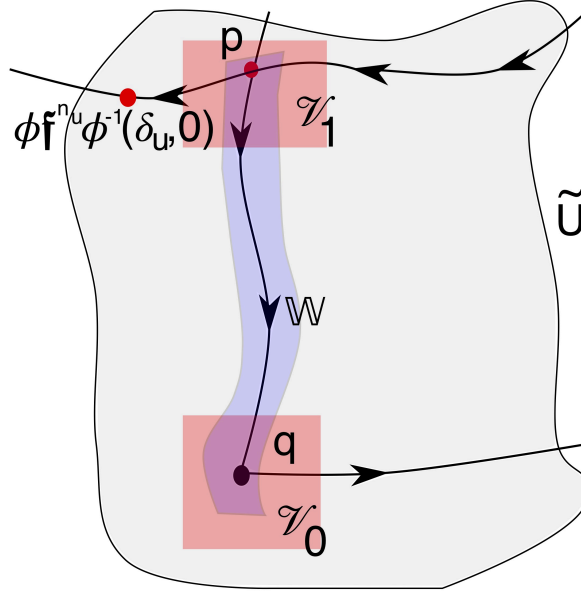


Figure 4.9 – The neighborhood  $\mathbb{W}$  of  $\phi \circ \mathfrak{f}^{-n_u} \circ \phi^{-1}(D_{\delta_s}^s)$ .

Let  $B_{\zeta'}(\mathfrak{f}^{-n_u}(0, \delta_s)) \subset \tilde{U}$  be a ball centered at  $\mathfrak{f}^{-n_u} \circ \phi^{-1}(0, \delta_s)$  and disjoint from  $CC(W^u(q) \cap \tilde{U}, p)$  (see Remark 4.3.4).

Since  $\mathfrak{f}^{-n_u}$  is a  $\mathcal{C}^1$  diffeomorphism and since the images through  $D\mathfrak{f}^{-1}$  of stable curves contained in  $\tilde{U}$  remain stable curves (see Lemma C.5.1), there exists  $\delta > 0$  so that if  $\gamma$  is a stable curve  $\delta$ - $\mathcal{C}^0$  close to  $\phi^{-1}(D_{\delta_s}^s)$ , then  $\mathfrak{f}^{-n_u} \circ \gamma$  is a stable curve in  $\mathbb{W}$  and the image through  $\mathfrak{f}^{-n_u}$  of the endpoint of  $\gamma$  that is  $\delta$ -close to  $(\phi^{-1}(0, \delta_s))$  is in  $B_{\zeta'}(\mathfrak{f}^{-n_u}(0, \delta_s))$ .

Consider now  $\mathfrak{f}^{n_u} \circ \phi^{-1}(D_{\delta_u}^u)$ . There exists a neighborhood  $\mathbb{U}$  of  $\mathfrak{f}^{n_u} \circ \phi^{-1}(D_{\delta_u}^u)$  such that (see Figure 4.10):

- $q$  and  $p$  belong to different connected components of  $\mathbb{U} \cap \tilde{U}$ ;

11. Recall that we are considering the coordinates of the chart  $\phi$ .



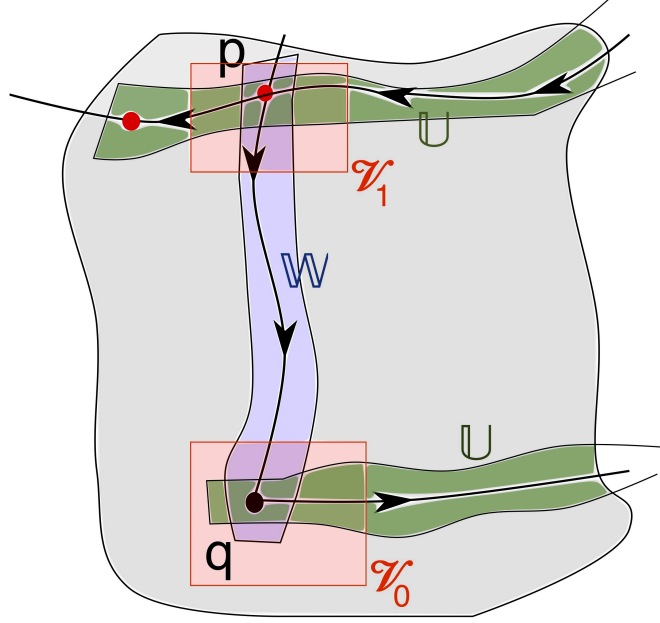


Figure 4.10 – The components of the neighborhood  $\mathbb{U}$  of  $\phi \circ \mathbf{f}^{n_u} \circ \phi^{-1}(D_{\delta_u}^u)$  in  $\tilde{U}$  containing  $q$  and  $p$ .

- the connected component of  $\mathbb{U} \cap \tilde{U}$  containing  $p$  is disjoint from  $B_{\zeta}(\mathbf{f}^{-n_u}(0, \delta_s))$ ;
- the projection over the second coordinate of  $CC(\mathbb{U} \cap \tilde{U}, p) \cap \mathbb{W}$  is contained in  $p_2(\mathcal{V}_1)$  and the projection over the second coordinate of  $CC(\mathbb{U} \cap \tilde{U}, q)$  is contained in  $p_2(\mathcal{V}_0)$ .

Let  $\rho > 0$  be such that if a curve  $\gamma$  contained in  $\tilde{U}$  is  $\rho$ - $\mathcal{C}^1$  close to either  $CC(W^u(q) \cap \tilde{U}, q)$  or to  $CC(W^u(q) \cap \tilde{U}, p)$ , then the vector tangent to the curve  $\gamma'$  belongs to the unstable cone of the corresponding point of  $\gamma$ . This comes from the fact that at points  $x$  of  $CC(W^u(q) \cap \tilde{U}, q)$  and of  $CC(W^u(q) \cap \tilde{U}, p)$  the subspace  $T_x W^u(q)$  is the unstable subspace of  $x$ .

Thus, if  $\gamma$  is a curve contained in  $\tilde{U}$  and it is  $\rho$ - $\mathcal{C}^1$  close to  $\mathbf{f}^{n_u} \circ \phi^{-1}(D_{\delta_u}^u)$ , then  $\gamma$  intersects a stable curve contained either in  $CC(\mathbb{U} \cap \mathbb{W}, p)$  or in  $CC(\mathbb{U} \cap \mathbb{W}, q)$  at most once and transversally because stable and unstable cones are disjoint.

Recall that  $\mathcal{W}$  is the fixed neighborhood of  $\phi^{-1}(D_{\delta_s}^s)$  where the vertical segments are stable curves. Let  $j_1 \in \mathbb{N}$  be the first integer such that  $\mathbf{f}^{-j_1} \circ \phi^{-1}(D_{\delta_u}^u) \subset \mathcal{W}$ . Since  $\mathbf{f}^{(j_1+n_u)}$  is a  $\mathcal{C}^1$  diffeomorphism, there exists  $\varepsilon > 0$  such that if an unstable curve  $\gamma$  is contained in  $\mathcal{W}$  and is  $\varepsilon$ - $\mathcal{C}^1$  close to  $\mathbf{f}^{-j_1} \circ \phi^{-1}(D_{\delta_u}^u)$ , then the curve  $\mathbf{f}^{(j_1+n_u)} \circ \gamma$  is contained in  $\mathbb{U}$  and is  $\rho$ - $\mathcal{C}^1$  close to  $\phi \circ \mathbf{f}^{n_u} \circ \phi^{-1}(D_{\delta_u}^u)$  and one of its endpoints is in  $B_{\zeta'}(\mathbf{f}^{n_u}(\delta_u, 0))$ .

Let us now build a rectangle  $Q = Q(j)$  within  $\mathcal{W}$ .

Let us start by building an horizontal foliation, made up of unstable curves. The horizontal curve  $\phi^{-1}(\mathbb{R} \times \{0\}) \cap \mathcal{W}$  is an unstable curve because

$$\phi^{-1}(\mathbb{R} \times \{0\}) \cap \mathcal{W} \subset CC(W^u(q) \cap \tilde{U}, q)$$

and for any  $z \in \phi^{-1}(\mathbb{R} \times \{0\}) \cap \mathcal{W}$  the vector  $e_z^u = D\phi^{-1}(\phi(z))(1, 0)$  belongs to  $\text{int } C_z^u$ .

**Claim 4.3.2.** There exists a  $\mathcal{C}^\infty$  vector field  $\tilde{e}^u : \mathcal{W} \rightarrow T_{\mathcal{W}}S$  such that:

- for any  $z \in \mathcal{W}$  the vector  $\tilde{e}_z^u \in C_z^u$ ;
- for any  $z \in \mathcal{W}$  the projection over the first coordinate  $D(p_1 \circ \phi(z))\tilde{e}_z^u = 1$ ;
- for any  $z \in \phi^{-1}(\mathbb{R} \times \{0\}) \cap \mathcal{W}$  it holds  $D\phi(z)\tilde{e}_z^u \in \mathbb{R}D\phi(z)e_z^u$ .

*Proof of Claim 4.3.2.* In the proof we omit the notations  $\phi, \phi^{-1}$  in order to lighten the text. Consider the function

$$\mathcal{W} \ni z \mapsto g(z) = \frac{Dp_2(z)e_z^u}{Dp_1(z)e_z^u} \in \mathbb{R}.$$

By definition of  $\mathcal{W}$ , it holds  $\mathbb{R}(0, 1) \in \text{int } C_z^s$  for any  $z \in \mathcal{W}$  and  $C_z^s \cap C_z^u = \{0\}$ . Consequently  $e_z^u \in E_z^u$  does not belong to  $\mathbb{R}(0, 1)$  and its projection over the first coordinate is not null. We deduce so that  $g$  is continuous. Since  $\mathcal{W}$  is compact, the function  $g$  is bounded.

By Stone-Weierstrass Theorem, the set of  $\mathcal{C}^\infty$  bounded functions on  $\mathcal{W}$  is dense in the set of continuous bounded functions on  $\mathcal{W}$ . Observe that for any  $z \in \mathcal{W}$  the vector  $(1, g(z)) \in E_z^u \subset C_z^u$ . We also remark that for any  $(x, 0) \in \mathcal{W}$  it holds  $g(x, 0) = 0$ . Thus for any  $(x, y) \in \mathcal{W}$  the vector  $(1, g(x, y) - g(x, 0))$  belongs to  $E_{(x, y)}^u \subset C_{(x, y)}^u$ . Consequently, there exists a  $\mathcal{C}^\infty$  bounded function  $\tilde{g}$  on  $\mathcal{W}$  which is a perturbation of  $g$  and such that at any point  $z \in \mathcal{W}$  the vector

$$\tilde{e}_z^u := (1, \tilde{g}(z) - \tilde{g}(p_1(z), 0)) \in C_z^u.$$

The first two conditions of the claim are satisfied. Moreover, for any  $z \in (\mathbb{R} \times \{0\}) \cap \mathcal{W}$  the vector  $\tilde{e}_z^u$  belongs to  $\mathbb{R}e_z^u = \mathbb{R}(1, 0)$ <sup>12</sup>.

□

Consider the local flow  $\Psi$  defined on an open neighborhood of  $D_{\delta_s}^s \times \{0\}$  contained in  $\mathcal{W} \times \mathbb{R}$  and determined by the vector field  $\tilde{e}^u$  of Claim 4.3.2.

For any  $\sigma \in [0, 1]$  consider the  $\mathcal{C}^\infty$  curve  $\Psi((0, \sigma\delta_s), \cdot) : \mathbb{R}_+ \rightarrow \tilde{U}$ . These curves provide us the required horizontal foliation into unstable curves of our rectangle. Every curve  $\Psi((0, \sigma\delta_s), \mathbb{R}_+) \cap \mathcal{W}$  is a graph with respect to the first coordinate (through  $\phi$ ) because at every point  $z$  of  $\mathcal{W}$  the vector  $\tilde{e}_z^u$  is transversal to  $D\phi^{-1}(\phi(z))\mathbb{R}(1, 0)$ .

By the compactness of  $[0, 1]$ , there exists an interval  $[0, \alpha]$  such that for any  $\sigma \in [0, 1]$  the curve  $\phi \circ \Psi((0, \sigma\delta_s), \cdot)$  contains the graph of a  $\mathcal{C}^\infty$  function  $\Gamma_\sigma : [0, \alpha] \rightarrow \mathbb{R}$ . In particular

$$[0, \alpha] \ni t \mapsto \phi^{-1}(t, \Gamma_\sigma(t)) = \Psi((0, s\delta_s), t) \in \mathcal{W}. \quad (4.14)$$

Apply now the  $\lambda$ -lemma (see Lemma 7.1 in [PdM82]) to the leaves of the horizontal foliation of  $Q$

$$(\phi^{-1}(\text{Graph}(\Gamma_\sigma)))_{\sigma \in [0, 1]}.$$

In particular there exists  $j_2 \in \mathbb{N}$  such that for any  $i \geq j_2$  for any  $\sigma \in [0, 1]$  the curve  $\mathfrak{f}^i \circ \phi^{-1}(\text{Graph}(\Gamma_\sigma))$  contains a curve  $\Omega_\sigma$  that is  $\varepsilon\text{-}\mathcal{C}^1$  close to  $\mathfrak{f}^{-j_1} \circ \phi^{-1}(D_{\delta_u}^u)$ , that is an unstable curve, because such curve is the image of an unstable curve (see Lemma C.5.1 in Appendix C) and that is contained in  $\mathcal{W}$ . From the definition of  $\mathcal{W}$ , the curve  $\phi(\Omega_\sigma)$  is transverse to  $\mathbb{R}(0, 1)$ : thus,  $\phi(\Omega_\sigma)$  is a graph with respect to the first coordinate.

---

12. Careful! We are omitting the chart  $\phi$ .

Up to modify the domain, assume that every curve  $\phi(\Omega_\sigma)$  is the graph of a function defined on  $p_1(\phi \circ \mathfrak{f}^{-j_1} \circ \phi^{-1}(D_{\delta_u}^u))$ .

Define

$$\begin{aligned} \mathbb{Q}_i &: p_1(\phi \circ \mathfrak{f}^{-j_1} \circ \phi^{-1}(D_{\delta_u}^u)) \times [0, 1] \rightarrow \mathcal{W} \\ (t, \sigma) &\mapsto \tilde{\Psi}(\phi \circ \mathfrak{f}^i \circ \phi^{-1}(0, \sigma\delta_s), t), \end{aligned}$$

where  $\tilde{\Psi}$  is the flow generated by the vector field

$$z \mapsto \frac{D\mathfrak{f}^i(z)\tilde{e}_z^u}{D(p_1 \circ \phi \circ \mathfrak{f}^i)(z)\tilde{e}_z^u}. \quad (4.15)$$

$\mathbb{Q}_i$  is a  $\mathcal{C}^1$  diffeomorphism. Observe that, by the choice of the vector field in (4.15), it holds that

$$p_1 \circ \phi \circ \tilde{\Psi}(\mathfrak{f}^i \circ \phi^{-1}(0, \sigma\delta_s), t) = t.$$

In particular  $\mathbb{Q}_i$  is a rectangle. Indeed, for any fixed  $\sigma$  the curve  $\mathbb{Q}_i(\cdot, \sigma)$  is an unstable curve by the definition of  $\tilde{e}^u$  and because images through  $D\mathfrak{f}^i$  of unstable curves remain unstable curves. For any fixed  $t$  the curve  $\mathbb{Q}_i(t, \cdot)$  is a vertical segment because

$$p_1 \circ \phi \circ \tilde{\Psi}(\mathfrak{f}^i \circ \phi^{-1}(0, \sigma\delta_s), t) = t$$

and, since we are in  $\mathcal{W}$ , any vertical segment is a stable curve.

Define now

$$Q_i(t, \sigma) = \mathfrak{f}^{-i}\mathbb{Q}_i(t, \sigma).$$

By our construction, also  $Q_i$  is a rectangle.

Observe that for any  $i$  we have that  $\partial_0^s Q_i = cl(D_{\delta_s}^s)$  and  $\partial_0^u Q_i = p_1 \circ \mathfrak{f}^{-(j_1+i)}(D_{\delta_u}^u)$ . The images of the vertical segment in  $\mathbb{Q}_i$ , which are stable curves, remain stable curves (thanks to Lemma C.5.1 in Appendix C).

Remark that the vertical foliation of  $\mathbb{Q}_i$  does not depend on  $i$ . Apply the  $\lambda$ -lemma (see Lemma 7.1 in [PdM82]) to these vertical leaves: there exists  $j_3 \in \mathbb{N}$  such that for any  $i \geq j_3$  the image of each vertical leaf through  $\mathfrak{f}^{-i}$  is  $\delta$ - $\mathcal{C}^1$  close to  $\phi^{-1}(D_{\delta_s}^s)$ .

Let  $J = \max\{j_2, j_3\}$  and denote as  $j = J + j_1$ . The rectangle  $Q = Q_j$  is so such that:

- $\mathfrak{f}^{-n_u}(Q) = R \subset \mathbb{W}$ , since every vertical leaf is  $\delta$ - $\mathcal{C}^1$  close to  $\phi^{-1}(D_{\delta_s}^s)$ ;
- $\mathfrak{f}^{(j+n_u)}(Q) = \mathfrak{f}^{(2n_u+j)}(R) \subset \mathbb{U}$ , since  $\mathfrak{f}^J$  of every horizontal leaf is  $\varepsilon$ - $\mathcal{C}^1$  close to  $\mathfrak{f}^{-j_1} \circ \phi^{-1}(D_{\delta_u}^u)$ .

The rectangle  $Q$  is well-defined.

In particular,  $q$  and  $p$  belong to different connected components of  $R \cap \mathfrak{f}^{(2n_u+j)}(R)$ . Recall that  $V_0(j), V_1(j)$  denote the connected components of  $R \cap \mathfrak{f}^{(2n_u+j)}(R)$  to which  $q$  and  $p$  belong, respectively. That is, Condition 4.3.1 holds.

We have that  $V_0(j) \subset CC(\mathbb{W} \cap \mathbb{U}, q)$  and  $V_1(j) \subset CC(\mathbb{W} \cap \mathbb{U}, p)$ . Consequently, since  $\mathcal{V}_0$  and  $\mathcal{V}_1$  are rectangle-shaped and since  $CC(\mathbb{W} \cap \mathbb{U}, q) \subset \mathcal{V}_0$  and  $CC(\mathbb{W} \cap \mathbb{U}, p) \subset \mathcal{V}_1$ , we deduce that also Condition 4.3.2 is satisfied.

The images through  $\mathfrak{f}^{(j+n_u)}$  of the unstable curves of  $Q$  are  $\rho$ - $\mathcal{C}^1$  close to  $\mathfrak{f}^{n_u} \circ \phi^{-1}(D_{\delta_u}^u)$ . From the choice of  $\rho$ , we conclude that Condition 4.3.3 holds.

□

**Remark 4.3.6.** Referring to Notation 4.2.2, observe that the rectangle  $Q_j$  that we have just built is contained in  $RV(n_u - 1) \cap RH(j - 1)$ .

**Definition 4.3.2** (Horseshoe  $H(U_\varepsilon, j)$ ). Let  $U_\varepsilon$  be an adapted neighborhood of  $\{q\} \cup \mathcal{O}(p, \mathfrak{f})$  with respect to  $0 < \varepsilon < \frac{1}{12}$ . Assume that  $p$  is the future-first-entry point for  $\tilde{U} = CC(U_\varepsilon, q)$  and that  $\mathfrak{f}^{-n_u}(p)$  is the past-first-entry point for  $\tilde{U} = CC(U_\varepsilon, q)$ . Let  $j \in \mathbb{N}$  satisfy Conditions 4.3.1 and 4.3.2.

The horseshoe  $H(U_\varepsilon, j)$  is

$$\bigcap_{n \in \mathbb{Z}} \mathfrak{f}^{(2n_u+j)n}(V_0(j) \cup V_1(j)) \subset \tilde{U}.$$

**Remark 4.3.7.** Observe that the horseshoe  $H(U_\varepsilon, j)$  is  $\mathfrak{f}^{(2n_u+j)}$ -invariant. Moreover, since  $\tilde{U}$  satisfies the cone field property from Condition 4.2.4, the set  $H(U_\varepsilon, j)$  is hyperbolic for  $\mathfrak{f}^{(2n_u+j)}$ .

**Lemma 4.3.2.** *The set*

$$\bigcup_{i=-n_u}^{n_u+j-1} \mathfrak{f}^i(V_0(j) \cup V_1(j))$$

*is contained in  $U_\varepsilon$ . In particular*

$$\mathfrak{f}^i(V_0(j) \cup V_1(j)) \subset \tilde{U} \quad \text{for any } i \in \llbracket 0, n_u + j - 1 \rrbracket,$$

$$\mathfrak{f}^{-i}(V_0(j)) \subset \tilde{U} \quad \text{for any } i \in \llbracket 1, n_u \rrbracket,$$

$$\mathfrak{f}^{-i}(V_1(j)) \subset U_i \quad \text{for any } i \in \llbracket 1, n_u - 1 \rrbracket$$

*and*

$$\mathfrak{f}^{-n_u}(V_1(j)) \subset \tilde{U}.$$

*Proof.* Clearly,  $V_0(j) \cup V_1(j) \subset R$  is contained in  $\tilde{U}$ . We remark that the set  $\mathfrak{f}^i(R)$  is contained in  $RV \subset \tilde{U}$  for any  $i \in \llbracket 1, n_u + j - 1 \rrbracket$  (see Remarks 4.2.4 and 4.3.6) where  $RV$  is the image through  $\phi^{-1}$  of the intersection of the following sets

- $\{x \geq 0, y \geq 0\}$ ;
- *Ipo-graph*( $u_0$ ) :=  $\{(x, y) : y \leq u_0(x)\}$ , i.e. the set of all points that lie on or under the graph of  $u_0$  which is  $\phi(CC(W^u(q) \cap \tilde{U}, p))$ ;
- *Left-graph*( $s_0$ ) :=  $\{(x, y) : x \leq s_0(y)\}$ , i.e. the set of all points that lie on or at the left of the graph of  $s_0$  which is  $\phi(CC(W^s(q) \cap \tilde{U}, \mathfrak{f}^{-n_u}(p)))$ .

By Condition 4.2.7 of  $\tilde{U}$  adapted neighborhood of  $q$  for  $\mathcal{O}(p, f^N)$ , the set  $RV$  is contained in  $\tilde{U}$ . Consequently,  $\mathfrak{f}^i(R)$  (and so also  $\mathfrak{f}^i(V_0(j) \cup V_1(j))$ ) is in  $\tilde{U}$  for any  $i \in \llbracket 1, n_u + j - 1 \rrbracket$ . Since by Condition 4.3.2 the set  $V_0(j)$  is contained in  $\mathcal{V}_0$  and  $\mathfrak{f}^{-i}(\mathcal{V}_0)$  is contained in  $\tilde{U}$  for any  $i \in \llbracket 0, n_u \rrbracket$ , we deduce that

$$\mathfrak{f}^{-i}(V_0(j)) \subset \tilde{U} \quad \text{for any } i \in \llbracket 0, n_u \rrbracket.$$

Similarly, since  $V_1(j)$  is contained in  $\mathcal{V}_1$  and since for any  $i \in \llbracket 1, n_u - 1 \rrbracket$  it holds  $\mathfrak{f}^{-i}(\mathcal{V}_1) \subset U_i$  and  $\mathfrak{f}^{-n_u}(\mathcal{V}_1) \subset \tilde{U}$ , we deduce that

$$\mathfrak{f}^{-i}(V_1(j)) \subset U_i \quad \forall i \in \llbracket 1, n_u - 1 \rrbracket, \quad \mathfrak{f}^{-n_u}(V_1(j)) \subset \tilde{U}.$$

That is,

$$\bigcup_{i=-n_u}^{n_u+j-1} \mathfrak{f}^i(V_0(j) \cup V_1(j)) \subset U_\varepsilon.$$

□

**Notation 4.3.5.** Let  $H(U_\varepsilon, j)$  be the horseshoe of Definition 4.3.2. Denote as

$$\Lambda := \bigcup_{i=0}^{2n_u+j-1} \mathfrak{f}^i(H(U_\varepsilon, j))$$

the  $\mathfrak{f}$ -orbit of  $H(U_\varepsilon, j)$ .

**Remark 4.3.8.** The set  $\Lambda$  introduced in Notation 4.3.5 is  $\mathfrak{f}$ -invariant. Observe that  $\Lambda$  is contained in  $\bigcup_{i=-n_u}^{n_u+j-1} \mathfrak{f}^i(V_0(j) \cup V_1(j))$ . By Lemma 4.3.2 we deduce that  $\Lambda$  is contained in  $U_\varepsilon$ . By Condition 4.2.9, the set  $\Lambda$  is hyperbolic for  $\mathfrak{f}$ . Moreover,  $\Lambda$  is contained in  $\Lambda(U_\varepsilon)$ , where  $\Lambda(U_\varepsilon)$  is the maximal  $\mathfrak{f}$ -invariant set contained in  $U_\varepsilon$  (see Remark 4.2.7).

**Lemma 4.3.3.** *Let  $U_\varepsilon$  be the fixed adapted neighborhood of  $\{q\} \cup \mathcal{O}(p, \mathfrak{f})$  with respect to  $\varepsilon$ . Let  $j \in \mathbb{N}$  satisfy Conditions 4.3.1 and 4.3.2. Let  $H(U_\varepsilon, j)$  be the horseshoe of Definition 4.3.2. Let  $x \in H(U_\varepsilon, j)$ . Then*

- (i) *if  $f^{(2n_u+j)N}(x) = \mathfrak{f}^{(2n_u+j)}(x) \in V_0(j)$  then  $f^{iN}(x) = \mathfrak{f}^i(x) \in \tilde{U}$  for any  $i \in \llbracket 0, 2n_u+j \rrbracket$ ;*
- (ii) *if  $f^{(2n_u+j)N}(x) = \mathfrak{f}^{(2n_u+j)}(x) \in V_1(j)$  then  $f^{iN}(x) = \mathfrak{f}^i(x) \in \tilde{U}$  for any  $i \in \llbracket 0, n_u+j \rrbracket$  and  $f^{(2n_u+j-i)N}(x) = \mathfrak{f}^{(2n_u+j-i)}(x) \in U_i$  for any  $i \in \llbracket 1, n_u-1 \rrbracket$ .*

*Proof.* By Lemma 4.3.2, for any  $i \in \llbracket 0, n_u+j-1 \rrbracket$  and for any point  $x \in H(U_\varepsilon, j)$  it holds  $\mathfrak{f}^i(x) \in \tilde{U}$ .

If  $\mathfrak{f}^{(2n_u+j)}(x) \in V_0(j) \subset \mathcal{V}_0$ , then, by Lemma 4.3.2, for any  $i \in \llbracket 0, n_u \rrbracket$  we have that

$$\mathfrak{f}^{(2n_u+j-i)}(x) \in \mathfrak{f}^{-i}(\mathcal{V}_0) \subset \tilde{U}.$$

If  $\mathfrak{f}^{(2n_u+j)}(x) \in V_1(j) \subset \mathcal{V}_1$ , then, by Lemma 4.3.2, for any  $i \in \llbracket 1, n_u-1 \rrbracket$  we have that

$$\mathfrak{f}^{(2n_u+j-i)}(x) \in \mathfrak{f}^{-i}(\mathcal{V}_1) \subset U_i,$$

and

$$\mathfrak{f}^{(n_u+j)}(x) \in \mathfrak{f}^{-n_u}(\mathcal{V}_1) \subset \tilde{U}.$$

□

We now briefly recall how to build the homeomorphism to link the dynamics of the horseshoe  $H(U_\varepsilon, j)$  to the symbolic dynamics. For an accurate treatment of the argument we refer to [Sma65], [Shu87], [Rob99] and [Dev03].

Let  $\{0, 1\}^{\mathbb{Z}}$  denote the set of bi-infinite sequences of 0's and 1's symbols. The dynamics over  $\{0, 1\}^{\mathbb{Z}}$  is given by the shift map (to the left), i.e.

$$S : \{0, 1\}^{\mathbb{Z}} \rightarrow \{0, 1\}^{\mathbb{Z}}$$

$$(s_i)_{i \in \mathbb{Z}} \mapsto S((s_i)_{i \in \mathbb{Z}}) = (s_{i+1})_{i \in \mathbb{Z}}.$$

We consider the metric space  $(\{0, 1\}^{\mathbb{Z}}, \bar{d})$ , where the distance is defined as

$$\bar{d}((s_i)_{i \in \mathbb{Z}}, (t_i)_{i \in \mathbb{Z}}) = \sum_{i=-\infty}^{+\infty} \frac{|s_i - t_i|}{4^{|i|}}.$$

**Remark 4.3.9.** The metric space  $(\{0, 1\}^{\mathbb{Z}}, \bar{d})$  is a complete metric space (see Section 8.3 in [Rob99]). Actually, it is a complete metric space with respect to any metric

$$d_\lambda((s_i)_{i \in \mathbb{Z}}, (t_i)_{i \in \mathbb{Z}}) = \sum_{i=-\infty}^{+\infty} \frac{|s_i - t_i|}{\lambda^{|i|}},$$

where  $\lambda > 1$ . We highlight that, given  $(s_i)_{i \in \mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$  and  $k \geq 0$ , the cylinder

$$\mathbf{C}_k = \{(t_i)_{i \in \mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}} : t_i = s_i \ \forall |i| \leq k\}$$

coincides with the ball

$$B_{\frac{1}{\lambda^k}}((s_i)_{i \in \mathbb{Z}}) = \left\{ (t_i)_{i \in \mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}} : d_\lambda((s_i)_{i \in \mathbb{Z}}, (t_i)_{i \in \mathbb{Z}}) < \frac{1}{\lambda^k} \right\}$$

if and only if  $\lambda > 3$ .

**Proposition 4.3.2.** *Let  $H(U_\varepsilon, j)$  be the horseshoe with respect to the adapted neighborhood  $U_\varepsilon$  of  $\{q\} \cup \mathcal{O}(p, \mathbf{f})$ . There exists a homeomorphism*

$$h : H(U_\varepsilon, j) \rightarrow \{0, 1\}^{\mathbb{Z}}$$

such that

$$h \circ \mathbf{f}_{|H(U_\varepsilon, j)}^{(2n_u+j)} = S \circ h \quad \left( i.e. \ h \circ \mathbf{f}_{|H(U_\varepsilon, j)}^{(2n_u+j)N} = S \circ h \right).$$

This Proposition follows from the following Lemma 4.3.4 and from Proposition C.2.1 in Appendix C. Let us first introduce some definitions. We refer to Appendix C for a deeper discussion.

**Definition 4.3.3** (Definition C.1.3 in Appendix C). Let  $R, R'$  be rectangles. We say that  $R$  is a stable subrectangle of  $R'$  if  $R \subset R'$  and the stable boundary  $\partial^s R$  is contained in the stable boundary  $\partial^s R'$ .

**Definition 4.3.4** (Definition C.1.4 in Appendix C). Let  $R, R'$  be rectangles. We say that  $R$  is  $f$ -linked to  $R'$  and we write  $R \xrightarrow{f} R'$  if

- (i)  $f(R) \cap R' \neq \emptyset$ ;
- (ii)  $f(R) \cap R'$  is a stable subrectangle of  $R'$ ;
- (iii)  $f(\partial^s R) \cap (\text{int } R') = \emptyset$  and  $(\text{int } f(R)) \cap \partial^u R' = \emptyset$ .

**Lemma 4.3.4.** *The sets  $V_0(j), V_1(j)$  are rectangles and for  $l, m = 0, 1$  it holds*

$$V_l(j) \xrightarrow{\mathbf{f}^{(2n_u+j)}} V_m(j).$$

*Proof.* Denote as  $(\mathcal{H}_s)_{s \in [0,1]}$  the horizontal foliation of the rectangle  $Q$  and as  $(\mathcal{V}_t)_{t \in [0,1]}$  its vertical foliation. Then  $(\mathbf{f}^{-n_u}(\mathcal{V}_t))_{t \in [0,1]}$  is a vertical foliation made up of stable curves, because images through  $\mathbf{f}^{-1}$  of stable curves contained in  $\tilde{U}$  remain stable curves. By construction, for any  $s \in [0, 1]$  the curves  $\mathbf{f}^{(n_u+j)}(\mathcal{H}_s) \cap V_0(j)$  and  $\mathbf{f}^{n_u+j}(\mathcal{H}_s) \cap V_1(j)$  are unstable curves.

Let us show that  $V_0(j)$  is a rectangle. Let  $x \in V_0(j)$ : in particular  $x$  belongs to both  $\mathfrak{f}^{-n_u}(Q)$  and to  $\mathfrak{f}^{(n_u+j)}(Q)$ . Consider the function

$$V_0(j) \ni x \mapsto V_0(j)^{-1}(x) := (t(x), s(x)) = (p_1 \circ Q^{-1} \circ \mathfrak{f}^{n_u}(x), p_2 \circ Q^{-1} \circ \mathfrak{f}^{-(n_u+j)}(x)).$$

By Condition 4.3.3, every  $\mathfrak{f}^{n_u+j}(\mathcal{H}_s)$  intersects only once and transversally every  $\mathfrak{f}^{-n_u}(\mathcal{V}_t) \cap V_0(j)$ . Thus,  $V_0(j)^{-1}$  is bijective. Since stable and unstable cones are disjoint and since  $Q$  is a rectangle, it can be proved that  $V_0(j)^{-1}$  is a  $\mathcal{C}^1$  diffeomorphism (the strategy is that of the proof of Proposition C.5.3 in Appendix C).

The inverse function  $V_0(j) : [0, 1]^2 \rightarrow V_0(j)$  is the required  $\mathcal{C}^1$  diffeomorphism. Any  $V_0(j)(t, \cdot)$  is a curve contained in  $\mathfrak{f}^{-n_u}(\mathcal{V}_t)$  and so it is a stable curve. Any  $V_0(j)(\cdot, s)$  is the image through  $\mathfrak{f}^{(n_u+j)}$  of an unstable curve contained in  $\tilde{U}$  and so (by Lemma C.5.1 in Appendix C) it is an unstable curve. Thus,  $V_0(j)$  is a rectangle.

A similar argument proves that  $V_1(j)$  is a rectangle.

Let us show now that  $V_0(j) \xrightarrow{\mathfrak{f}^{(2n_u+j)}} V_0(j)$ . Thanks to the point  $q$ , it holds that

$$\mathfrak{f}^{(2n_u+j)}(V_0(j)) \cap V_0(j) \neq \emptyset.$$

Define the function

$$\mathfrak{f}^{(2n_u+j)}(V_0(j)) \cap V_0(j) \ni x \mapsto (p_1 \circ V_0(j)^{-1}(x), p_2 \circ V_0(j)^{-1} \circ \mathfrak{f}^{-(2n_u+j)}(x)) \in [0, 1]^2.$$

Using the same ideas as for  $V_0(j)^{-1}$ , it can be proved that it is a  $\mathcal{C}^1$  diffeomorphism and, through its inverse function, we can deduce that  $\mathfrak{f}^{(2n_u+j)}(V_0(j)) \cap V_0(j)$  is a stable subrectangle of  $V_0(j)$ .

Consider now the stable boundary of  $\mathfrak{f}^{(2n_u+j)}(V_0(j))$ . Its left component is contained in  $\partial_0^s(V_0(j))$ , so in particular it does not intersect the interior of  $V_0(j)$ .

Remark that  $\mathfrak{f}^{(n_u+j)}(\partial_1^s Q)$  is not contained in the interior of  $R$ , otherwise Condition 4.3.3 fails (not every image of horizontal leaf would intersect every image of vertical leaf). Thus, since  $\partial_1^s V_0(j) \subset \mathfrak{f}^{-n_u} \partial_1^s Q$ , we deduce that  $\mathfrak{f}^{(2n_u+j)}(\partial_1^s V_0(j))$  does not intersect the interior of  $V_0(j)$ .

Consider now the unstable boundary of  $V_0(j)$ . Its lower component  $\partial_0^u V_0(j)$  is contained in  $\partial_0^u(\mathfrak{f}^{(2n_u+j)}(V_0(j)))$ . In particular it does not intersect the interior of  $\mathfrak{f}^{(2n_u+j)}(V_0(j))$ .

Observe that  $\partial_1^u V_0(j)$  is contained in  $\mathfrak{f}^{n_u+j}(\partial_1^u Q)$  and that  $\mathfrak{f}^{n_u}(V_0(j)) \subset Q$ . In particular  $\text{int}(\mathfrak{f}^{(2n_u+j)}(V_0(j)))$  is contained in  $\mathfrak{f}^{n_u+j}(\text{int} Q)$ . Equivalently,  $\partial_1^u V_0(j)$  cannot intersect  $\text{int}(\mathfrak{f}^{(2n_u+j)}(V_0(j)))$ .

In conclusion

$$V_0(j) \xrightarrow{\mathfrak{f}^{(2n_u+j)N}} V_0(j).$$

The other relations can be shown with similar arguments. □

*Proof of Proposition 4.3.2.* The function  $h : H(U_\varepsilon, j) \rightarrow \{0, 1\}^\mathbb{Z}$  is defined as follows: for any  $x \in H(U_\varepsilon, j)$

$$h(x) := (s_i)_{i \in \mathbb{Z}}, \tag{4.16}$$

where for any  $i \in \mathbb{Z}$  it holds

$$\mathfrak{f}^{(2n_u+j)i}(x) = \mathfrak{f}^{(2n_u+j)Ni}(x) \in V_{s_i}(j).$$

That is, the value  $s_i$  tells us which connected component among  $V_0(j)$  and  $V_1(j)$  the  $i$ -th iterate through  $\mathfrak{f}^{(2n_u+j)}$  of  $x$  belongs at.

The function  $h$  is continuous because both  $V_0(j)$  and  $V_1(j)$  are open disjoint sets with respect to the topology induced by  $H(U_\varepsilon, j)$ .

Referring to Definitions C.1.5 and C.1.6 in Appendix C, by Lemma 4.3.4 it holds that  $\{V_0(j), V_1(j)\}$  is a geometric Markov partition and that any bi-infinite sequence in  $\{0, 1\}^{\mathbb{Z}}$  is admissible.

From Proposition C.2.1 in Appendix C we deduce that  $h$  is bijective. Since  $h$  is a continuous bijection on a compact set, we conclude that it is a homeomorphism.  $\square$

## 4.4 Symbolic dynamics and torsion

Let  $U_\varepsilon$  be a fixed adapted neighborhood of  $\{q\} \cup \mathcal{O}(p, \mathfrak{f})$  with respect to  $0 < \varepsilon < \frac{1}{12}$  (see Definition 4.2.4). Let  $H(U_\varepsilon, j)$  be the horseshoe introduced in Definition 4.3.2.

We are going to calculate (finite-time) torsion at points of the horseshoe  $H(U_\varepsilon, j)$  for  $\mathfrak{f} = f^N$ . In particular, as a first step, we look at the torsion at finite-time  $2n_u + j$  for  $\mathfrak{f} = f^N$ , where  $2n_u + j$  is the iterate of  $\mathfrak{f}$  with respect to which we have obtained the horseshoe dynamics in Section 4.3.

The following result concerns the link between symbolic dynamics and finite-time torsion. In particular it explains how to estimate  $(2n_u + j)$ -finite time torsion for  $\mathfrak{f}$  at  $x \in H(U_\varepsilon, j)$  from  $h(x)_1$ . Equivalently, it determines the  $(2n_u + j)N$ -finite time torsion for  $f$  at  $x \in H(U_\varepsilon, j)$ .

**Notation 4.4.1.** In the sequel, we always refer to the diffeomorphism  $f$  (instead of  $\mathfrak{f} = f^N$ ).

**Theorem 4.4.1.** *Let  $f : S \rightarrow S$  be a  $C^1$  diffeomorphism isotopic to the identity<sup>13</sup> and let  $q \in S$  be a hyperbolic periodic point for  $f$  of period  $N$ . Let  $p \in (W^s(q) \cap W^u(q)) \setminus \{q\}$ . Denote*

$$k = N \text{ Torsion}_N(f, q, v) \in \mathbb{Z}$$

*for  $v \in E_q^u$  or  $v \in E_q^s$ . Let  $0 < \varepsilon < \frac{1}{12}$ .*

*Fix  $U_\varepsilon$  adapted neighborhood of  $\{q\} \cup \mathcal{O}(p, f^N)$  with respect to  $\varepsilon$ . Denote as  $p$  and  $f^{-n_u N}(p)$  the future-first-entry and past-first-entry points of  $\tilde{U} = CC(U_\varepsilon, q)$  respectively.*

*Let  $H(U_\varepsilon, j)$  be the horseshoe of Definition 4.3.2. Let  $h : H(U_\varepsilon, j) \rightarrow \{0, 1\}^{\mathbb{Z}}$  be the homeomorphism (see Proposition 4.3.2) such that*

$$h \circ f_{|H(U_\varepsilon, j)}^{(2n_u + j)N} = S \circ h.$$

*Then there exists  $m \in \mathbb{Z}$  such that for any  $x \in H(U_\varepsilon, j)$*

$$\left| (2n_u + j)N \text{ Torsion}_{(2n_u + j)N}(f, x, w) - \left[ (2n_u + j)k + h(x)_1 \frac{m}{2} \right] \right| < \varepsilon. \quad (4.17)$$

*for  $w \in E_x^u$ .*

**Remark 4.4.1.** We do not make explicit the isotopy joining the identity  $\text{Id}_S$  to  $f$ . Indeed, if  $S = \mathbb{T}^2$  or  $S = \mathbb{A}$ , the finite-time torsion does not depend on the choice of the isotopy. If  $S = \mathbb{R}^2$  and  $f$  has compact support, the torsion does not depend on the choice of the isotopy.

---

13. If  $S = \mathbb{R}^2$  assume that  $f$  has compact support to guarantee the independence of the torsion from the choice of the isotopy.



Recall that  $U_\varepsilon$  is of the form (see Condition 4.2.8)

$$U_\varepsilon = \tilde{U} \cup \bigcup_{i=1}^{n_u-1} U_i,$$

where  $\tilde{U}$  is an adapted neighborhood of  $q$  for  $\mathcal{O}(p, \mathfrak{f})$  and each connected component  $U_i$  meets  $\mathcal{O}(p, \mathfrak{f})$  only at  $\mathfrak{f}^i(p) = f^{-iN}(p)$ . The set  $\Lambda(U_\varepsilon)$  is the maximal  $f^N$ -invariant set contained in  $U_\varepsilon$ .

The proof of Theorem 4.4.1 is divided into two cases. First, we calculate the torsion for  $f$  at finite-time  $(2n_u + j)N$  of any  $x \in H(U_\varepsilon, j)$  such that  $f^{(2n_u+j)N}(x) \in V_0(j)$ , i.e.  $h(x)_1 = 0$ . Then, we estimate the  $(2n_u + j)N$ -finite-time torsion for  $f$  at  $x \in H(U_\varepsilon, j)$  so that  $f^{(2n_u+j)N}(x) \in V_1(j)$ , i.e.  $h(x)_1 = 1$ .

**Notation 4.4.2.** The functions  $\gamma_s$  and  $\gamma_u$  presented respectively in (4.4) and (4.5) are parametrizations of the local stable and unstable manifolds

$$[0, t_s] \ni t \mapsto \gamma_s(t) = \phi^{-1}((0, t_s - t)) \in W_{loc, \delta}^s(q)$$

and

$$[0, t_u] \ni t \mapsto \gamma_u(t) = \phi^{-1}((t, 0)) \in W_{loc, \delta}^u(q),$$

so that  $p = \gamma_s(0)$  and  $f^{-n_u N}(p) = \gamma_u(t_u)$ .

Moreover, denote

$$e_q^u = \frac{\gamma_u'(0)}{\|\gamma_u'(0)\|} \quad \text{and} \quad e_q^s = \frac{\gamma_s'(t_s)}{\|\gamma_s'(t_s)\|}.$$

We are going to prove the following Lemma that will be largely used within this Section.

**Lemma 4.4.1.** *Let  $x_1, x_2 \in \Lambda(U_\varepsilon)$ . Suppose there exist  $\alpha, \beta \in \mathbb{R}_+$  such that  $\alpha + \beta < \frac{1}{2}$ ,  $i \in \mathbb{N}^*$  and  $l \in \mathbb{Z}$  so that*

$$(i) \quad \left| iN \text{ Torsion}_{iN}(f, x_1, e_{x_1}^u) - iN \text{ Torsion}_{iN}(f, x_2, e_{x_2}^u) + \frac{l}{2} \right| < \frac{1}{2} - (\alpha + \beta);$$

$$(ii) \quad \theta(E_{x_1}^u, E_{x_2}^u) < \alpha;$$

$$(iii) \quad \theta(E_{f^{iN}(x_1)}^u, E_{f^{iN}(x_2)}^u) < \beta.$$

Then

$$\left| iN \text{ Torsion}_{iN}(f, x_1, e_{x_1}^u) - iN \text{ Torsion}_{iN}(f, x_2, e_{x_2}^u) + \frac{l}{2} \right| < \alpha + \beta.$$

*Proof.* Since

$$e_{x_1}^u \in E_{x_1}^u \cap \mathbb{S}^1 = \{e_{x_1}^u, -e_{x_1}^u\} \quad \text{and} \quad e_{x_2}^u \in E_{x_2}^u \cap \mathbb{S}^1 = \{e_{x_2}^u, -e_{x_2}^u\}$$

and since (see Notation 4.2.1)

$$\theta(E_{x_1}^u, E_{x_2}^u) < \alpha,$$

the oriented angle  $\theta(e_{x_1}^u, e_{x_2}^u)$  admits a measure either in  $(-\alpha, \alpha)$  or in  $(\frac{1}{2} - \alpha, \frac{1}{2} + \alpha)$ .

Denote as  $\bar{\theta}(e_{x_1}^u, e_{x_2}^u)$  such a measure. By the invariance of the unstable bundle,

$$Df^{iN}(x_1)e_{x_1}^u \in E_{f^{iN}(x_1)}^u \quad \text{and} \quad Df^{iN}(x_2)e_{x_2}^u \in E_{f^{iN}(x_2)}^u.$$

As remarked above for  $\theta(e_{x_1}^u, e_{x_2}^u)$ , since

$$\theta(E_{f^{iN}(x_1)}^u, E_{f^{iN}(x_2)}^u) < \beta,$$

the oriented angle  $\theta(Df^{iN}(x_1)e_{x_1}^u, Df^{iN}(x_2)e_{x_2}^u)$  admits a measure either in  $(-\beta, \beta)$  or in  $(\frac{1}{2} - \beta, \frac{1}{2} + \beta)$ .

Denote as  $\bar{\theta}(Df^{iN}(x_1)e_{x_1}^u, Df^{iN}(x_2)e_{x_2}^u)$  such a measure. Look now at the quantity

$$iN \text{ Torsion}_{iN}(f, x_1, e_{x_1}^u) - iN \text{ Torsion}_{iN}(f, x_2, e_{x_2}^u) + \frac{l}{2}.$$

Observe that  $iN \text{ Torsion}_{iN}(f, x_1, e_{x_1}^u) - iN \text{ Torsion}_{iN}(f, x_2, e_{x_2}^u)$  is a measure of the angle

$$\theta(e_{x_1}^u, Df^{iN}(x_1)e_{x_1}^u) - \theta(e_{x_2}^u, Df^{iN}(x_2)e_{x_2}^u).$$

Equivalently, it is a measure of the angle

$$\theta(e_{x_1}^u, e_{x_2}^u) - \theta(Df^{iN}(x_1)e_{x_1}^u, Df^{iN}(x_2)e_{x_2}^u).$$

Hence

$$\begin{aligned} iN \text{ Torsion}_{iN}(f, x_1, e_{x_1}^u) - iN \text{ Torsion}_{iN}(f, x_2, e_{x_2}^u) &= \\ &= \bar{\theta}(e_{x_1}^u, e_{x_2}^u) - \bar{\theta}(Df^{iN}(x_1)e_{x_1}^u, Df^{iN}(x_2)e_{x_2}^u) + n \end{aligned}$$

for some  $n \in \mathbb{Z}$ .

From the definition of  $\bar{\theta}(e_{x_1}^u, e_{x_2}^u)$  and of  $\bar{\theta}(Df^{iN}(x_1)e_{x_1}^u, Df^{iN}(x_2)e_{x_2}^u)$ , we have that

$$iN \text{ Torsion}_{iN}(f, x_1, e_{x_1}^u) - iN \text{ Torsion}_{iN}(f, x_2, e_{x_2}^u) + \frac{l}{2} \subset \bigcup_{n \in \mathbb{Z}} \left( -\alpha - \beta + \frac{n}{2}, \alpha + \beta + \frac{n}{2} \right).$$

This observation and hypothesis (i) imply that the quantity

$$iN \text{ Torsion}_{iN}(f, x_1, e_{x_1}^u) - iN \text{ Torsion}_{iN}(f, x_2, e_{x_2}^u) + \frac{l}{2}$$

is contained in

$$\left( -\frac{1}{2} + (\alpha + \beta), \frac{1}{2} - (\alpha + \beta) \right) \cap \left( -(\alpha + \beta) + \frac{j}{2}, \alpha + \beta + \frac{j}{2} \right)$$

for some  $j \in \mathbb{Z}$ . Such an intersection is not empty if and only if  $j = 0$  and so we conclude that

$$\left| iN \text{ Torsion}_{iN}(f, x_1, e_{x_1}^u) - iN \text{ Torsion}_{iN}(f, x_2, e_{x_2}^u) + \frac{l}{2} \right| < \alpha + \beta.$$

□

#### 4.4.1 Torsion at finite-time $(2n_u + j)N$ for $h(x)_1 = 0$

The main result of this Subsection is the following

**Lemma 4.4.2.** *Let  $x \in H(U_\varepsilon, j)$  be such that  $f^{(2n_u + j)N}(x) \in V_0(j)$ , that is  $h(x)_1 = 0$ . Then*

$$(2n_u + j)N \text{ Torsion}_{(2n_u + j)N}(f, x, e_x^u) \in ((2n_u + j)k - \varepsilon, (2n_u + j)k + \varepsilon). \quad (4.18)$$

Lemma 4.4.2 is an outcome of the following result.

**Lemma 4.4.3.** *Let  $x \in \Lambda(U_\varepsilon) \cap \tilde{U}$ , where  $\Lambda(U_\varepsilon)$  is the maximal  $f^N$ -invariant subset contained in  $U_\varepsilon$ . Assume that there exists  $n \in \mathbb{N}^*$  such that  $f^{iN}(x) \in \Lambda(U_\varepsilon) \cap \tilde{U}$  for any  $i \in \llbracket 0, n \rrbracket$ .*

*Then*

$$|nN \text{ Torsion}_{nN}(f, x, e_x^u) - nN \text{ Torsion}_{nN}(f, q, e_q^u)| < \varepsilon,$$

*equivalently*

$$|nN \text{ Torsion}_{nN}(f, x, e_x^u) - nk| < \varepsilon.$$

We postpone the proof of Lemma 4.4.3 and we now show how Lemma 4.4.2 follows from it.

*Proof of Lemma 4.4.2.* By hypothesis, the point  $x \in H(U_\varepsilon, j)$  is such that  $f^{(2n_u+j)N}(x) \in V_0(j)$ . By Lemma 4.3.3 it holds that  $f^{iN}(x) \in \tilde{U}$  for any  $i \in \llbracket 0, 2n_u + j \rrbracket$ . Applying then Lemma 4.4.3 for  $n = 2n_u + j$  we have

$$|(2n_u + j)N \text{ Torsion}_{(2n_u+j)N}(f, x, e_x^u) - (2n_u + j)k| < \varepsilon,$$

that is

$$(2n_u + j)N \text{ Torsion}_{(2n_u+j)N}(f, x, e_x^u) \in ((2n_u + j)k - \varepsilon, (2n_u + j)k + \varepsilon).$$

□

*Proof of Lemma 4.4.3.* We are going to prove the result by induction over  $n \in \mathbb{N}^*$ . The case  $n = 1$  is actually Condition 4.2.11 of  $U_\varepsilon$  adapted neighborhood of  $\{q\} \cup \mathcal{O}(p, f^N)$  (recall that  $\mathfrak{f} = f^N$ ). Indeed, by this condition for any  $x \in \Lambda(U_\varepsilon) \cap \tilde{U}$  it holds

$$|N \text{ Torsion}_N(f, x, e_x^u) - N \text{ Torsion}_N(f, q, e_q^u)| < \frac{\varepsilon}{2}.$$

That is

$$|N \text{ Torsion}_N(f, x, e_x^u) - k| < \frac{\varepsilon}{2} < \varepsilon.$$

Let  $n \in \mathbb{N}, n \geq 2$ . Assume that  $f^{iN}(x) \in \Lambda(U_\varepsilon) \cap \tilde{U}$  for any  $i \in \llbracket 0, n \rrbracket$  and suppose by induction that

$$|(n-1)N \text{ Torsion}_{(n-1)N}(f, x, e_x^u) - (n-1)N \text{ Torsion}_{(n-1)N}(f, q, e_q^u)| < \varepsilon. \quad (4.19)$$

We remark that  $f^{(n-1)N}(x) \in \Lambda(U_\varepsilon) \cap \tilde{U}$  and, by the invariance of the unstable bundle, it holds

$$Df^{(n-1)N}(x)e_x^u \in E_{f^{(n-1)N}(x)}^u.$$

In particular, we have that

$$\frac{Df^{(n-1)N}(x)e_x^u}{\|Df^{(n-1)N}(x)e_x^u\|} \in \{e_{f^{(n-1)N}(x)}^u, -e_{f^{(n-1)N}(x)}^u\}. \quad (4.20)$$

From (4.20), we have so

$$N \text{ Torsion}_N(f, f^{(n-1)N}(x), Df^{(n-1)N}(x)e_x^u) = N \text{ Torsion}_N(f, f^{(n-1)N}(x), e_{f^{(n-1)N}(x)}^u).$$

Then, by Condition 4.2.11 of  $U_\varepsilon$  adapted neighborhood of  $\{q\} \cup \mathcal{O}(p, f^N)$ , we deduce that

$$|N \text{Torsion}_N(f, f^{(n-1)N}(x), Df^{(n-1)N}(x)e_x^u) - N \text{Torsion}_N(f, q, e_q^u)| < \frac{\varepsilon}{2}. \quad (4.21)$$

Consequently, by (4.19) and (4.21), we obtain

$$\begin{aligned} & |nN \text{Torsion}_{nN}(f, x, e_x^u) - nN \text{Torsion}_{nN}(f, q, e_q^u)| \leq \\ & \leq |(n-1)N \text{Torsion}_{(n-1)N}(f, x, e_x^u) - (n-1)N \text{Torsion}_{(n-1)N}(f, q, e_q^u)| + \\ & + |N \text{Torsion}_N(f, f^{(n-1)N}(x), Df^{(n-1)N}(x)e_x^u) - N \text{Torsion}_N(f, q, e_q^u)| < \frac{3}{2}\varepsilon. \end{aligned} \quad (4.22)$$

We can improve the inequality (4.22) by applying Lemma 4.4.1 at the points  $x, q$  with respect to  $\frac{\varepsilon}{2}$  (both as  $\alpha$  and as  $\beta$ ), to  $n \in \mathbb{N}^*$  as  $i$  and to 0 as  $l \in \mathbb{Z}$ . Indeed, since  $\frac{3}{2}\varepsilon < \frac{1}{2} - \varepsilon$ , from (4.22) hypothesis (i) of Lemma 4.4.1 is satisfied. We obtain so

$$|nN \text{Torsion}_{nN}(f, x, e_x^u) - nN \text{Torsion}_{nN}(f, q, e_q^u)| < \varepsilon.$$

□

#### 4.4.2 Torsion at finite-time $(2n_u + j)N$ for $h(x)_1 = 1$

The main result of this subsection is the following

**Lemma 4.4.4.** *Let  $x \in H(U_\varepsilon, j)$  be such that  $f^{(2n_u+j)N}(x) \in V_1(j)$ , that is  $h(x)_1 = 1$ . Hence*

$$\begin{aligned} & |(2n_u + j)N \text{Torsion}_{(2n_u+j)N}(f, x, e_x^u) - \\ & - (2n_u + j)N \text{Torsion}_{(2n_u+j)N}(f, f^{-(2n_u+j)N}(p), e_{f^{-(2n_u+j)N}(p)}^u)| < 2\varepsilon. \end{aligned} \quad (4.23)$$

Thanks to Lemma 4.4.4, in order to estimate finite-time torsion at points of the horseshoe for which  $h(x)_1 = 1$ , we deduce that it is sufficient to calculate the  $(2n_u + j)N$ -finite-time torsion at the point  $f^{-(2n_u+j)N}(p) \in \mathcal{O}(p, f^N)$ .

Lemma 4.4.4 is an outcome of Lemma 4.4.3 and of the following

**Lemma 4.4.5.** *Let  $x \in \Lambda(U_\varepsilon)$  be such that  $x, f^{n_u N}(x) \in \Lambda(U_\varepsilon) \cap \tilde{U}$  and such that  $f^{(n_u-i)N}(x) \in U_i$  for any  $i \in \llbracket 1, n_u - 1 \rrbracket$ .*

*Then it holds*

$$|n_u N \text{Torsion}_{n_u N}(f, x, e_x^u) - n_u N \text{Torsion}_{n_u N}(f, f^{-n_u N}(p), e_{f^{-n_u N}(p)}^u)| < 2\varepsilon.$$

We postpone the proof of Lemma 4.4.5 and we now show how Lemma 4.4.4 follows from it and from Lemma 4.4.3.

*Proof of Lemma 4.4.4.* We are assuming that  $p$  and  $f^{-n_u N}(p)$  are future-first-entry and past-first-entry points for  $\tilde{U} = CC(U_\varepsilon, q)$ . In particular it holds that

- $f^{-(n_u+i)N}(p) \in \tilde{U}$  for any  $i \in \llbracket 0, n_u + j \rrbracket$ ;
- $f^{-iN}(p) \in U_i$  for any  $i \in \llbracket 1, n_u - 1 \rrbracket$ .

From Lemma 4.3.3, since  $f^{(2n_u+j)N}(x) \in V_1(j)$ , it holds

- $f^{iN}(x) \in \tilde{U}$  for any  $i \in \llbracket 0, n_u + j \rrbracket$ ;
- $f^{(2n_u+j-i)N}(x) \in U_i$  for any  $i \in \llbracket 1, n_u - 1 \rrbracket$ .

By applying Lemma 4.4.3 to both  $x$  and  $f^{-(2n_u+j)N}(p)$  for  $n = n_u + j$ , we deduce that

$$\begin{aligned} & \left| (n_u + j)N \text{ Torsion}_{(n_u+j)N}(f, x, e_x^u) - (n_u + j)N \text{ Torsion}_{(n_u+j)N}(f, f^{-(2n_u+j)N}(p), e_{f^{-(2n_u+j)N}(p)}}^u) \right| \leq \\ & \leq \left| (n_u + j)N \text{ Torsion}_{(n_u+j)N}(f, x, e_x^u) - (n_u + j)N \text{ Torsion}_{(n_u+j)N}(f, q, e_q^u) \right| + \\ & + \left| (n_u + j)N \text{ Torsion}_{(n_u+j)N}(f, q, e_q^u) - (n_u + j)N \text{ Torsion}_{(n_u+j)N}(f, f^{-(2n_u+j)N}(p), e_{f^{-(2n_u+j)N}(p)}}^u) \right| \\ & < 2\varepsilon. \end{aligned}$$

We will now apply Lemma 4.4.5 at  $f^{(n_u+j)N}(x)$ . By the  $Df^N$ -invariance of the unstable bundle and since  $E_{f^{(n_u+j)N}(x)}}^u = \{e_{f^{(n_u+j)N}(x)}}^u, -e_{f^{(n_u+j)N}(x)}}^u\}$  we have that

$$\frac{Df^{(n_u+j)N}(x)e_x^u}{\|Df^{(n_u+j)N}(x)e_x^u\|} \in \{e_{f^{(n_u+j)N}(x)}}^u, -e_{f^{(n_u+j)N}(x)}}^u\}.$$

Therefore

$$n_u N \text{ Torsion}_{n_u N}(f, f^{(n_u+j)N}(x), Df^{(n_u+j)N}(x)e_x^u) = n_u N \text{ Torsion}_{n_u N}(f, f^{(n_u+j)N}(x), e_{f^{(n_u+j)N}(x)}}^u).$$

The same argument for  $Df^{(n_u+j)N}(f^{-(2n_u+j)N}(p))e_{f^{-(2n_u+j)N}(p)}}^u$  tells us that

$$\begin{aligned} & n_u N \text{ Torsion}_{n_u N}(f, f^{-n_u N}(p), Df^{(n_u+j)N}(f^{-(2n_u+j)N}(p))e_{f^{-(2n_u+j)N}(p)}}^u) = \\ & = n_u N \text{ Torsion}_{n_u N}(f, f^{-n_u N}(p), e_{f^{-n_u N}(p)}}^u). \end{aligned}$$

We so obtain, from Lemma 4.4.5, that

$$\begin{aligned} & \left| n_u N \text{ Torsion}_{n_u N}(f, f^{(n_u+j)N}(x), Df^{(n_u+j)N}(x)e_x^u) - \right. \\ & \left. - n_u N \text{ Torsion}_{n_u N}(f, f^{-n_u N}(p), Df^{(n_u+j)N}(f^{-(2n_u+j)N}(p))e_{f^{-(2n_u+j)N}(p)}}^u) \right| < 2\varepsilon. \end{aligned}$$

Finally then, it holds

$$\begin{aligned} & \left| (2n_u + j)N \text{ Torsion}_{(2n_u+j)N}(f, x, e_x^u) - \right. \\ & \left. - (2n_u + j)N \text{ Torsion}_{(2n_u+j)N}(f, f^{-(2n_u+j)N}(p), e_{f^{-(2n_u+j)N}(p)}}^u) \right| \leq \\ & \leq \left| (n_u + j)N \text{ Torsion}_{(n_u+j)N}(f, x, e_x^u) - \right. \\ & \left. - (n_u + j)N \text{ Torsion}_{(n_u+j)N}(f, f^{-(2n_u+j)N}(p), e_{f^{-(2n_u+j)N}(p)}}^u) \right| + \\ & + \left| n_u N \text{ Torsion}_{n_u N}(f, f^{(n_u+j)N}(x), e_{f^{(n_u+j)N}(x)}}^u) - n_u N \text{ Torsion}_{n_u N}(f, f^{-n_u N}(p), e_{f^{-n_u N}(p)}}^u) \right| < 4\varepsilon. \end{aligned} \tag{4.24}$$

Observe that, since  $x, f^{(2n_u+j)N}(x), f^{-(2n_u+j)N}(p)$  and  $p$  all belong to  $\Lambda(U_\varepsilon) \cap \tilde{U}$ , by Condition 4.2.10 it holds (see Notation 4.2.1)

$$\theta(E_x^u, E_{f^{-(2n_u+j)N}(p)}}^u) \leq \theta(E_x^u, E_q^u) + \theta(E_q^u, E_{f^{-(2n_u+j)N}(p)}}^u) < \varepsilon$$

and

$$\theta(E_{f^{(2n_u+j)N}(x)}^u, E_p^u) \leq \theta(E_{f^{(2n_u+j)N}(x)}^u, E_q^u) + \theta(E_q^u, E_p^u) < \varepsilon.$$

Since  $4\varepsilon < \frac{1}{2} - 2\varepsilon$ , we can improve the inequality (4.24) by applying Lemma 4.4.1 at the points  $x, f^{-(2n_u+j)N}(p)$  with respect to  $\varepsilon$  (both as  $\alpha$  and as  $\beta$ ), to  $(2n_u + j)$  as  $i$  and 0 as  $l \in \mathbb{Z}$ . We deduce so that

$$\begin{aligned} & \left| (2n_u + j)N \text{ Torsion}_{(2n_u+j)N}(f, x, e_x^u) - \right. \\ & \left. - (2n_u + j)N \text{ Torsion}_{(2n_u+j)N}(f, f^{-(2n_u+j)N}(p), e_{f^{-(2n_u+j)N}(p)}}^u) \right| < 2\varepsilon. \end{aligned}$$

□

*Proof of Lemma 4.4.5.* The first step of the proof is showing the following

**Claim 4.4.1.** Assume that  $x \in \tilde{U}$  and that  $f^{iN}(x) \in U_{n_u-i}$  for  $i \in \llbracket 1, n_u - 1 \rrbracket$ . Then

$$\left| iN \text{ Torsion}_{iN}(f, x, e_x^u) - iN \text{ Torsion}_{iN}(f, f^{-n_uN}(p), e_{f^{-n_uN}(p)}}^u) \right| < 2\varepsilon \quad (4.25)$$

for  $i \in \llbracket 1, n_u - 1 \rrbracket$ .

*Proof of Claim 4.4.1.* We proceed by induction.

— *Case  $n = 1$ :* both  $x$  and  $f^{-n_uN}(p)$  belong to  $\Lambda(U_\varepsilon) \cap \tilde{U}$ . Hence, by Condition 4.2.11 of  $U_\varepsilon$  adapted neighborhood, it holds

$$\begin{aligned} & \left| N \text{ Torsion}_N(f, x, e_x^u) - N \text{ Torsion}_N(f, f^{-n_uN}(p), e_{f^{-n_uN}(p)}}^u) \right| \leq \\ & \leq \left| N \text{ Torsion}_N(f, x, e_x^u) - N \text{ Torsion}_N(f, q, e_q^u) \right| \\ & + \left| N \text{ Torsion}_N(f, q, e_q^u) - N \text{ Torsion}_N(f, f^{-n_uN}(p), e_{f^{-n_uN}(p)}}^u) \right| < \varepsilon < 2\varepsilon. \end{aligned}$$

— *Inductive step:* let now  $n \in \llbracket 2, n_u - 1 \rrbracket$ . By inductive hypothesis we have that

$$\left| (n-1)N \text{ Torsion}_{(n-1)N}(f, x, e_x^u) - (n-1)N \text{ Torsion}_{(n-1)N}(f, f^{-n_uN}(p), e_{f^{-n_uN}(p)}}^u) \right| < 2\varepsilon.$$

We consider now

$$\begin{aligned} & \left| nN \text{ Torsion}_{nN}(f, x, e_x^u) - nN \text{ Torsion}_{nN}(f, f^{-n_uN}(p), e_{f^{-n_uN}(p)}}^u) \right| \leq \\ & \leq \left| (n-1)N \text{ Torsion}_{(n-1)N}(f, x, e_x^u) - (n-1)N \text{ Torsion}_{(n-1)N}(f, f^{-n_uN}(p), e_{f^{-n_uN}(p)}}^u) \right| + \\ & \quad + \left| N \text{ Torsion}_N(f, f^{(n-1)N}(x), Df^{(n-1)N}(x)e_x^u) - \right. \\ & \quad \left. - N \text{ Torsion}_N(f, f^{-(n_u-n+1)N}(p), Df^{(n-1)N}(f^{-n_uN}(p))e_{f^{-n_uN}(p)}}^u) \right| < \\ & < 2\varepsilon + \frac{\varepsilon}{2}. \end{aligned} \quad (4.26)$$

The last inequality is an outcome of the inductive hypothesis and of Condition 4.2.11 of  $U_\varepsilon$  adapted neighborhood. Indeed, by hypothesis, the point  $f^{(n-1)N}(x) \in U_{n_u-n+1}$  and, by definition of  $U_{n_u-n+1}$  (recall that  $p$  is the future-first-entry point for  $\tilde{U}$ ), also

$f^{-n_u N + (n-1)N}(p)$  belongs to this connected component of  $U_\varepsilon$ . By the  $Df^N$ -invariance of the unstable bundle it holds that

$$Df^{(n-1)N}(x)e_x^u \in E_{f^{(n-1)N}(x)}^u$$

and

$$Df^{(n-1)N}(f^{-n_u N}(p))e_{f^{-n_u N}(p)}^u \in E_{f^{-n_u N + (n-1)N}(p)}^u.$$

In particular we have that

$$\frac{Df^{(n-1)N}(x)e_x^u}{\|Df^{(n-1)N}(x)e_x^u\|} \in \{e_{f^{(n-1)N}(x)}^u, -e_{f^{(n-1)N}(x)}^u\}.$$

Similarly, it holds

$$\frac{Df^{(n-1)N}(f^{-n_u N}(p))e_{f^{-n_u N}(p)}^u}{\|Df^{(n-1)N}(f^{-n_u N}(p))e_{f^{-n_u N}(p)}^u\|} \in \{e_{f^{-(n_u - n + 1)N}(p)}^u, -e_{f^{-(n_u - n + 1)N}(p)}^u\}.$$

So we have that

$$\begin{aligned} & |N \text{Torsion}_N(f, f^{(n-1)N}(x), Df^{(n-1)N}(x)e_x^u) - \\ & - N \text{Torsion}_N(f, f^{-(n_u - n + 1)N}(p), Df^{(n-1)N}(f^{-n_u N}(p))e_{f^{-n_u N}(p)}^u)| = \\ & = \left| N \text{Torsion}_N(f, f^{(n-1)N}(x), e_{f^{(n-1)N}(x)}^u) - N \text{Torsion}_N(f, f^{-(n_u - n + 1)N}(p), e_{f^{-(n_u - n + 1)N}(p)}^u) \right| \end{aligned}$$

is smaller than  $\frac{\varepsilon}{2}$  by Condition 4.2.11.

Observe now that, since both  $x$  and  $f^{-n_u N}(p)$  belong to  $\tilde{U}$ , by Condition 4.2.10, it holds

$$\theta(E_x^u, E_{f^{-n_u N}(p)}^u) \leq \theta(E_x^u, E_q^u) + \theta(E_q^u, E_{f^{-n_u N}(p)}^u) < \varepsilon. \quad (4.27)$$

Since  $f^{(n-1)N}(x)$  is in  $\Lambda(U_\varepsilon) \cap U_{n_u - n + 1}$ , by Condition 4.2.10 it holds

$$\theta(E_{f^{(n-1)N}(x)}^u, E_{f^{-(n_u - n + 1)N}(p)}^u) < \frac{\varepsilon}{2}. \quad (4.28)$$

Thus, we improve inequality 4.26 by applying Lemma 4.4.1 to the points  $x, f^{-n_u N}(p)$  with respect to  $\varepsilon$  as  $\alpha$ ,  $\frac{\varepsilon}{2}$  as  $\beta$ ,  $n$  as  $i$  and  $0$  as  $l \in \mathbb{Z}$ . Indeed, since  $2\varepsilon + \frac{\varepsilon}{2} < \frac{1}{2} - (\varepsilon + \frac{\varepsilon}{2})$ , from (4.26) hypothesis (i) of Lemma 4.4.1 holds. We deduce so that

$$\left| nN \text{Torsion}_{nN}(f, x, e_x^u) - nN \text{Torsion}_{nN}(f, f^{-n_u N}(p), e_{f^{-n_u N}(p)}^u) \right| < \varepsilon + \frac{\varepsilon}{2} < 2\varepsilon.$$

The proof is so ended by induction. □

In order to end the proof of Lemma 4.4.5, we recall that  $x, f^{n_u N}(x) \in \Lambda(U_\varepsilon) \cap \tilde{U}$  and  $f^{iN}(x) \in U_{n_u - i}$  for any  $i \in \llbracket 1, n_u - 1 \rrbracket$ . Look now at

$$\begin{aligned} & \left| n_u N \text{Torsion}_{n_u N}(f, x, e_x^u) - n_u N \text{Torsion}_{n_u N}(f, f^{-n_u N}(p), e_{f^{-n_u N}(p)}^u) \right| \leq \\ & \leq \left| (n_u - 1)N \text{Torsion}_{(n_u - 1)N}(f, x, e_x^u) - (n_u - 1)N \text{Torsion}_{(n_u - 1)N}(f, f^{-n_u N}(p), e_{f^{-n_u N}(p)}^u) \right| + \\ & \quad + \left| N \text{Torsion}_N(f, f^{(n_u - 1)N}(x), Df^{(n_u - 1)N}(x)e_x^u) - \right. \end{aligned}$$

$$-N \text{Torsion}_N(f, f^{-N}(p), Df^{(n_u-1)N}(f^{-n_uN}(p))e_{f^{-n_uN}(p)}^u) \Big|. \quad (4.29)$$

By Claim 4.4.1 for  $i = n_u - 1$  it holds

$$\left| (n_u - 1)N \text{Torsion}_{(n_u-1)N}(f, x, e_x^u) - (n_u - 1)N \text{Torsion}_{(n_u-1)N}(f, f^{-n_uN}(p), e_{f^{-n_uN}(p)}^u) \right| < 2\varepsilon.$$

By hypothesis  $f^{(n_u-1)N}(x) \in \Lambda(U_\varepsilon) \cap U_1$ . Moreover, by the invariance of the unstable bundle, it holds  $Df^{(n_u-1)N}(x)e_x^u \in E_{f^{(n_u-1)N}(x)}^u$ . Recalling that

$$E_{f^{(n_u-1)N}(x)}^u \cap \mathbb{S}^1 = \{e_{f^{(n_u-1)N}(x)}^u, -e_{f^{(n_u-1)N}(x)}^u\},$$

we have that

$$N \text{Torsion}_N(f, f^{(n_u-1)N}(x), Df^{(n_u-1)N}(x)e_x^u) = N \text{Torsion}_N(f, f^{(n_u-1)N}(x), e_{f^{(n_u-1)N}(x)}^u).$$

Similarly

$$N \text{Torsion}_N(f, f^{-N}(p), Df^{(n_u-1)N}(f^{-n_uN}(p))e_{f^{-n_uN}(p)}^u) = N \text{Torsion}_N(f, f^{-N}(p), e_{f^{-N}(p)}^u).$$

Consequently, it holds

$$\begin{aligned} & \left| N \text{Torsion}_N(f, f^{(n_u-1)N}(x), Df^{(n_u-1)N}(x)e_x^u) - \right. \\ & \quad \left. - N \text{Torsion}_N(f, f^{-N}(p), Df^{(n_u-1)N}(f^{-n_uN}(p))e_{f^{-n_uN}(p)}^u) \right| = \\ & = \left| N \text{Torsion}_N(f, f^{(n_u-1)N}(x), e_{f^{(n_u-1)N}(x)}^u) - N \text{Torsion}_N(f, f^{-N}(p), e_{f^{-N}(p)}^u) \right|. \end{aligned}$$

By Condition 4.2.11 of  $U_\varepsilon$  adapted neighborhood of  $\{q\} \cup \mathcal{O}(p, f^N)$ , we then deduce that

$$\begin{aligned} & \left| N \text{Torsion}_N(f, f^{(n_u-1)N}(x), Df^{(n_u-1)N}(x)e_x^u) - \right. \\ & \quad \left. - N \text{Torsion}_N(f, f^{-N}(p), Df^{(n_u-1)N}(f^{-n_uN}(p))e_{f^{-n_uN}(p)}^u) \right| < \frac{\varepsilon}{2}. \end{aligned}$$

Consequently we estimate inequality (4.29) and we have

$$\left| n_u N \text{Torsion}_{n_uN}(f, x, e_x^u) - n_u N \text{Torsion}_{n_uN}(f, f^{-n_uN}(p), e_{f^{-n_uN}(p)}^u) \right| < 2\varepsilon + \frac{\varepsilon}{2}. \quad (4.30)$$

Observe that, since  $x, f^{n_uN}(x), f^{-n_uN}(p)$  and  $p$  belong to  $\tilde{U}$ , it holds by Condition 4.2.10 that

$$\theta(E_x^u, E_{f^{-n_uN}(p)}^u) \leq \theta(E_x^u, E_q^u) + \theta(E_q^u, E_{f^{-n_uN}(p)}^u) < \varepsilon$$

and

$$\theta(E_{f^{n_uN}(x)}^u, E_p^u) \leq \theta(E_{f^{n_uN}(x)}^u, E_q^u) + \theta(E_q^u, E_p^u).$$

We improve inequality (4.30) by applying Lemma 4.4.1 at the points  $x, f^{-n_uN}(p)$  with respect to  $\varepsilon$  both as  $\alpha$  and as  $\beta$ , to  $n_u$  as  $i$  and to 0 as  $l \in \mathbb{Z}$ . Indeed, since  $2\varepsilon + \frac{\varepsilon}{2} < \frac{1}{2} - 2\varepsilon$ , from (4.30) hypothesis (i) of Lemma 4.4.1 holds. We conclude that

$$\left| n_u N \text{Torsion}_{n_uN}(f, x, e_x^u) - n_u N \text{Torsion}_{n_uN}(f, f^{-n_uN}(p), e_{f^{-n_uN}(p)}^u) \right| < 2\varepsilon,$$

ending so our proof. □



As remarked above, Lemma 4.4.4 implies that it is sufficient to calculate the finite-time torsion at point  $f^{-n_u N}(p)$  to estimate finite-time torsion at any point  $x \in H(U_\varepsilon, j)$  such that  $h(x)_1 = 1$ . In the sequel we are going to show how to calculate finite-time torsion at  $f^{-n_u N}(p)$  through the angle variation of the vector tangent to the unstable manifold. We recall the notations presented in (4.5) and (4.7). See also Notation 4.4.2. The curves  $\gamma_u$  and  $\Gamma_u$  are defined as

$$[0, t_u] \ni t \mapsto \gamma_u(t) = \phi^{-1}(t, 0) \in W_{loc, \delta}^u(q),$$

$$[0, t_u] \ni t \mapsto \Gamma_u(t) = f^{n_u N}(\gamma_u(t)) \in W^u(q).$$

Observe that  $\gamma_u(0) = \Gamma_u(0) = q$ ,  $\gamma_u(t_u) = f^{-n_u N}(p)$  and  $\Gamma_u(t_u) = p$ .

**Notation 4.4.3.** Introduce the following angle function

$$[0, t_u] \ni t \mapsto \theta(\mathcal{H}, \Gamma'_u(t)) \in \mathbb{T},$$

where  $\mathcal{H}$  is the horizontal vector  $(1, 0)$  and  $\theta(u, v)$  is the oriented angle between the two non zero vectos  $u, v$ . Denote as

$$[0, t_u] \ni t \mapsto \tilde{\theta}(\mathcal{H}, \Gamma_u(t)) \in \mathbb{R}$$

a continuous determination of the previous angle function. Define the following continuous function

$$[0, t_u] \ni t \mapsto \tau(t) := \gamma_u^{-1} \circ f^{-n_u N} \circ \gamma_u(t) \in [0, t_u].$$

That is, for  $t \in [0, t_u]$  the function  $\tau(t)$  denotes the parameter which correponds to the point  $f^{-n_u N}(\gamma_u(t))$  on the local unstable manifold. In particular, it holds  $\Gamma_u(\tau(t)) = \gamma_u(t)$ .

We will focus then on the angle variation of the vector tangent to the unstable manifold between  $f^{-n_u N}(p)$  and  $p$ , that is we will be interested in

$$\tilde{\theta}(\mathcal{H}, \Gamma'_u(t_u)) - \tilde{\theta}(\mathcal{H}, \Gamma'_u(\tau(t_u))) = \tilde{\theta}(\mathcal{H}, \Gamma'_u(t_u)) - \tilde{\theta}(\mathcal{H}, \gamma'_u(t_u)).$$

The following lemma allows us to make explicit the relation between the  $n_u N$ -finite-time torsion at  $f^{-n_u N}(p)$  and the angle variation along the unstable manifold between  $f^{-n_u N}(p)$  and  $p$ .

**Lemma 4.4.6.** *Let  $q \in S$  be a hyperbolic periodic point of period  $N$  for  $f$  and let  $p \in (W^s(q) \cap W^u(q)) \setminus \{q\}$ . Denote*

$$k = N \text{ Torsion}_N(f, q, v) \in \mathbb{Z}$$

for  $v \in E_q^u$  or  $v \in E_q^s$ .

Let  $O_\varepsilon$  be an adapted neighborhood of  $q$  for  $\mathcal{O}(p, f^N)$  with respect to  $0 < \varepsilon < \frac{1}{12}$ . Denote as  $p, f^{-n_u N}(p)$  the future-first-entry and the past-first-entry points for  $O_\varepsilon$  respectively. Then

$$n_u N \text{ Torsion}_{n_u N}(f, f^{-n_u N}(p), e_{f^{-n_u N}(p)}^u) = \tilde{\theta}(\mathcal{H}, \Gamma'_u(t_u)) - \tilde{\theta}(\mathcal{H}, \Gamma'_u(\tau(t_u))) + kn_u. \quad (4.31)$$

*Proof.* Let us define the following continuous function

$$\Theta : [0, t_u] \longrightarrow \mathbb{R}$$

$$t \mapsto \Theta(t) := n_u N \text{ Torsion}_{n_u N}(f, \gamma_u(t), \gamma'_u(t)) - \left( \tilde{\theta}(\mathcal{H}, \Gamma'_u(t)) - \tilde{\theta}(\mathcal{H}, \Gamma'_u(\tau(t))) \right).$$

For any  $t \in [0, t_u]$  both  $n_u N \text{ Torsion}_{n_u N}(f, \gamma_u(t), \gamma'_u(t))$  and  $\tilde{\theta}(\mathcal{H}, \Gamma'_u(t)) - \tilde{\theta}(\mathcal{H}, \Gamma'_u(\tau(t)))$  are measures of the same angle

$$\theta(\gamma'_u(t), Df^{n_u N}(f^{-n_u N}(p))\gamma'_u(t)).$$

Therefore, the function  $\Theta$  takes values in  $\mathbb{Z}$ . By the continuity of the function and since  $\Gamma_u([0, t_u])$  is connected, we deduce that

$$n_u N \text{ Torsion}_{n_u N}(f, \gamma_u(t), \gamma'_u(t)) - \left( \tilde{\theta}(\mathcal{H}, \Gamma'_u(t)) - \tilde{\theta}(\mathcal{H}, \Gamma'_u(\tau(t))) \right) = l,$$

for some  $l \in \mathbb{Z}$  which does not depend on  $t$ . In particular

$$\begin{aligned} n_u N \text{ Torsion}_{n_u N}(f, \gamma_u(t_u), \gamma'_u(t_u)) - \left( \tilde{\theta}(\mathcal{H}, \Gamma'_u(t_u)) - \tilde{\theta}(\mathcal{H}, \Gamma'_u(\tau(t_u))) \right) = \\ n_u N \text{ Torsion}_{n_u N}(f, f^{-n_u N}(p), e_{f^{-n_u N}(p)}^u) - \left( \tilde{\theta}(\mathcal{H}, \Gamma'_u(t_u)) - \tilde{\theta}(\mathcal{H}, \Gamma'_u(\tau(t_u))) \right) = l. \end{aligned}$$

Observe that

$$n_u N \text{ Torsion}_{n_u N}(f, \gamma_u(0), \gamma'_u(0)) = n_u N \text{ Torsion}_{n_u N}(f, q, e_q^u) = kn_u$$

and clearly

$$\tilde{\theta}(\mathcal{H}, \Gamma'_u(0)) - \tilde{\theta}(\mathcal{H}, \Gamma'_u(\tau(0))) = 0.$$

That is,  $\Theta(0) = kn_u$ . Since the function  $\Theta$  is constant, we conclude that

$$\Theta(t_u) = n_u N \text{ Torsion}_{n_u N}(f, f^{-n_u N}(p), e_{f^{-n_u N}(p)}^u) - \left( \tilde{\theta}(\mathcal{H}, \Gamma'_u(t_u)) - \tilde{\theta}(\mathcal{H}, \Gamma'_u(\tau(t_u))) \right) = kn_u.$$

□

**Remark 4.4.2.** Observe that there exists  $m \in \mathbb{Z}$  such that

$$\tilde{\theta}(\mathcal{H}, \Gamma'_u(t_u)) - \tilde{\theta}(\mathcal{H}, \Gamma'_u(\tau(t_u))) \in \left( \frac{m}{2} - \varepsilon, \frac{m}{2} + \varepsilon \right). \quad (4.32)$$

Indeed, since  $p$  and  $f^{-n_u N}(p)$  belong to  $O_\varepsilon$ , by Conditions 4.2.1 and 4.2.3 it holds that

$$\theta(E_p^u, E_{f^{-n_u N}(p)}^u) \leq \theta(E_p^u, E_q^u) + \theta(E_q^u, E_{f^{-n_u N}(p)}^u) < \varepsilon. \quad (4.33)$$

Recall that

$$\Gamma'_u(t_u) \in E_p^u \quad \text{and} \quad \Gamma'_u(\tau(t_u)) \in E_{f^{-n_u N}(p)}^u. \quad (4.34)$$

Since

$$E_p^u \cap \mathbb{S}^1 = \{e_p^u, -e_p^u\} \quad \text{and} \quad E_{f^{-n_u N}(p)}^u \cap \mathbb{S}^1 = \{e_{f^{-n_u N}(p)}^u, -e_{f^{-n_u N}(p)}^u\},$$

the quantity  $\tilde{\theta}(\mathcal{H}, \Gamma'_u(t_u)) - \tilde{\theta}(\mathcal{H}, \Gamma'_u(\tau(t_u)))$  is a measure of the oriented angle  $\theta(\Gamma'_u(\tau(t_u)), \Gamma'_u(t_u))$ . From (4.33) and (4.34), this angle admits either a measure in  $(-\varepsilon, \varepsilon)$  or in  $(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon)$ .

Thus

$$\tilde{\theta}(\mathcal{H}, \Gamma'_u(t_u)) - \tilde{\theta}(\mathcal{H}, \Gamma'_u(\tau(t_u))) \in \left( \frac{m}{2} - \varepsilon, \frac{m}{2} + \varepsilon \right),$$

for some  $m \in \mathbb{Z}$ . Such an integer  $m \in \mathbb{Z}$  can be determined by calculating the angle variation of the tangent vector along the unstable manifold from  $q$  to  $p$ , which will be contained in  $(\frac{m}{2} - \frac{\varepsilon}{2}, \frac{m}{2} + \frac{\varepsilon}{2})$ .

From (4.32) and from Lemma 4.4.6, we deduce that

$$\left| n_u N \text{Torsion}_{n_u N}(f, f^{-n_u N}(p), e_{f^{-n_u N}(p)}^u) - kn_u - \frac{m}{2} \right| < \varepsilon. \quad (4.35)$$

Let us introduce the following

**Definition 4.4.1.** Let  $q$  be a hyperbolic periodic point for  $f$  of period  $N$  and let  $\mathcal{O}(p, f^N)$  be the  $f^N$ -orbit of a transverse homoclinic point in  $(W^u(q) \cap W^s(q)) \setminus \{q\}$ . Let  $U_\varepsilon$  be an adapted neighborhood of  $\{q\} \cup \mathcal{O}(p, f^N)$  with respect to  $0 < \varepsilon < \frac{1}{12}$  (see Definition 4.5). Let  $p$  be the future-first-entry point for the connected component of  $U_\varepsilon$  containing  $q$ .

The **unstable angle variation of  $(q, p)$**  is the integer  $m \in \mathbb{Z}$  such that

$$\left( \tilde{\theta}(\mathcal{H}, \Gamma'_u(t_u)) - \tilde{\theta}(\mathcal{H}, \Gamma'_u(0)) \right) - \frac{m}{2} = \min \left\{ \left( \tilde{\theta}(\mathcal{H}, \Gamma'_u(t_u)) - \tilde{\theta}(\mathcal{H}, \Gamma'_u(0)) \right) - \frac{k}{2} : k \in \mathbb{Z} \right\}.$$

That is,  $m$  is the integer that is the closest to  $2 \left( \tilde{\theta}(\mathcal{H}, \Gamma'_u(t_u)) - \tilde{\theta}(\mathcal{H}, \Gamma'_u(0)) \right)$ .

See Figure 4.11 for examples of different values of the unstable angle variation of  $(q, p)$ .

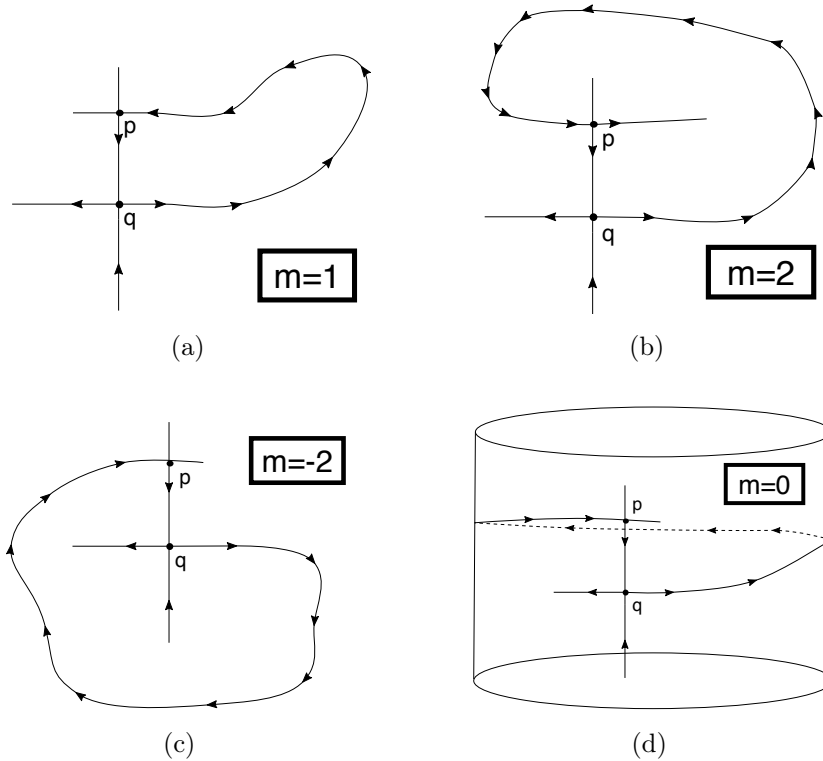


Figure 4.11 – Examples of possible values of the unstable angle variation of  $(q, p)$ .

**Remark 4.4.3.** By Lemma 4.4.3, Lemma 4.4.6 and Remark 4.4.2 we deduce that

$$\left| (2n_u + j)N \text{Torsion}_{(2n_u + j)N}(f, f^{-(2n_u + j)N}(p), e_{f^{-(2n_u + j)N}(p)}^u) - k(2n_u + j) - \frac{m}{2} \right| < 2\varepsilon. \quad (4.36)$$

The following Lemma is a refinement of Lemma 4.4.4, i.e. of the estimation of  $(2n_u + j)N$ -finite time torsion at  $x \in H(U_\varepsilon, j)$  such that  $h(x)_1 = 1$ .

**Lemma 4.4.7.** *Let  $x \in H(U_\varepsilon, j)$  be such that  $h(x)_1 = 1$ . Then it holds*

$$\left| (2n_u + j)N \text{ Torsion}_{(2n_u + j)N}(f, x, e_x^u) - k(2n_u + j) - \frac{m}{2} \right| < \varepsilon. \quad (4.37)$$

*Proof.* By Lemma 4.4.4 and by inequality (4.36) it holds

$$\left| (2n_u + j)N \text{ Torsion}_{(2n_u + j)N}(f, x, e_x^u) - k(2n_u + j) - \frac{m}{2} \right| < 4\varepsilon. \quad (4.38)$$

Recalling that  $N \text{ Torsion}_N(f, q, e_q^u) = k$ , we have

$$\left| (2n_u + j)N \text{ Torsion}_{(2n_u + j)N}(f, x, e_x^u) - (2n_u + j)N \text{ Torsion}_{(2n_u + j)N}(f, q, e_q^u) - \frac{m}{2} \right| < 4\varepsilon. \quad (4.39)$$

Both  $x$  and  $f^{(2n_u + j)N}(x)$  belong to  $\Lambda(U_\varepsilon) \cap \tilde{U}$  and so, by Condition 4.2.10, we have

$$\theta(E_x^u, E_q^u) < \frac{\varepsilon}{2} \quad \text{and} \quad \theta(E_{f^{(2n_u + j)N}(q)}^u, E_{f^{(2n_u + j)N}(x)}^u) = \theta(E_q^u, E_{f^{(2n_u + j)N}(x)}^u) < \frac{\varepsilon}{2}. \quad (4.40)$$

We want to apply Lemma 4.4.1 at the points  $x, q$  with respect to  $\frac{\varepsilon}{2}$  as both  $\alpha$  and  $\beta$ , to  $2n_u + j$  as  $i$  and to  $-m$  as  $l \in \mathbb{Z}$ . Indeed, since  $4\varepsilon < \frac{1}{2} - \varepsilon$ , from inequality (4.39), hypothesis (i) of Lemma 4.4.1 holds. Thus, we conclude that

$$\begin{aligned} & \left| (2n_u + j)N \text{ Torsion}_{(2n_u + j)N}(f, x, e_x^u) - (2n_u + j)N \text{ Torsion}_{(2n_u + j)N}(f, q, e_q^u) - \frac{m}{2} \right| = \\ & = \left| (2n_u + j)N \text{ Torsion}_{(2n_u + j)N}(f, x, e_x^u) - k(2n_u + j) - \frac{m}{2} \right| < \varepsilon. \end{aligned}$$

□

The proof of Theorem 4.4.1 follows then immediately from the above lemmas (Lemmas 4.4.2 and 4.4.7).

*Proof of Theorem 4.4.1.* Let  $x \in H(U_\varepsilon, j)$ . Then

— either  $f^{(2n_u + j)N}(x) \in V_0(j)$ , i.e.  $h(x)_1 = 0$ . Then from Lemma 4.4.2

$$(2n_u + j)N \text{ Torsion}_{(2n_u + j)N}(f, x, e_x^u) \in (k(2n_u + j) - \varepsilon, k(2n_u + j) + \varepsilon);$$

— or  $f^{(2n_u + j)N}(x) \in V_1(j)$ , i.e.  $h(x)_1 = 1$ . Then from Lemma 4.4.7

$$(2n_u + j)N \text{ Torsion}_{(2n_u + j)N}(f, x, e_x^u) \in \left( k(2n_u + j) + \frac{m}{2} - \varepsilon, k(2n_u + j) + \frac{m}{2} + \varepsilon \right).$$

Summing up, we so conclude that for any  $x \in H(U_\varepsilon, j)$

$$\begin{aligned} & (2n_u + j)N \text{ Torsion}_{(2n_u + j)N}(f, x, e_x^u) \in \\ & \in \left( k(2n_u + j) + \frac{m}{2}h(x)_1 - \varepsilon, k(2n_u + j) + \frac{m}{2}h(x)_1 + \varepsilon \right). \end{aligned}$$

□

### 4.4.3 On asymptotic torsion of points of $H(U_\varepsilon, j)$

From Theorem 4.4.1 we can calculate the  $n(2n_u + j)N$ -finite-time torsion at points of the horseshoe  $H(U_\varepsilon, j)$  for  $n \in \mathbb{N}^*$ .

**Proposition 4.4.1.** *Let  $x \in H(U_\varepsilon, j)$  and let  $(h(x)_i)_{i \in \mathbb{Z}}$  be its associated sequence in  $\{0, 1\}^{\mathbb{Z}}$ . Then for any  $n \in \mathbb{N}^*$  it holds*

$$\left| n(2n_u + j)N \text{ Torsion}_{n(2n_u + j)N}(f, x, e_x^u) - \left( kn(2n_u + j) + \frac{m}{2} \sum_{i=1}^n h(x)_i \right) \right| < \varepsilon, \quad (4.41)$$

where  $m \in \mathbb{Z}$  is the unstable angle variation of  $(q, p)$ .

*Proof.* The proof is made by induction. The case  $n = 1$  is Theorem 4.4.1.

Assume now that (4.41) holds for  $n \in \mathbb{N}^*$  and let us prove it for  $n + 1$ . Remark that  $f^{n(2n_u + j)N}(x) \in H(U_\varepsilon, j)$  and by the invariance of the unstable bundle

$$Df^{n(2n_u + j)N}(x)e_x^u \in E_{f^{n(2n_u + j)N}(x)}^u.$$

Since  $E_{f^{n(2n_u + j)N}(x)}^u \cap \mathbb{S}^1 = \{e_{f^{n(2n_u + j)N}(x)}^u, -e_{f^{n(2n_u + j)N}(x)}^u\}$ , we have that

$$\begin{aligned} (2n_u + j)N \text{ Torsion}_{(2n_u + j)N}(f, f^{n(2n_u + j)N}(x), Df^{n(2n_u + j)N}(x)e_x^u) &= \\ &= (2n_u + j)N \text{ Torsion}_{(2n_u + j)N}(f, f^{n(2n_u + j)N}(x), e_{f^{n(2n_u + j)N}(x)}^u). \end{aligned}$$

Using the inductive hypothesis, applying Theorem 4.4.1 at the point  $f^{n(2n_u + j)N}(x)$  and recalling that  $h(x)_{n+1} = h(f^{n(2n_u + j)N}(x))_1$ , we obtain that

$$\begin{aligned} &\left| (n+1)(2n_u + j)N \text{ Torsion}_{(n+1)(2n_u + j)N}(f, x, e_x^u) - \left( k(n+1)(2n_u + j) + \frac{m}{2} \sum_{i=1}^{n+1} h(x)_i \right) \right| \leq \\ &\leq \left| n(2n_u + j)N \text{ Torsion}_{n(2n_u + j)N}(f, x, e_x^u) - \left( kn(2n_u + j) + \frac{m}{2} \sum_{i=1}^n h(x)_i \right) \right| + \\ &\quad + \left| (2n_u + j)N \text{ Torsion}_{(2n_u + j)N}(f, f^{n(2n_u + j)N}(x), Df^{n(2n_u + j)N}(x)e_x^u) - \right. \\ &\quad \left. - \left( k(2n_u + j) + \frac{m}{2} h(x)_{n+1} \right) \right| < 2\varepsilon. \end{aligned} \quad (4.42)$$

Recall that

$$k(n+1)(2n_u + j) = (n+1)(2n_u + j)N \text{ Torsion}_{(n+1)(2n_u + j)N}(f, q, e_q^u).$$

Both  $x$  and  $f^{(n+1)(2n_u + j)N}(x)$  belong to  $\Lambda(U_\varepsilon) \cap \tilde{U}$ . Consequently by Condition 4.2.10 we have that

$$\theta(E_q^u, E_x^u) < \frac{\varepsilon}{2} \quad \text{and} \quad \theta(E_{f^{(n+1)(2n_u + j)N}(q)}^u, E_{f^{(n+1)(2n_u + j)N}(x)}^u) = \theta(E_q^u, E_{f^{(n+1)(2n_u + j)N}(x)}^u) < \frac{\varepsilon}{2}.$$

We want to apply Lemma 4.4.1 at the points  $x, q$  with respect to  $\frac{\varepsilon}{2}$  as both  $\alpha$  and  $\beta$ , to  $(n+1)(2n_u + j)$  as  $i$  and to

$$-m \sum_{i=1}^{n+1} h(x)_i$$

as  $l \in \mathbb{Z}$ . Since  $2\varepsilon < \frac{1}{2} - \varepsilon$ , inequality (4.42) implies hypothesis (i) of Lemma 4.4.1. We conclude that

$$\left| (n+1)(2n_u+j)N \text{Torsion}_{(n+1)(2n_u+j)N}(f, x, e_x^u) - \left( k(n+1)(2n_u+j) + \frac{m}{2} \sum_{i=1}^{n+1} h(x)_i \right) \right| < \varepsilon.$$

□

As an outcome of Proposition 4.4.1 we can discuss the asymptotic torsion for  $f$  at points of the horseshoe  $H(U_\varepsilon, j)$ . Recall that  $S$  is either  $\mathbb{R}^2$  or  $\mathbb{A}$  or  $\mathbb{T}^2$ .

**Corollary 4.4.1.** *Let  $f : S \rightarrow S$  be a  $\mathcal{C}^1$  diffeomorphism isotopic to the identity. Let  $q \in S$  be a periodic hyperbolic point for  $f$  of period  $N$ . Denote*

$$k = N \text{Torsion}_N(f, q, v) \in \mathbb{Z}$$

for any  $v \in E_q^u$ . Let  $p \in (W^u(q) \cap W^s(q)) \setminus \{q\}$ .

Let  $0 < \varepsilon < \frac{1}{12}$ . Let  $U_\varepsilon$  be an adapted neighborhood of  $\{q\} \cup \mathcal{O}(p, f^N)$  with respect to  $\varepsilon$ .

Let  $H(U_\varepsilon, j)$  be the horseshoe of Definition 4.3.2. Let  $h : H(U_\varepsilon, j) \rightarrow \{0, 1\}^{\mathbb{Z}}$  be the homeomorphism of Proposition 4.3.2. Let  $m \in \mathbb{Z}$  be the unstable angle variation of  $(q, p)$  (see Definition 4.4.1).

For any  $x \in H(U_\varepsilon, j)$  the torsion of  $f$  at  $x$  exists if and only if the limit

$$\lim_{n \rightarrow +\infty} \frac{\sum_{i=1}^n h(x)_i}{n}$$

exists. Whenever it exists, it holds

$$\text{Torsion}(f, x) = \lim_{n \rightarrow +\infty} \left( \frac{k}{N} + \frac{m}{2(2n_u+j)N} \frac{\sum_{i=1}^n h(x)_i}{n} \right).$$

*Proof.* The torsion at  $x$  is, when the limit exists,

$$\lim_{n \rightarrow +\infty} \text{Torsion}_n(f, x, v) = \lim_{n \rightarrow +\infty} \text{Torsion}_{n(2n_u+j)N}(f, x, v)$$

where  $v \in E_x^u$  (the asymptotic torsion does not depend on the choice of the tangent vector, see Lemma 1.1.3). By Proposition 4.4.1 it holds for any  $n \in \mathbb{N}^*$

$$\text{Torsion}_{n(2n_u+j)N}(f, x, v) \in$$

$$\left( \frac{k}{N} + \frac{m}{2(2n_u+j)N} \frac{\sum_{i=1}^n h(x)_i}{n} - \frac{\varepsilon}{n(2n_u+j)N}, \frac{k}{N} + \frac{m}{2(2n_u+j)N} \frac{\sum_{i=1}^n h(x)_i}{n} + \frac{\varepsilon}{n(2n_u+j)N} \right).$$

Since  $\lim_{n \rightarrow +\infty} \frac{\varepsilon}{n(2n_u+j)N} = 0$ , we conclude that the torsion at  $x$  exists if and only if the limit  $\lim_{n \rightarrow +\infty} \frac{\sum_{i=1}^n h(x)_i}{n}$  exists. Moreover, whenever the limit exists, we have

$$\text{Torsion}(f, x) = \lim_{n \rightarrow +\infty} \left( \frac{k}{N} + \frac{m}{2(2n_u+j)N} \frac{\sum_{i=1}^n h(x)_i}{n} \right).$$

□

## 4.5 On triviality and non triviality of the torsion

From Corollary 4.4.1 we observe that the unstable angle variation of  $(q, p)$  plays a fundamental role in determining the torsion of points of the horseshoe  $H(U_\varepsilon, j)$ . Indeed

**Corollary 4.5.1.** *In the hypothesis of Corollary 4.4.1 let  $m \in \mathbb{Z}$  be the unstable angle variation of  $(q, p)$ .*

— *If  $m = 0$ , then the torsion exists at every  $x \in H(U_\varepsilon, j)$  and for any  $x \in H(U_\varepsilon, j)$*

$$\text{Torsion}(f, x) = \frac{k}{N}.$$

— *If  $m \neq 0$ , then the torsion at  $x \in H(U_\varepsilon, j)$  exists if and only if the limit  $\lim_{n \rightarrow +\infty} \frac{\sum_{i=1}^n h(x)_i}{n}$  exists and*

$$\text{Torsion}(f, x) = \frac{k}{N} + \frac{m}{2(2n_u + j)N} \lim_{n \rightarrow +\infty} \frac{\sum_{i=1}^n h(x)_i}{n}.$$

When the unstable angle variation  $m$  is null, then the torsion is trivial, i.e. the torsion exists at every point and it is constantly equal to  $\frac{k}{N}$ .

Observe that if  $f \in \mathcal{C}^2$  then, through Livšic's periodic point theorem, we can deduce that

$$x \mapsto (2n_u + j)N \text{Torsion}_{(2n_u + j)N}(f, x, e_x^u) - (2n_u + j)k$$

is cohomologous to the zero constant if and only if the angle variation  $m$  is null. This remark arises from a question of J. Buzzi.

We recall now Livšic's periodic point theorem (see [CPW98]) in our framework.

**Theorem 4.5.1.** *Consider the horseshoe dynamical system  $(H(U_\varepsilon, j), f^{(2n_u + j)N})$ . Let  $T : H(U_\varepsilon, j) \rightarrow \mathbb{R}$  be Hölder continuous. Then the following conditions are equivalent.*

- (i) *There exists a continuous map  $g : H(U_\varepsilon, j) \rightarrow \mathbb{R}$  such that  $T = g \circ f^{(2n_u + j)N} - g$ , i.e.  $T$  is cohomologous to the zero constant function.*
- (ii) *For every periodic  $x \in H(U_\varepsilon, j)$ , i.e. such that  $(f^{(2n_u + j)N})^l(x) = x$  for some  $l = l(x) \in \mathbb{N}$ , it holds*

$$\sum_{i=0}^{l-1} T \circ f^{(2n_u + j)Ni}(x) = T(x) + \dots + T(f^{(2n_u + j)N(l-1)}(x)) = 0.$$

The function  $T$  in our case is

$$H(U_\varepsilon, j) \ni x \mapsto T(x) := (2n_u + j)N \text{Torsion}_{(2n_u + j)N}(f, x, e_x^u) - (2n_u + j)k \in \mathbb{R}.$$

In order to apply Livšic's theorem, we first need to check that the finite-time torsion is Hölder continuous. Here is the point that demands some further regularity hypothesis.

**Proposition 4.5.1.** *Let  $f$  be  $\mathcal{C}^2$ . Then the function*

$$H(U_\varepsilon, j) \ni x \mapsto (2n_u + j)N \text{ Torsion}_{(2n_u + j)N}(f, x, e_x^u) \in \mathbb{R}$$

*is Hölder continuous.*

*Proof.* We are going to prove that the  $(2n_u + j)N$ -finite time torsion is locally Hölder. This will enable us to conclude because a locally Hölder continuous function defined on a compact metric space (the horseshoe  $H(U_\varepsilon, j)$  in our case) is Hölder continuous (see Theorem 4.4.2 in [Fio16]).

Let  $\eta > 0$  be such that if  $x, y \in H(U_\varepsilon, j)$  and  $d(x, y) < \eta$  then  $h(x)_1 = h(y)_1$  where  $(h(x))_{i \in \mathbb{Z}}, (h(y))_{i \in \mathbb{Z}}$  are the sequences associated to  $x, y$  respectively.

For  $x, y \in H(U_\varepsilon, j)$  such that  $d(x, y) < \eta$  we have from Proposition 4.4.1 that

$$\left| (2n_u + j)N \text{ Torsion}_{(2n_u + j)N}(f, x, e_x^u) - (2n_u + j)N \text{ Torsion}_{(2n_u + j)N}(f, y, e_y^u) \right| < 2\varepsilon. \quad (4.43)$$

Up to replace  $e_y^u$  with  $-e_y^u$ , assume that

$$\|e_x^u - e_y^u\| = \min_{v \in E_y^u \cap \mathbb{S}^1} \|e_x^u - v\|. \quad (4.44)$$

Observe now that

$$\begin{aligned} & \left| (2n_u + j)N \text{ Torsion}_{(2n_u + j)N}(f, x, e_x^u) - (2n_u + j)N \text{ Torsion}_{(2n_u + j)N}(f, y, e_y^u) \right| = \\ & = \left| \bar{\theta}(e_x^u, e_y^u) - \tilde{\theta}(e_{f^{(2n_u + j)N}(x)}^u, e_{f^{(2n_u + j)N}(y)}^u) \right|, \end{aligned}$$

where  $\bar{\theta}(e_x^u, e_y^u)$  is the measure of the oriented angle  $\theta(e_x^u, e_y^u)$  contained in  $(-\varepsilon, \varepsilon)$  (thanks to (4.44)). The notation  $\tilde{\theta}(e_{f^{(2n_u + j)N}(x)}^u, e_{f^{(2n_u + j)N}(y)}^u)$  refers to a measure of the oriented angle

$$\theta(e_{f^{(2n_u + j)N}(x)}^u, e_{f^{(2n_u + j)N}(y)}^u).$$

Consequently, from (4.43),

$$\tilde{\theta}(e_{f^{(2n_u + j)N}(x)}^u, e_{f^{(2n_u + j)N}(y)}^u) \in (-3\varepsilon, 3\varepsilon).$$

By the invariance of the unstable bundle and since  $\varepsilon < \frac{1}{12}$ , we deduce that

$$\left\| e_{f^{(2n_u + j)N}(x)}^u - e_{f^{(2n_u + j)N}(y)}^u \right\| = \min_{v \in E_{f^{(2n_u + j)N}(y)}^u \cap \mathbb{S}^1} \left\| e_{f^{(2n_u + j)N}(x)}^u - v \right\|. \quad (4.45)$$

Indeed, since  $\theta(e_{f^{(2n_u + j)N}(x)}^u, e_{f^{(2n_u + j)N}(y)}^u)$  admits a measure in  $(-3\varepsilon, 3\varepsilon)$  and since  $\varepsilon \in (0, \frac{1}{12})$ , we have that

$$\cos \left( 2\pi \theta(e_{f^{(2n_u + j)N}(x)}^u, e_{f^{(2n_u + j)N}(y)}^u) \right) > 0.$$

Consequently it holds

$$\begin{aligned} & \left\| e_{f^{(2n_u + j)N}(x)}^u + e_{f^{(2n_u + j)N}(y)}^u \right\|^2 = \\ & = \left\| e_{f^{(2n_u + j)N}(x)}^u - e_{f^{(2n_u + j)N}(y)}^u \right\|^2 + 4 \cos \left( 2\pi \theta(e_{f^{(2n_u + j)N}(x)}^u, e_{f^{(2n_u + j)N}(y)}^u) \right) > \end{aligned}$$



$$> \left\| e_{f(2n_u+j)N(x)}^u - e_{f(2n_u+j)N(y)}^u \right\|^2,$$

implying (4.45). We then continue and have<sup>14</sup>

$$\begin{aligned} \left| \bar{\theta}(e_x^u, e_y^u) - \tilde{\theta}(e_{f(2n_u+j)N(x)}^u, e_{f(2n_u+j)N(y)}^u) \right| &\leq \left| \bar{\theta}(e_x^u, e_y^u) \right| + \left| \tilde{\theta}(e_{f(2n_u+j)N(x)}^u, e_{f(2n_u+j)N(y)}^u) \right| = \\ &= \frac{1}{2\pi} \arccos \left( 1 - \frac{\|e_x^u - e_y^u\|^2}{2} \right) + \frac{1}{2\pi} \arccos \left( 1 - \frac{\|e_{f(2n_u+j)N(x)}^u - e_{f(2n_u+j)N(y)}^u\|^2}{2} \right). \end{aligned}$$

The arc cosinus is well-defined thanks to (4.45). Thus

$$\begin{aligned} \left| (2n_u + j)N \text{ Torsion}_{(2n_u+j)N}(f, x, e_x^u) - (2n_u + j)N \text{ Torsion}_{(2n_u+j)N}(f, y, e_y^u) \right| &\leq \\ &\leq \frac{1}{2\pi} \arccos \left( 1 - \frac{\|e_x^u - e_y^u\|^2}{2} \right) - \frac{1}{2\pi} \arccos(1) + \\ &+ \frac{1}{2\pi} \arccos \left( 1 - \frac{\|e_{f(2n_u+j)N(x)}^u - e_{f(2n_u+j)N(y)}^u\|^2}{2} \right) - \frac{1}{2\pi} \arccos(1) \leq \\ &\leq C \|e_x^u - e_y^u\|^\alpha + C \left\| e_{f(2n_u+j)N(x)}^u - e_{f(2n_u+j)N(y)}^u \right\|^\alpha, \end{aligned}$$

for some  $C > 0, \alpha > 0$ , where the last inequality comes from the fact that  $\arccos$  is Hölder and  $\arccos(1) = 0$ .

From (4.44) and (4.45) we deduce that

$$\begin{aligned} \left| (2n_u + j)N \text{ Torsion}_{(2n_u+j)N}(f, x, e_x^u) - (2n_u + j)N \text{ Torsion}_{(2n_u+j)N}(f, y, e_y^u) \right| &\leq \\ &\leq Cd_H (E_x^u \cap \mathbb{S}^1, E_y^u \cap \mathbb{S}^1)^\alpha + Cd_H (E_{f(2n_u+j)N(x)}^u \cap \mathbb{S}^1, E_{f(2n_u+j)N(y)}^u \cap \mathbb{S}^1)^\alpha. \end{aligned}$$

If  $f \in \mathcal{C}^2$ , then the function<sup>15</sup>

$$H(U_\varepsilon, j) \ni x \mapsto E_x^u \in \bigcup_{x \in H(U_\varepsilon, j)} \mathcal{G}_1(T_x S),$$

is Hölder continuous (see Corollary 2.1 in [BP74] or Theorem 5.18 in [Shu87]).

Consequently, for suitable  $\bar{C} > 0, \beta > 0$ ,

$$\begin{aligned} \left| (2n_u + j)N \text{ Torsion}_{(2n_u+j)N}(f, x, e_x^u) - (2n_u + j)N \text{ Torsion}_{(2n_u+j)N}(f, y, e_y^u) \right| &\leq \\ &\leq \bar{C} d(x, y)^\beta + \bar{C} d(f^{(2n_u+j)N}(x), f^{(2n_u+j)N}(y))^\beta \end{aligned}$$

and, since  $f$  is in particular  $\mathcal{C}^1$  and the horseshoe is compact, we conclude that

$$\begin{aligned} \left| (2n_u + j)N \text{ Torsion}_{(2n_u+j)N}(f, x, e_x^u) - (2n_u + j)N \text{ Torsion}_{(2n_u+j)N}(f, y, e_y^u) \right| &\leq \\ &\leq \bar{C}(1 + \tilde{C}) d(x, y)^\beta, \end{aligned}$$

that is the  $(2n_u + j)N$ -finite time torsion is locally Hölder continuous. □

14. We are using the fact that the length  $\|v\|^2$  of the chord of an angle  $\theta$  is  $2 \sin(\frac{\theta}{2})$  and that  $2 \sin^2(\frac{\theta}{2}) = 1 - \cos(\theta)$ .

15.  $\mathcal{G}_1(T_x S)$  is the Grassmannian of the 1-dimensional subspaces in  $T_x S$ .

We can now show the following

**Lemma 4.5.1.** *Let  $f$  be  $\mathcal{C}^2$ . Then the function*

$$H(U_\varepsilon, j) \ni x \mapsto (2n_u + j)N \text{ Torsion}_{(2n_u + j)N}(f, x, e_x^u) \in \mathbb{R}$$

*is cohomologous to the constant  $(2n_u + j)k$  if and only if the unstable angle variation of  $(q, p)$  is null.*

*Proof.* Let us discuss the two possible cases.

- Case  $m = 0$ . Let  $x \in H(U_\varepsilon, j)$  be a periodic point of period  $l \in \mathbb{N}$ . Since the unstable bundle is invariant and  $f^{(2n_u + j)Nl}(x) = x$ , we deduce that

$$(2n_u + j)Nl \text{ Torsion}_{(2n_u + j)Nl}(f, x, e_x^u) - (2n_u + j)lk \in \frac{1}{2} + \mathbb{Z}. \quad (4.46)$$

From Proposition 4.4.1 since  $m = 0$  we have that

$$(2n_u + j)Nl \text{ Torsion}_{(2n_u + j)Nl}(f, x, e_x^u) - (2n_u + j)lk \in (-\varepsilon, \varepsilon). \quad (4.47)$$

Since  $\varepsilon \in (0, \frac{1}{12})$ , from (4.46) and (4.47), it holds

$$(2n_u + j)Nl \text{ Torsion}_{(2n_u + j)Nl}(f, x, e_x^u) - (2n_u + j)lk = 0.$$

Thanks to the arbitrariness of  $x \in H(U_\varepsilon, j)$  periodic point, we can apply Livšic's periodic point theorem and conclude that if  $m = 0$  then

$$H(U_\varepsilon, j) \ni x \mapsto (2n_u + j)N \text{ Torsion}_{(2n_u + j)N}(f, x, e_x^u) - (2n_u + j)k \in \mathbb{R}$$

is cohomologous to the zero constant function.

- Case  $m \neq 0$ . We want to show that

$$x \mapsto (2n_u + j)N \text{ Torsion}_{(2n_u + j)N}(f, x, e_x^u)$$

is not cohomologous to a constant function. Thanks again to Livšic's periodic point theorem, it is sufficient to exhibit two periodic points  $x, y \in H(U_\varepsilon, j)$  of periods  $l_1, l_2 \in \mathbb{N}$  respectively such that

$$(2n_u + j)Nl_1 \text{ Torsion}_{(2n_u + j)Nl_1}(f, x, e_x^u) \neq (2n_u + j)Nl_2 \text{ Torsion}_{(2n_u + j)Nl_2}(f, y, e_y^u).$$

Consider the points  $x, y$  of the horseshoe corresponding to the symbolic sequences  $(s_i)_{i \in \mathbb{Z}}, (t_i)_{i \in \mathbb{Z}}$  so that for any  $i$  we have  $s_i = 0, t_i = 1$ . They are fixed points with respect to  $f^{(2n_u + j)N}$ . From Proposition 4.4.1 and the invariance of the unstable bundle we deduce that

$$\begin{aligned} (2n_u + j)N \text{ Torsion}_{(2n_u + j)N}(f, x, e_x^u) &= (2n_u + j)k \neq \\ &\neq (2n_u + j)k + \frac{m}{2} = (2n_u + j)N \text{ Torsion}_{(2n_u + j)N}(f, y, e_y^u). \end{aligned}$$

Equivalently, from Livšic's periodic point theorem, if  $m \neq 0$  then the  $(2n_u + j)N$ -finite time torsion is not cohomologous to a constant function. □

### 4.5.1 Sufficient conditions for the non triviality of torsion

In the sequel, we are going to give sufficient conditions to assure that the unstable angle variation of  $(q, p)$   $m$  is not null.

**Reminder 4.5.1.** Recall that  $\Gamma_u$  refers to the parametrization of  $[q, p]^u$  (see (4.7)), that is the subset of  $W^u(q)$  connecting  $q$  to  $p$ . Remark that  $\Gamma_u(0) = q, \Gamma_u(t_u) = p$ . The angle function

$$[0, t_u] \ni t \mapsto \theta(\mathcal{H}, \Gamma'_u(t)) \in \mathbb{T}$$

is the oriented angle between the constant horizontal vector and the vector tangent to  $W^u(q)$  at  $\Gamma_u(t)$ . Let

$$[0, t_u] \ni t \mapsto \tilde{\theta}(\mathcal{H}, \Gamma'_u(t)) \in \mathbb{R}$$

be a continuous determination of such angle function.

The angle variation of the tangent vector along the unstable manifold between  $q$  and  $p$  is so

$$\tilde{\theta}(\mathcal{H}, \Gamma'_u(t_u)) - \tilde{\theta}(\mathcal{H}, \Gamma'_u(0)).$$

Observe that such an angle variation does not depend on the chosen continuous determination.

We recall that  $[0, t_s] \ni t \mapsto \gamma_s(t) \in CC(W^s(q), q)$  denotes a parametrization of  $[p, q]^s$ , i.e. the subset of the local stable manifold of  $q$  connecting  $p$  to  $q$ , such that  $\gamma_s(0) = p, \gamma_s(t_s) = q$ .

**Definition 4.5.1.** Let  $p \in (W^u(q) \cap W^s(q)) \setminus \{q\}$  be the future-first-entry point for  $CC(U_\varepsilon, q)$ . The point  $p$  is zero homoclinic if the closed curve  $\gamma_s([0, t_s]) \cup \Gamma_u([0, t_u])$  does not have self-intersections and it is homotopic to a point.

**Definition 4.5.2.** Let  $p \in (W^u(q) \cap W^s(q)) \setminus \{q\}$  be the first-entry point for  $CC(U_\varepsilon, q)$ . The point  $p \in S$  has a good orientation with respect to  $q$  if the orientation determined by

$$(\Gamma'_u(t_u), \gamma'_s(0))$$

is opposite with respect to the orientation determined by  $(\Gamma'_u(0), \gamma'_s(t_s))$ .

The point  $p \in S$  has a bad orientation with respect to  $q$  if  $p$  has not a good orientation (with respect to  $q$ ).

Remark that by the invariance of the unstable manifold it holds  $\frac{\Gamma'_u(0)}{\|\Gamma'_u(0)\|} = \frac{\gamma'_u(0)}{|\gamma'_u(0)|} = e_q^u$  (see (4.6)) and  $\Gamma'_u(t_u) \in E_p^u$ .

The following Proposition sums up sufficient conditions to assure  $m \in \mathbb{Z}$  not null.

**Proposition 4.5.2.** *Let  $S$  be a surface among  $\mathbb{R}^2, \mathbb{A}$  and  $\mathbb{T}^2$ . Let  $f : S \rightarrow S$  be a  $\mathcal{C}^1$  diffeomorphism isotopic to the identity<sup>16</sup>. Let  $q \in S$  be a periodic hyperbolic point for  $f$  of period  $N$ . Let  $\mathcal{O}(p, f^N) \subset (W^u(q) \cap W^s(q)) \setminus \{q\}$ . Let  $U_\varepsilon$  be an adapted neighborhood of  $\{q\} \cup \mathcal{O}(p, f^N)$  with respect to  $0 < \varepsilon < \frac{1}{12}$ . Suppose that  $p$  is the first-entry point for  $CC(U_\varepsilon, q)$ . Assume that*

- *the point  $p$  is zero homoclinic (see Definition 4.5.1)*

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16. If  $S = \mathbb{R}^2$  then assume also that  $f$  has compact support.

or

— the point  $p$  has a good orientation with respect to  $q$  (see Definition 4.5.2).

Then  $m \in \mathbb{Z}$  is not zero.

The proof of Proposition 4.5.2 is an outcome of the following two Lemmas.

**Lemma 4.5.2.** *If the point  $p$  is zero homoclinic, then*

$$\tilde{\theta}(\mathcal{H}, \Gamma'_u(t_u)) - \tilde{\theta}(\mathcal{H}, \Gamma'_u(0)) \in \left( \frac{m}{2} - \frac{\varepsilon}{2}, \frac{m}{2} + \frac{\varepsilon}{2} \right)$$

for some  $m \in \mathbb{Z} \setminus \{0\}$ .

*Proof of Lemma 4.5.2.* Consider  $\Gamma$  a lift on  $\mathbb{R}^2$  of  $\gamma_s([0, t_s]) \cup \Gamma_u([0, t_u])$ . It is a  $\mathcal{C}^1$  piecewise closed curve which does not have self-intersections since  $p$  is zero homoclinic (see Definition 4.5.1).

We apply the Turning Tangent Theorem to  $\Gamma$  (see [DC76], Chapter 4, Section 5) to determine the angle variation we are interested in. By Condition 4.2.1 of  $CC(U_\varepsilon, q)$  adapted neighborhood of  $q$  for  $\mathcal{O}(p, f^N)$  with respect to  $0 < \varepsilon < \frac{1}{12}$ , for any  $t \in [0, t_s]$  the angle

$$\theta(\gamma'_s(0), \gamma'_s(t)) = \theta(\mathcal{H}, \gamma'_s(t)) - \theta(\mathcal{H}, \gamma'_s(0))$$

admits a measure in  $(-\frac{\varepsilon}{4}, \frac{\varepsilon}{4})$  and we denote such measure as  $\bar{\theta}(\gamma'_s(0), \gamma'_s(t))$ .

Consequently, the variation of any continuous determination of the angle function  $t \mapsto \theta(\mathcal{H}, \gamma'_s(t))$  between 0 and  $t_s$  along the local stable manifold, that we denote as  $\bar{\theta}(\mathcal{H}, \gamma'_s(t_s)) - \bar{\theta}(\mathcal{H}, \gamma'_s(0))$ , is in  $(-\frac{\varepsilon}{4}, \frac{\varepsilon}{4})$  and so equal to  $\bar{\theta}(\gamma'_s(0), \gamma'_s(t_s))$ .

By Condition 4.2.3 of  $CC(U_\varepsilon, q)$  adapted neighborhood of  $q$  for  $\mathcal{O}(p, f^N)$ , by Remark 4.2.3 and by the invariance of the unstable manifold, the angle

$$\theta(\Gamma'_u(0), \Gamma'_u(t_u)) = \theta(\mathcal{H}, \Gamma'_u(t_u)) - \theta(\mathcal{H}, \Gamma'_u(0))$$

admits a measure either in  $(\frac{1}{2} - \frac{\varepsilon}{2}, \frac{1}{2} + \frac{\varepsilon}{2})$  (if  $p$  has a good orientation) or in  $(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$  (if  $p$  has a bad orientation) and we denote such a measure as  $\bar{\theta}(\Gamma'_u(0), \Gamma'_u(t_u))$ .

The angle variation of the tangent vector along the unstable manifold

$$\tilde{\theta}(\mathcal{H}, \Gamma'_u(t_u)) - \tilde{\theta}(\mathcal{H}, \Gamma'_u(0))$$

is a measure of the angle  $\theta(\Gamma'_u(0), \Gamma'_u(t_u))$  and so it is equal to

$$\bar{\theta}((\Gamma'_u(0), \Gamma'_u(t_u)) + n \quad \text{for some } n \in \mathbb{Z}.$$

Denote as

- (i)  $\bar{\theta}(\gamma'_s(t_s), \Gamma'_u(0))$  the measure of the angle  $\theta(\gamma'_s(t_s), \Gamma'_u(0))$  contained in  $(-\frac{1}{2}, \frac{1}{2})$ ;
- (ii)  $\bar{\theta}(\Gamma'_u(t_u), \gamma'_s(0))$  the measure of the angle  $\theta(\Gamma'_u(t_u), \gamma'_s(0))$  contained in  $(-\frac{1}{2}, \frac{1}{2})$ .

The Turning Tangent Theorem implies that

$$\begin{aligned} & \left( \tilde{\theta}(\mathcal{H}, \gamma'_s(t_s)) - \tilde{\theta}(\mathcal{H}, \gamma'_s(0)) \right) + \bar{\theta}(\gamma'_s(t_s), \Gamma'_u(0)) + \\ & \left( \tilde{\theta}(\mathcal{H}, \Gamma'_u(t_u)) - \tilde{\theta}(\mathcal{H}, \Gamma'_u(0)) \right) + \bar{\theta}(\Gamma'_u(t_u), \gamma'_s(0)) = \delta, \end{aligned}$$

where  $\delta$  is either 1 or  $-1$ , depending on the orientation of the lifted curve  $\Gamma$ .

Consequently

$$\bar{\theta}(\gamma'_s(0), \gamma'_s(t_s)) + \bar{\theta}(\gamma'_s(t_s), \Gamma'_u(0)) + \bar{\theta}(\Gamma'_u(0), \Gamma'_u(t_u)) + n + \bar{\theta}(\Gamma'_u(t_u), \gamma'_s(0)) = \delta$$

and so  $n = \delta$ , that is  $n$  is either 1 or  $-1$ .

If  $p$  has a good orientation, then

$$\tilde{\theta}(\mathcal{H}, \Gamma'_u(t_u)) - \tilde{\theta}(\mathcal{H}, \Gamma'_u(0)) \in \left( \frac{1}{2} + \delta - \frac{\varepsilon}{2}, \frac{1}{2} + \delta + \frac{\varepsilon}{2} \right).$$

If  $p$  has a bad orientation, then

$$\tilde{\theta}(\mathcal{H}, \Gamma'_u(t_u)) - \tilde{\theta}(\mathcal{H}, \Gamma'_u(0)) \in \left( \delta - \frac{\varepsilon}{2}, \delta + \frac{\varepsilon}{2} \right).$$

In both cases we have that

$$\tilde{\theta}(\Gamma'_u(0), \Gamma'_u(t_u)) \in \left( \frac{m}{2} - \frac{\varepsilon}{2}, \frac{m}{2} + \frac{\varepsilon}{2} \right)$$

for some  $m \in \mathbb{Z}, m \in \{-2, -1, 2, 3\}$ .

□

**Lemma 4.5.3.** *Let  $p \in (W^u(q) \cap W^s(q)) \setminus \{q\}$  be the first-entry point for  $CC(U_\varepsilon, q)$ . If the point  $p$  has a good orientation with respect to  $q$ , then*

$$\tilde{\theta}(\mathcal{H}, \Gamma'_u(t_u)) - \tilde{\theta}(\mathcal{H}, \Gamma'_u(0)) \in \left( \frac{m}{2} - \frac{\varepsilon}{2}, \frac{m}{2} + \frac{\varepsilon}{2} \right) \quad \text{for some } m \in \mathbb{Z} \setminus \{0\}.$$

*Proof of Lemma 4.5.3.* By Condition 4.2.3 of  $CC(U_\varepsilon, q)$  adapted neighborhood of  $q$  for  $\mathcal{O}(p, f^N)$  with respect to  $0 < \varepsilon < \frac{1}{12}$ , by Remark 4.2.3 and since  $p$  has a good orientation with respect to  $q$ , the angle

$$\theta(\Gamma'_u(0), \Gamma'_u(t_u)) = \theta(\mathcal{H}, \Gamma'_u(t_u)) - \theta(\mathcal{H}, \Gamma'_u(0))$$

admits a measure in  $(\frac{1}{2} - \frac{\varepsilon}{2}, \frac{1}{2} + \frac{\varepsilon}{2})$  and we denote such a measure as  $\bar{\theta}(\Gamma'_u(0), \Gamma'_u(t_u))$ .

Since the variation between 0 and  $t_u$  of any continuous determination of the angle function  $[0, t_u] \ni t \mapsto \theta(\mathcal{H}, \Gamma'_u(t)) \in \mathbb{T}$ , that is

$$\tilde{\theta}(\mathcal{H}, \Gamma'_u(t_u)) - \tilde{\theta}(\mathcal{H}, \Gamma'_u(0)),$$

is a measure of the angle  $\theta(\Gamma'_u(0), \Gamma'_u(t_u))$ , we have that

$$\tilde{\theta}(\mathcal{H}, \Gamma'_u(t_u)) - \tilde{\theta}(\mathcal{H}, \Gamma'_u(0)) \in \left( \frac{m}{2} - \frac{\varepsilon}{2}, \frac{m}{2} + \frac{\varepsilon}{2} \right)$$

for some  $m \in 2\mathbb{Z} + 1$ , in particular  $m \neq 0$ .

□

In the framework of  $\mathbb{R}^2$  and of  $\mathbb{A}$ , the following Lemma gives us sufficient conditions to obtain the hypothesis of (and so apply) Proposition 4.5.2.

**Lemma 4.5.4.** *Assume that the curve  $[p, q]^s$  intersects transversally the curve  $[q, f^N(p)]^u$ . That is, for any  $x \in [p, q]^s \cap [q, f^N(p)]^u$  it holds  $T_x W^s(q) + T_x W^u(q) = T_x S$ .*

- (i) *If  $S = \mathbb{R}^2$  then there exists a transverse homoclinic point  $r \in (W^u(q) \cap W^s(q)) \setminus \{q\}$  which is zero homoclinic.*
- (ii) *If  $S = \mathbb{A}$  then there exists a transverse homoclinic point  $r \in (W^u(q) \cap W^s(q)) \setminus \{q\}$  which has a good orientation with respect to  $q$ .*

*Proof.* Let  $S = \mathbb{R}^2$ . Since  $[q, p]^u \subset [q, f^N(p)]^u$  and  $[p, q]^s \cap [q, f^N(p)]^u$ , we deduce that also  $[p, q]^s \cap [q, p]^u$  and the intersection is not empty since  $p \in [p, q]^s \cap [q, p]^u$ . Consider the parametrization of  $[q, p]^u$

$$[0, t_u] \ni t \mapsto \Gamma_u(t) \in \mathbb{R}^2.$$

Let  $\bar{t} \in (0, t_u]$  be the first point of intersection between  $[q, p]^u$  and  $[p, q]^s$  (not considering  $q$ ) and denote as  $r$  the corresponding point  $\Gamma_u(\bar{t})$ . Since  $[p, q]^s \cap [q, p]^u$ , the point  $r$  is a point of transverse homoclinic intersection. Because of the choice of  $r$  as first intersection point, the closed curve  $[r, q]^s \cup [q, r]^u$  does not have self-intersections. Moreover, since we are considering as surface  $\mathbb{R}^2$ , the closed curve  $[r, q]^s \cup [q, r]^u$  is homotopic to a point. That is,  $r \in (W^u(q) \cap W^s(q)) \setminus \{q\}$  is zero homoclinic.

Let  $S = \mathbb{A}$ . If  $p$  has a good orientation with respect to  $q$ , then there is nothing to prove.

Assume that  $p$  has a bad orientation with respect to  $q$ . Let  $r \in (W^u(q) \cap W^s(q)) \setminus \{q\}$  be the first point of intersection between  $[p, q]^s$  and  $[q, p]^u$ , not considering the point  $q$ . That is, let  $t \in (0, t_u]$  be such that

- $r = \Gamma_u(t) \in (W^s(q) \cap W^u(q)) \setminus \{q\}$ ;
- for any  $s \in (0, t)$  the point  $\Gamma_u(s)$  does not belong to  $[p, q]^s$ .

The point  $r$  is a point of transverse homoclinic intersection because  $[q, r]^u \subset [q, p]^u \subset [q, f^N(p)]^u$  and by hypothesis  $[p, q]^s \cap [q, f^N(p)]^u$ . Observe that, since  $r$  is the first point of intersection between  $[p, q]^s$  and  $[q, p]^u$  (after  $q$ ), the closed curve  $[r, q]^s \cup [q, r]^u$  does not have self-intersections.

Let us parametrize the curves  $[r, q]^s$  and  $[q, r]^u$  as follows:

$$[0, \tau_u] \ni t \mapsto \Psi_u(t) \in [q, r]^u,$$

$$[0, \tau_s] \ni t \mapsto \psi_s(t) \in [r, q]^s$$

such that  $\Psi_u(0) = \psi_s(\tau_s) = q, \Psi_u(\tau_u) = \psi_s(0) = r$ .

If  $r$  has a good orientation with respect to  $q$ , then there is nothing to prove. Assume that  $r$  has a bad orientation with respect to  $q$ . Remark that the curve  $[r, q]^s \cup [q, r]^u$  separates the annulus into two connected components, say  $\mathbb{A}_1, \mathbb{A}_2$ . Since  $\lambda_1, \lambda_2$  eigenvalues of  $Df^N(q)$  are both positive<sup>17</sup>, the points  $r$  and  $f^N(r)$  belong to the same connected component of  $W^s(q) \setminus \{q\}$ . In particular  $f^N(r) \in [r, q]^s$ .

Recall that  $[q, r]^u \subset f^N([q, r]^u) = [q, f^N(r)]^u$ .

---

17. Recall that at the beginning of Chapter 4 we have assumed that the eigenvalues of  $Df^N(q)$  are in  $\mathbb{R}_+$ , up to replace  $f$  with  $f^2$ .

We have already remarked that the closed curve  $[r, q]^s \cup [q, r]^u$  separates the annulus. Since  $f^N$  preserves the orientation, the vector  $Df^N(r)\Psi'_u(\tau_u)$  points towards the same connected component (either  $\mathbb{A}_1$  or  $\mathbb{A}_2$ ) as  $\Psi'_u(\tau_u)$ . Thus, the curve  $f^N([q, r]^u) \setminus [q, r]^u$  intersects the closed curve  $[r, q]^s \cup [q, r]^u$ .

Since the unstable manifold cannot intersect itself, we deduce that  $f^N([q, r]^u) \setminus [q, r]^u$  meets  $[r, q]^s$ . Because  $[r, q]^s \subset [p, q]^s$ ,  $f^N([q, r]^u) \subset [q, f^N(p)]^u$  and since  $[p, q]^s$  intersects  $[q, f^N(p)]^u$  transversally, the points of intersection between  $f^N([q, r]^u) \setminus [q, r]^u$  and  $[r, q]^s$  are of transverse intersection. In particular, there exists a point  $z \in (f^N([q, r]^u) \setminus [q, r]^u) \pitchfork [r, q]^s$  (see Figure 4.12) such that the tangent vector at  $z$  to the unstable manifold  $W^u(q)$  points towards the same connected component as  $\psi'_s(\tau_s)$ , which is the vector tangent to the parametrized curve  $[p, q]^s$  at  $q$ . That is, the point  $z$  is a point with a good orientation with respect to  $q$ .

□

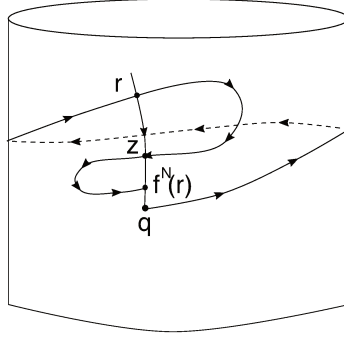


Figure 4.12 – The point  $z \in \mathbb{A}$  of transverse homoclinic intersection with a good orientation with respect to  $q$ .

**Remark 4.5.1.** In Lemma 4.5.4, the hypothesis that  $[p, q]^s \pitchfork [q, f^N(p)]^u$  is too strong. Indeed, it is not a necessary condition to have points which are zero homoclinic or with good orientation, that is so that Proposition 4.5.2 holds. Nevertheless, the required hypothesis  $[p, q]^s \pitchfork [q, f^N(p)]^u$  is an open condition within the set of  $\mathcal{C}^1$  diffeomorphisms.

**Remark 4.5.2.** Observe that the presence of tangent homoclinic points could prevent from having a not zero unstable angle variation of  $(q, p)$  (see Definition 4.4.1), that is from having a transverse homoclinic intersection with  $m \neq 0$ . See Figure 4.13.

Nevertheless, if  $f$  is  $\mathcal{C}^\infty$  (with no further hypothesis on stable and unstable manifolds), then a remark by S. Crovisier provides us points of transverse homoclinic intersection with good orientation.

Consider the situation of Figure 4.13. Let  $p$  be a point of transverse homoclinic intersection with a bad orientation with respect to  $q$ . Let  $z \in [q, p]^u$  be the first point of intersection between  $[p, q]^s$  and  $[q, p]^u$ . Then,  $[q, z]^u \cup [z, q]^s$  separates the annulus (or plane). Assume  $z$  is a point of tangent intersection.

Then, we can build a horseshoe with respect to the homoclinic transverse point  $p$  and then look at the leaves of the unstable foliation of the horseshoe itself. Through the dynamical Sard's Theorem of Buzzi–Crovisier–Sarig (see Theorem 4.2 in [BCS]), since  $f$  is  $\mathcal{C}^\infty$ , there exist leaves of the unstable foliation that cross transversally  $W^s(q)$  both near the periodic point  $q$  and near the tangent intersection point  $z$ , with different orientation (with respect

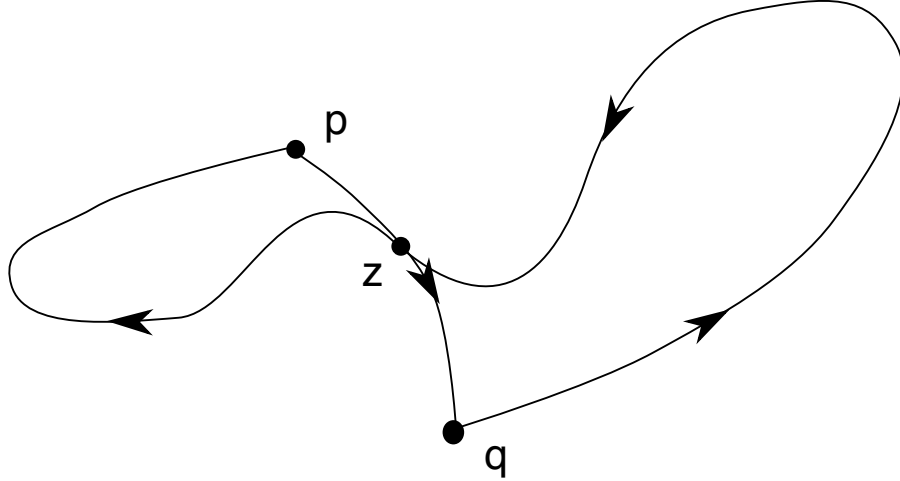


Figure 4.13 – A transverse homoclinic point  $p$  with  $m = 0$ . The point  $z$  is a tangent homoclinic point.

to  $q$ ) at the two points of intersection (see Figure 4.13). Then, by the  $\lambda$ -lemma, the unstable manifold  $W^u(q)$  also intersects transversally the stable manifold in two points with different orientations. This provides us the required transverse homoclinic point with a good orientation with respect to  $q$ .

## 4.6 Results on torsion in the non trivial case

The results presented in Section 4.4 enable us to deduce some interesting consequences over the asymptotic torsion at points of the horseshoe in the non trivial case.

We assume that the unstable angle variation  $m$  is **not null** and, using the symbolic dynamics, we deduce some consequences over torsion at points of  $H(U_\varepsilon, n_u, j)$ .

**Assumption 4.6.1.** In this Section we assume that the unstable angle variation of  $(q, p)$   $m \in \mathbb{Z}$  is not null.

**Proposition 4.6.1.** For any  $A \leq B$ ,  $A, B \in [0, 1]$  there exists  $(\delta_i)_{i \in \mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$  such that

$$\liminf_{n \rightarrow +\infty} \frac{\sum_{i=1}^n \delta_i}{n} = A \leq B = \limsup_{n \rightarrow +\infty} \frac{\sum_{i=1}^n \delta_i}{n}. \quad (4.48)$$

From Proposition 4.6.1 it immediately descends the following

**Corollary 4.6.1.** Let  $m \neq 0$ . For any  $\alpha \leq \beta$ ,  $\alpha, \beta \in \left[ \frac{k}{N}, \frac{k}{N} + \frac{m}{2(2n_u + j)N} \right]$  there exists  $x \in H(U_\varepsilon, j)$  such that

$$\liminf_{n \rightarrow +\infty} \text{Torsion}_n(f, x, v) = \alpha \leq \beta = \limsup_{n \rightarrow +\infty} \text{Torsion}_n(f, x, v), \quad (4.49)$$

where  $v \in T_x S^{18}$ .

---

18. The asymptotic torsion does not depend on the tangent vector.



*Proof.* Define

$$A := \frac{2(2n_u + j)N}{m} \left( \alpha - \frac{k}{N} \right) \quad \text{and} \quad B := \frac{2(2n_u + j)N}{m} \left( \beta - \frac{k}{N} \right).$$

Observe that  $A, B \in [0, 1]$ .

From Proposition 4.6.1 there exists  $(\delta_i)_{i \in \mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$  such that

$$\liminf_{n \rightarrow +\infty} \frac{\sum_{i=1}^n \delta_i}{n} = \alpha \leq \beta = \limsup_{n \rightarrow +\infty} \frac{\sum_{i=1}^n \delta_i}{n}.$$

Let  $x \in H(U_\varepsilon, j)$  be such that  $h(x) = (\delta_i)_{i \in \mathbb{Z}}$ . Such  $x$  exists because  $h$  is a homeomorphism on the horseshoe  $H(U_\varepsilon, j)$ . From Corollary 4.4.1 it holds that

$$\liminf_{n \rightarrow +\infty} \text{Torsion}_n(f, x, v) = \frac{k}{N} + \frac{m}{2(2n_u + j)N} \liminf_{n \rightarrow +\infty} \frac{\sum_{i=1}^n \delta_i}{n} = \frac{k}{N} + \frac{m}{2(2n_u + j)N} A = \alpha$$

and similarly

$$\limsup_{n \rightarrow +\infty} \text{Torsion}_n(f, x, v) = \frac{k}{N} + \frac{m}{2(2n_u + j)N} \limsup_{n \rightarrow +\infty} \frac{\sum_{i=1}^n \delta_i}{n} = \frac{k}{N} + \frac{m}{2(2n_u + j)N} B = \beta.$$

□

*Proof of Proposition 4.6.1.* Assume as a first case that  $A < B$ . Let  $\left(\frac{p_n}{q_n}\right)_{n \in \mathbb{N}}, \left(\frac{r_n}{t_n}\right)_{n \in \mathbb{N}}$  be sequences in  $\mathbb{Q}^{\mathbb{N}}$  converging decreasingly to  $A$  and increasingly to  $B$ , respectively. That is

$$\frac{p_n}{q_n} \xrightarrow{n \rightarrow +\infty} A \quad \text{and} \quad A \leq \frac{p_{n+1}}{q_{n+1}} \leq \frac{p_n}{q_n} \quad \forall n \in \mathbb{N}$$

and

$$\frac{r_n}{t_n} \xrightarrow{n \rightarrow +\infty} B \quad \text{and} \quad B \geq \frac{r_{n+1}}{t_{n+1}} \geq \frac{r_n}{t_n} \quad \forall n \in \mathbb{N}.$$

Since  $A < B$  we can suppose that  $\frac{p_0}{q_0} \leq \frac{r_0}{t_0}$ .

**Remark 4.6.1.** If  $A$  or  $B$  is in  $\mathbb{Q}$ , then we simply consider the constant sequence  $\frac{p_n}{q_n} = A$  for any  $n \in \mathbb{N}$  or  $\frac{r_n}{t_n} = B$  for any  $n \in \mathbb{N}$ .

Define now the sequences  $\left(\frac{a_n}{b_n}\right)_{n \in \mathbb{N}} \in \mathbb{Q}^{\mathbb{N}}$  and  $(\varepsilon_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$  as follows

$$\frac{a_n}{b_n} := \begin{cases} \frac{p_{\frac{n}{2}}}{q_{\frac{n}{2}}} & \text{if } n \text{ is even} \\ \frac{r_{\frac{n-1}{2}}}{t_{\frac{n-1}{2}}} & \text{if } n \text{ is odd} \end{cases} \quad (4.50)$$

and

$$\varepsilon_n := \begin{cases} \frac{p_{\frac{n}{2}}}{q_{\frac{n}{2}}} - A & \text{if } n \text{ is even} \\ B - \frac{r_{\frac{n-1}{2}}}{t_{\frac{n-1}{2}}} & \text{if } n \text{ is odd.} \end{cases} \quad (4.51)$$

Observe that  $(\varepsilon_n)_{n \in \mathbb{N}}$  converges to 0. In particular both  $(\varepsilon_{2n})_{n \in \mathbb{N}}$  and  $(\varepsilon_{2n+1})_{n \in \mathbb{N}}$  converge decreasingly to 0.

Let  $n \in \mathbb{N}$  and consider a finite sequence of 0s and 1s of length  $b_n$  as follows. Let  $\rho_n, \tau_n \in \mathbb{N}$  be such that

$$b_n = \rho_n a_n + \tau_n$$

where  $\tau_n \in \llbracket 0, a_n - 1 \rrbracket$  and  $\rho_n \leq \frac{b_n}{a_n}$ .

**Definition 4.6.1.** Let  $a \in \mathbb{N}, b \in \mathbb{N}^*$  such that  $a \leq b$ . Let  $\rho, \tau \in \mathbb{N}$  be such that  $b = \rho a + \tau$  and  $\tau \in \llbracket 0, a - 1 \rrbracket$ . An *adapted sequence* for  $\frac{a}{b}$  is a sequence of finite length  $b$  obtained by alternating a 1-symbol and then  $(\rho - 1)$  0-symbols for  $a$  times and by ending the sequence with  $\tau$  0-symbols. That is

$$\overbrace{\underbrace{1 \overbrace{0 \dots 0}^{\rho-1}}_{\rho} 1 0 \dots 0 1 0 \dots 0 \dots 1 0 \dots 0 \underbrace{0 \dots 0}_{\tau}}^{b}$$

$\rho a$

**Remark 4.6.2.** Consider an adapted sequence for  $\frac{a}{b}$  according to Definition 4.6.1. Observe that the total number of 1-symbols contained in such a sequence is exactly  $a$ .

Let us define the sequence  $(N_n, \tilde{N}_n)_{n \in \mathbb{N}}$  recursively as follows. Let  $(N_0, \tilde{N}_0)$  be  $(1, 1)$ . Choose now  $N_1 \in \mathbb{N}$  so that for any  $N \geq N_1$  it holds

$$\frac{a_0 + N a_1}{b_0 + N b_1} = \frac{b_0}{b_0 + N b_1} \frac{a_0}{b_0} + \frac{N b_1}{b_0 + N b_1} \frac{a_1}{b_1} \geq \frac{a_1}{b_1} - \varepsilon_1 = B - 2\varepsilon_1.$$

The integer  $\tilde{N}_1$  is so  $\max\{N_1, b_2\}$ . Let  $n > 1$  and assume now that we have defined the couples  $(N_i, \tilde{N}_i)$  for all  $i \in \llbracket 1, n - 1 \rrbracket$ . Define  $N_n \in \mathbb{N}$  so that for any  $N \geq N_n$  it holds

$$\begin{aligned} & \frac{a_0 + \tilde{N}_1 a_1 + \dots + \tilde{N}_{n-1} a_{n-1} + N a_n}{b_0 + \tilde{N}_1 b_1 + \dots + \tilde{N}_{n-1} b_{n-1} + N b_n} = \\ &= \frac{b_0}{\sum_{i=0}^{n-1} \tilde{N}_i b_i + N b_n} \frac{a_0}{b_0} + \frac{\tilde{N}_1 b_1}{\sum_{i=0}^{n-1} \tilde{N}_i b_i + N b_n} \frac{a_1}{b_1} + \dots + \frac{N b_n}{\sum_{i=0}^{n-1} \tilde{N}_i b_i + N b_n} \frac{a_n}{b_n} \end{aligned}$$

is:

- $\geq \frac{a_n}{b_n} - \varepsilon_n = B - 2\varepsilon_n$  if  $n$  is odd;
- $\leq \frac{a_n}{b_n} + \varepsilon_n = A + 2\varepsilon_n$  if  $n$  is even.

Define  $\tilde{N}_n$  as  $\max\{N_n, n b_{n+1}\}$ .

We now construct a sequence  $(\delta_i)_{i \in \mathbb{Z}}$  as follows. For  $i \in \mathbb{Z}, i \leq 0$ , that is for a non positive index, the value  $\delta_i$  is no matter which value among 0 and 1.

Starting from the 1-th entry, we concatenate (according to Definition 4.6.1 of adapted sequence):

- one adapted sequence for  $\frac{a_0}{b_0}$  (determining so  $\delta_i$  for  $i \in \llbracket 1, b_0 \rrbracket$ ) ;
- $\tilde{N}_1$  adapted sequences for  $\frac{a_1}{b_1}$  (determining so  $\delta_i$  for  $i \in \llbracket b_0 + 1, b_0 + \tilde{N}_1 b_1 \rrbracket$ );

- $\tilde{N}_2$  adapted sequences for  $\frac{a_2}{b_2}$  (determining  $\delta_i$  for  $i \in \llbracket b_0 + \tilde{N}_1 b_1 + 1, b_0 + \tilde{N}_1 b_1 + \tilde{N}_2 b_2 \rrbracket$ );
- $\dots$
- $\tilde{N}_n$  adapted sequences for  $\frac{a_n}{b_n}$  (determining  $\delta_i$  for  $i \in \llbracket \sum_{j=0}^{n-1} \tilde{N}_j b_j + 1, \sum_{j=0}^n \tilde{N}_j b_j \rrbracket$ );
- $\dots$

We are now going to show that

$$\limsup_{m \rightarrow +\infty} \frac{\sum_{i=1}^m \delta_i}{m} = B \quad \text{and} \quad \liminf_{m \rightarrow +\infty} \frac{\sum_{i=1}^m \delta_i}{m} = A.$$

Since we are using adapted sequences (see Definition 4.6.1), from Remark 4.6.2 we are able to estimate  $\sum_{i=1}^m \delta_i$ .

Fix  $\delta > 0$ . Consider  $m \in \mathbb{N}$ : there exists  $n \in \mathbb{N}^*$  such that

$$m = b_0 + \tilde{N}_1 b_1 + \dots + \tilde{N}_{n-1} b_{n-1} + l b_n + j \rho_n + k,$$

where  $l \in \llbracket 0, \tilde{N}_n - 1 \rrbracket$ ,  $j \in \llbracket 0, a_n \rrbracket$  and  $k \in \llbracket 0, \rho_n - 1 \rrbracket$ . In particular

$$j \rho_n + k < b_n. \quad (4.52)$$

If  $m \in \mathbb{N}$  is large enough then  $n \geq 2$  and

$$\frac{2}{n-1} < \delta. \quad (4.53)$$

Then

$$\begin{aligned} \frac{\sum_{i=1}^m \delta_i}{m} &= \frac{m - j \rho_n - k}{m} \frac{\sum_{i=1}^{m-j\rho_n-k} \delta_i}{m - j \rho_n - k} + \frac{\sum_{i=m-j\rho_n-k+1}^m \delta_i}{m} \leq \\ &\leq \frac{\sum_{i=1}^{m-j\rho_n-k} \delta_i}{m - j \rho_n - k} + \frac{j+1}{m}. \end{aligned}$$

On one hand, since (see Remark 4.6.2 for the following equality)

$$\begin{aligned} &\frac{\sum_{i=1}^{m-j\rho_n-k} \delta_i}{m - j \rho_n - k} = \\ &= \frac{b_0}{m - j \rho_n - k} \frac{a_0}{b_0} + \dots + \frac{\tilde{N}_{n-1} b_{n-1}}{m - j \rho_n - k} \frac{a_{n-1}}{b_{n-1}} + \frac{l b_n}{m - j \rho_n - k} \frac{a_n}{b_n} \end{aligned}$$

is a convex combination of  $\frac{a_0}{b_0}, \dots, \frac{a_n}{b_n}$  and for any  $i \in \mathbb{N}$  it holds  $\frac{a_i}{b_i} \leq B$ , we have that

$$\frac{\sum_{i=1}^{m-j\rho_n-k} \delta_i}{m - j \rho_n - k} \leq B.$$

On the other hand, from the choice of  $\tilde{N}_{n-1} = \max\{N_{n-1}, (n-1)b_n\}$  and from (4.53), it holds

$$\frac{j+1}{m} \leq \frac{b_n + 1}{\tilde{N}_{n-1} b_{n-1}} \leq \frac{b_n + 1}{(n-1)b_n b_{n-1}} \leq \frac{2}{n-1} < \delta.$$

Therefore, we can deduce that  $\frac{\sum_{i=1}^m \delta_i}{m} \leq B + \delta$ . So by the arbitrariness of  $\delta$  we conclude that

$$\limsup_{m \rightarrow +\infty} \frac{\sum_{i=1}^m \delta_i}{m} \leq B. \quad (4.54)$$

Consider now the subsequence  $(\sum_{j=0}^{2n+1} \tilde{N}_j b_j)_{n \in \mathbb{N}}$ : for any  $n \in \mathbb{N}$  it holds, from the construction of  $(\delta_i)_{i \in \mathbb{Z}}$  and the choice of  $\tilde{N}_n$ , that

$$\begin{aligned} \frac{\sum_{i=1}^{\sum_{j=0}^{2n+1} \tilde{N}_j b_j} \delta_i}{\sum_{j=0}^{2n+1} \tilde{N}_j b_j} &= \frac{b_0}{\sum_{j=0}^{2n+1} \tilde{N}_j b_j} \frac{a_0}{b_0} + \frac{\tilde{N}_1 b_1}{\sum_{j=0}^{2n+1} \tilde{N}_j b_j} \frac{a_1}{b_1} + \cdots + \frac{\tilde{N}_{2n+1} b_{2n+1}}{\sum_{j=0}^{2n+1} \tilde{N}_j b_j} \frac{a_{2n+1}}{b_{2n+1}} \geq \\ &\geq \frac{a_{2n+1}}{b_{2n+1}} - \varepsilon_{2n+1} = B - 2\varepsilon_{2n+1}, \end{aligned}$$

where the last equality comes from (4.50) and (4.51).

Consequently, since  $\varepsilon_{2n+1} \xrightarrow{n \rightarrow +\infty} 0$ , we have

$$\liminf_{n \rightarrow +\infty} \frac{\sum_{i=1}^{\sum_{j=0}^{2n+1} \tilde{N}_{2n+1} b_{2n+1}} \delta_i}{\sum_{j=0}^{2n+1} \tilde{N}_j b_j} \geq B$$

and so from (4.54)

$$\lim_{n \rightarrow +\infty} \frac{\sum_{i=1}^{\sum_{j=0}^{2n+1} \tilde{N}_{2n+1} b_{2n+1}} \delta_i}{\sum_{j=0}^{2n+1} \tilde{N}_j b_j} = B.$$

We conclude that

$$\limsup_{m \rightarrow +\infty} \frac{\sum_{i=1}^m \delta_i}{m} = B.$$

Adapting the previous argument, for  $m \in \mathbb{N}$  large enough it holds

$$\begin{aligned} \frac{\sum_{i=1}^m \delta_i}{m} &= \left(1 - \frac{j\rho_n + k}{m}\right) \frac{\sum_{i=1}^{m-j\rho_n-k} \delta_i}{m-j\rho_n-k} + \frac{\sum_{i=m-j\rho_n-k+1}^m \delta_i}{m} \geq \left(1 - \frac{j\rho_n + k}{m}\right) A \geq \\ &\geq \left(1 - \frac{2}{n-1}\right) A > A - \delta, \end{aligned}$$

since  $A \in [0, 1]$  and, from (4.52) and from (4.53),

$$\frac{j\rho_n + k}{m} \leq \frac{b_n}{\tilde{N}_{n-1} b_{n-1}} \leq \frac{2}{n-1} < \delta.$$

That is, by the arbitrariness of  $\delta$ ,

$$\liminf_{m \rightarrow +\infty} \frac{\sum_{i=1}^m \delta_i}{m} \geq A. \quad (4.55)$$

Consider now the subsequence  $(\sum_{j=0}^{2n} \tilde{N}_j b_j)_{n \in \mathbb{N}}$ : for any  $n \in \mathbb{N}$  it holds, from the construction of  $(\delta_i)_{i \in \mathbb{Z}}$  and the choice of  $\tilde{N}_n$ ,

$$\frac{\sum_{i=1}^{\sum_{j=0}^{2n} \tilde{N}_j b_j} \delta_i}{\sum_{j=0}^{2n} \tilde{N}_j b_j} = \frac{b_0}{\sum_{j=0}^{2n} \tilde{N}_j b_j} \frac{a_0}{b_0} + \cdots + \frac{\tilde{N}_{2n} b_{2n}}{\sum_{j=0}^{2n} \tilde{N}_j b_j} \frac{a_{2n}}{b_{2n}} \leq \frac{a_{2n}}{b_{2n}} + \varepsilon_{2n} = A + 2\varepsilon_{2n}.$$

Consequently, since  $\varepsilon_{2n} \xrightarrow{n \rightarrow +\infty} 0$ , we have

$$\limsup_{n \rightarrow +\infty} \frac{\sum_{i=1}^{\sum_{j=0}^{2n} \tilde{N}_j b_j} \delta_i}{\sum_{j=0}^{2n} \tilde{N}_j b_j} \leq A$$

and so from (4.55)

$$\lim_{n \rightarrow +\infty} \frac{\sum_{i=1}^{\sum_{j=0}^{2n} \tilde{N}_j b_j} \delta_i}{\sum_{j=0}^{2n} \tilde{N}_j b_j} = A.$$

We deduce that

$$\liminf_{m \rightarrow +\infty} \frac{\sum_{i=1}^m \delta_i}{m} = A.$$

Let now  $A = B$ . Assume that  $A > 0$ : if this is not the case, then the sequence  $(\delta_i)_{i \in \mathbb{Z}}$  whose entries are all zeroes is a sequence such that  $\lim_{m \rightarrow +\infty} \frac{\sum_{i=1}^m \delta_i}{m} = 0$ . Let  $(\frac{p_n}{q_n})_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{Q}^{\mathbb{N}}$  converging increasingly to  $A$ , that is

$$\lim_{n \rightarrow +\infty} \frac{p_n}{q_n} = A \quad \text{and} \quad \frac{p_{n-1}}{q_{n-1}} \leq \frac{p_n}{q_n} \leq A \quad \forall n \in \mathbb{N}^*.$$

Define the sequence  $(\varepsilon_n)_{n \in \mathbb{N}}$  so that

$$\varepsilon_n := A - \frac{p_n}{q_n}. \quad (4.56)$$

Define the sequence  $(N_n, \tilde{N}_n)_{n \in \mathbb{N}}$  recursively as follows. Let  $(N_0, \tilde{N}_0)$  be  $(1, 1)$ . Let  $n > 1$  and assume we have already defined the couples  $(N_i, \tilde{N}_i)$  for  $i \in \llbracket 1, n-1 \rrbracket$ . Let  $N_n \in \mathbb{N}$  be such that for any  $N \geq N_n$  it holds

$$\begin{aligned} & \frac{q_0}{\sum_{j=0}^{n-1} \tilde{N}_j q_j + N_n q_n} \frac{p_0}{q_0} + \frac{\tilde{N}_1 q_1}{\sum_{j=0}^{n-1} \tilde{N}_j q_j + N_n q_n} \frac{p_1}{q_1} + \dots + \frac{N_n q_n}{\sum_{j=0}^{n-1} \tilde{N}_j q_j + N_n q_n} \frac{p_n}{q_n} \\ & \in \left[ \frac{p_n}{q_n} - \varepsilon_n, \frac{p_n}{q_n} + \varepsilon_n \right] = [A - 2\varepsilon_n, A + 2\varepsilon_n], \end{aligned} \quad (4.57)$$

where the last equality comes from the definition of  $\varepsilon_n$  in (4.56). Define  $\tilde{N}_n$  as  $\max\{nq_{n+1}, N_n\}$ . Construct the sequence  $(\delta_i)_{i \in \mathbb{Z}}$  as follows. For any  $i \in \mathbb{Z}, i \leq 0$ , choose no matter which value of  $\delta_i$  in  $\{0, 1\}$ . Starting from the 1-th entry and refering to Definition 4.6.1, concatenate: an adapted sequence for  $\frac{p_0}{q_0}$ ,  $\tilde{N}_1$  adapted sequences for  $\frac{p_1}{q_1}, \dots, \tilde{N}_n$  adapted sequences for  $\frac{p_n}{q_n}$  and so on. We immediately remark that the sequence  $\left( \frac{\sum_{i=1}^{\sum_{j=0}^n \tilde{N}_j q_j} \delta_i}{\sum_{j=0}^n \tilde{N}_j q_j} \right)_{n \in \mathbb{N}}$  converges to  $A$ .

For any  $n \in \mathbb{N}$  we denote as  $\rho_n \in \llbracket 1, q_n \rrbracket$  the integer such that

$$q_n = p_n \rho_n + \tau_n,$$

where  $\tau_n \in \llbracket 0, p_n - 1 \rrbracket$ .

Fix now  $0 < \delta < 2A$ . Let  $m \in \mathbb{N}$ . There exists  $n \in \mathbb{N}^*$  so that

$$m = q_0 + \tilde{N}_1 q_1 + \dots + \tilde{N}_{n-1} q_{n-1} + l q_n + j \rho_n + k,$$

where  $l \in \llbracket 0, \tilde{N}_n - 1 \rrbracket, j \in \llbracket 0, p_n \rrbracket$  and  $k \in \llbracket 0, \rho_n - 1 \rrbracket$ . If  $m \in \mathbb{N}^*$  (and so  $n \in \mathbb{N}^*$ ) is large enough, then

$$\frac{2}{n-1} < \frac{\delta}{2} < A \quad \text{and} \quad \varepsilon_{n-1} < \frac{\delta}{4}. \quad (4.58)$$

Observe that

$$\frac{\sum_{i=1}^m \delta_i}{m} = \left(1 - \frac{j\rho_n + k}{m}\right) \frac{\sum_{i=1}^{m-j\rho_n-k} \delta_i}{m-j\rho-k} + \frac{\sum_{i=m-j\rho_n-k+1}^m \delta_i}{m}.$$

First, we remark that

$$\frac{\sum_{i=1}^{m-j\rho_n-k} \delta_i}{m-j\rho-k} \in [\min\{A - 2\varepsilon_{n-1}, A - \varepsilon_n\}, A + 2\varepsilon_{n-1}] = [A - 2\varepsilon_{n-1}, A + 2\varepsilon_{n-1}]$$

since  $\varepsilon_n \leq \varepsilon_{n-1}$  and since  $\frac{\sum_{i=1}^{m-j\rho_n-k} \delta_i}{m-j\rho-k}$  is a convex combination of

$$\frac{\sum_{i=1}^{\sum_{j=0}^{n-1} \tilde{N}_j q_j} \delta_i}{\sum_{j=0}^{n-1} \tilde{N}_j q_j} \quad \text{and} \quad \frac{p_n}{q_n},$$

where  $\frac{p_n}{q_n} = A - \varepsilon_n$  from (4.56) and

$$\frac{\sum_{i=1}^{\sum_{j=0}^{n-1} \tilde{N}_j q_j} \delta_i}{\sum_{j=0}^{n-1} \tilde{N}_j q_j} \in [A - 2\varepsilon_{n-1}, A + 2\varepsilon_{n-1}]$$

thanks to the choice of  $\tilde{N}_{n-1}$  (see (4.57)).

Moreover, we have that

$$0 \leq \frac{\sum_{i=m-j\rho_n-k+1}^m \delta_i}{m} \leq \frac{j+1}{m} \leq \frac{q_n+1}{\tilde{N}_{n-1}q_{n-1}} \leq \frac{2}{(n-1)} < \frac{\delta}{2}.$$

Therefore it holds (from the second inequality in (4.58))

$$\frac{\sum_{i=1}^m \delta_i}{m} \leq A + 2\varepsilon_{n-1} + \frac{\delta}{2} < A + \delta. \quad (4.59)$$

Concerning the lower bound, we have that

$$\frac{\sum_{i=1}^m \delta_i}{m} \geq \left(1 - \frac{j\rho_n + k}{m}\right) \frac{\sum_{i=1}^{m-j\rho_n-k} \delta_i}{m-j\rho_n-k} \geq \left(1 - \frac{j\rho_n + k}{m}\right) (A - 2\varepsilon_{n-1}).$$

Since  $m > \tilde{N}_{n-1}q_{n-1} \geq (n-1)q_nq_{n-1}$  and from (4.58) it holds  $\frac{2}{n-1} < \frac{\delta}{2}$  and  $\varepsilon_{n-1} < \frac{\delta}{4}$ , we deduce that

$$\left(1 - \frac{j\rho_n + k}{m}\right) (A - 2\varepsilon_{n-1}) > \left(1 - \frac{\delta}{2}\right) \left(A - \frac{\delta}{2}\right) = A - \frac{\delta}{2}A - \frac{\delta}{2} + \frac{\delta^2}{4}.$$

Recalling that  $A \leq 1$  we have that  $\frac{A}{2} + \frac{1}{2} - \frac{\delta}{4} < 1$  and we conclude that

$$\frac{\sum_{i=1}^m \delta_i}{m} > A - \delta. \quad (4.60)$$

By (4.59) and (4.60) and by the arbitrariness of  $\delta > 0$  we conclude that

$$\lim_{m \rightarrow +\infty} \frac{\sum_{i=1}^m \delta_i}{m} = A.$$

□

Immediate consequences of Corollary 4.6.1 are the following results. Recall that  $m \in \mathbb{Z}$  is the unstable angle variation of  $(q, p)$  (see Definition 4.4.1).

**Corollary 4.6.2.** For any  $\alpha \in \left[ \frac{k}{N}, \frac{k}{N} + \frac{m}{2(2n_u+j)N} \right]$  there exists a point  $x \in H(U_\varepsilon, j)$  so that  $\text{Torsion}(f, x) = \alpha$ .

**Corollary 4.6.3.** There exist points in  $H(U_\varepsilon, j)$  where the torsion does not exist.

Actually we can say something more concerning the set of points with prescribed torsion value using the symbolic dynamics of  $\{0, 1\}^{\mathbb{Z}}$ .

**Claim 4.6.1.** For any  $\alpha \leq \beta, \alpha, \beta \in \left[ \frac{k}{N}, \frac{k}{N} + \frac{m}{2(2n_u+j)N} \right]$  the set of points  $x \in H(U_\varepsilon, j)$  such that

$$\liminf_{n \rightarrow +\infty} \text{Torsion}_n(f, x, v) = \alpha \leq \beta = \limsup_{n \rightarrow +\infty} \text{Torsion}_n(f, x, v)$$

is dense in  $H(U_\varepsilon, j)$ .

*Proof.* Let  $\alpha \leq \beta, \alpha, \beta \in \left[ \frac{k}{N}, \frac{k}{N} + \frac{m}{2(2n_u+j)N} \right]$ . From Corollary 4.6.1 there exists  $\bar{x} \in H(U_\varepsilon, j)$  such that

$$\liminf_{n \rightarrow +\infty} \text{Torsion}_n(f, \bar{x}, v) = \alpha \quad \text{and} \quad \limsup_{n \rightarrow +\infty} \text{Torsion}_n(f, \bar{x}, v) = \beta.$$

Fix  $x \in H(U_\varepsilon, j)$  and  $\eta > 0$ . Denote  $(\delta_i)_{i \in \mathbb{Z}} = h(x)$  and  $(\bar{\delta}_i)_{i \in \mathbb{Z}} = h(\bar{x})$ . Since  $h$  is a homeomorphism, there exists  $\zeta > 0$  such that if  $\bar{d}(\delta_i)_{i \in \mathbb{Z}}, (\bar{\delta}_i)_{i \in \mathbb{Z}} < \zeta$  then  $d(x, h^{-1}((\bar{\delta}_i)_{i \in \mathbb{Z}})) < \eta$ . Let  $\bar{n} \in \mathbb{N}$  be such that  $2 \sum_{i=\bar{n}}^{+\infty} \frac{1}{2^i} < \zeta$ . Define the sequence  $(\tilde{\delta}_i)_{i \in \mathbb{Z}}$  so that

- for  $i \in \llbracket -\bar{n} + 1, \bar{n} - 1 \rrbracket$  it holds  $\tilde{\delta}_i = \delta_i$ ;
- for  $|i| \geq \bar{n}$  it holds  $\tilde{\delta}_i = \bar{\delta}_i$ .

Therefore we have that  $d(x, h^{-1}((\tilde{\delta}_i)_{i \in \mathbb{Z}})) < \eta$  and

$$\liminf_{n \rightarrow +\infty} \text{Torsion}_n(f, h^{-1}((\tilde{\delta}_i)_{i \in \mathbb{Z}}), v) = \liminf_{n \rightarrow +\infty} \text{Torsion}_n(f, \bar{x}, w) = \alpha,$$

$$\limsup_{n \rightarrow +\infty} \text{Torsion}_n(f, h^{-1}((\tilde{\delta}_i)_{i \in \mathbb{Z}}), v) = \limsup_{n \rightarrow +\infty} \text{Torsion}_n(f, \bar{x}, w) = \beta,$$

where  $v$  denotes no matter which tangent vector in  $T_{h^{-1}((\tilde{\delta}_i)_{i \in \mathbb{Z}})}S$  and  $w$  no matter which vector in  $T_{\bar{x}}^1S$ .

By the arbitrariness of  $\zeta > 0$  and of  $x \in H(U_\varepsilon, j)$ , the set of points whose inferior limit of torsion is  $\alpha$  and whose superior limit of torsion is  $\beta$  is dense in  $H(U_\varepsilon, j)$ . □

We can actually say more concerning the level sets of the torsion function. We recall the definition of Cantor set.

**Definition 4.6.2.** A Cantor set  $C$  is a non empty, compact, totally disconnected set with no isolated points.

**Claim 4.6.2.** Let  $\alpha \in \left[ \frac{k}{N}, \frac{k}{N} + \frac{m}{2(2n_u+j)N} \right] \setminus \mathbb{Q}$ . There exist uncountably many disjoint invariant Cantor sets  $C_\alpha$  so that  $\text{Torsion}(f, x) = \alpha$  for every  $x \in C_\alpha$ .

Claim 4.6.2 is an outcome of Proposition 4.4.1 and of the following result over symbolic dynamics due to K. Hockett and P. Holmes (see Section 3.3, Lemmas C, D and E in [HH86]) applied with respect to  $\rho = \left( \alpha - \frac{k}{N} \right) \frac{2(2n_u+j)N}{m} \in [0, 1] \setminus \mathbb{Q}$ .

**Lemma 4.6.1.** *Let  $\rho \in [0, 1] \setminus \mathbb{Q}$ . There exist uncountable many disjoint  $S$ -invariant Cantor sets  $\Omega_\rho$  in  $\{0, 1\}^{\mathbb{Z}}$  such that for any  $(s_i)_{i \in \mathbb{Z}} \in \Omega_\rho$  it holds*

$$\lim_{n \rightarrow +\infty} \frac{\sum_{i=1}^n s_i}{n} = \rho.$$

Thus, since the coding map  $h$  is a homeomorphism, the image through  $h^{-1}$  of the set  $\Omega_\rho$  (with  $\rho = (\alpha - \frac{k}{N}) \frac{2(2n_u+j)N}{m}$ ) is the union of uncountably many disjoint invariant Cantor sets and it is contained in the  $\alpha$ -level set of the torsion function.

In [HH86], Hockett and Holmes work in the framework of some homoclinic transverse intersection and use this result to show the existence of uncountably many disjoint Cantor sets whose points have prescribed irrational rotation number.

**Claim 4.6.3.** The set of points in  $H(U_\varepsilon, j)$  where the torsion does not exist contains a dense  $\mathcal{G}_\delta$ -set, i.e. it is residual.

*Proof.* The set of points of  $H(U_\varepsilon, j)$  at which the torsion does not exist can be described as the set

$$\mathcal{N}\mathcal{T} := \bigcup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \bigcap_{N \in \mathbb{N}} \bigcup_{i > j \geq N} \left\{ x \in H(U_\varepsilon, j) : |\text{Torsion}_i(f, x, e_x^u) - \text{Torsion}_j(f, x, e_x^u)| > \frac{1}{n} \right\},$$

where  $e_x^u \in T_x^1 S$  belongs to the unstable subspace of  $x$ .

Fix  $n \in \mathbb{N}^*, N \in \mathbb{N}$ . For any fixed  $i > j \geq N$  the set

$$\left\{ x \in H(U_\varepsilon, j) : |\text{Torsion}_i(f, x, e_x^u) - \text{Torsion}_j(f, x, e_x^u)| > \frac{1}{n} \right\}$$

is open. Then, the set

$$\bigcup_{i > j \geq N} \left\{ x \in H(U_\varepsilon, j) : |\text{Torsion}_i(f, x, e_x^u) - \text{Torsion}_j(f, x, e_x^u)| > \frac{1}{n} \right\},$$

i.e. the set of points admitting subsequences converging to different limit values, is an open set (since it is a countable union of open sets) containing a dense set (see Claim 4.6.1). So, in particular, it is dense. By Baire's theory (since  $H(U_\varepsilon, j)$  is a Baire space), the set  $\mathcal{N}\mathcal{T}$  is a union of  $\mathcal{G}_\delta$ -dense sets. □

**Claim 4.6.4.** There exists a point  $x \in H(U_\varepsilon, j)$  at which the torsion does not exist and which has dense orbit in  $H(U_\varepsilon, j)$ .

*Proof.* Claim 4.6.4 is an outcome of Claim 4.6.3. Indeed, Claim 4.6.3 says that the set of points at which the torsion does not exist is residual.

The horseshoe is a transitive dynamical system, i.e. there exists a point whose orbit is dense in  $H(U_\varepsilon, j)$ . By Birkhoff Transitive Theorem (see Theorem 2.1 in Chapter 8 in [Rob99]) the set of points whose orbit is dense in the horseshoe is a  $G_\delta$ -dense set.

Consequently, the intersection of these two sets is dense, so in particular not empty. That is, there exist points whose orbit is dense in  $H(U_\varepsilon, j)$  and at which the torsion does not exist. □



**Remark 4.6.3.** We can also make explicit the point of Claim 4.6.4. Indeed, in Corollary 4.6.3 we have remarked that there exist points in  $H(U_\varepsilon, j)$  at which the torsion does not exist. Let  $x \in H(U_\varepsilon, j)$  be one of those points. Denote  $(\delta_i)_{i \in \mathbb{Z}} = h(x)$  and build a sequence  $(\bar{\delta}_i)_{i \in \mathbb{Z}}$  so that

- for  $i \geq 0$  it holds  $\bar{\delta}_i = \delta_i$ ;
- for  $i < 0$  the sequence is completed by successively concatenating all blocks of 0-symbols and 1-symbols of all the possible lengths, i.e.

$$\left( \dots \mid \underbrace{11 \ 10 \ 01 \ 00}_{\text{2-lengthed blocks}} \mid \underbrace{1 \ 0}_{\text{1-lengthed blocks}} \mid (\delta_i)_{i \in \mathbb{N}} \right).$$

At the point  $\bar{x} := h^{-1}((\bar{\delta}_i)_{i \in \mathbb{Z}})$  the torsion does not exist. At the same time, we remark that the  $S^{-1}$ -orbit of  $(\bar{\delta}_i)_{i \in \mathbb{Z}}$  (i.e. the orbit of the shift to the right) is dense in  $(\{0, 1\}^{\mathbb{Z}}, S)$  and consequently the  $f^{(2n_u+j)N}$ -orbit of  $\bar{x}$  is dense in  $H(U_\varepsilon, j)$ .

#### 4.6.1 Consequences for torsion of invariant measures of the horseshoe

In this Subsection we are interested in torsion values of  $f$ -invariant measures whose support is contained in the horseshoe. We start by recalling the definition of the torsion of a  $f$ -invariant measure.

**Definition 4.6.3.** Let  $f : S \rightarrow S$  be a  $\mathcal{C}^1$  diffeomorphism isotopic to the identity. Let  $I = (f_t)_t \in \text{Diff}^1(S)$  be an isotopy joining the identity to  $f$  and let  $\mu$  be a  $f$ -invariant Borel probability measure on  $S$ . Assume that  $\mu$  or  $I$  has compact support. Then the torsion of the measure  $\mu$  is

$$\text{Torsion}(I, \mu) = \int_S \text{Torsion}(I, x) d\mu(x).$$

The main result of this section is the following

**Theorem 4.6.1.** *Let  $S$  be either  $\mathbb{R}^2$  or  $\mathbb{A}$  or  $\mathbb{T}^2$ . Let  $f : S \rightarrow S$  be a  $\mathcal{C}^1$  diffeomorphism isotopic to the identity<sup>19</sup> and let  $q \in S$  be a hyperbolic periodic point for  $f$  of period  $N$ . Let  $p \in (W^s(q) \cap W^u(q)) \setminus \{q\}$ . Denote as*

$$k = N \text{Torsion}_N(f, q, v) \in \mathbb{Z}$$

*for  $v \in E_q^u$ . Let  $0 < \varepsilon < \frac{1}{12}$ . Let  $U_\varepsilon$  be an adapted neighborhood of  $\{q\} \cup \mathcal{O}(p, f^N)$  with respect to  $\varepsilon$ . Let  $H(U_\varepsilon, j)$  be the horseshoe of Definition 4.3.2 and let  $m \in \mathbb{Z}$  be the unstable angle variation of  $(q, p)$  (see Definition 4.4.1).*

*For any  $\alpha \in \left[ \frac{k}{N}, \frac{k}{N} + \frac{m}{2(2n_u+j)N} \right]$  there exists an ergodic  $f$ -invariant measure  $\Gamma$  whose support is contained in  $\mathcal{O}(H(U_\varepsilon, j), f)$  and such that*

$$\text{Torsion}(f, \Gamma) = \int_S \text{Torsion}(f, x) d\Gamma(x) = \alpha.$$

---

19. If  $S = \mathbb{R}^2$  then assume also that  $f$  has compact support.

**Reminder 4.6.1.** The torsion does depend on the chosen isotopy neither on  $\mathbb{A}$  nor on  $\mathbb{T}^2$ . It is independent of the isotopy also on  $\mathbb{R}^2$  up to consider diffeomorphisms with compact support. For this reason we denote the torsion of a measure  $\mu$  for  $f$  as  $\text{Torsion}(f, \mu)$ .

In order to prove Theorem 4.6.1 we are going to use Arnaud's result in [Arn18] to insert Denjoy dynamics within the horseshoe and obtain  $f$ -invariant ergodic measures with prescribed torsion value.

### Coding of Denjoy counterexample

We are going to provide a coding of a Denjoy's counterexample  $F$  with rotation number  $\rho \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1)$  with one wandering interval  $I \subset \mathbb{T}$ . We refer to [Her79] for a detailed construction of such a Denjoy's counterexample.

**Notation 4.6.1.** Let  $F : \mathbb{T} \rightarrow \mathbb{T}$  be the Denjoy's counterexample with one wandering interval  $I$  with rotation number  $\rho \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1)$ . Denote the non wandering set  $K$ . Then,  $F$  is semiconjugated to the irrational rotation  $R_\rho$ . Denote as  $g : \mathbb{T} \rightarrow \mathbb{T}$  the semiconjugation such that  $g(I) = 0$  and

$$g \circ F = R_\rho \circ g.$$

The next Proposition is due to M.-C. Arnaud (see [Arn18]).

**Proposition 4.6.2.** *Let  $\rho \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1)$  and let  $F : \mathbb{T} \rightarrow \mathbb{T}$  be the Denjoy's counterexample of rotation number  $\rho$  with one wandering interval  $I$ . Let  $K$  be the non-wandering set of  $(\mathbb{T}, F)$ . There exists a continuous injective map*

$$H : K \rightarrow H(K) \subset \{0, 1\}^{\mathbb{Z}}$$

such that

$$H \circ F|_K = S|_{H(K)} \circ H,$$

where  $S$  is the shift (to the left) map on  $\{0, 1\}^{\mathbb{Z}}$ .

*Proof.* Consider the wandering interval  $I$  and its image  $F(I)$ , which are connected components of the complementary set of  $K$ . Denote as  $\Delta_0, \Delta_1$  the closed intervals between (running clockwise along the circle)  $I$  and  $F(I)$  and between  $F(I)$  and  $I$ , respectively (see Figure 4.14).

Define

$$\begin{aligned} H : K = \mathbb{T} \setminus \bigcup_{n \in \mathbb{Z}} \text{int } F^n(I) &\longrightarrow \{0, 1\}^{\mathbb{Z}} \\ x &\longmapsto H(x) = (s_i)_{i \in \mathbb{Z}} \end{aligned} \tag{4.61}$$

where

$$F^i(x) \in \Delta_{s_i}.$$

The function  $H$  is well-defined since  $K$  is  $F$ -invariant. The function  $H$  is continuous because  $\mathbb{T} \setminus (\text{int } I \cup \text{int } F(I))$  is open in  $K$  and  $F$  is continuous.

In order to show the injectivity of  $H$  we distinguish two cases.

- (a) Let  $x_1, x_2 \in K, x_1 \neq x_2$  be points on the boundary of the same  $F^l(I)$ , for some  $l \in \mathbb{Z}$ . Then, either  $F^{-l}(x_1) \in \Delta_0$  and  $F^{-l}(x_2) \in \Delta_1$  or  $F^{-l}(x_1) \in \Delta_1$  and  $F^{-l}(x_2) \in \Delta_0$ . In particular  $(H(x_1))_{-l} \neq (H(x_2))_{-l}$ , that is  $H(x_1) \neq H(x_2)$ .

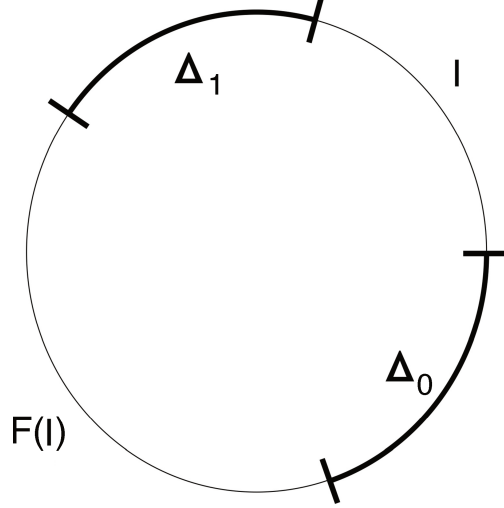


Figure 4.14 – Coding of Denjoy counterexample.

- (b) Let  $x_1, x_2 \in K, x_1 \neq x_2$  be points not on the boundary of the same connected component of  $\mathbb{T} \setminus K$ . Consequently it holds  $g(x_1) \neq g(x_2)$  where  $g$  is the semiconjugation of  $F$  with  $R_\rho$  (see Notation 4.6.1).

Consider a lift of  $(\mathbb{T}, R_\rho)$  and let  $Y, Y + \rho \in \mathbb{R}$  be lifts of  $g(I), g(F(I))$  respectively. Assume that  $\rho \leq 1 - \rho$ : if this is not the case, in the sequel consider  $Y + \rho$  instead of  $Y$  and  $Y + 1$  instead of  $Y + \rho$ .

Let  $G(x_1), G(x_2) \in \mathbb{R}$  be lifts of  $g(x_1), g(x_2)$  respectively and assume without loss of generality (up to invert the roles of  $x_1, x_2$ ) that  $0 < G(x_2) - G(x_1) = \delta < 1$ . Suppose also that  $\delta \leq \frac{1}{2}$ : if this is not the case, then consider as  $\delta$  the quantity  $G(x_1) + 1 - G(x_2)$  and in the sequel replace  $G(x_1)$  with  $G(x_2)$  and replace  $G(x_2)$  with  $G(x_1) + 1$ .

Fix  $0 < \varepsilon < \min(\delta, \rho, 1 - \rho)$ . Since the system  $(\mathbb{T}, R_\rho)$  is minimal, there exist  $n \in \mathbb{Z}, p \in \mathbb{Z}$  such that

$$Y + \rho + p - \varepsilon < G(x_1) + n\rho < Y + \rho + p. \quad (4.62)$$

So, from the choice of  $\varepsilon$  (since  $\rho - \varepsilon > 0$ ) it holds

$$Y + p < G(x_1) + n\rho < Y + \rho + p. \quad (4.63)$$

Since  $G(x_2) + n\rho = G(x_1) + n\rho + \delta$  (because  $R_\rho$  preserves the orientation and the distance), we have from (4.62)

$$Y + \rho + p + \delta - \varepsilon < G(x_2) + n\rho < Y + \rho + p + \delta,$$

that is, from the choice of  $\varepsilon$  (since  $\delta - \varepsilon > 0$ ) and from our assumptions (since  $\rho \leq \frac{1}{2}, \delta \leq \frac{1}{2}$ ),

$$Y + \rho + p < G(x_2) + n\rho < Y + p + 1. \quad (4.64)$$

From (4.63) and (4.64), considering on  $\mathbb{T}$  the points  $R_\rho^n(g(x_1))$  and  $R_\rho^n(g(x_2))$  and since  $g \circ F^n = R_\rho^n \circ g$ , the images  $F^n(x_1)$  and  $F^n(x_2)$  belong to different intervals  $\Delta_0, \Delta_1$ . Equivalently, it holds  $(H(x_1))_n \neq (H(x_2))_n$ , i.e.  $H(x_1) \neq H(x_2)$ .

Since  $H(K)$  is compact, we conclude that  $H : K \rightarrow H(K) \subset \{0, 1\}^\mathbb{Z}$  is a homeomorphism to its image and we have that  $H \circ F = S \circ H$ .

□

**Remark 4.6.4.** The image  $H(K)$  is strictly contained in  $\{0, 1\}^{\mathbb{Z}}$  since the Denjoy counterexample  $(\mathbb{T}, F)$  has zero topological entropy (see Example 2 in [AKM65]), while the system  $(\{0, 1\}^{\mathbb{Z}}, S)$  has positive entropy.

**Remark 4.6.5.** The Denjoy's counterexample is uniquely ergodic (see Theorem 11.2.9 in [KH95]). Denote as  $\mu$  the unique  $F$ -invariant measure of the system. The support of  $\mu$  is the non wandering set  $K$ .

**Claim 4.6.5.**  $\mu(\Delta_1) = 1 - \rho$ .

*Proof.* The irrational rotation  $R_\rho$  is uniquely ergodic and its unique invariant measure is the Lebesgue measure  $Leb$  on  $\mathbb{T}$ . The rotation  $R_\rho$  is semiconjugated to the Denjoy's counterexample  $(\mathbb{T}, F)$  through  $g$ . Since  $g_*\mu$  is  $R_\rho$ -invariant, we deduce that  $g_*\mu = Leb$ . Denote as  $cl(F(I)), cl(I)$  the closure of the intervals  $F(I), I$  respectively. Recall that  $g(I) = 0$  and  $g(F(I)) = \rho$  and observe now that

$$(\Delta_1 \cup cl(F(I)) \cup cl(I)) \cap K = g^{-1}([\rho, 1]) \cap K.$$

Consequently, since  $\mu(\Delta_1) = \mu((\Delta_1 \cup cl(F(I)) \cup cl(I)) \cap K)$  because  $\mu$  is absolutely continuous (see Proposition 12.4.1 in [KH95]) and it is supported on  $K$ , it holds

$$\mu(\Delta_1) = \mu((\Delta_1 \cup cl(F(I)) \cup cl(I)) \cap K) = \mu(g^{-1}([\rho, 1])) = g_*\mu([\rho, 1]) = Leb([\rho, 1]) = 1 - \rho.$$

□

**Notation 4.6.2.** Denote the function

$$\begin{aligned} Pr : \{0, 1\}^{\mathbb{Z}} &\rightarrow \{0, 1\} \\ (s_i)_{i \in \mathbb{Z}} &\mapsto s_0. \end{aligned}$$

For any irrational  $\mathcal{A} \in [0, 1]$  there exists an ergodic measure  $\nu$  on  $\{0, 1\}^{\mathbb{Z}}$  invariant for the shift such that the integral of the function  $Pr$  with respect to  $\nu$  over  $\{0, 1\}^{\mathbb{Z}}$  is the fixed value  $\mathcal{A}$ . The key idea is using the unique ergodic measure on the Denjoy's counterexample of rotation number  $1 - \mathcal{A}$  and the coding of such system presented in Proposition 4.6.2.

**Lemma 4.6.2.** *Let  $\mathcal{A} \in [0, 1] \setminus \mathbb{Q}$ . There exists an ergodic  $S$ -invariant measure  $\nu$  whose support is strictly contained in  $\{0, 1\}^{\mathbb{Z}}$  and such that*

$$\int_{\{0, 1\}^{\mathbb{Z}}} Pr((s_i)_{i \in \mathbb{Z}}) d\nu((s_i)_{i \in \mathbb{Z}}) = \mathcal{A}.$$

*Proof.* Consider the Denjoy's counterexample with one wandering interval  $(\mathbb{T}, F)$  with irrational rotation number  $\rho = 1 - \mathcal{A}$ . Let  $\mu$  be the unique  $F$ -invariant measure in  $\mathbb{T}$  (see Remark 4.6.5). Its support is  $K$ .

Let  $H : K \rightarrow \{0, 1\}^{\mathbb{Z}}$  be the coding homeomorphism (to its image) of such Denjoy's counterexample given by Proposition 4.6.2. Let  $\nu := H_*\mu$ . The support of the measure  $\nu$  is  $H(K)$  and it is  $S$ -invariant. Since  $\mu$  is ergodic, then also  $\nu$  is ergodic. Let us calculate now  $\int_{\{0, 1\}^{\mathbb{Z}}} Pr((s_i)_{i \in \mathbb{Z}}) d\nu((s_i)_{i \in \mathbb{Z}})$ . It holds

$$\int_{\{0, 1\}^{\mathbb{Z}}} Pr((s_i)_{i \in \mathbb{Z}}) d\nu((s_i)_{i \in \mathbb{Z}}) = \int_{H(K)} Pr((s_i)_{i \in \mathbb{Z}}) dH_*\mu((s_i)_{i \in \mathbb{Z}}) = \int_K \mathbb{I}_{\Delta_1}(x) d\mu(x) = \mu(\Delta_1).$$

From Claim 4.6.5, we have  $\mu(\Delta_1) = 1 - \rho = \mathcal{A}$  and so we conclude that

$$\int_{\{0, 1\}^{\mathbb{Z}}} Pr((s_i)_{i \in \mathbb{Z}}) d\nu((s_i)_{i \in \mathbb{Z}}) = \mu(\Delta_1) = \mathcal{A}.$$

□

**Notation 4.6.3.** By the coding of the Denjoy counterexample, it is possible to insert such a dynamics within the horseshoe  $H(U_\varepsilon, j)$  through the topological conjugacies  $h, H$ . Denote as

$$\zeta : K \rightarrow \zeta(K) \subset H(U_\varepsilon, j)$$

the map  $h_{|H(K)}^{-1} \circ H$ .

### Measures with prescribed torsion value

The main result of the section is Theorem 4.6.1. In order to prove it we need the following Lemma. Recall that  $m \in \mathbb{Z}$  is the unstable angle variation of  $(q, p)$  (see Definition 4.4.1).

**Lemma 4.6.3.** *In the hypothesis of Theorem 4.6.1 for any  $A \in [k(2n_u + j), k(2n_u + j) + \frac{m}{2}]$  there exists an ergodic  $f^{(2n_u + j)N}$ -invariant measure  $\lambda$  whose support is contained in  $H(U_\varepsilon, j)$  and such that  $\text{Torsion}(f^{(2n_u + j)N}, \lambda) = A$ .*

We postpone the proof of Lemma 4.6.3 and we now show how Theorem 4.6.1 follows from it.

*Proof of Theorem 4.6.1.* Fix  $\alpha \in [\frac{k}{N}, \frac{k}{N} + \frac{m}{2(2n_u + j)N}]$  and denote  $A = (2n_u + j)N\alpha$ . By Lemma 4.6.3 there exists an ergodic  $f^{(2n_u + j)N}$ -invariant measure  $\lambda$  such that

$$\text{Torsion}(f^{(2n_u + j)N}, \lambda) = A.$$

Define then

$$\Gamma = \frac{1}{(2n_u + j)N} \sum_{i=0}^{(2n_u + j)N-1} f_*^i \lambda,$$

where  $f_*^i \lambda$  is the push-forward of the measure with respect to  $f^i$ .

Observe that the measure  $\Gamma$  is  $f$ -invariant, indeed

$$f_* \Gamma = \frac{1}{(2n_u + j)N} \left( \sum_{i=1}^{(2n_u + j)N-1} f_*^i \lambda + f_*^{(2n_u + j)N} \lambda \right) = \frac{1}{(2n_u + j)N} \sum_{i=0}^{(2n_u + j)N} f_*^i \lambda = \Gamma,$$

since  $\lambda$  is  $f^{(2n_u + j)N}$ -invariant.

Let us show that the measure  $\Gamma$  is ergodic with respect to  $f$ . Let  $B$  be a  $f$ -invariant measurable set. Then,  $B$  is also  $f^{(2n_u + j)N}$ -invariant. Since  $\lambda$  is an ergodic measure with respect to  $f^{(2n_u + j)N}$ , we have that either  $\lambda(B) = 0$  or  $\lambda(B) = 1$ . Consider now

$$\Gamma(B) = \frac{1}{(2n_u + j)N} \sum_{i=0}^{(2n_u + j)N-1} f_*^i \lambda(B).$$

It follows that  $\Gamma(B)$  is either 0 or 1, that is  $\Gamma$  is ergodic with respect to  $f$ .

Finally, we look at the torsion of  $\Gamma$  with respect to  $f$ . It holds

$$\begin{aligned} \text{Torsion}(f, \Gamma) &= \int_S \text{Torsion}(f, x) d\Gamma(x) = \frac{1}{(2n_u + j)N} \sum_{i=0}^{(2n_u + j)N-1} \int_S \text{Torsion}(f, x) df_*^i \lambda(x) = \\ &= \frac{1}{(2n_u + j)N} \sum_{i=0}^{(2n_u + j)N-1} \int_S \text{Torsion}(f, x) d\lambda(x), \end{aligned}$$

by the definition of  $\Gamma$  and since the asymptotic torsion is invariant along the  $f$ -orbit of a point.

By Lemma 4.6.3 we have that

$$\text{Torsion}(f^{(2n_u+j)N}, \lambda) = \int_S \text{Torsion}(f^{(2n_u+j)N}, x) d\lambda(x) = A.$$

Consequently

$$\begin{aligned} \text{Torsion}(f, \Gamma) &= \int_S \text{Torsion}(f, x) d\lambda(x) = \int_S \frac{1}{(2n_u+j)N} \text{Torsion}(f^{(2n_u+j)N}, x) d\lambda(x) = \\ &= \frac{\text{Torsion}(f^{(2n_u+j)N}, \lambda)}{(2n_u+j)N} = \frac{A}{(2n_u+j)N} = \alpha. \end{aligned}$$

□

Let us now prove Lemma 4.6.3. We are going to use the results presented in Section 4.6.

*Proof of Lemma 4.6.3.* Fix  $A \in [k(2n_u+j), k(2n_u+j) + \frac{m}{2}]$ . If  $m = 0$ , then the measure

$$\lambda = \frac{1}{N} \sum_{i=0}^{N-1} \delta_{f^i(q)},$$

that is the measure supported on the orbit of the periodic point  $q$ , is the required ergodic  $f^{(2n_u+j)N}$ -invariant measure such that  $\text{Torsion}(f^{(2n_u+j)N}, \lambda) = k(2n_u+j)N = A$ .

Assume now that  $m \neq 0$  and denote as

$$\mathcal{A} = \frac{2(A - k(2n_u+j))}{m} \in [0, 1].$$

As a first case, suppose that  $\mathcal{A} \in \mathbb{R} \setminus \mathbb{Q}$ . From Lemma 4.6.2, there exists a  $S$ -invariant ergodic measure  $\nu$  whose support is a subset  $H(K)$  of  $\{0, 1\}^{\mathbb{Z}}$  and such that

$$\int_{H(K)} \text{Pr}((s_i)_{i \in \mathbb{Z}}) d\nu((s_i)_{i \in \mathbb{Z}}) = \mathcal{A}. \quad (4.65)$$

Consider then the measure  $\lambda := (h_{|H(K)}^{-1})_* \nu$ . It is  $f^{(2n_u+j)N}$ -invariant and its support  $h^{-1} \circ H(K)$  is contained in the horseshoe  $H(U_\varepsilon, j)$ . Moreover,  $\lambda$  is an ergodic measure for  $f^{(2n_u+j)N}$ . Finally, we have

$$\begin{aligned} \text{Torsion}(f^{(2n_u+j)N}, \lambda) &= \int_{H(U_\varepsilon, j)} \text{Torsion}(f^{(2n_u+j)N}, z) d\lambda(z) = \\ &= \int_{h^{-1} \circ H(K)} \text{Torsion}(f^{(2n_u+j)N}, z) d(h_{|H(K)}^{-1})_* \nu(z). \end{aligned}$$

Since  $h^{-1} \circ H(K) \subset H(U_\varepsilon, j)$ , from Corollary 4.4.1 and since

$$\text{Torsion}(f^{(2n_u+j)N}, z) = (2n_u+j)N \text{Torsion}(f, z),$$

we obtain that

$$\int_{h^{-1} \circ H(K)} \text{Torsion}(f^{(2n_u+j)N}, z) d(h_{|H(K)}^{-1})_* \nu(z) =$$

$$\begin{aligned}
&= (2n_u + j)N \int_{h^{-1} \circ H(K)} \text{Torsion}(f, z) d(h_{|H(K)}^{-1})_* \nu(z) = \\
&= k(2n_u + j) + \frac{m}{2} \lim_{n \rightarrow +\infty} \int_{h^{-1} \circ H(K)} \frac{\sum_{i=1}^n h(z)_i}{n} d(h_{|H(K)}^{-1})_* \nu(z) = \\
&= k(2n_u + j) + \frac{m}{2} \lim_{n \rightarrow +\infty} \int_{H(K)} \frac{\sum_{i=1}^n \text{Pr}(S^i((s_j)_{j \in \mathbb{Z}}))}{n} d\nu((s_j)_{j \in \mathbb{Z}}).
\end{aligned}$$

Now, since  $\nu$  is  $S$ -invariant, it holds that for any  $i \in \mathbb{Z}$

$$\int_{H(K)} \text{Pr}(S^i((s_j)_{j \in \mathbb{Z}})) d\nu((s_j)_{j \in \mathbb{Z}}) = \int_{H(K)} \text{Pr}((s_j)_{j \in \mathbb{Z}}) d\nu((s_j)_{j \in \mathbb{Z}}).$$

Consequently

$$\begin{aligned}
&k(2n_u + j) + \frac{m}{2} \lim_{n \rightarrow +\infty} \int_{H(K)} \frac{\sum_{i=1}^n \text{Pr}(S^i((s_j)_{j \in \mathbb{Z}}))}{n} d\nu((s_j)_{j \in \mathbb{Z}}) = \\
&= k(2n_u + j) + \frac{m}{2} \int_{H(K)} \text{Pr}((s_j)_{j \in \mathbb{Z}}) d\nu((s_j)_{j \in \mathbb{Z}}) = k(2n_u + j) + \frac{m\mathcal{A}}{2},
\end{aligned}$$

where the last equality is (4.65) (from Lemma 4.6.2). We so conclude that

$$\text{Torsion}(f^{(2n_u+j)N}, \lambda) = k(2n_u + j) + \frac{m\mathcal{A}}{2} = A.$$

As second and last case, assume that  $\mathcal{A} \in \mathbb{Q}$ , i.e.  $\mathcal{A} = \frac{p}{q}$  for some  $p \in \mathbb{Z}, q \in \mathbb{N}^*$ . Consequently  $A = k(2n_u + j) + \frac{m}{2} \frac{p}{q}$ .

Let  $\alpha = \frac{k}{N} + \frac{m}{2(2n_u+j)N} \frac{p}{q}$ . In particular it holds  $(2n_u + j)N\alpha = A$ . From Corollary 4.6.2 there exists  $x \in H(U_\varepsilon, j)$  such that

$$\text{Torsion}(f, x) = \alpha.$$

Therefore

$$\text{Torsion}(f^{(2n_u+j)N}, x) = (2n_u + j)N \text{Torsion}(f, x) = (2n_u + j)N\alpha = A.$$

Let  $x \in H(U_\varepsilon, j)$  be such that its torsion value is  $\alpha = \frac{k}{N} + \frac{m}{2(2n_u+j)N} \frac{p}{q} \in \mathbb{Q}$  and such that it is periodic for  $f$  of period  $(2n_u + j)Nq$ : it is sufficient to select the point which corresponds to the periodic sequence in  $\{0, 1\}^{\mathbb{Z}}$  obtained by repeating

$$\underbrace{1 \dots 1}_{p \text{ entries}} \underbrace{0 \dots 0}_{q \text{ entries}}.$$

Let  $\lambda$  be the  $f^{(2n_u+j)N}$ -invariant measure defined as

$$\lambda = \frac{1}{(2n_u + j)Nq} \sum_{i=0}^{(2n_u+j)Nq-1} \delta_{f^i(x)},$$

where  $\delta_{f^i(x)}$  is the Dirac measure at  $f^i(x)$ . Hence

$$\text{Torsion}(f^{(2n_u+j)N}, \lambda) = \int_S \text{Torsion}(f^{(2n_u+j)N}, z) d\lambda(z) =$$

$$= \frac{1}{(2n_u + j)Nq} \sum_{i=0}^{(2n_u+j)Nq-1} \text{Torsion}(f^{(2n_u+j)N}, f^i(x)) = \text{Torsion}(f^{(2n_u+j)N}, x) = A,$$

that is  $\lambda$  is a ergodic  $f^{(2n_u+j)N}$ -invariant probability measure with the prescribed  $A$  as torsion value.

□





# Appendix A

## Reminders on hyperbolic sets and stable/unstable manifolds

Let  $M$  be a  $m$ -dimensional smooth Riemannian manifold and let  $f : M \rightarrow M$  be a  $\mathcal{C}^1$  diffeomorphism. A subset  $\Lambda \subset M$  is  $f$ -invariant if  $f(\Lambda) = \Lambda$ . Fix a metric on  $M$  and denote as  $\|\cdot\|$  the corresponding norm.

**Definition A.0.1** (Hyperbolic set). A subset  $\Lambda \subset M$  is hyperbolic for  $f$  if  $\Lambda$  is compact,  $f$ -invariant and there exist a splitting  $T_\Lambda M = E^s \oplus E^u$ , a norm  $\|\cdot\|$  and constants  $\lambda \in (0, 1), C > 0$  such that

- (i) for any  $x \in \Lambda$  it holds  $Df(x)E_x^u = E_{f(x)}^u, Df(x)E_x^s = E_{f(x)}^s$ ;
- (ii) for any  $n > 0$  and for any  $v \in E^s$  it holds  $\|Df^n(x)v\| \leq C\lambda^n\|v\|$ ;
- (iii) for any  $n > 0$  and for any  $w \in E^u$  it holds  $\|Df^{-n}(x)w\| \leq C\lambda^n\|w\|$ .

**Remark A.0.1.** Let  $\Lambda$  be a hyperbolic set for  $f$ . Since  $\Lambda$  is compact and since in a finite-dimensional vector space all norms are equivalent, Definition A.0.1 holds with respect to any norm, up to modify the constant  $C > 0$ .

**Remark A.0.2.** Let  $\Lambda$  be a hyperbolic set for  $f$ . Then there exists an adapted norm  $\|\cdot\|_a$  for  $\Lambda$  such that Definition A.0.1 holds with  $C = 1$  and for any  $x \in \Lambda$  and for any  $v \in T_x M$ ,  $v = v^s + v^u$ ,  $v^s \in E_x^s, v^u \in E_x^u$  it holds  $\|v\|_a = \max(\|v^s\|_a, \|v^u\|_a)$  (see [Mat68] or Section 1.4.1 in [Yoc95]).

**Remark A.0.3.** Observe that in Definition A.0.1 we do not ask that the splitting  $E^s \oplus E^u$  is continuous. Nevertheless, the continuity of the splitting is an outcome of Definition A.0.1. Consequently the functions

$$\Lambda \ni x \mapsto \delta^u(x) := \dim E_x^u \in \mathbb{N} \quad \text{and} \quad \Lambda \ni x \mapsto \delta^s(x) := \dim E_x^s \in \mathbb{N}$$

are continuous, locally constant and  $f$ -invariant (i.e.  $\delta^u(x) = \delta^u(f(x)), \delta^s(x) = \delta^s(f(x))$ ).

Let  $\Lambda \subset M$  be a compact  $f$ -invariant subset and let  $E^1 \oplus E^2 = T_\Lambda M$  be a splitting (not necessarily continuous).

A cone field in  $T_\Lambda M = E^1 \oplus E^2$  with respect to  $\|\cdot\|$  and  $\eta > 0$  is a family  $(C_x^\eta)_{x \in \Lambda}$  where

$$C_x^\eta := \{v \in T_x M, v = v^1 + v^2, v^1 \in E_x^1, v^2 \in E_x^2 : \|v^1\| \leq \eta\|v^2\|\}.$$

The parameter  $\eta > 0$  is the size of the cone field.

**Definition A.0.2** (Cone field property). A compact  $f$ -invariant subset  $\Lambda \subset M$  satisfies the cone field property with respect to  $\|\cdot\|$  if there exist a splitting  $E^1 \oplus E^2 = T_\Lambda M$  and a cone field  $(C_x^\eta)_{x \in \Lambda}$  of size  $\eta > 0$  (with respect to  $\|\cdot\|$ ) such that

- (i) for any  $x \in \Lambda$  it holds  $\dim E_x^1 = \dim E_{f(x)}^1$  and  $\dim E_x^2 = \dim E_{f(x)}^2$ ;
- (ii) there exist  $\xi, \delta \in (0, 1)$  and  $m \in \mathbb{N}^*$  such that for any  $x \in \Lambda$ 
  - $Df^m(x)C_x^\eta \subset C_{f^m(x)}^{\eta^\delta}$ ;
  - for any  $v \in C_x^\eta$  it holds  $\|Df^m(x)v\| \geq \frac{1}{\xi}\|v\|$ ;
  - for any  $w \notin \text{int}(C_x^\eta)$  it holds  $\|Df^{-m}(x)w\| \geq \frac{1}{\xi}\|w\|$ .

**Proposition A.0.1.** *Let  $\Lambda \subset M$  be a compact  $f$ -invariant set in  $M$ . Then  $\Lambda$  is hyperbolic if and only if  $\Lambda$  satisfies the cone field property.*

For a proof of Proposition A.0.1 we refer to Section 1.5 in [Yoc95].

In the sequel we recall the definitions of (local) stable/unstable manifolds and their main properties. We refer to [HP70], [New72], [PdM82] and [Shu87] for an exhaustive treatment of the subject.

Let  $\Lambda \subset M$  be a hyperbolic set for  $f$  and let  $d$  be the fixed metric.

**Definition A.0.3** (Local stable/unstable manifolds). Let  $x \in \Lambda$  and  $\varepsilon > 0$ . The  $\varepsilon$ -local stable manifold at  $x$  is

$$W_{loc,\varepsilon}^s(x) := \{y \in M : d(f^n(y), f^n(x)) < \varepsilon \forall n \geq 0\}.$$

The  $\varepsilon$ -local unstable manifold at  $x$  is

$$W_{loc,\varepsilon}^u(x) := \{y \in M : d(f^{-n}(y), f^{-n}(x)) < \varepsilon \forall n \geq 0\}.$$

The  $\varepsilon$ -local stable manifold system over  $\Lambda$ <sup>1</sup> is the family  $\{W_{loc,\varepsilon}^s(x)\}_{x \in \Lambda}$ . Similarly the  $\varepsilon$ -local unstable manifold system over  $\Lambda$  is the family  $\{W_{loc,\varepsilon}^u(x)\}_{x \in \Lambda}$ .

We recall the Local Stable Manifold Theorem, first for a hyperbolic point (see [Shu87]) and then for a hyperbolic set (see [HP70]).

**Definition A.0.4** ( $u$ - $s$  chart). Let  $q \in M$  be a fixed hyperbolic point for  $f$ . Then,  $(U, \phi)$  is a  $u$ - $s$  chart for  $q$  with respect to  $f$  if  $U$  is an open neighborhood of  $q$  and  $\phi : U \rightarrow \mathbb{R}^m$  is a chart such that  $\phi(q) = 0 \in \mathbb{R}^m$  and there exists  $r > 0$  such that  $\phi(U) = B_r^s(0) \times B_r^u(0)$ , where  $s$  is the dimension of  $E_q^s$ ,  $u$  is the dimension of  $E_q^u$  and  $B_r^s(0)$  (respectively  $B_r^u(0)$ ) is the ball in  $\mathbb{R}^s$  (respectively in  $\mathbb{R}^u$ ) centered at 0 of radius  $r$ .

**Notation A.0.1.** Let  $q \in M$  be a fixed hyperbolic point for  $f$  with respect to  $\|\cdot\|$  and to constants  $\lambda \in (0, 1), C > 0$ . Let  $(U, \phi)$  be a  $u$ - $s$  chart for  $q$  with respect to  $f$ . Denote as  $\tilde{f} : V \rightarrow B_r^s(0) \times B_r^u(0)$  the map

$$\phi \circ f \circ \phi^{-1} : V \longrightarrow B_r^s(0) \times B_r^u(0)$$

where  $V = \phi(U \cap f^{-1}(U))$ . Denote as  $T : \mathbb{R}^m \rightarrow \mathbb{R}^m$  the linear hyperbolic homeomorphism

$$D\phi(q) \circ Df(q) \circ (D\phi(q))^{-1} : \mathbb{R}^m \longrightarrow \mathbb{R}^m$$

such that for any  $n \in \mathbb{N}^*$

$$\|T_{|\mathbb{R}^s \times \{0\}}^n\| \leq C\lambda^n, \quad \left\|T_{\{0\} \times \mathbb{R}^u}^{-n}\right\| \leq C\lambda^n.$$

---

1. Such definition is presented in [HP70].

**Notation A.0.2.** Let  $N \subset M$  and let  $x \in N$ . Denote as  $CC(N, x)$  the connected component of  $N$  containing  $x$ .

**Notation A.0.3.** Let  $x \in M$  and denote as  $\mathcal{O}(x, f)$  the orbit of  $x$  with respect to  $f$ , i.e.

$$\mathcal{O}(x, f) = \{f^n(x) : n \in \mathbb{Z}\}.$$

**Theorem A.0.1** (Local Stable Manifold Theorem for a fixed hyperbolic point). *Let  $f : M \rightarrow M$  be a  $\mathcal{C}^1$  diffeomorphism and let  $q \in M$  be a fixed hyperbolic point for  $f$ . There exist  $\varepsilon, r > 0$  small enough and a  $u$ - $s$  chart  $(U, \phi)$  for  $q$  such that  $U \subset B_\varepsilon(q)$ ,  $\phi(U) = B_r^s(0) \times B_r^u(0)$  and there exists a  $\mathcal{C}^1$  map  $g : B_r^s(0) \rightarrow B_r^u(0)$  so that*

$$\phi^{-1}(\text{graph}(g)) = W_{loc, \varepsilon}^s(q) \subset U.$$

We recall some further properties of the map  $g$  and hence of the  $\varepsilon$ -local stable manifold of  $q$ .

**Properties of the  $\varepsilon$ -local stable manifold of  $q$ :**

- (i) The function  $g$  is Lipschitz with Lipschitz constant  $L(g) < 1$ .
- (ii) The image through  $\phi^{-1}$  of the graph of  $g$  is  $f$ -forward invariant. That is

$$f^n(\phi^{-1}(\text{graph}(g))) \subset \phi^{-1}(\text{graph}(g)) \quad \forall n > 0.$$

- (iii) The function  $\tilde{f}$  is a contraction on  $\text{graph}(g)$ . Then there exists  $\zeta \in (0, 1)$  such that for any  $x_1, x_2 \in B_r^s(0)$  and for any  $n > 0$  it holds

$$\left\| \tilde{f}^n(x_1, g(x_1)) - \tilde{f}^n(x_2, g(x_2)) \right\| \leq \zeta^n \|x_1 - x_2\|.$$

- (iv) The point  $q \in \phi^{-1}(\text{graph}(g))$ , i.e.  $g(0_s) = 0_u$  (where  $0_s \in \mathbb{R}^s$ ,  $0_u \in \mathbb{R}^u$ ). Moreover the graph of  $g$  is tangent to  $\mathbb{R}^s \times \{0\}$  at 0, i.e.  $Dg(0)$  is null. Consequently  $\phi^{-1}(\text{graph}(g)) = W_{loc, \varepsilon}^s(q)$  is tangent to  $E_q^s$  at  $q$ .
- (v) If  $f$  is  $\mathcal{C}^k$  then the function  $g$  is  $\mathcal{C}^k$ .

**Remark A.0.4.** A similar result holds for the local unstable manifold  $W_{loc, \varepsilon}^u(q)$ .

This implies that  $W_{loc, \varepsilon}^s(q)$  and  $W_{loc, \varepsilon}^u(q)$  are  $\mathcal{C}^1$  submanifolds of  $M$ .

We want now to present the Local Stable Manifold Theorem for a hyperbolic set (see [HP70] and [Yoc95]). For the sake of clarity, let us recall the definition of two submanifolds that are  $\mathcal{C}^1$  close.

**Definition A.0.5.** Fix an atlas  $\{U_j, \phi_j\}_j$  of  $M$ . Let  $S, S' \subset M$  be differentiable submanifolds. Denote as  $i : S \hookrightarrow M, i' : S' \hookrightarrow M$  the corresponding inclusions. Let  $\{U_j \cap S, \Pi_s \circ \phi_j\}_j$  be an atlas for  $S$  such that  $\phi_j(U_j \cap S) = \mathbb{R}^s \times \{0\}$ , where  $s$  is the dimension of  $S$  and  $\Pi_s : \mathbb{R}^m \rightarrow \mathbb{R}^s$  is the projection over the first  $s$  coordinates. Assume that each change of coordinates  $\phi_j \circ \phi_l^{-1}$  has bounded  $\mathcal{C}^1$  norm.

Fix  $\varepsilon > 0$ . The submanifold  $S'$  is  $\varepsilon$ -close to  $S$  with respect to  $\{U_j, \phi_j\}_j$  if:

- (a)  $S' \subset \bigcup_{j \in J} U_j =: \mathcal{U}$ , where  $J = \{j : U_j \cap S \neq \emptyset\}$ ;

- (b) there exists a  $\mathcal{C}^1$  diffeomorphism  $h : S' \rightarrow S$  such that for any  $p \in S'$  there exists  $j = j(p)$  so that  $p, h(p) \in U_j$  and

$$\|\phi_j \circ i'(p) - \phi_j \circ i \circ h(p)\| < \varepsilon;$$

- (c) for any  $p \in S'$  let us consider

$$D\phi_j(i'(p)) \circ Di'(p) : T_p S' \rightarrow \mathbb{R}^m$$

and

$$D\phi_j(i \circ h(p)) \circ D(i \circ h)(p) : T_p S' \rightarrow \mathbb{R}^m.$$

Then it holds for any  $v \in T_p S'$

$$\|D\phi_j(i'(p)) \circ Di'(p)v - D\phi_j(i \circ h(p)) \circ D(i \circ h)(p)v\| < \varepsilon \|v\|.$$

**Theorem A.0.2** (Local Stable Manifold Theorem for a hyperbolic set). *Let  $\Lambda \subset M$  be a hyperbolic set for  $f$ . Then*

- (i) *there exists  $\varepsilon > 0$  small enough such that for any  $x \in \Lambda$*

$$W_{loc,\varepsilon}^s(x) = \{y \in M : d(f^n(y), f^n(x)) < \varepsilon \ \forall n \geq 0\}$$

*is a  $\mathcal{C}^1$  submanifold of  $M$ ;*

- (ii)  *$\{W_{loc,\varepsilon}^s(x)\}_{x \in \Lambda}$  is a continuous family of  $\mathcal{C}^1$  submanifolds, i.e. for any  $x \in \Lambda$  for any  $\delta > 0$  there exists  $U$  neighborhood of  $x$  such that for any  $y \in U \cap \Lambda$  the submanifold  $W_{loc,\varepsilon}^s(y)$  is  $\delta$ - $\mathcal{C}^1$  close to  $W_{loc,\varepsilon}^s(x)$ ;*

- (iii) *the  $\varepsilon$ -local stable manifold system  $\{W_{loc,\varepsilon}^s(x)\}_{x \in \Lambda}$  is  $f$ -forward invariant, i.e.*

$$f^n(W_{loc,\varepsilon}^s(x)) \subset W_{loc,\varepsilon}^s(f^n(x)) \quad \forall n > 0.$$

*In particular, there exist  $\zeta \in (0, 1)$  and  $K > 0$  such that for any  $x \in \Lambda$  for any  $y, z \in W_{loc,\varepsilon}^s(x)$  it holds*

$$d(f^n(y), f^n(z)) \leq K\zeta^n d(y, z) \quad \forall n > 0;$$

- (iv) *every  $W_{loc,\varepsilon}^s(x)$  is tangent to  $E_x^s$  at  $x$ .*

We now give the definition of (global) stable/unstable manifolds for a hyperbolic set.

**Definition A.0.6** (Global stable/unstable manifolds). Let  $\Lambda \subset M$  be a hyperbolic set for  $f$ . Let  $\varepsilon > 0$  be the parameter given by point (i) of Theorem A.0.2. The stable (unstable) manifold at  $x \in \Lambda$  is

$$W^s(x) = \bigcup_{n \geq 0} f^{-n}(W_{loc,\varepsilon}^s(x)) \quad \left( W^u(x) = \bigcup_{n \geq 0} f^n(W_{loc,\varepsilon}^u(x)) \right).$$

The stable manifold system of  $\Lambda$  is  $\{W^s(x)\}_{x \in \Lambda}$  (and the unstable one is  $\{W^u(x)\}_{x \in \Lambda}$ ).

Observe that the stable (unstable) manifold at  $x$  can be defined as

$$W^s(x) = \{y \in M : \lim_{n \rightarrow +\infty} d(f^n(y), f^n(x)) = 0\}$$

$$\left( W^u(x) = \{y \in M : \lim_{n \rightarrow +\infty} d(f^{-n}(y), f^{-n}(x)) = 0\} \right).$$

The stable (respectively unstable) manifold at  $x$ , i.e.  $W^s(x)$  (respectively  $W^u(x)$ ), is an injectively immersed submanifold of  $M$ .

We recall now another fundamental result: the  $\lambda$ -lemma. We refer to [PdM82] for the proof of the result.

**Theorem A.0.3** ( $\lambda$ -lemma). *Let  $f : M \rightarrow M$  be a  $\mathcal{C}^1$  diffeomorphism and let  $q$  be a fixed hyperbolic point for  $f$  (with respect to  $\|\cdot\|$ ). Let  $p \in W^s(q) \setminus \{q\}$  and let  $i : B_1^u(0) \rightarrow M$  be an embedding of the  $u$ -dimensional ball of radius 1 (where  $u$  is the dimension of  $E_q^u$ ) such that  $i(0) = p$ ,  $i(B_1^u(0)) = D^u$  and  $D^u$  is transverse to  $W^s(q)$  at  $p$ , i.e.*

$$Di(0)T_0B_1^u(0) + T_pW^s(q) = T_pM.$$

Let  $(U, \phi)$  be a  $u$ -s chart for  $q$  with respect to  $f$  such that there exist  $\epsilon > 0$  so that

$$CC(W^u(q) \cap U, q) = W_{loc, \epsilon}^u(q).$$

Given  $\delta > 0$  there exists  $n_0$  so that for any  $n > n_0$

$$CC(f^n(D^u) \cap U, f^n(p))$$

is  $\delta$ - $\mathcal{C}^1$  close to  $W_{loc, \epsilon}^u(q)$ .

We can adapt this statement to a periodic hyperbolic point in the following way.

**Theorem A.0.4** ( $\lambda$ -lemma for a hyperbolic periodic point). *Let  $q$  be a periodic hyperbolic point for  $f$  of period  $N$ . Let  $p \in W^s(q) \setminus \mathcal{O}(q)$ . Let  $D^u$  be the image of the embedding of a  $u$ -dimensional ball<sup>2</sup> of radius 1 such that  $D^u$  is transverse to  $W^s(q)$  at  $p$ . For any  $i \in \llbracket 0, N-1 \rrbracket$ , let  $(U_i, \phi_i)$  be a  $u$ -s chart for  $f^i(q)$  with respect to  $f^N$  such that there exists  $\epsilon > 0$  so that*

$$CC(W^u(f^i(q)) \cap U_i, f^i(q)) = W_{loc, \epsilon}^u(f^i(q)).$$

Given  $\delta > 0$  there exists  $n_0$  such that for any  $n > n_0$

$$CC(f^{nN}(D^u) \cap U_0, f^{nN}(p))$$

is  $\delta$ - $\mathcal{C}^1$  close to  $W_{loc, \epsilon}^u(q)$ .

Analogous statements hold with respect to the local stable manifold and embedding of  $s$ -dimensional ball transverse to the unstable manifold (where  $s$  is the dimension of the stable manifold).

The following Corollary is an outcome of the  $\lambda$ -lemma.

---

2.  $u$  is the dimension of the unstable manifold.

**Corollary A.0.1.** *Let  $q_1, q_2, q_3 \in M$  be hyperbolic periodic points for  $f$ . Assume that*

- (i)  $W^u(q_1)$  has a point of transverse intersection with  $W^s(q_2)$  which does not belong to  $\mathcal{O}(q_1)$ ;
- (ii)  $W^u(q_2)$  has a point of transverse intersection with  $W^s(q_3)$  which does not belong to  $\mathcal{O}(q_2)$ .

*Then  $W^u(q_1)$  has a point of transverse intersection with  $W^s(q_3)$  which does not belong to  $\mathcal{O}(q_1)$ .*

Let us consider the set of hyperbolic periodic points in  $M$  for  $f$ : we denote it as  $HP(f)$  and suppose it is not empty. Define the relation  $\dashv$  on  $HP(f)$  as follows.

**Definition A.0.7** (Relation  $\dashv$ ). Let  $p, q \in HP(f)$ . Then  $p \dashv q$  if  $W^u(p)$  has a point of transverse intersection with  $W^s(q)$  which does not belong to  $\mathcal{O}(p)$ .

**Remark A.0.5.** The relation  $\dashv$  is transitive thanks to Corollary A.0.1. In particular the point of transverse intersection does not belong to the orbit of the involved periodic points.

Using this relation  $\dashv$  on  $HP(f)$  we deduce the following

**Fact A.0.1.** *Let  $q \in HP(f)$ . Then*

$$(W^s(\mathcal{O}(q)) \cap W^u(\mathcal{O}(q))) \setminus \mathcal{O}(q) \neq \emptyset \Leftrightarrow (W^s(q) \cap W^u(q)) \setminus \{q\} \neq \emptyset.$$

*Proof of Fact A.0.1.* Let us show the two implications.

( $\Leftarrow$ ) If  $p \in (W^s(q) \cap W^u(q)) \setminus \{q\}$ , then  $p \notin \mathcal{O}(q)$  and clearly  $p \in W^s(\mathcal{O}(q)) \cap W^u(\mathcal{O}(q))$ .

( $\Rightarrow$ ) Let  $p \in (W^s(f^i(q)) \cap W^u(f^j(q))) \setminus \mathcal{O}(q)$  for some  $i, j \in \llbracket 0, N-1 \rrbracket$  where  $N$  is the period of  $q$ . Up to consider  $f^{-i}(p)$  instead of  $p$ , we assume that  $p \in W^s(q) \cap W^u(f^j(q))$ , that is  $q \dashv f^j(q)$ .

By considering iterates of  $p$ , it holds  $f^k(q) \dashv f^{j+k}(q)$  for any  $k \in \mathbb{Z}$ . In particular for any  $k \in \mathbb{Z}$  we have  $f^{kj}(q) \dashv f^{(k+1)j}(q)$ . Since the relation is transitive, we deduce that  $q \dashv f^{kj}(q)$  for any  $k \in \mathbb{N}$ .

Hence, choosing  $k = N$ , there exists a point of transverse intersection between  $W^s(q)$  and  $W^u(q)$  which does not belong to the orbit of  $q$ , i.e.

$$(W^s(q) \cap W^u(q)) \setminus \{q\} \neq \emptyset.$$

□

Let now consider  $q \in M$  hyperbolic periodic point for  $f$  and let

$$p \in (W^s(q) \cap W^u(q)) \setminus \{q\}.$$

**Fact A.0.2.** *The set  $\mathcal{O}(q) \cup \mathcal{O}(p)$  is a hyperbolic set for  $f$ .*

*Proof of Fact A.0.2.* Denote as  $\|\cdot\|$  the adapted norm for  $\mathcal{O}(q)$  (see Remark A.0.2). The set  $\mathcal{O}(q) \cup \mathcal{O}(p)$  is  $f$ -invariant. Since  $\mathcal{O}(q)$  is a finite set and since the limit set of  $(f^n(p))_{n \in \mathbb{N}}, (f^{-n}(p))_{n \in \mathbb{N}}$  is contained in  $\mathcal{O}(q)$ , the set  $\mathcal{O}(q) \cup \mathcal{O}(p)$  is compact.

We are going to show that there exist a splitting  $E^u \oplus E^s = T_{\mathcal{O}(q) \cup \mathcal{O}(p)}M$ , constants  $C > 0, \lambda \in (0, 1)$  such that conditions (i), (ii) and (iii) of Definition A.0.1 are satisfied with respect to  $\|\cdot\|$ .

For a point in  $\mathcal{O}(q)$ , the splitting is the one given by the definition of hyperbolicity of  $\mathcal{O}(q)$ . For  $P \in \mathcal{O}(p)$  the candidate splitting is

$$E_P^u = T_P W^u(q), \quad E_P^s = T_P W^s(q).$$

Since the intersection is transverse at  $p$ , it holds

$$T_P W^s(q) + T_P W^u(q) = T_P M \quad \forall P \in \mathcal{O}(p).$$

The candidate splitting is  $f$ -invariant thanks to the  $f$ -invariance of the stable and unstable manifolds. By the hyperbolicity of  $\mathcal{O}(q)$ <sup>3</sup>, there exists  $\xi \in (0, 1)$  such that for any  $n > 0$

$$\left\| Df^n|_{E_{\mathcal{O}(q)}^s} \right\| \leq \xi^n, \quad \left\| Df^{-n}|_{E_{\mathcal{O}(q)}^u} \right\| \leq \xi^n$$

where  $E_{\mathcal{O}(q)}^s, E_{\mathcal{O}(q)}^u$  are respectively the stable and unstable bundle restricted to  $\mathcal{O}(q)$ . The candidate splitting  $E^u \oplus E^s$  over  $\mathcal{O}(q) \cup \mathcal{O}(p)$  is continuous thanks to the Local (Un)Stable Manifold Theorem and to the  $\lambda$ -lemma.

Fix  $\varepsilon > 0$  such that  $\xi + \varepsilon < 1$ . By the continuity of  $Df$  and  $Df^{-1}$  and by the continuity of the splitting  $E^u \oplus E^s$  over  $\mathcal{O}(q) \cup \mathcal{O}(p)$  there exists a neighborhood  $U$  of  $\mathcal{O}(q)$  such that for any  $y \in U \cap (\mathcal{O}(q) \cup \mathcal{O}(p))$

$$\left\| Df(y)|_{E_y^s} \right\| \leq \xi + \varepsilon \quad \text{and} \quad \left\| Df^{-1}(y)|_{E_y^u} \right\| \leq \xi + \varepsilon.$$

Since  $(f^n(p))_{n \in \mathbb{N}}, (f^{-n}(p))_{n \in \mathbb{N}}$  converge to  $q$ , there is a finite number  $n_0$  of points in  $\mathcal{O}(p) \setminus U$ .

We now show that there exist  $C > 0$  and  $\lambda \in (0, 1)$  such that for any  $n > 0$

$$\left\| Df^n|_{E^s} \right\| \leq C\lambda^n.$$

A similar argument holds for the unstable bundle.

Define

$$D := \max \left( 1, \max_{y \in \mathcal{O}(p) \setminus U} \left\| Df(y)|_{E_y^s} \right\| \right) > 0.$$

For any  $n > 0$  we have

$$\left\| Df^n|_{E^s} \right\| \leq D^{n_0} (\xi + \varepsilon)^{n-n_0} = C\lambda^n,$$

where  $C := \frac{D^{n_0}}{(\xi + \varepsilon)^{n_0}} > 0$  and  $\lambda = (\xi + \varepsilon) \in (0, 1)$ .

□

We end this appendix by recalling the definitions of locally maximal hyperbolic set and of local product structure. Referring to [Yoc95] and [KH95], we recall also the equivalence of these notions.

---

3. Recall that we are considering an adapted norm for  $\mathcal{O}(q)$ .



**Definition A.0.8** (Locally maximal hyperbolic set). A hyperbolic set  $\Lambda \subset M$  for  $f$  is locally maximal (or isolated) if there exists a neighborhood  $V$  of  $\Lambda$  in  $M$  such that

$$\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(V).$$

Before introducing the definition of local product structure, we highlight an outcome of the Local (Un)Stable Manifold Theorem.

**Proposition A.0.2** (Proposition 7.2 in [Shu87]). *Let  $\Lambda \subset M$  be a hyperbolic set for  $f$ . For any  $\varepsilon > 0$  (such that the  $\varepsilon$ -local stable (unstable) manifold is well-defined) there exists  $\delta > 0$  such that if  $x, y \in \Lambda$  and  $d(x, y) < \delta$  then  $W_{loc, \varepsilon}^s(x) \cap W_{loc, \varepsilon}^u(y) = \{z\}$  and the unique point of intersection  $z$  is a point of transverse intersection.*

Proposition A.0.2 is an outcome of the continuity of the family of local stable (unstable) manifold system  $\{W_{loc, \varepsilon}^s(x)\}_{x \in \Lambda}$  and of the fact that  $W_{loc}^s(x) \pitchfork W_{loc}^u(x) = \{x\}$ .

**Definition A.0.9** (Local product structure). A hyperbolic set  $\Lambda \subset M$  has a local product structure if for any  $\varepsilon > 0$ <sup>4</sup> there exists  $\delta > 0$  such that if  $x, y \in \Lambda$  and  $d(x, y) < \delta$  then  $W_{loc, \varepsilon}^s(x) \pitchfork W_{loc, \varepsilon}^u(y) = \{z\} \subset \Lambda$ .

That is,  $\Lambda$  has a local product structure if the points of transverse intersection of local stable/unstable manifolds presented in Proposition A.0.2 are points of  $\Lambda$ .

Definitions A.0.8 and A.0.9 are equivalent, that is

**Theorem A.0.5** (Section 4.1 in [Yoc95]). *Let  $\Lambda \subset M$  be a hyperbolic set. Then  $\Lambda$  is locally maximal if and only if  $\Lambda$  has a local product structure.*

For a detailed proof of this result we refer to [Yoc95] (see also Proposition 8.22 in [Shu87]).

We focus now on homoclinic transverse intersections. We then remark the following

**Fact A.0.3.** *Let  $q$  be a hyperbolic periodic point for  $f$  and let  $p \in (W^s(q) \pitchfork W^u(q)) \setminus \{q\}$ . Then  $\mathcal{O}(q) \cup \mathcal{O}(p)$  is not a locally maximal hyperbolic set.*

*Proof of Fact A.0.3.* By the hyperbolicity of  $\mathcal{O}(q) \cup \mathcal{O}(p)$  (see Fact A.0.2), there exists  $\varepsilon > 0$  such that we can extend the open cone field property on  $B_\varepsilon = \{x \in M : d(x, \mathcal{O}(q) \cup \mathcal{O}(p)) < \varepsilon\}$  (see Appendix B), where

$$d(x, \mathcal{O}(q) \cup \mathcal{O}(p)) = \inf_{z \in \mathcal{O}(q) \cup \mathcal{O}(p)} d(x, z).$$

Being  $p$  a homoclinic point, the sequences  $(f^{nN}(p))_{n \in \mathbb{N}}, (f^{-nN}(p))_{n \in \mathbb{N}}$  both converge to  $q$ <sup>5</sup>.

By Proposition A.0.2 and by this last remark on  $(f^{nN}(p))_{n \in \mathbb{N}}, (f^{-nN}(p))_{n \in \mathbb{N}}$ , for any  $\varepsilon > 0$  there exists  $n_0 \in \mathbb{N}$  such that for any  $n \geq n_0$  it holds

$$W_{loc, \varepsilon}^s(f^{-nN}(p)) \pitchfork W_{loc, \varepsilon}^u(f^{nN}(p)) = \{z_n\}.$$

We observe that  $z_n$  does not belong to  $\mathcal{O}(q) \cup \mathcal{O}(p)$ , but  $\mathcal{O}(q) \cup \mathcal{O}(p) \cup \mathcal{O}(z_n)$  is compact,  $f$ -invariant and contained in  $B_\varepsilon$ .

By the choice of  $B_\varepsilon$  and by the cone field criterion (see Proposition A.0.1),  $\mathcal{O}(q) \cup \mathcal{O}(p) \cup \mathcal{O}(z_n)$  is a hyperbolic set. As just observed,  $\mathcal{O}(q) \cup \mathcal{O}(p) \cup \mathcal{O}(z_n)$  strictly contains  $\mathcal{O}(q) \cup \mathcal{O}(p)$ . We then deduce that  $\mathcal{O}(q) \cup \mathcal{O}(p)$  is not locally maximal. □

---

4. Sufficiently small so that the  $\varepsilon$ -local stable (unstable) manifold is well-defined.

5. Recall that  $q = f^N(q)$ .

# Appendix B

## Extension of the Cone Field Property

In this appendix we refer to [CP15] and we prove the following

**Fact B.0.1.** *Let  $M$  be a  $m$ -dimensional smooth manifold. Let  $f : M \rightarrow M$  be a  $\mathcal{C}^1$  diffeomorphism. Let  $\Lambda \subset M$  be a hyperbolic set for  $f$ . There exists a neighborhood  $U$  of  $\Lambda$  which satisfies the cone field property.*

In this Appendix we fix a Riemannian norm  $\|\cdot\|$ , but we remark that, since  $M$  is finite-dimensional and  $\Lambda$  is compact, all the norms are equivalent. Let us recall the cone field property.

**Definition B.0.1.** Let  $f : M \rightarrow M$  be a  $\mathcal{C}^1$  diffeomorphism on a  $m$ -dimensional manifold  $M$ . A set  $U \subset M$  satisfies the cone field property for  $\eta \in \mathbb{R}_+$ ,  $\xi, \delta \in (0, 1) \cap \mathbb{R}$ ,  $m \in \mathbb{N}^*$  if there exist a splitting  $E^1 \oplus E^2 = T_U M$  and a cone field  $(C_x^\eta)_{x \in U}$  where

$$C_x^\eta = \{v \in T_x M : v = v^1 + v^2, v^1 \in E_x^1, v^2 \in E_x^2, \|v^2\| \leq \eta \|v^1\|\}$$

such that

- (i) for any  $x \in U$  it holds  $\dim E_x^1 = d_1$  and  $\dim E_x^2 = d_2$ ;
- (ii) for any  $x \in U \cap f^{-1}(U)$  it holds  $Df(x)C_x^\eta \subset C_{f(x)}^{\eta^\delta}$ ;
- (iii) for any  $x \in U$

- for any  $v \in C_x^\eta$  it holds  $\|Df^m(x)v\| \geq \frac{1}{\xi}\|v\|$ ;
- for any  $w \notin \text{int}(C_x^\eta)$  it holds  $\|Df^{-m}(x)w\| \geq \frac{1}{\xi}\|w\|$ .

Let  $\Lambda$  be a hyperbolic set. From Proposition A.0.1 in Appendix A, it satisfies the cone field property with respect to some constants  $\eta \in \mathbb{R}_+$ ,  $\delta, \xi \in (0, 1)$ ,  $m \in \mathbb{N}^*$ . Then the proof of Fact B.0.1 is the proof of the following

**Proposition B.0.1.** *For any  $0 < \zeta < \min(1 - \delta, 1 - \xi)$  there exists a neighborhood  $U$  of  $\Lambda$  such that  $U$  satisfies the cone field property for  $\eta, \xi + \zeta, \delta + \zeta, m$ .*

Denote as  $E^u \oplus E^s$  the hyperbolic splitting of  $\Lambda$ . Remark that such splitting is continuous (see Remark A.0.3). In order to prove Proposition B.0.1, we first extend continuously the splitting  $E^u \oplus E^s$  on a neighborhood of  $\Lambda$ .

**Lemma B.0.1.** *There exist a neighborhood  $W$  of  $\Lambda$  and continuous functions*

$$W \ni x \mapsto E_x^u \in T_x S \quad \text{and} \quad W \ni x \mapsto E_x^s \in T_x S$$

*which coincide with the unstable and stable subspaces respectively on the hyperbolic set  $\Lambda$ .*

Lemma B.0.1 is then a particular case of the following

**Proposition B.0.2** (Proposition 2.7 in [CP15]). *Any continuous linear bundle  $E \subset T_\Lambda M$  over a compact set  $\Lambda \subset M$  admits a continuous extension to a neighborhood of  $\Lambda$ .*

*Proof.* Denote as  $d$  the dimension of the subspace  $E(x)$  for any  $x \in \Lambda$ . Up to discuss each set of components of  $\Lambda$  with the same dimension of  $E$ , we can assume that the dimension  $d$  is the same for any  $x \in \Lambda$ .

For each  $x \in \Lambda$  there exists  $\varepsilon = \varepsilon(x) > 0$  and a chart  $\phi_x : B_\varepsilon^m(x) \rightarrow \mathbb{R}^m$  from the  $m$ -dimensional ball centered at  $x$  of radius  $\varepsilon$  such that for any  $y \in B_\varepsilon^m(x) \cap \Lambda$

$$D\phi_x(y)E(y)$$

is transverse to  $\{0\} \times \mathbb{R}^{m-d}$ . For any  $y \in B_\varepsilon^m \cap \Lambda$  denote as

$$L_y^x : \mathbb{R}^d \rightarrow \mathbb{R}^{m-d}$$

the linear map such that  $D\phi_x(y)E(y) = \text{Graph}(L_y^x)$ . By the continuity of the linear bundle  $E$ , the linear maps  $L_y^x$  depend continuously on  $y \in B_\varepsilon^m(x) \cap \Lambda$ .

Since  $\Lambda$  is compact, by Tietze-Urysohn's Theorem (see [Die81]), we can extend continuously  $y \mapsto L_y^x$  on  $B_\varepsilon^m(x)$  by extending each of the coordinate of the linear map.

In particular, we are extending it continuously on  $\overline{B_{\frac{\varepsilon}{2}}^m(x)}$ . Thus, we have extended the bundle  $E$  on  $\overline{B_{\frac{\varepsilon}{2}}^m(x)}$  in a continuous way so that it remains a graph with respect to the coordinates determined by the chart  $\phi_x$ .

Consider for any  $x \in \Lambda$  the ball  $B_{\frac{\varepsilon(x)}{4}}^m(x)$ . Since  $\Lambda$  is compact, we extract a finite covering of  $\Lambda$

$$\bigcup_{i=1}^N B_{\frac{\varepsilon_i}{4}}^m(x_i) \supset \Lambda,$$

where  $\varepsilon_i = \varepsilon(x_i)$ . In particular

$$\Lambda \subset \bigcup_{i=1}^N \Lambda_i,$$

where  $\Lambda_i := B_{\frac{\varepsilon_i}{4}}^m(x_i) \cap \Lambda$ . For any  $i \in \llbracket 1, N \rrbracket$  we highlight the corresponding chart

$$\phi_{x_i} : B_{\varepsilon_i}^m(x_i) \rightarrow \mathbb{R}^m$$

and observe that  $\Lambda_i \subset \Lambda \cap B_{\frac{\varepsilon_i}{2}}^m(x_i)$ .

Let us show by induction that we can extend continuously the linear bundle  $E$  on a neighborhood of  $\Lambda_1 \cup \dots \cup \Lambda_n$ .

Let  $n = 1$  and consider  $\Lambda_1$ . Since  $\Lambda_1 \subset B_{\varepsilon_1}^m(x_1)$ , we have already extended continuously  $E(y)$  for any  $y \in B_{\varepsilon_1}^m(x_1)$  as graph of the linear map  $L_y^{x_1}$ .

Let now  $1 \leq n < N$ . Assume by induction hypothesis that we have extended continuously the bundle  $E$  on an open neighborhood  $V$  of  $\Lambda_1 \cup \dots \cup \Lambda_n$ . Denote as  $E' : V \rightarrow T_V M$

such an extension and observe that  $E'_{|\Lambda_1 \cup \dots \cup \Lambda_n} = E$ .

Consider now  $\Lambda_{n+1}$ . If  $\Lambda_{n+1} \cap (\Lambda_1 \cup \dots \cup \Lambda_n) = \emptyset$ , then, up to restrict  $V$ , we find a neighborhood  $U$  of  $\Lambda_{n+1}$  contained in  $B_{\varepsilon_{n+1}}^m(x_{n+1})$  and such that  $\bar{V} \cap \bar{U} = \emptyset$ .

We extend  $E$  continuously on the neighborhood  $V \cup U$  of  $\Lambda_1 \cup \dots \cup \Lambda_{n+1}$  as follows:

$$E(y) = \begin{cases} E'(y) & \text{if } y \in V \\ \text{Graph}(L_y^{x_{n+1}}) & \text{if } y \in U. \end{cases}$$

Assume now that  $\Lambda_{n+1} \cap (\Lambda_1 \cup \dots \cup \Lambda_n) \neq \emptyset$ . Up to restrict the neighborhood  $V$ , we can assume that for any  $y \in V \cap \overline{B_{\frac{\varepsilon_{n+1}}{2}}^m(x_{n+1})}$  the subspace  $E'(y)$  (i.e. the continuous extension of the bundle on  $V$ ) remains a graph with respect to the coordinates determined by the chart  $\phi_{x_{n+1}}$ . Denote so as  $\text{Graph}(\mathfrak{L}_y^{x_{n+1}})$  the subspace  $E'(y)$  for any  $y \in V \cap \overline{B_{\frac{\varepsilon_{n+1}}{2}}^m(x_{n+1})}$ . Denote as  $\mathcal{V} \subset V$  and as  $\mathcal{W}$  open neighborhoods of  $\Lambda_1 \cup \dots \cup \Lambda_n$  and of  $\partial V$  respectively such that their closures are disjoint. Define a continuous function  $\psi_V : M \rightarrow [0, 1]$  so that

- (i)  $\psi_V(y) = 1$  for any  $y \in \mathcal{V}$ ;
- (ii)  $\psi_V(y) = 0$  for any  $y \in (M \setminus V) \cup \mathcal{W}$ ;
- (iii)  $\psi_V(y) \in (0, 1)$  for any other  $y$ .

Denote as  $\mathcal{B}$  an open neighborhood of  $\partial B_{\frac{\varepsilon_{n+1}}{2}}^m(x_{n+1})$  whose closure is disjoint from  $\Lambda_{n+1}$ . Define a continuous function  $\psi_{n+1} : M \rightarrow [0, 1]$  so that

- (i)  $\psi_{n+1}(y) = 1$  for any  $y \in \Lambda_{n+1}$ ;
- (ii)  $\psi_{n+1}(y) = 0$  for any  $y \in (M \setminus B_{\frac{\varepsilon_{n+1}}{2}}^m(x_{n+1})) \cup \mathcal{B}$ ;
- (iii)  $\psi_{n+1}(y) \in (0, 1)$  for any other  $y$ .

Consider then, when defined, the linear map

$$\mathbb{L}_y := \frac{1}{\psi_{n+1}(y) + \psi_V(y)} [\psi_{n+1}(y)L_y^{x_{n+1}} + \psi_V(y)\mathfrak{L}_y^{x_{n+1}}].$$

Extend so the bundle  $E$  as  $E''$  on  $\tilde{V} := (V \cup B_{\frac{\varepsilon_{n+1}}{2}}^m(x_{n+1})) \setminus (\overline{\mathcal{W}} \cap \overline{\mathcal{B}})$  as follows

$$E''(y) := \begin{cases} E'(y) & \text{if } y \in \tilde{V} \setminus B_{\frac{\varepsilon_{n+1}}{2}}^m(x_{n+1}) \\ \text{Graph}(L_y^{x_{n+1}}) & \text{if } y \in \tilde{V} \setminus V \\ \text{Graph}(\mathbb{L}_y) & \text{if } y \in \tilde{V} \cap V \cap B_{\frac{\varepsilon_{n+1}}{2}}^m(x_{n+1}). \end{cases}$$

The set  $\tilde{V}$  is a neighborhood of  $\Lambda_1 \cup \dots \cup \Lambda_{n+1}$ . The extension  $E''$  is continuous because for  $y \in \tilde{V} \cap V \cap B_{\frac{\varepsilon}{2}}^m(x_{n+1})$  it holds  $\psi_{n+1}(y) + \psi_V(y) \neq 0$  and because for  $y \in \partial V \cap B_{\frac{\varepsilon}{2}}^m(x_{n+1})$  (respectively for  $y \in \partial B_{\frac{\varepsilon}{2}}^m(x_{n+1}) \cap V$ ) it holds that  $\text{Graph}(\mathbb{L}_y) = \text{Graph}(L_y^{x_{n+1}})$  (respectively  $\text{Graph}(\mathbb{L}_y) = \text{Graph}(\mathfrak{L}_y^{x_{n+1}})$ ). Moreover for  $y \in \Lambda \cap V \cap B_{\frac{\varepsilon}{2}}^m(x_{n+1})$  we have that  $L_y^{x_{n+1}} = \mathfrak{L}_y^{x_{n+1}}$  and so  $\mathbb{L}_y = L_y^{x_{n+1}}$ .

The proof is so ended by induction. □

Thanks to Proposition B.0.2, there exists an open neighborhood of  $\Lambda$  where we extend continuously both the unstable and the stable bundle. Since the splitting is hyperbolic at  $\Lambda$  and transversality is an open condition, we find a neighborhood  $W$  of  $\Lambda$  where the splitting  $E^u + E^s$  is continuous and transversal.

**Remark B.0.1.** Since the splitting  $E^u \oplus E^s$  is continuous on  $W$ , we deduce that the functions

$$W \ni x \mapsto C_x^\eta \cap \mathbb{S}^{m-1} \in T_x M$$

and

$$W \ni x \mapsto \overline{\mathbb{S}^{m-1} \setminus C_x^{\eta\delta}} \in T_x M$$

are continuous (for any  $\eta \in \mathbb{R}_+, \delta \in (0, 1)$ ).

We proceed now with the proof of Proposition B.0.1.

*Proof of Proposition B.0.1.* The hyperbolic set  $\Lambda$  satisfies the cone field property with respect to  $\eta \in \mathbb{R}_+, \delta \in (0, 1), \xi \in (0, 1), m \in \mathbb{N}^*$ . Let  $W$  be an open neighborhood of  $\Lambda$  on which we extend continuously the hyperbolic splitting  $E^u \oplus E^s$  (see Lemma B.0.1). In particular condition (i) of Definition B.0.1 is satisfied.

From Remark B.0.1 we deduce that the function

$$W \cap f^{-1}(W) \ni x \mapsto \mathcal{H}(x) := \min_{v \in Df(x)C_x^\eta \cap \mathbb{S}^{m-1}} d\left(v, \overline{\mathbb{S}^{m-1} \setminus C_{f(x)}^{\eta(\delta+\zeta)}}\right) \in \mathbb{R}$$

is continuous. From condition (ii) of Definition B.0.1 for  $\Lambda$  it holds that

$$\min_{x \in \Lambda} \mathcal{H}(x) =: \rho > 0.$$

By the continuity of the function there exists  $V \subset W$  such that for any  $x \in V \cap f^{-1}(V)$  we have that  $\mathcal{H}(x) > \frac{\rho}{2} > 0$ . That is, for any  $x \in V \cap f^{-1}(V)$  it holds that

$$Df(x)C_x^\eta \subset C_{f(x)}^{\eta(\delta+\zeta)},$$

i.e. condition (ii) of Definition B.0.1 is satisfied.

Because of the continuity of  $Df^m$  with respect to the point and the continuity of the cones, there exists an open neighborhood  $\mathcal{U}$  of  $\Lambda$  contained in  $V$  such that for any  $x \in \mathcal{U}$  there exists  $y \in \Lambda$  so that  $x, y$  belong to the domain of a same chart,

$$\|Df^m(x) - Df^m(y)\| < \frac{\zeta}{2(\xi + \zeta)}$$

and

$$d_H(C_x^\eta \cap \mathbb{S}^{m-1}, C_y^\eta \cap \mathbb{S}^{m-1}) < \frac{1}{R} \frac{\zeta}{2(\xi + \zeta)},$$

where  $R := \max_{y \in \Lambda} \|Df^m(y)\| > 0$ .

So, identifying their tangent spaces through the chart, for any  $v \in C_x^\eta \cap \mathbb{S}^{m-1}$  there exists  $w \in C_y^\eta \cap \mathbb{S}^{m-1}$  so that  $\|v - w\| < \frac{1}{R} \frac{\zeta}{2(\xi + \zeta)}$  and consequently

$$\begin{aligned} \|Df^m(x)v\| &\geq \|Df^m(y)w\| - \|Df^m(x) - Df^m(y)\|\|v\| - \|Df^m(y)\|\|v - w\| \geq \\ &\geq \frac{1}{\xi}\|w\| - \frac{\zeta}{2(\xi + \zeta)}\|v\| - \|Df^m(y)\|\frac{1}{R} \frac{\zeta}{2(\xi + \zeta)} \geq \frac{1}{\xi} - \frac{\zeta}{\xi + \zeta} = \end{aligned}$$

$$= \frac{1}{\xi + \zeta} \|w\| = \frac{1}{\xi + \zeta} \|v\|.$$

That is the first condition of (iii) of Definition B.0.1 holds. Arguing similarly, we obtain a neighborhood  $U$  of  $\Lambda$  contained in  $\mathcal{U}$  so that also the second condition of (iii) of Definition B.0.1 holds.

The set  $U$  is so a neighborhood of  $\Lambda$  which satisfies the cone field property for  $\eta \in \mathbb{R}_+$ ,  $\xi + \zeta \in (0, 1)$ ,  $\delta + \zeta \in (0, 1)$ ,  $m \in \mathbb{N}^*$ . □

### B.0.1 Extension of the Cone Field Property on $\{q\} \cup \mathcal{O}(p, f^N)$

In Chapter 4 we work with a surface diffeomorphism  $f^N : S \rightarrow S$  and the hyperbolic set  $\Lambda = \{q\} \cup \mathcal{O}(p, f^N)$ , where  $q$  is a fixed hyperbolic point for  $f^N$  and  $p \in (W^u(q) \cap W^s(q)) \setminus \{q\}$ . In such a framework we can prove something more.

Let  $\eta \in \mathbb{R}_+$ ,  $\delta, \xi \in (0, 1)$ ,  $m \in \mathbb{N}^*$  be the constants with respect to which  $\Lambda = \{q\} \cup \mathcal{O}(p, f^N)$  satisfies the cone field property. Denote as  $E^u \oplus E^s = T_\Lambda S$  the hyperbolic splitting of  $\Lambda$ .

**Proposition B.0.3.** *There exists a continuous non singular vector field  $e^u : \Lambda \rightarrow E_\Lambda^u \cap \mathbb{S}^1$  (respectively  $e^s : \Lambda \rightarrow E_\Lambda^s \cap \mathbb{S}^1$ ).*

**Notation B.0.1.** In the proof of Lemma B.0.3 we will use the standard Euclidean norm  $\|\cdot\|$  and denote as  $\langle \cdot, \cdot \rangle$  the standard scalar product.

*Proof.* Let us show the result for the unstable subspace.

Fix  $0 < \varepsilon < \sqrt{2} - 1$ . By the continuity of the unstable subspace, there exists  $\delta > 0$  such that for any  $x \in \Lambda$ ,  $d(x, q) < \delta$  it holds

$$d_H(E_q^u \cap \mathbb{S}^1, E_x^u \cap \mathbb{S}^1) < \varepsilon.$$

Consider the closed ball  $\overline{B_{\frac{\delta}{2}}(q)}$ . There is a finite number  $n \in \mathbb{N}$  of points of  $\Lambda$  which do not belong to this closed ball. Denote those points as  $p_i, i = 1, \dots, n$ . Let  $U_i$  for  $i = 0, \dots, n$  be open disjoint neighborhoods of  $\overline{B_{\frac{\delta}{2}}(q)}$  and  $p_i$  respectively, in particular such that  $U_i \cap \Lambda = \{p_i\}$  for any  $i \in \llbracket 1, n \rrbracket$ .

Choose  $e_q^u \in E_q^u \cap \mathbb{S}^1$  and  $e_{p_i}^u \in E_{p_i}^u \cap \mathbb{S}^1$  for any  $i = 1, \dots, n$ . The continuity of the vector field  $e^u$  at  $p_i$  for any  $i$  follows immediately.

Let  $x \in \Lambda \cap \overline{B_{\frac{\delta}{2}}(q)} = \Lambda \cap U_0$ . Define the vector  $e_x^u$  so that

$$\|e_x^u - e_q^u\| = \min_{v \in E_x^u \cap \mathbb{S}^1} \|v - e_q^u\|.$$

Observe that  $\|e_x^u - e_q^u\| < \varepsilon$ . The vector field  $e^u$  is uniquely defined. Indeed,  $\min_{v \in E_x^u \cap \mathbb{S}^1} \|v - e_q^u\| = \min\{\|e_x^u - e_q^u\|, \|e_x^u + e_q^u\|\}$  and

$$\|e_x^u + e_q^u\| = \|e_x^u - e_q^u\| + 4 \langle e_x^u, e_q^u \rangle$$

Since

$$\langle e_x^u, e_q^u \rangle \geq \|e_q^u\|^2 - |\langle e_x^u - e_q^u, e_q^u \rangle| \geq 1 - \|e_x^u - e_q^u\| > 1 - \varepsilon > 0,$$

we deduce that  $\|e_x^u + e_q^u\| > \|e_x^u - e_q^u\|$ . This shows that  $e_x^u$  is uniquely defined.

Let us show the continuity of  $e^u$  at  $x \in \Lambda \cap U_0$ . Let  $x \in \Lambda \cap U_0$  and fix  $\varepsilon' > 0$ . By the continuity of  $E^u$  there exists  $0 < \delta' < \frac{\delta}{3}$  such that for any  $y \in \Lambda$ ,  $d(x, y) < \delta'$  it holds

$$d_H(E_x^u \cap \mathbb{S}^1, E_y^u \cap \mathbb{S}^1) < \varepsilon'.$$

Observe that  $d(y, q) < \delta$  (since  $d(x, y) < \frac{\delta}{3}$  and  $d(q, x) < \frac{\delta}{2}$ ) so it holds

$$d_H(E_y^u \cap \mathbb{S}^1, E_q^u \cap \mathbb{S}^1) < \varepsilon.$$

By showing that  $\|e_x^u - e_y^u\| < \|e_x^u + e_y^u\|$  we deduce the continuity of the vector field  $e^u$  at  $x$ .

We have

$$\|e_y^u + e_x^u\| = \|e_y^u - e_x^u\| + 4 \langle e_x^u, e_y^u \rangle.$$

In particular, since  $\varepsilon < \sqrt{2} - 1$ ,

$$\begin{aligned} \langle e_x^u, e_y^u \rangle &= \|e_q^u\|^2 + \langle e_x^u - e_q^u, e_q^u \rangle + \langle e_y^u - e_q^u, e_q^u \rangle + \langle e_x^u - e_q^u, e_y^u - e_q^u \rangle \geq \\ &\geq 1 - \|e_x^u - e_q^u\| - \|e_y^u - e_q^u\| - \|e_x^u - e_q^u\| \|e_y^u - e_q^u\| > 1 - 2\varepsilon - \varepsilon^2 > 0. \end{aligned}$$

That is,  $\|e_y^u + e_x^u\| > \|e_y^u - e_x^u\|$  and so

$$\|e_y^u - e_x^u\| = \min_{v \in E_y^u \cap \mathbb{S}^1} \|v - e_x^u\| \leq d_H(E_y^u \cap \mathbb{S}^1, E_x^u \cap \mathbb{S}^1) < \varepsilon'.$$

□

Consequently, combining Proposition B.0.2 and Proposition B.0.3, we obtain the following

**Lemma B.0.2.** *There exist a neighborhood  $W$  of  $\Lambda = \{q\} \cup \mathcal{O}(p, f^N)$  which admits a continuous extension of  $E^u$  and  $E^s$  and continuous non singular vector fields*

$$e^1 : W \rightarrow E_W^u \cap \mathbb{S}^1 \quad \text{and} \quad e^2 : W \rightarrow E_W^s \cap \mathbb{S}^1$$

*such that  $e_{|\Lambda}^1 = e^u, e_{|\Lambda}^2 = e^s$ .*

The proof follows the ideas of the proofs of Proposition B.0.3 and we omit it.

# Appendix C

## Geometric Markov partition

In this Appendix we define geometric Markov partitions and discuss the symbolic dynamics associated to them. Our main references will be [PT93] and [GZ04]. Let  $S$  be a surface among  $\mathbb{R}^2, \mathbb{A}$  and  $\mathbb{T}^2$  and let  $f : S \rightarrow S$  be a  $\mathcal{C}^1$  diffeomorphism. Let  $\|\cdot\|$  be a norm on  $S$ . Let  $\Lambda \subset S$  be a hyperbolic set for  $f$ . Denote as  $E^s \oplus E^u = T_\Lambda S$  its continuous hyperbolic splitting. Let  $(C_x^{u,\eta})_{x \in \Lambda}, (C_x^{s,\eta})_{x \in \Lambda}$  with  $0 < \eta < 1$  be the unstable and stable cone fields

$$C_x^{u,\eta} = \{v \in T_x S : v = v^s + v^u, v^s \in E_x^s, v^u \in E_x^u, \|v^s\| \leq \eta \|v^u\|\},$$

$$C_x^{s,\eta} = \{v \in T_x S : v = v^s + v^u, v^s \in E_x^s, v^u \in E_x^u, \|v^u\| \leq \eta \|v^s\|\}$$

such that  $\Lambda$  satisfies the cone field property for  $f$  with respect to  $\eta \in (0, 1), \delta, \mu \in (0, 1), m \in \mathbb{N}^*$  and  $(C_x^{u,\eta})_{x \in \Lambda}$  and such that  $\Lambda$  satisfies the cone field property for  $f^{-1}$  with respect to  $\eta \in (0, 1), \mu \in (0, 1), l \in \mathbb{N}^*$  and  $(C_x^{s,\eta})_{x \in \Lambda}$  (see Definition B.0.1 in Appendix C).

Observe that for any  $x \in \Lambda$  it holds  $C_x^{u,\eta} \cap C_x^{s,\eta} = \{0\}$ .

Fix  $0 < \xi < \min(1 - \delta, 1 - \mu)$ . By Lemma B.0.1, let  $U$  be an open neighborhood of  $\Lambda$  on which we extend continuously the splitting  $E^s \oplus E^u$  and the cone fields  $(C_x^{u,\eta})_{x \in U}, (C_x^{s,\eta})_{x \in U}$  so that  $U$  satisfies the cone field property for  $f$  with respect to  $\eta \in (0, 1), \delta + \xi, \mu + \xi \in (0, 1), m \in \mathbb{N}^*$  and  $(C_x^{u,\eta})_{x \in U}$  and so that  $U$  satisfies the cone field property for  $f^{-1}$  with respect to  $\eta \in (0, 1), \delta + \xi, \mu + \xi \in (0, 1), l \in \mathbb{N}^*$  and  $(C_x^{s,\eta})_{x \in U}$  (see Definition B.0.1 in Appendix C).

We will call  $(C_x^{u,\eta})_{x \in U}, (C_x^{s,\eta})_{x \in U}$  the unstable and stable cone fields respectively. Remark that for any  $x \in U$  it holds  $C_x^{u,\eta} \cap C_x^{s,\eta} = \{0\}$ .

### C.1 Definition of geometric Markov partition

**Definition C.1.1.** Let  $\gamma : [0, 1] \rightarrow U$  be a  $\mathcal{C}^1$  embedding. Then  $\gamma$  is an unstable curve if for any  $t \in [0, 1]$  it holds

$$\gamma'(t) \in C_{\gamma(t)}^{u,\eta}.$$

Similarly,  $\gamma$  is a stable curve if for any  $t \in [0, 1]$  it holds

$$\gamma'(t) \in C_{\gamma(t)}^{s,\eta}.$$



**Definition C.1.2.** A rectangle  $R$  is a  $\mathcal{C}^1$  embedding

$$R : [0, 1]^2 \rightarrow U$$

such that for any  $t \in [0, 1]$  the curve  $R(\{t\} \times [0, 1])$  is stable and the curve  $R([0, 1] \times \{t\})$  is unstable.

**Notation C.1.1.** Denote as  $\partial_0^s R$  the set  $R(\{0\} \times [0, 1])$  and as  $\partial_1^s R$  the set  $R(\{1\} \times [0, 1])$ . We call them the left and right stable boundaries of  $R$ , respectively. The stable boundary of  $R$  is

$$\partial^s R = \partial_0^s R \cup \partial_1^s R.$$

Similarly, denote as  $\partial_0^u R$  the set  $R([0, 1] \times \{0\})$  and as  $\partial_1^u R$  the set  $R([0, 1] \times \{1\})$ . We call them the lower and upper unstable boundaries of  $R$ , respectively. The unstable boundary of  $R$  is

$$\partial^u R = \partial_0^u R \cup \partial_1^u R.$$

With an abuse of notation, we denote as  $R$  the image  $R([0, 1]^2)$ .

**Definition C.1.3.** Let  $R, R'$  be rectangles. Then  $R$  is a stable subrectangle of  $R'$  if  $R([0, 1]^2) \subset R'([0, 1]^2)$  and  $\partial^u R \subset \partial^u R'$ . Similarly,  $R$  is an unstable subrectangle of  $R'$  if  $R([0, 1]^2) \subset R'([0, 1]^2)$  and  $\partial^s R \subset \partial^s R'$  (see Figure C.1).

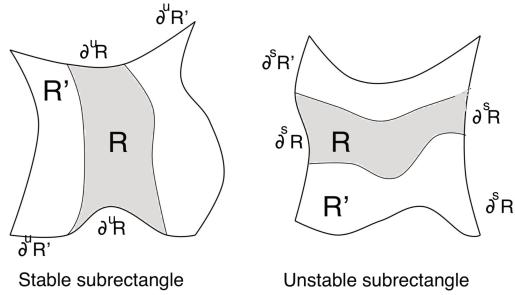


Figure C.1 – Examples of stable and unstable subrectangles.

Let  $\mathcal{R} = \{R_1, \dots, R_N\}$  be a finite family of rectangles such that  $\bigcup_{i=1}^N R_i \subset U$ .

**Definition C.1.4.** Let  $R_j, R_k$  be rectangles of  $\mathcal{R}$ . Then  $R_j$  is  $f$ -linked to  $R_k$  and we write  $R_j \xrightarrow{f} R_k$  if

- (i)  $R_{jk}^f := f(R_j) \cap R_k \neq \emptyset$ ;
- (ii)  $R_{jk}^f$  is an unstable subrectangle of  $R_k$ ;
- (iii)  $f(\partial^s R_j) \cap (\text{int } R_k) = \emptyset$  and  $(\text{int } f(R_j)) \cap \partial^u R_k = \emptyset$ .

Similarly,  $R_j$  is  $f^{-1}$ -linked to  $R_k$  and we write  $R_j \xrightarrow{f^{-1}} R_k$  if

- (i)  $R_{jk}^{f^{-1}} := f^{-1}(R_j) \cap R_k \neq \emptyset$ ;

(ii)  $R_{jk}^{f^{-1}}$  is a stable subrectangle of  $R_k$ ;

(iii)  $f^{-1}(\partial^u R_j) \cap (\text{int } R_k) = \emptyset$  and  $(\text{int } f^{-1}(R_j)) \cap \partial^s R_k = \emptyset$ .

**Fact C.1.1.** Let  $R_j, R_k \in \mathcal{R}$ . Then,  $R_j \xrightarrow{f} R_k$  if and only if  $R_k \xrightarrow{f^{-1}} R_j$ .

**Definition C.1.5.** A geometric Markov partition (or a partition through covering relations) is a finite set  $\mathcal{R}$  of rectangles  $\{R_1, \dots, R_N\}$  so that  $\bigcup_{i=1}^N R_i \subset U$  and such that for any  $j, k \in \{1, \dots, N\}$  either  $R_{jk}^f = \emptyset$  or  $R_j \xrightarrow{f} R_k$ .

**Definition C.1.6.** Let  $\mathcal{R} = \{R_1, \dots, R_N\}$  be a geometric Markov partition. An admissible sequence  $(j_i)_{i \in \mathbb{Z}} \in \{1, \dots, N\}^{\mathbb{Z}}$  for  $\mathcal{R}$  is a bi-infinite sequence such that for any  $i \in \mathbb{Z}$  it holds

$$R_{j_i} \xrightarrow{f} R_{j_{i+1}}.$$

## C.2 Link between symbolic dynamics and geometric Markov partition

The main result concerning the relation between symbolic dynamics and geometric Markov partition is the following Theorem.

**Theorem C.2.1.** The sequence  $(j_i)_{i \in \mathbb{Z}}$  is an admissible sequence for  $\mathcal{R}$  if and only if there exists a unique  $x \in U$  such that  $f^i(x) \in R_{j_i}$  for any  $i \in \mathbb{Z}$ .

*Proof of ( $\Leftarrow$ ) of Theorem C.2.1.* If there exists a point  $x$  such that  $f^i(x) \in R_{j_i}$  for any  $i \in \mathbb{Z}$ , then the sequence  $(j_i)_{i \in \mathbb{Z}}$  is an admissible one since, from the definition of geometric Markov partition and because  $R_{j_i j_{i+1}}^f \neq \emptyset$  for any  $i \in \mathbb{Z}$ , it holds that  $R_{j_i} \xrightarrow{f} R_{j_{i+1}}$  for any  $i \in \mathbb{Z}$ . □

In order to show the other implication of Theorem C.2.1, we first need to introduce some notations.

**Notation C.2.1.** Let  $(j_i)_{i \in \mathbb{Z}}$  be an admissible sequence for  $\mathcal{R}$ . For any  $k \in \mathbb{N}$  let us denote

$$D_k^s((j_i)_{i \in \mathbb{Z}}) = D_k^s := \bigcap_{i=0}^k f^{-i}(R_{j_i}) \quad \text{and} \quad D_k^u((j_i)_{i \in \mathbb{Z}}) = D_k^u := \bigcap_{i=0}^k f^i(R_{j_{-i}}).$$

Denote as

$$D_\infty^s := \lim_{k \rightarrow +\infty} D_k^s = \bigcap_{i=0}^{+\infty} f^{-i}(R_{j_i}) \quad \text{and} \quad D_\infty^u := \lim_{k \rightarrow +\infty} D_k^u = \bigcap_{i=0}^{+\infty} f^i(R_{j_{-i}}).$$

The sequence  $(D_k^s)_{k \in \mathbb{N}}$  is non increasing, that is for any  $k \in \mathbb{N}$  it holds

$$D_{k+1}^s \subseteq D_k^s.$$

Moreover, all the sets  $D_k^s$  are compact. It follows then that  $D_\infty^s$  is not empty. Similarly, also  $D_\infty^u$  is not empty.

The proof of Theorem C.2.1 will be a consequence of the following propositions.

**Proposition C.2.1.** *Let  $R$  be a rectangle. Let  $\gamma : [0, 1] \rightarrow R$  be a stable curve such that  $\gamma(0) \in \partial_0^u R$  and  $\gamma(1) \in \partial_1^u R$ . Let  $\Gamma : [0, 1] \rightarrow R$  be an unstable curve such that  $\Gamma(0) \in \partial_0^s R$  and  $\Gamma(1) \in \partial_1^s R$ . Then there exists a unique  $x \in R$  such that  $\gamma \cap \Gamma = \{x\}$ .*

**Proposition C.2.2.** *The curve  $D_\infty^s$  (respectively  $D_\infty^u$ ) is  $C^1$  and it is a stable (respectively unstable) curve joining  $\partial_0^u R_{j_0}$  and  $\partial_1^u R_{j_0}$  (respectively  $\partial_0^s R_{j_0}$  and  $\partial_1^s R_{j_0}$ ).*

*Proof of ( $\Rightarrow$ ) of Theorem C.2.1.* Let  $(j_i)_{i \in \mathbb{Z}}$  be an admissible sequence for  $\mathcal{R}$ . From Propositions C.2.2 and C.2.1, there exists a unique point  $x \in D_\infty^s((j_i)_{i \in \mathbb{Z}}) \cap D_\infty^u((j_i)_{i \in \mathbb{Z}})$ . From the definition of  $D_\infty^s((j_i)_{i \in \mathbb{Z}})$  and of  $D_\infty^u((j_i)_{i \in \mathbb{Z}})$ , we conclude that  $f^i(x) \in R_{j_i}$  for any  $i \in \mathbb{Z}$ . □

The following result will be used throughout the whole appendix.

**Lemma C.2.1.** *Let  $R$  be a rectangle and let  $x \in R$ . Then for any  $v \in C_x^{u,\eta} \setminus \{0\}$  (respectively  $v \in C_x^{s,\eta} \setminus \{0\}$ ) it holds*

$$Dp_1(R^{-1}(x))DR^{-1}(x)v \neq 0 \quad (\text{respectively } Dp_2(R^{-1}(x))DR^{-1}(x)v \neq 0).$$

*Proof.* If by contradiction there exists  $x \in R$  and  $v \in C_x^{u,\eta} \setminus \{0\}$  such that

$$Dp_1(R^{-1}(x))DR^{-1}(x)v = 0,$$

then  $DR^{-1}(x)v$  belongs to  $\mathbb{R}(0, 1)$  and so, by the definition of rectangle (see Definition C.1.2), the vector  $v$  belongs to the stable cone  $C_x^{s,\eta}$ . Since  $C_x^{s,\eta} \cap C_x^{u,\eta} = \{0\}$  and  $v \neq 0$ , we obtain the required contradiction. □

## C.3 Proof of Proposition C.2.1

A key point for the proof of Proposition C.2.1 is an immediate outcome of Lemma C.2.1.

**Lemma C.3.1.** *Let  $R$  be a rectangle and let  $\gamma : [0, 1] \rightarrow R$  be an unstable (respectively stable) curve. Then for any  $t \in [0, 1]$  the vector*

$$DR^{-1}(\gamma(t))\gamma'(t)$$

*does not belong to  $\mathbb{R}(0, 1)$  (respectively  $\mathbb{R}(1, 0)$ ).*

From Lemma C.3.1 we deduce

**Lemma C.3.2.** *Let  $R$  be a rectangle. Let  $\gamma : [0, 1] \rightarrow R$  be a stable curve (respectively an unstable curve). Then, the curve  $R^{-1} \circ \gamma$  is the graph of a  $C^1$  function of the second coordinate (respectively of the first coordinate) in  $[0, 1]^2$  defined on  $p_2 \circ R^{-1} \circ \gamma([0, 1])$  (respectively defined on  $p_1 \circ R^{-1} \circ \gamma([0, 1])$ ).*

*Proof.* Let  $\gamma$  be a stable curve in  $R$  and consider the curve

$$[0, 1] \ni t \mapsto R^{-1}(\gamma(t)) \in [0, 1]^2.$$

From Lemma C.3.1, we deduce that the function

$$[0, 1] \ni t \mapsto y(t) := p_2 \circ R^{-1} \circ \gamma(t) \in [0, 1]$$

is a  $\mathcal{C}^1$  diffeomorphism to its image, that is to  $p_2 \circ R^{-1} \circ \gamma([0, 1])$ . Consequently, the curve  $R^{-1} \circ \gamma$  is the graph of the  $\mathcal{C}^1$  function

$$p_2 \circ R^{-1} \circ \gamma([0, 1]) \ni s \mapsto p_1 \circ R^{-1} \circ \gamma(y^{-1}(s)) \in [0, 1].$$

□

**Remark C.3.1.** Let  $\gamma$  be a stable curve in  $R$  such that  $\gamma(0) \in \partial_0^u R$  and  $\gamma(1) \in \partial_1^u R$ . Therefore, the projection  $p_2 \circ R^{-1} \circ \gamma([0, 1])$  is  $[0, 1]$ . By Lemma C.3.2,  $R^{-1} \circ \gamma$  is a graph of a function of the second coordinate, defined on  $[0, 1]$ . In particular

$$\gamma([0, 1]) \cap \partial_0^u R = \{\gamma(0)\} \quad \text{and} \quad \gamma([0, 1]) \cap \partial_1^u R = \{\gamma(1)\}.$$

Let us introduce now the notions of half-cones on a rectangle  $R$ .

**Notation C.3.1.** Let  $R$  be a rectangle in  $U$ . The positive unstable half-cone at  $x \in R([0, 1]^2)$  is denoted as  $C_x^{u, \eta, +}$  and it is the connected component of  $C_x^{u, \eta} \setminus \{0\}$  to which the vector  $DR(R^{-1}(x))(1, 0)$  belongs. The negative unstable half-cone at  $x \in R([0, 1]^2)$  is denoted as  $C_x^{u, \eta, -}$  and it is  $C_x^{u, \eta} \setminus (\{0\} \cup C_x^{u, \eta, +})$ .

The positive stable half-cone at  $x \in R([0, 1]^2)$  is denoted as  $C_x^{s, \eta, +}$  and it is the connected component of  $C_x^{s, \eta} \setminus \{0\}$  to which the vector  $DR(R^{-1}(x))(0, 1)$  belongs. The negative stable half-cone at  $x \in R([0, 1]^2)$  is denoted as  $C_x^{s, \eta, -}$  and it is  $C_x^{s, \eta} \setminus (\{0\} \cup C_x^{s, \eta, +})$ .

**Proposition C.3.1.** *Let  $\gamma$  be an unstable curve contained in  $R$  such that  $\gamma(0) \in \partial_0^s R$  (respectively  $\gamma(0) \in \partial_1^s R$ ). Then for any  $t \in [0, 1]$  the vector  $\gamma'(t)$  belongs to the positive unstable half-cone  $C_{\gamma(t)}^{u, \eta, +}$  (respectively to the negative unstable half-cone  $C_{\gamma(t)}^{u, \eta, -}$ ).*

*Let  $\gamma$  be a stable curve contained in  $R$  such that  $\gamma(0) \in \partial_0^u R$  (respectively  $\gamma(0) \in \partial_1^u R$ ). Then for any  $t \in [0, 1]$  the vector  $\gamma'(t)$  belongs to the positive stable half-cone  $C_{\gamma(t)}^{s, \eta, +}$  (respectively to the negative stable half-cone  $C_{\gamma(t)}^{s, \eta, -}$ ).*

*Proof.* We are going to prove the statement for unstable curves assuming that  $\gamma(0) \in \partial_0^s R$ . The unstable curve is contained in  $R$ . By Lemma C.3.1, for any  $t \in [0, 1]$  the image

$$\frac{d}{dt} p_1 \circ R^{-1} \circ \gamma(t) \neq 0. \tag{C.1}$$

Observe that for any  $x \in R$  the image  $Dp_1(R^{-1}(x))DR^{-1}(x)(C_x^{u, \eta} \setminus \{0\}) \subset \mathbb{R}$  does not contain 0. Otherwise we would have  $C_x^{u, \eta} \cap C_x^{s, \eta} \neq \{0\}$  which is not possible. Since  $C_x^{u, \eta, +}$  is connected and by definition it contains  $DR(R^{-1}(x))(1, 0)$ , we deduce that for any  $x \in R$

$$Dp_1(R^{-1}(x))DR^{-1}(x)C_x^{u, \eta, +} \subset \mathbb{R}_+. \tag{C.2}$$

Similarly

$$Dp_1(R^{-1}(x))DR^{-1}(x)C_x^{u, \eta, -} \subset \mathbb{R}_-. \tag{C.3}$$

Consider the function

$$[0, 1] \ni t \mapsto Dp_1(R^{-1} \circ \gamma(t))DR^{-1}(\gamma(t))\gamma'(t) \in \mathbb{R}.$$

It is continuous and never vanishes because of (C.1). In particular, it has constant sign. From (C.2) and (C.3), we deduce that for any  $t \in [0, 1]$  the vector  $\gamma'(t)$  belongs to the same unstable half-cone as  $\gamma'(0)$ . It is so sufficient showing that  $\gamma'(0) \in C_{\gamma(0)}^{u, \eta, +}$ .

The point  $R^{-1} \circ \gamma(0)$  belongs to  $\{0\} \times [0, 1]$  because  $\gamma(0) \in \partial_0^s R$ . Observe that for any  $t \in (0, 1]$  it holds  $p_1 \circ R^{-1} \circ \gamma(t) > 0$ . Consequently

$$Dp_1(R^{-1}(\gamma(0)))(DR^{-1}(\gamma(0))\gamma'(0)) = \lim_{t \rightarrow 0} \frac{p_1 \circ R^{-1} \circ \gamma(t) - p_1 \circ R^{-1} \circ \gamma(0)}{t} > 0,$$

where the strict inequality comes again from (C.1). From (C.2) and (C.3), we deduce that  $\gamma'(0) \in C_{\gamma(0)}^{u, \eta, +}$ , concluding so the proof.  $\square$

The following result is an outcome of Proposition C.3.1.

**Corollary C.3.1.** *There does not exist  $\gamma : [0, 1] \rightarrow R$  unstable curve such that  $\gamma(0), \gamma(1) \in \partial_0^s R$  (or  $\partial_1^s R$ ).*

*There does not exist  $\Gamma : [0, 1] \rightarrow R$  stable curve such that  $\Gamma(0), \Gamma(1) \in \partial_0^u R$  (or  $\partial_1^u R$ ).*

*Proof.* We prove the result for  $\gamma$  unstable curve. Argue by contradiction and assume there exists an unstable curve contained in  $R$  such that  $\gamma(0), \gamma(1) \in \partial_0^s R$ . The vector  $\gamma'(0)$  points inside the rectangle  $R$ , while  $\gamma'(1)$  points outside  $R$ . In particular,  $\gamma'(0)$  and  $\gamma'(1)$  belong to different unstable half-cone fields. This contradicts Proposition C.3.1 and we conclude.  $\square$

We can now show Proposition C.2.1.

*Proof of Proposition C.2.1.* By Lemma C.3.2 and Remark C.3.1 the curve  $R^{-1} \circ \gamma$  is the graph of a  $\mathcal{C}^1$  function with respect to the second coordinate defined on  $[0, 1]$  and  $R^{-1} \circ \Gamma$  is the graph of a  $\mathcal{C}^1$  function with respect to the first coordinate defined on  $[0, 1]$ . In particular,  $\partial^u R \cap \gamma = \{\gamma(0), \gamma(1)\}$ . The curve  $\gamma$  separates  $R$  into two connected components. Observe that  $\partial_0^s R$  is a subset of one component while  $\partial_1^s R$  is a subset of the other component. Since  $\Gamma$  joins  $\partial_0^s R$  and  $\partial_1^s R$ , the intersection  $\gamma \cap \Gamma$  is not empty.

Let now show that  $\gamma$  and  $\Gamma$  intersect only once. Argue by contradiction and assume there exist  $x_1 \neq x_2$  belonging to  $\gamma \cap \Gamma$ . Denote for some  $t_1 < t_2$  and for some  $s_1 \neq s_2$  the points  $\Gamma(t_1) = \gamma(s_1) = x_1, \Gamma(t_2) = \gamma(s_2) = x_2$ .

Denote as

$$t_3 = \min\{t \in (t_1, t_2] : \Gamma(t) \in \gamma\}.$$

Such  $t_3$  is well-defined because, by Lemma C.3.1, the curve  $\Gamma$  is transversal to the curve  $\gamma$  at  $\Gamma(t_1)$ . In particular,  $\Gamma_{[t_1, t_3]}$  is a subset of one of the two components of  $R$  determined by  $\gamma$ . Since by Lemma C.3.1 the curves  $\Gamma$  and  $\gamma$  are transversal at points of intersection, the vectors  $\Gamma'(t_1)$  and  $\Gamma'(t_3)$  point towards different components determined by  $\gamma$ . That is, the couples of vectors  $(\gamma(s_1), \Gamma(t_1))$  and  $(\gamma(s_2), \Gamma(t_2))$  determine opposite orientation. By Proposition C.3.1, both  $\gamma(s_1)$  and  $\gamma(s_2)$  belong to the same stable half-cone field and we so deduce that  $\Gamma(t_1)$  and  $\Gamma(t_2)$  belong to different unstable half-cone fields. This contradicts Proposition C.3.1 and we conclude.  $\square$

## C.4 Proof of Proposition C.2.2

We will deduce Proposition C.2.2 from the following results. We state these propositions for  $D_\infty^s$ , but the analogous statements holds for  $D_\infty^u$ .

**Proposition C.4.1.** *Let  $(j_i)_{i \in \mathbb{Z}}$  be an admissible sequence for  $\mathcal{R}$ . The set  $D_\infty^s = D_\infty^s((j_i)_{i \in \mathbb{Z}})$  is the image through  $R_{j_0}$  of the graph of a Lipschitz function of the second coordinate in  $[0, 1]^2$ . That is*

$$D_\infty^s = R_{j_0}(\{(h_\infty(y), y) : y \in [0, 1]\}),$$

where  $h_\infty : [0, 1] \rightarrow [0, 1]$  is a Lipschitz function.

**Proposition C.4.2.** *Let  $x \in D_\infty^s$  be a point of differentiability of  $D_\infty^s$ . Then the tangent vector to  $D_\infty^s$  at  $x$  belongs to the stable cone  $C_x^{s, \eta}$ .*

**Remark C.4.1.** By Proposition C.4.1, since  $D_\infty^s$  is the image of the graph of a Lipschitz function, almost every point of  $D_\infty^s$  is a point of differentiability at which we can apply Proposition C.4.2.

Actually,  $D_\infty^s$  is contained in the stable manifold of its same points and this implies that every point of  $D_\infty^s$  is of differentiability. Denote as  $d$  the distance determined by the norm  $\|\cdot\|$ .

**Proposition C.4.3.** *Let  $(j_i)_{i \in \mathbb{Z}}$  be an admissible sequence and let  $x \in D_\infty^s = D_\infty^s((j_i)_{i \in \mathbb{Z}})$ . There exists  $N \in \mathbb{N}$  such that*

$$D_\infty^s \subset f^{-N}(W_{loc, \varepsilon}^s(f^N(x))),$$

where  $W_{loc, \varepsilon}^s(f^N(x)) = \{y \in S : d(f^k(y), f^{N+k}(x)) < \varepsilon, \forall k \in \mathbb{N}\}$ .

Assuming these results hold, we deduce the proof of Proposition C.2.2.

*Proof of Proposition C.2.2.* By Proposition C.4.1,  $D_\infty^s$  is the image through  $R_{j_0}$  of the graph of a Lipschitz function joining  $[0, 1] \times \{0\}$  to  $[0, 1] \times \{1\}$ . Since by Proposition C.4.3 the curve  $D_\infty^s$  is contained in  $f^{-N}(W_{loc, \varepsilon}^s(f^N(x)))$  for some  $n \in \mathbb{N}$ , since each local stable manifold is  $\mathcal{C}^1$  and  $f^{-N}$  is  $\mathcal{C}^1$  too, we deduce that  $D_\infty^s$  is a  $\mathcal{C}^1$  curve. In particular, the curve  $D_\infty^s$  is differentiable at every point. By Proposition C.4.2, the tangent vector to  $D_\infty^s$  at any point belongs to the stable cone. That is,  $D_\infty^s$  is a stable curve joining  $\partial_0^u R_{j_0}$  and  $\partial_1^u R_{j_0}$ . □

## C.5 Properties of $D_\infty^s$

The rest of the Appendix concerns the proofs of Propositions C.4.1, C.4.2 and C.4.3, which describes the structure of  $D_\infty^s$ . We start by introducing notations and results that will be used in all the proofs.

**Lemma C.5.1.** *Let  $\gamma : [0, 1] \rightarrow U$  be an unstable curve (respectively a stable curve). Assume that  $f \circ \gamma([0, 1]) \subset U$  (respectively  $f^{-1} \circ \gamma([0, 1]) \subset U$ ). Then,  $f \circ \gamma$  is an unstable curve (respectively  $f^{-1} \circ \gamma$  is a stable curve).*

*Proof.* We prove the statement for  $\gamma$  unstable curve. The curve  $f \circ \gamma$  is a  $\mathcal{C}^1$  embedding. For any  $t \in [0, 1]$  the vector  $\gamma'(t)$  belongs to the unstable cone  $C_{\gamma(t)}^{u, \eta}$ . Since  $U$  satisfies the cone field property for  $f$  with respect to  $\eta, \delta + \xi, \mu + \xi \in (0, 1), m \in \mathbb{N}^*$  and  $(C_x^{u, \eta})_{x \in U}$  and since  $\gamma(t) \in U \cap f^{-1}(U)$  for any  $t \in [0, 1]$ , we have that  $Df(\gamma(t))\gamma'(t) \in C_{f(\gamma(t))}^{u, (\delta+\xi)\eta} \subset C_{f(\gamma(t))}^{u, \eta}$ . That is,  $f \circ \gamma$  is an unstable curve.  $\square$

**Notation C.5.1.** Let  $R_i$  be a rectangle in  $\mathcal{R}$  and let  $x, y \in R_i$ . Let  $\gamma : [0, 1] \rightarrow R_i$  be a stable (unstable) curve such that  $\gamma(t_1) = x, \gamma(t_2) = y$  for some  $0 \leq t_1 < t_2 \leq 1$ . Then

$$L_\gamma(x, y) := \int_{t_1}^{t_2} \|\gamma'(s)\| ds.$$

### C.5.1 About the intersection of stable and unstable boundaries

**Proposition C.5.1.** *Let  $R_1, R_2$  be rectangles such that  $R_1 \xrightarrow{f} R_2$  (respectively  $R_1 \xrightarrow{f^{-1}} R_2$ ). Then both  $f(\partial_0^u R_1) \cap R_2$  and  $f(\partial_1^u R_1) \cap R_2$  (respectively both  $f^{-1}(\partial_0^s R_1) \cap R_2$  and  $f^{-1}(\partial_1^s R_1) \cap R_2$ ) are connected.*

*Proof.* Let us show the result for  $R_1 \xrightarrow{f} R_2$  and for  $f(\partial_0^u R_1) \cap R_2$ . Since  $f(R_1) \cap R_2$  is not empty and since  $(\text{int } f(R_1)) \cap \partial^u R_2$  is empty, at least one among  $f(\partial_0^u R_1)$  and  $f(\partial_1^u R_1)$  intersects  $\partial^s R_2$ . Assume that  $f(\partial_0^u R_1)$  intersects  $\partial^s R_2$ . Since  $f(\partial^s R_1) \cap (\text{int } R_2)$  is empty,  $f(\partial_0^u R_1)$  intersects  $\partial^s R_2$  at least twice.

We say that two points of intersection between  $f(\partial_0^u R_1)$  and  $\partial^s R_2$  are successive if, denoting them as  $f \circ R_1(x_1, 0), f \circ R_1(x_2, 0)$  with  $x_1 < x_2$ , it holds  $f \circ R_1((x_1, x_2) \times \{0\}) \cap \partial^s R_2 = \emptyset$ . We call them internally successive if  $f \circ R_1([x_1, x_2] \times \{0\}) \subset R_2$  and externally successive otherwise.

Each curve contained in  $f \circ R_1([0, 1] \times \{0\})$  is unstable and so, by Corollary C.3.1, two internally successive points of intersection between  $f(\partial_0^u R_1)$  and  $\partial^s R_2$  belong to different components of  $\partial^s R_2$ .

Also  $f(\partial_1^u R_1)$  intersects the stable boundary of  $R_2$ . Argue by contradiction and assume  $f(\partial_1^u R_1) \cap \partial^s R_2 = \emptyset$ . Consider

$$x_{\min} = \min\{t \in [0, 1] : f \circ R_1(t, 0) \in \partial^s R_2\},$$

$$x_{\max} = \max\{t \in [0, 1] : f \circ R_1(t, 0) \in \partial^s R_2\}.$$

- If  $f \circ R_1(x_{\min}, 0)$  and  $f \circ R_1(x_{\max}, 0)$  lie on different components of the stable boundary of  $R_2$ , then  $(\text{int } f(R_1)) \cap \partial^u R_2$  would not be empty, contradicting condition (iii) of Definition C.1.4 (see (a) of Figure C.2).
- Assume that  $f \circ R_1(x_{\min}, 0)$  and  $f \circ R_1(x_{\max}, 0)$  lie on the same component of  $\partial^s R_2$ . Then either  $(\text{int } f(R_1)) \cap \partial^u R_2$  is not empty (see the (b) of Figure C.2) or there exists a stable curve  $\gamma$  in  $R_2$  connecting two points on  $f(\partial_0^u R_1)$  and whose interior is contained in the interior of  $f(R_1)$  (see (c) of Figure C.2).

In the case (b) of Figure C.2 we contradict condition (iii) of Definition C.1.4. In case (c) of Figure C.2  $f^{-1} \circ \gamma$  would be a stable curve in  $R_1$  whose endpoints lie on the same component of  $\partial^u R_1$ , contradicting Corollary C.3.1.

Consequently, also  $f(\partial_1^u R_1)$  intersects  $\partial^s R_2$ . For the same reason of  $f(\partial_0^u R_1)$ ,  $f(\partial_1^u R_1)$  intersects  $\partial^s R_2$  at least twice.

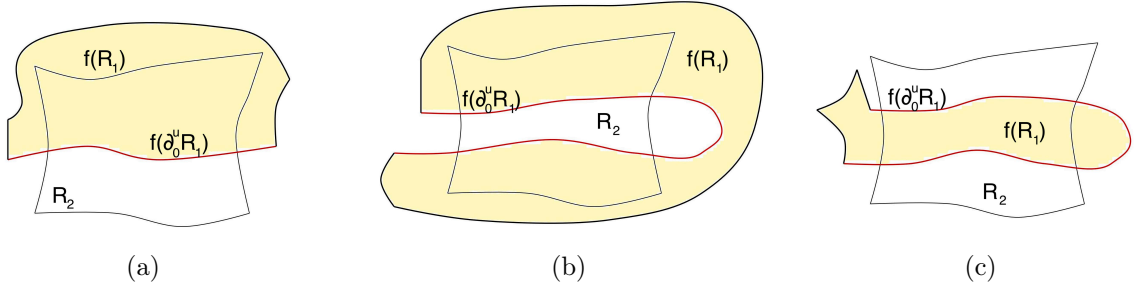


Figure C.2

Argue now by contradiction and assume that  $f(\partial_0^u R_1) \cap R_2$  has two connected components and denote them as  $C_1, C_2$ . Since  $f(R_1) \cap R_2$  is connected,  $f(R_1) \cap R_2$  is contained in the connected component of  $R_2$  included between  $C_1$  and  $C_2$ . Observe that  $C_1$  and  $C_2$  are separated by a connected component of  $f(\partial_1^u R_1) \cap R_2$ . If this is not true, then we would find a stable curve connecting two points of  $f(\partial_0^u R_1)$  whose interior is contained in the interior of  $R_2$ : its image through  $f^{-1}$  would be a stable curve connecting the same component of  $\partial^u R_1$ , contradicting Corollary C.3.1. Consequently,  $f(R_1) \cap R_2$  is not connected, contradicting the fact that  $f(R_1) \cap R_2$  is a subrectangle.  $\square$

**Proposition C.5.2.** *Let  $R_1, R_2$  be rectangles such that  $R_1 \xrightarrow{f} R_2$ . Let  $\gamma : [0, 1] \rightarrow R_1$  be an unstable curve such that  $\gamma(0) \in \partial_0^s R_1$  and  $\gamma(1) \in \partial_1^s R_1$ . Then  $f \circ \gamma([0, 1]) \cap R_2$  is a connected unstable curve joining  $\partial_0^s R_2$  and  $\partial_1^s R_2$ .*

*Similarly, let  $R_1, R_2$  be rectangles such that  $R_1 \xrightarrow{f^{-1}} R_2$  and let  $\gamma : [0, 1] \rightarrow R_1$  be a stable curve such that  $\gamma(0) \in \partial_0^u R_1$  and  $\gamma(1) \in \partial_1^u R_1$ . Then  $f^{-1} \circ \gamma([0, 1]) \cap R_2$  is a connected stable curve joining  $\partial_0^u R_2$  and  $\partial_1^u R_2$ .*

*Proof.* We prove the first statement. Let  $R_1, R_2$  be such that  $R_1 \xrightarrow{f} R_2$  and let  $\gamma$  be an unstable curve in  $R_1$  joining  $\partial_0^s R_1$  and  $\partial_1^s R_1$ . By Lemma C.5.1 any connected component of the curve  $f \circ \gamma([0, 1]) \cap R_2$  is an unstable curve. Since  $\gamma$  joins the two components of  $\partial^s R_1$  and since  $f(\partial^s R_1) \cap (\text{int } R_2) = \emptyset$  by condition (iii) of Definition C.1.4, the curve  $f \circ \gamma$  exits the rectangle  $R_2$ . Since  $(\text{int } f(R_1)) \cap \partial^u R_2 = \emptyset$  by condition (iii) of Definition C.1.4, the curve  $f \circ \gamma$  can exit the rectangle  $R_2$  only through the stable boundary  $\partial^s R_2$ . By Corollary C.3.1, every connected component  $f \circ \gamma([0, 1]) \cap R_2$  is an unstable curve joining the opposite components of  $\partial^s R_2$ . Denote  $R_{12}^f = f(R_1) \cap R_2$ .

Up to invert the roles of the components of the stable boundary, assume that  $\partial_0^s R_{12}^f \subset \partial_0^s R_2$  and  $\partial_1^s R_{12}^f \subset \partial_1^s R_2$  and denote

$$\partial_0^s R_{12}^f = R_2(\{0\} \times [a, b]) \quad \text{and} \quad \partial_1^s R_{12}^f = R_2(\{1\} \times [c, d]),$$

with  $[a, b] \subset [0, 1], [c, d] \subset [0, 1]$ . From Proposition C.5.1, both  $f \circ R_1([0, 1] \times \{0\})$  and  $f \circ R_1([0, 1] \times \{1\})$  intersect both  $\partial_0^s R_{12}^f$  and  $\partial_1^s R_{12}^f$  in just one point. Without loss of generality (since  $f(\partial_0^u R_1)$  and  $f(\partial_1^u R_1)$  cannot intersect), denote

$$f \circ R_1([0, 1] \times \{0\}) \cap \partial_0^s R_{12}^f = R_2(0, a), \quad f \circ R_1([0, 1] \times \{0\}) \cap \partial_1^s R_{12}^f = R_2(1, c),$$

$$f \circ R_1([0, 1] \times \{1\}) \cap \partial_0^s R_{12}^f = R_2(0, b), \quad f \circ R_1([0, 1] \times \{1\}) \cap \partial_1^s R_{12}^f = R_2(1, d).$$



We want now to build a first-time-intersection map between the  $f$ -image of every horizontal curve in  $R_1$  and  $\partial_1^s R_2$ . Define the  $\mathcal{C}^1$  function

$$[a, b] \times [0, 1] \ni (x, \tau) \mapsto G(x, \tau) := p_1 \circ R_2^{-1} \circ f \circ R_1(t(x) + \tau, s(x)) - 1 \in \mathbb{R},$$

where  $(t(x), s(x)) = R_1^{-1} \circ f^{-1} \circ R_2(0, x)$ . We are interested into the first  $\tau$  such that  $G(x, \tau) = 0$ . The partial derivative

$$\frac{\partial}{\partial \tau} G(x, \tau) = (1, 0) D(R_2^{-1} \circ f \circ R_1)(t(x) + \tau, s(x))(1, 0)$$

is always non zero. Indeed, the vector  $DR_1(t(x) + \tau, s(x))(1, 0)$  belongs to the unstable cone of  $R_1(t(x) + \tau, s(x))$  and so its image through  $Df(R_1(t(x) + \tau, s(x)))$  is in the unstable cone at  $f \circ R_1(t(x) + \tau, s(x))$ . The vector  $D(R_2^{-1} \circ f \circ R_1)(t(x) + \tau, s(x))(1, 0)$  has non zero first coordinate by Lemma C.2.1.

By the implicit function theorem, there exists a unique  $\mathcal{C}^1$  function  $[a, b] \ni x \mapsto \tau(x) \in \mathbb{R}$  such that  $G(x, \tau(x)) = 0$ . In particular

$$[a, b] \ni x \mapsto S(x) := p_2 \circ R_2^{-1} \circ f \circ R_1(t(x) + \tau(x), s(x)) \in [c, d]$$

is a continuous function. Since  $S(a) = c, S(b) = d$  and since the function is continuous, the image  $S([a, b])$  is  $[c, d]$ .

We deduce that for any  $s \in [0, 1]$  the curve  $f \circ R_1([0, 1] \times \{s\})$  intersects only once  $\partial_1^s R_2$ . Indeed, if by contradiction there is a horizontal leaf  $R_1([0, 1] \times \{s_1\})$  whose image intersects twice  $\partial_1^s R_2$ , then denoting as  $f \circ R_1(t_1, s_1)$  and  $f \circ R_1(t_2, s_1)$  the first and second points of intersection ( $t_1 \neq t_2$ ), we have  $f \circ R_1(t_2, s_1) \in R_2(\{1\} \times [c, d])$ . Since  $S([a, b]) = [c, d]$ , we would find  $(\bar{t}, \bar{s})$  with  $\bar{s} \neq s_1$  such that  $f \circ R_1(\bar{t}, \bar{s}) = f \circ R_1(t_2, s_1)$ : this contradicts the injectivity of  $f \circ R_1$ .

Consequently, the stable curve  $f^{-1} \circ R_2(\{1\} \times [c, d]) \cap R_1$  is a connected stable curve joining  $\partial_0^u R_1$  and  $\partial_1^u R_1$ .

In order to conclude the proof of Proposition C.5.2, argue by contradiction and assume there exists an unstable curve  $\gamma$  in  $R_1$  such that  $f \circ \gamma \cap R_2$  is not connected. In particular  $f \circ \gamma$  intersects twice  $\partial_1^s R_2$ . Equivalently, the curve  $\gamma$ , which joins  $\partial_0^s R_1$  and  $\partial_1^s R_1$ , intersects twice the stable connected curve  $f^{-1} \circ R_2(\{1\} \times [c, d])$  which joins  $\partial_0^u R_1$  and  $\partial_1^u R_1$ . This contradicts Proposition C.2.1. □

In particular, Proposition C.5.2 implies the following result.

**Lemma C.5.2.** *Let  $R_1 \xrightarrow{f^{-1}} R_2 \xrightarrow{f^{-1}} \dots \xrightarrow{f^{-1}} R_{n-1} \xrightarrow{f^{-1}} R_n$ . Assume that  $x, y \in R_1$  are such that  $f^{-i}(x), f^{-i}(y) \in R_i$  for any  $i \in \llbracket 0, n \rrbracket$ . Let  $\gamma : [0, 1] \rightarrow R_1$  be a stable curve such that  $\gamma(0) = x, \gamma(1) = y$ . Then  $f^{-i} \circ \gamma([0, 1])$  is contained in  $R_i$  for every  $i \in \llbracket 0, n \rrbracket$ .*

**Remark C.5.1.** The adapted result holds for unstable curves.

*Proof.* From Lemma C.5.1 each image  $f^{-i} \circ \gamma([0, 1]) \cap R_i$  remains a stable curve joining  $f^{-i}(x)$  and  $f^{-i}(y)$ . If by contradiction there exists  $\bar{i} \in \llbracket 0, n \rrbracket$  such that there exists a point in  $f^{-\bar{i}} \circ \gamma([0, 1])$  not in  $R_{\bar{i}}$ , then  $f^{-\bar{i}} \circ \gamma([0, 1]) \cap R_{\bar{i}}$  should have at least two connected components because its endpoints,  $f^{-\bar{i}}(x), f^{-\bar{i}}(y)$ , are in  $R_{\bar{i}}$ . This contradicts Proposition C.5.2 and we conclude. □

### C.5.2 $D_\infty^s$ as graph of a function: proof of Proposition C.4.1

In order to prove Proposition C.4.1, we focus our attention on  $(D_k^s)_{k \in \mathbb{N}}$ . We are going to prove that it is an increasing sequence of stable subrectangles contained in  $R_{j_0}$ . Let  $(j_i)_{i \in \mathbb{Z}}$  be an admissible sequence.

**Lemma C.5.3.** *Let  $i \in \mathbb{N}$ . Let  $\gamma$  be a stable curve in  $R_{j_i}$  joining  $\partial_0^u R_{j_i}$  and  $\partial_1^u R_{j_i}$ . Then the curve  $f^{-i} \circ \gamma([0, 1]) \cap \bigcap_{l=0}^i f^{-l}(R_{j_l})$  is a connected stable curve joining  $\partial_0^u R_{j_0}$  and  $\partial_1^u R_{j_0}$ .*

*Proof.* We proceed by induction. For  $i = 0$  there is nothing to prove. Assume now that  $\mathcal{R}_i := f^{-(i-1)} \circ \gamma([0, 1]) \cap \bigcap_{l=1}^i f^{-(l-1)}(R_{j_l})$  is a connected stable curve joining  $\partial_0^u R_{j_1}$  and  $\partial_1^u R_{j_1}$ . Since  $R_{j_0} \xrightarrow{f} R_{j_1}$ , then  $R_{j_1} \xrightarrow{f^{-1}} R_{j_0}$  (see Fact C.1.1). From Proposition C.5.2 it holds that

$$f^{-1}(\mathcal{R}_i) \cap R_{j_0} = f^{-i} \circ \gamma([0, 1]) \cap \bigcap_{l=0}^i f^{-l}(R_{j_l})$$

is a connected stable curve that joins opposite components of the unstable boundary  $\partial^u R_{j_0}$ . □

**Remark C.5.2.** For any  $i \in \mathbb{N}$  from Lemma C.5.3 and Lemma C.3.2 the curve

$$R_{j_0}^{-1} \left( f^{-i} \circ \gamma([0, 1]) \cap \bigcap_{l=0}^i f^{-l}(R_{j_l}) \right)$$

is the graph of a function with respect to the second coordinate in  $[0, 1]^2$ .

For any  $i \in \mathbb{N}$  consider the stable curves  $R_{j_i}(\{0\} \times [0, 1])$  and  $R_{j_i}(\{1\} \times [0, 1])$ . From Lemma C.5.3

$$\partial_0^s D_i^s := f^{-i}(R_{j_i}(\{0\} \times [0, 1])) \cap \bigcap_{l=0}^{i-1} f^{-l}(R_{j_l}) \quad \text{and} \quad \partial_1^s D_i^s := f^{-i}(R_{j_i}(\{1\} \times [0, 1])) \cap \bigcap_{l=0}^{i-1} f^{-l}(R_{j_l})$$

are both connected stable curves. So, they are the image through  $R_{j_0}$  of the graphs of  $\mathcal{C}^1$  functions with respect to the second coordinate (see Remark C.5.2). Observe that for any  $i \in \mathbb{N}$  it holds  $\partial_0^s D_i^s \cap \partial_1^s D_i^s = \emptyset$ .

Up to invert the notations of  $\partial_0^s D_i^s$  and  $\partial_1^s D_i^s$ , we denote

$$\partial_0^s D_i^s = R_{j_0}(\text{Graph}(h_{j_i})) \quad \text{and} \quad \partial_1^s D_i^s = R_{j_0}(\text{Graph}(g_{j_i}))$$

so that for any  $t \in [0, 1]$  it holds

$$h_{j_i}(t) < g_{j_i}(t).$$

**Proposition C.5.3.** *Let  $(j_i)_{i \in \mathbb{Z}}$  be an admissible sequence. For any  $i \in \mathbb{N}$  the set  $D_i^s$  is a stable subrectangle of  $R_{j_0}$  and its stable boundary is  $\partial_0^s D_i^s \cup \partial_1^s D_i^s$ .*

*Proof.* We proceed by induction. For  $i = 0$  there is nothing to prove. Assume that the result holds for  $i \in \mathbb{N}$ . Observe that also  $(j_{i+1})_{i \in \mathbb{Z}}$  is an admissible sequence. By inductive hypothesis we have that

$$\bigcap_{l=0}^i f^{-l}(R_{j_{l+1}}) = f \left( \bigcap_{l=1}^{i+1} f^{-l}(R_{j_l}) \right)$$

is a stable subrectangle of  $R_{j_1}$ . Its stable boundary is the union of

$$f^{-i}(R_{j_{i+1}}(\{0\} \times [0, 1])) \cap \bigcap_{l=0}^{i-1} f^{-l}(R_{j_{l+1}}) \quad \text{and} \quad f^{-i}(R_{j_{i+1}}(\{1\} \times [0, 1])) \cap \bigcap_{l=0}^{i-1} f^{-l}(R_{j_{l+1}}),$$

which contain  $f(\partial_0^s D_i^s)$  and  $f(\partial_1^s D_i^s)$ . Denote as  $\mathcal{R} : [0, 1]^2 \rightarrow \bigcap_{l=0}^i f^{-l}(R_{j_{l+1}})$  the corresponding  $\mathcal{C}^1$  embedding.

Consider then

$$f^{-1} \left( \bigcap_{l=0}^i f^{-l}(R_{j_{l+1}}) \right) \cap R_{j_0} = D_{i+1}^s.$$

Clearly,  $D_{i+1}^s \subset R_{j_0}$ . Observe that

$$f(D_{i+1}^s) = \bigcap_{l=0}^i f^{-l}(R_{j_{l+1}}) \cap f(R_{j_0}) \subset \bigcap_{l=0}^i f^{-l}(R_{j_{l+1}}) = \mathcal{R}.$$

Define the function

$$D_{i+1}^s \ni x \mapsto (p_1 \circ \mathcal{R}^{-1} \circ f(x), p_2 \circ R_{j_0}^{-1}(x)) \in [0, 1]^2.$$

This function is a  $\mathcal{C}^1$  diffeomorphism and its inverse is the required embedding with respect to which  $D_{i+1}^s$  is a rectangle. Indeed, if

$$\begin{pmatrix} D(p_1 \circ \mathcal{R}^{-1} \circ f)(x) \\ D(p_2 \circ R_{j_0}^{-1})(x) \end{pmatrix} v = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

then the differential of the inverse function sends  $(1, 0)$  in  $v$ . Since  $D(p_2 \circ R_{j_0}^{-1})v = 0$  and since  $R_{j_0}$  is a rectangle, then from Lemma C.2.1 we deduce that  $v \in C_x^{u, \eta}$ . Similarly, if

$$\begin{pmatrix} D(p_1 \circ \mathcal{R}^{-1} \circ f)(x) \\ D(p_2 \circ R_{j_0}^{-1})(x) \end{pmatrix} v = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

then the differential of the inverse functions sends  $(0, 1)$  in  $v$ . Since  $Dp_1(\mathcal{R}^{-1} \circ f(x)) \circ D\mathcal{R}^{-1}(f(x)) \circ Df(x)v = 0$  and since  $\mathcal{R}$  is a rectangle, then from Lemma C.2.1  $Df(x)v \in C_{f(x)}^{s, \eta}$ . In particular  $v \in C_x^{s, \eta}$  from Lemma C.5.1.

Observe that  $\partial^u D_{i+1}^s$  is the set

$$\{x \in D_{i+1}^s : p_2 \circ R_{j_0}^{-1}(x) \in \{0, 1\}\}$$

and so it is contained in  $\partial^u R_{j_0}$ . That is,  $D_{i+1}^s$  is a stable subrectangle of  $R_{j_0}$ .

Concerning the stable boundary, it is the union of points such that either  $p_1 \circ \mathcal{R}^{-1} \circ f(x) = 0$  or  $p_1 \circ \mathcal{R}^{-1} \circ f(x) = 1$ . Equivalently, it is the set of points  $x \in D_{i+1}^s$  such that  $f(x) \in \partial^s \bigcap_{l=0}^i f^{-l}(R_{j_{l+1}}) = \partial^s \mathcal{R}$ . Finally, since  $f^{-1}(\partial^s \mathcal{R}) \cap R_{j_0} = \partial_0^s D_{i+1}^s \cup \partial_1^s D_{i+1}^s$ , we conclude that the stable boundary of  $D_{i+1}^s$  is  $\partial_0^s D_{i+1}^s \cup \partial_1^s D_{i+1}^s$ . □

Summing up, the sequence  $(D_i^s)_{i \in \mathbb{N}}$  is a non increasing sequence of stable subrectangles contained in  $R_{j_0}$ . Recall that the left and right stable boundaries of each  $D_i^s$  are images through  $R_{j_0}$  of the graphs of  $h_{j_i}, g_{j_i}$ .

**Lemma C.5.4.** *Let  $i \in \mathbb{N}$ . A point  $x$  belongs to  $D_i^s$  if and only if*

$$p_1 \circ R_{j_0}^{-1}(x) \in [h_{j_i}(p_2 \circ R_{j_0}^{-1}(x)), g_{j_i}(p_2 \circ R_{j_0}^{-1}(x))].$$

*Proof.* Let  $s \in [h_{j_i}(t), g_{j_i}(t)]$ . The point  $x = R_{j_0}(t, s)$  belongs to the curve  $R_{j_0}([h_{j_i}(t), g_{j_i}(t)] \times \{t\})$ , which is an unstable curve in  $R_{j_0}$ . Since for any  $n \in \llbracket 0, i \rrbracket$  the points  $f^n \circ R_{j_0}(t, h_{j_i}(t))$ ,  $f^n \circ R_{j_0}(t, g_{j_i}(t))$  belong to  $R_{j_n}$ , by Lemma C.5.2 the image  $f^n \circ R_{j_0}([h_{j_i}(t), g_{j_i}(t)] \times \{t\})$  is contained in  $R_{j_n}$ . In particular the point  $x$  belongs to  $f^{-n}(R_{j_n})$  for any  $n \in \llbracket 0, i \rrbracket$ , that is  $x \in D_i^s$ .

Let  $x \in D_i^s$ , that is  $f^n(x) \in R_{j_n}$  for any  $n \in \llbracket 0, i \rrbracket$ . Assume by contradiction that  $R_{j_0}^{-1}(x) = (t, s)$  does not lie between the two graphs of  $h_{j_i}$  and  $g_{j_i}$ . Without loss of generality, assume that  $0 \leq s < h_{j_i}(t)$ . The points  $x$ ,  $R_{j_0}(h_{j_i}(t), t)$ ,  $R_{j_0}(g_{j_i}(t), t)$  belong to the unstable curve  $R_{j_0}([0, 1] \times \{t\})$ .

Let  $\bar{n} \in \llbracket 0, i \rrbracket$  be the maximum  $n \in \llbracket 0, i \rrbracket$  such that for any  $m \in \llbracket 0, n \rrbracket$  the point  $f^m(x)$  belongs to the same connected component of  $(f^m \circ R_{j_0}([0, 1] \times \{t\})) \cap R_{j_m}$  as  $f^n \circ R_{j_0}(h_{j_i}(y), t)$  and  $f^n \circ R_{j_0}(g_{j_i}(t), t)$ .

Observe that  $\bar{n} \leq i - 1$ . Indeed,  $f^i \circ R_{j_0}([h_{j_i}(t), g_{j_i}(t)] \times \{t\})$  is a curve joining the opposite components of  $\partial^s R_{j_i}$  and  $f^i(x) \notin f^i \circ R_{j_0}([h_{j_i}(t), g_{j_i}(t)] \times \{t\})$  (since  $x \notin R_{j_0}([h_{j_i}(t), g_{j_i}(t)] \times \{t\})$ ).

Denote now the connected component of  $f^{\bar{n}} \circ R_{j_0}([h_{j_i}(t), g_{j_i}(t)] \times \{t\})$  containing  $f^{\bar{n}}(x)$  as  $\Gamma$ : it is an unstable curve in  $R_{j_{\bar{n}}}$ . By definition of  $\bar{n}$  it holds that  $f^{\bar{n}+1}(x)$  does not belong to the same connected component of  $f^{\bar{n}+1} \circ R_{j_0}([h_{j_i}(t), g_{j_i}(t)] \times \{t\}) \cap R_{j_{\bar{n}+1}}$  as  $f^{\bar{n}+1} \circ R_{j_0}(h_{j_i}(t), t)$  and  $f^{\bar{n}+1} \circ R_{j_0}(g_{j_i}(t), t)$ .

Since by Proposition C.5.2 the image  $f \circ \Gamma \cap R_{j_{\bar{n}+1}}$  is connected, we deduce that  $f^{\bar{n}+1}(x) \notin R_{j_{\bar{n}+1}}$  and, since  $\bar{n} + 1 \leq i$ , we contradict that  $x \in D_i^s$  and we conclude.  $\square$

**Lemma C.5.5.** *For any  $i \in \mathbb{N}$  for any  $t \in [0, 1]$*

$$h_{j_i}(t) \leq h_{j_{i+1}}(t) < g_{j_{i+1}}(t) \leq g_{j_i}(t).$$

*Proof.* Argue by contradiction and assume there exists  $i \in \mathbb{N}$  such that there exists  $t \in [0, 1]$  so that  $h_{j_{i+1}}(t) < h_{j_i}(t)$ . In particular the point  $(h_{j_{i+1}}(t), t)$  does not lie between the graphs of  $h_{j_i}$  and  $g_{j_i}$ . From Lemma C.5.4  $R_{j_0}(h_{j_{i+1}}(t), t)$  does not belong to  $D_i^s$  and consequently it does not belong to  $D_{i+1}^s \subset D_i^s$ . This provides the required contradiction.  $\square$

**Claim C.5.1.** The sequences  $(h_n)_{n \in \mathbb{N}}$  and  $(g_n)_{n \in \mathbb{N}}$  converge uniformly respectively to Lipschitz functions (so almost everywhere differentiable)  $h_\infty$  and  $g_\infty$  such that for any  $t \in [0, 1]$  it holds  $h_\infty(t) \leq g_\infty(t)$ .

*Proof.* The sequences of functions  $(h_{j_i})_{i \in \mathbb{N}}$ ,  $(g_{j_i})_{i \in \mathbb{N}}$  defined on  $[0, 1]$  are non-decreasing and non-increasing, respectively, from Lemma C.5.5. By Dini's Theorem, the sequences converge uniformly to functions  $h_\infty : [0, 1] \rightarrow [0, 1]$ ,  $g_\infty : [0, 1] \rightarrow [0, 1]$  respectively that are continuous. In particular, it holds for any  $i \in \mathbb{N}$  that  $h_{j_i} \leq h_\infty \leq g_\infty \leq g_{j_i}$ .

Observe that the sequence  $(h_{j_i})_{i \in \mathbb{N}}$  is equi-Lipschitz with Lipschitz constant bounded above by

$$\mathcal{K} = \max_{x \in R_{j_0}} \max_{v \in C_x^{s, \eta} \cap \mathbb{S}^1} \frac{Dp_1(R_{j_0}^{-1}(x))DR_{j_0}^{-1}(x)v}{Dp_2(R_{j_0}^{-1}(x))DR_{j_0}^{-1}(x)v} < +\infty.$$

Such  $\mathcal{K}$  is finite because of Lemma C.2.1 and because  $R_{j_0}$  is compact. Consequently, the function  $h_\infty$  is Lipschitz with Lipschitz constant bounded above by  $\mathcal{K}$ . By Rademacher's

theorem, the function  $h_\infty$  is almost everywhere differentiable.

Similar arguments imply that also  $g_\infty$  is a Lipschitz function with Lipschitz constant bounded above by

$$\mathcal{L} = \max_{x \in R_{j_0}} \max_{v \in C_x^{u,\eta} \cap \mathbb{S}^1} \frac{Dp_2(R_{j_0}^{-1}(x))DR_{j_0}^{-1}(x)v}{Dp_1(R_{j_0}^{-1}(x))DR_{j_0}^{-1}(x)v} < +\infty$$

and moreover it is almost everywhere differentiable. □

**Proposition C.5.4.** *For any  $t \in [0, 1]$  it holds  $h_\infty(t) = g_\infty(t)$ .*

*Proof.* Assume by contradiction that there exists  $t \in [0, 1]$  such that  $\rho = g_\infty(t) - h_\infty(t) > 0$ . Then for any  $i \in \mathbb{N}$  it holds from Claim C.5.1 that

$$g_{j_i}(t) - h_{j_i}(t) \geq \rho.$$

The horizontal curve  $[0, 1] \ni s \mapsto \mathcal{H}_i(s) = (sg_{j_i}(t) + (1-s)h_{j_i}(t), t)$  is sent by  $R_{j_0}$  into an unstable curve that joins the points  $x_i := R_{j_0}(h_{j_i}(t), t) \in f^{-i}(\partial_0^s R_{j_i})$  and  $y_i := R_{j_0}(g_{j_i}(t), t) \in f^{-i}(\partial_1^s R_{j_i})$ . In particular (see Notation C.5.1)

$$\begin{aligned} L_{R_{j_0} \circ \mathcal{H}_i}(x_i, y_i) &= \int_0^1 \left\| \frac{d}{ds} R_{j_0}(\mathcal{H}_i(s)) \right\| ds \geq \\ &\geq \min_{s \in [0, 1]} \|DR_{j_0}(s, t)|_{\mathbb{R}(1,0)}\| (g_{j_i}(t) - h_{j_i}(t)) > \mathbf{K} \rho > 0, \end{aligned}$$

where  $\mathbf{K} := \min_{s \in [0, 1]} \|DR_{j_0}(s, t)|_{\mathbb{R}(1,0)}\| > 0$ . This length is bounded from below uniformly in  $i \in \mathbb{N}$ .

Up to choose a subsequence, assume that  $R_{j_{im}} = R_{j_0}$  for any  $i \in \mathbb{N}$ , where  $m \in \mathbb{N}^*$  is the integer with respect to which  $U$  satisfies the cone field property (see Definition B.0.1).

For any  $i \in \mathbb{N}$  the points  $x_{im}, y_{im}$  belong to  $\bigcap_{l=0}^{im} f^{-l}(R_{j_l})$ . By Proposition C.5.2 the curve  $f^l \circ R_{j_0} \circ \mathcal{H}_{im}$  is contained in  $R_{j_l}$  for any  $l \in \llbracket 0, im \rrbracket$ .

It holds

$$\begin{aligned} L_{f^{im} \circ R_{j_0} \circ \mathcal{H}_{im}}(f^{im}(x_{im}), f^{im}(y_{im})) &= \int_0^1 \left\| \frac{d}{ds} f^{im} \circ R_{j_0} \circ \mathcal{H}_{im}(s) \right\| ds = \\ &= \int_0^1 \left\| \left[ \prod_{l=1}^i Df^m(f^{(l-1)m}(R_{j_0}(\mathcal{H}_{im}(s)))) \right] DR_{j_0}(\mathcal{H}_{im}(s))(g_{j_{im}}(t) - h_{j_{im}}(t), 0) \right\| ds \geq \\ &\geq \frac{1}{(\mu + \xi)^i} \int_0^1 \|DR_{j_0}(\mathcal{H}_{im}(s))(g_{j_{im}}(t) - h_{j_{im}}(t), 0)\| ds = \frac{1}{(\mu + \xi)^i} L_{R_{j_0} \circ \mathcal{H}_{im}}(x_{im}, y_{im}), \end{aligned}$$

because of Condition (iv) of the cone field criterion on vectors of the unstable cone (see Definition B.0.1). That is

$$L_{f^{im} \circ R_{j_0} \circ \mathcal{H}_{im}}(f^{im}(x_{im}), f^{im}(y_{im})) \geq \frac{1}{(\mu + \xi)^i} L_{R_{j_0} \circ \mathcal{H}_{im}}(x_{im}, y_{im}) > \frac{1}{(\mu + \xi)^i} \mathbf{K} \rho. \quad (\text{C.4})$$

We show now that the length  $L_\gamma(x, y)$  of any unstable curve  $\gamma \subset R_{j_0}$  such that  $\gamma(0) = x \in R_{j_0}, \gamma(1) = y \in R_{j_0}$  is bounded.

The curve  $R_{j_0}^{-1} \circ \gamma([0, 1])$  is the graph of a function  $\Gamma : [p_1 \circ R_{j_0}^{-1} \circ \gamma(0), p_1 \circ R_{j_0}^{-1} \circ \gamma(1)] \rightarrow [0, 1]$

with respect to the first coordinate from Lemma C.3.2. Observe that  $\Gamma$  is a Lipschitz function and its Lipschitz constant is bounded above by

$$\mathcal{L} := \max_{x \in R_{j_0}} \max_{v \in C_x^{u, \eta} \cap \mathbb{S}^1} \frac{Dp_2(R_{j_0}^{-1}(x)) \circ DR_{j_0}^{-1}(x)v}{Dp_1(R_{j_0}^{-1}(x)) \circ DR_{j_0}^{-1}(x)v} < +\infty,$$

that is finite because of Lemma C.2.1 and of the compactness of  $R_{j_0}$ .

The function  $[0, 1] \ni s \mapsto p_1 \circ R_{j_0}^{-1} \circ \gamma(s) =: t(s) \in [0, 1]$  is a  $\mathcal{C}^1$  diffeomorphism. Assume without loss of generality that  $t'(s) > 0$  for any  $s$ . So for any  $s \in [0, 1]$  it holds  $R_{j_0}^{-1} \circ \gamma(s) = (t(s), \Gamma(t(s)))$ . Denote as  $\|\cdot\|_e$  the standard euclidean norm and let  $\mathcal{P} > 0$  be such that  $\|\cdot\| \leq \mathcal{P}\|\cdot\|_e$ . Consequently

$$\begin{aligned} L_\gamma(x, y) &= \int_0^1 \|\gamma'(s)\| ds = \int_0^1 \left\| \frac{d}{ds} R_{j_0}(t(s), \Gamma(t(s))) \right\| ds = \\ &= \int_0^1 \left\| DR_{j_0}(t(s), \Gamma(t(s))) \left( 1, \frac{d}{dt} \Gamma(t(s)) \right) t'(s) \right\| ds \leq \\ &\leq \max_{x \in [0, 1]^2} \|DR_{j_0}(x)\| \int_0^1 \mathcal{P} \left\| \left( 1, \frac{d}{dt} \Gamma(t(s)) \right) \right\|_e |t'(s)| ds = \\ &= \max_{x \in [0, 1]^2} \|DR_{j_0}(x)\| \mathcal{P} \int_0^1 \sqrt{1 + \left| \frac{d}{dt} \Gamma(t(s)) \right|^2} t'(s) ds \leq \max_{x \in [0, 1]^2} \|DR_{j_0}(x)\| \mathcal{P} \sqrt{1 + \mathcal{L}^2} < +\infty. \end{aligned} \tag{C.5}$$

Choose  $i \in \mathbb{N}$  such that

$$\frac{1}{(\mu + \xi)^i} \mathbf{K}\rho > \max_{x \in [0, 1]^2} \|DR_{j_0}(x)\| \mathcal{P} \sqrt{1 + \mathcal{L}^2}.$$

The curve  $f^{im} \circ R_{j_0} \circ \mathcal{H}_{im}$  is an unstable curve contained in  $R_{j_{im}} = R_{j_0}$  joining  $f^{im}(x_{im})$  and  $f^{im}(y_{im})$ . We have so, by (C.4) and (C.5) and by the choice of  $i$ ,

$$\frac{1}{(\mu + \xi)^i} \mathbf{K}\rho < L_{f^{im} \circ R_{j_0} \circ \mathcal{H}_{im}}(f^{im}(x_{im}), f^{im}(y_{im})) \leq \max_{x \in [0, 1]^2} \|DR_{j_0}(x)\| \mathcal{P} \sqrt{1 + \mathcal{L}^2} < \frac{1}{(\mu + \xi)^i} \mathbf{K}\rho.$$

This is the required contradiction.

We so conclude that for any  $t \in [0, 1]$

$$h_\infty(t) = g_\infty(t).$$

□

*Proof of Proposition C.4.1.* From Lemma C.5.4, it holds that  $D_\infty^s \subset R_{j_0}(\text{Graph}(h_\infty))$ . Actually,  $D_\infty^s = R_{j_0}(\text{Graph}(h_\infty))$ . Indeed, argue by contradiction and assume there exists  $t \in [0, 1]$  such that  $(h_\infty(t), t) \notin R_{j_0}^{-1}(D_\infty^s)$ . This means that there exists  $i \in \mathbb{N}$  so that  $(h_\infty(t), t) \notin R_{j_0}^{-1}(D_i^s)$ . Equivalently, there exists  $i \in \mathbb{N}$  such that  $h_\infty(t) \notin [h_{j_i}(t), g_{j_i}(t)]$ , which is the required contradiction. In particular, from Proposition C.5.4,  $D_\infty^s$  is the image of the graph of a Lipschitz function  $h_\infty$  and it joins  $\partial_0^u R_{j_0}$  and  $\partial_1^u R_{j_0}$ .

□

Remark that if  $(j_i)_{i \in \mathbb{Z}}$  is an admissible sequence, then for any  $n \in \mathbb{Z}$  the sequence  $(j_{i+n})_{i \in \mathbb{Z}}$  is admissible too. The following result concerns the relation between  $D_\infty^s$  associated to a sequence  $(j_i)_{i \in \mathbb{N}}$  and to a shifted one  $(j_{i+n})_{i \in \mathbb{N}}$ .

**Lemma C.5.6.** *Let  $D_\infty^s = D_\infty^s((j_i)_{i \in \mathbb{Z}})$ . Then for any  $n \in \mathbb{N}$*

$$f^n(D_\infty^s((j_i)_{i \in \mathbb{Z}})) \subset D_\infty^s((j_{i+n})_{i \in \mathbb{Z}}) = \bigcap_{i=0}^{\infty} f^{-i}(R_{j_{i+n}}).$$

*Proof.* We proceed by induction. For  $n = 0$  there is nothing to prove.

Assume now that  $f^{n-1}(D_\infty^s((j_i)_{i \in \mathbb{Z}})) \subset D_\infty^s((j_{i+n-1})_{i \in \mathbb{Z}})$  for  $n \geq 1$ . Consider then

$$f(D_\infty^s((j_{i+n-1})_{i \in \mathbb{Z}})) = f\left(\bigcap_{i=0}^{\infty} f^{-i}(R_{j_{i+n-1}})\right) = f\left(\bigcap_{i=1}^{\infty} f^{-i}(R_{j_{i+n-1}})\right) \cap f(R_{j_{n-1}}).$$

Since  $R_{j_n} \cap f(R_{j_{n-1}}) \subset R_{j_n}$ , we have

$$f\left(\bigcap_{i=1}^{\infty} f^{-i}(R_{j_{i+n-1}})\right) \cap f(R_{j_{n-1}}) \subset \bigcap_{i=1}^{\infty} f^{-i+1}(R_{j_{i+n-1}}) = \bigcap_{i=0}^{\infty} f^{-i}(R_{j_{i+n}}) = D_\infty^s((j_{i+n})_{i \in \mathbb{Z}}).$$

That is,  $f(D_\infty^s((j_{i+n-1})_{i \in \mathbb{Z}})) \subset D_\infty^s((j_{i+n})_{i \in \mathbb{Z}})$ . By inductive hypothesis we have

$$f^{n-1}(D_\infty^s((j_i)_{i \in \mathbb{Z}})) \subset D_\infty^s((j_{i+n-1})_{i \in \mathbb{Z}})$$

and so we conclude

$$f^n(D_\infty^s((j_i)_{i \in \mathbb{Z}})) = f(f^{n-1}(D_\infty^s((j_i)_{i \in \mathbb{Z}}))) \subset f(D_\infty^s((j_{i+n-1})_{i \in \mathbb{Z}})) \subset D_\infty^s((j_{i+n})_{i \in \mathbb{Z}}).$$

□

### C.5.3 On points of differentiability of $D_\infty^s$ : proof of Proposition C.4.2

We recall, from Proposition C.4.1, that  $D_\infty^s$  is the image through  $R_{j_0}$  of the graph of  $h_\infty$ , where  $h_\infty$  is the limit (in the uniform convergence) of the sequence  $(h_n)_{n \in \mathbb{N}}$  (see Claim C.5.1).

*Proof of Proposition C.4.2.* Let  $x \in D_\infty^s$ . Denote as  $(h_\infty(\tau), \tau) = R_{j_0}^{-1}(x) \in [0, 1]^2$ . We are assuming that  $x$  is a point of differentiability, equivalently that there exists

$$h'_\infty(\tau) = \lim_{h \rightarrow 0} \frac{h_\infty(\tau + h) - h_\infty(\tau)}{h}.$$

On  $[0, 1]^2$  we use the trivialization given by the projections over the first and the second coordinates. Define the functions

$$[0, 1]^2 \ni (t, s) \mapsto \mathbb{C}_{(t,s)}^s = DR_{j_0}^{-1}(R_{j_0}(t, s))C_{R_{j_0}(t,s)}^{\eta,s,+},$$

that is the image through  $DR_{j_0}^{-1}$  of the positive stable half-cone at  $R_{j_0}(t, s)$  (see Notation C.3.1. There exist  $m = m(t, s), M = M(t, s) \in \mathbb{R}$  such that <sup>1</sup>

$$\mathbb{C}_{(t,s)}^s = \{(v_1, v_2) \in T_{(t,s)}[0, 1]^2 : v_2 > 0, m \leq \frac{v_1}{v_2} \leq M\}.$$

---

1. By using the trivialization through the projections over the first and second coordinates in  $[0, 1]^2$ .

In particular

$$m(t, s) = \min_{v \in \mathbb{C}_{(t,s)}^s} \frac{p_1(v)}{p_2(v)} \quad \text{and} \quad M(t, s) = \max_{v \in \mathbb{C}_{(t,s)}^s} \frac{p_1(v)}{p_2(v)}.$$

The function  $\frac{p_1}{p_2}$  is continuous on stable cones of points in  $R_{j_0}$  since, from Lemma C.2.1,  $p_2$  does not vanish on images of stable cones. By the continuity of the cone fields, of  $DR_{j_0}^{-1}$  and of the function  $\frac{p_1}{p_2}$  restricted to the images of stable cones, the functions

$$[0, 1]^2 \ni (t, s) \mapsto m(t, s) \in \mathbb{R} \quad \text{and} \quad [0, 1]^2 \ni (t, s) \mapsto M(t, s) \in \mathbb{R}$$

are continuous.

Fix  $\varepsilon > 0$ . There exists  $\bar{h} > 0$  such that for any  $0 < \xi \leq \bar{h}$  it holds

$$h'_\infty(\tau) < \frac{h_\infty(\tau + \xi) - h_\infty(\tau)}{\xi} + \frac{\varepsilon}{4}.$$

By the continuity of  $M = M(t, s)$ , there exists a neighborhood  $U$  of  $(h_\infty(\tau), \tau)$  such that for any  $(t, s) \in U$  it holds

$$M(t, s) < M(h_\infty(\tau), \tau) + \frac{\varepsilon}{4}.$$

Fix now  $0 < \bar{\xi} < \bar{h}$  such that

$$\text{Graph}(h_{\infty|_{[\tau, \tau + \bar{\xi}]}}) := \{(h_\infty(\tau + \xi), \tau + \xi) : \xi \in [0, \bar{\xi}]\} \subset U.$$

Since  $U$  is open, let  $\rho > 0$  be such that

$$\{(t, s) \in [0, 1]^2 : d((t, s), \text{Graph}(h_{\infty|_{[\tau, \tau + \bar{\xi}]}})) < \rho\} \subset U. \quad (\text{C.6})$$

Recall that the sequence  $(h_n)_{n \in \mathbb{N}}$  converges uniformly to  $h_\infty$ . Therefore, there exists  $\bar{n} \in \mathbb{N}$  such that for any  $n \geq \bar{n}$  it holds

$$\|h_n - h_\infty\|_{\mathcal{C}^0} < \min \left\{ \frac{\bar{\xi}\varepsilon}{4}, \rho \right\}. \quad (\text{C.7})$$

Observe in particular that, by (C.6) and (C.7), for any  $n \geq \bar{n}$  for any  $\xi \in [0, \bar{\xi}]$  we have that

$$(h_n(\tau + \xi), \tau + \xi) \in U. \quad (\text{C.8})$$

Consequently for any  $n \geq \bar{n}$  it holds

$$\begin{aligned} h'_\infty(\tau) &< \frac{h_\infty(\tau + \bar{\xi}) - h_\infty(\tau)}{\bar{\xi}} + \frac{\varepsilon}{4} < \frac{h_n(\tau + \bar{\xi}) - h_n(\tau) + \frac{\bar{\xi}\varepsilon}{2}}{\bar{\xi}} + \frac{\varepsilon}{4} = \\ &= \frac{h_n(\tau + \bar{\xi}) - h_n(\tau)}{\bar{\xi}} + \frac{3}{4}\varepsilon = h'_n(\tau + \bar{\xi}) + \frac{3}{4}\varepsilon, \end{aligned}$$

for some  $\xi \in [0, \bar{\xi}]$ . Since  $h_n$  is a stable curve and since  $(h_n(\tau + \xi), \tau + \xi) \in U$  from (C.8), it holds that

$$h'_n(\tau + \xi) \leq M(h_n(\tau + \xi), \tau + \xi) < M(h_\infty(\tau), \tau) + \frac{\varepsilon}{4}.$$

So we obtain

$$h'_\infty(\tau) < h'_n(\tau + \xi) + \frac{3}{4}\varepsilon < M(h_\infty(\tau), \tau) + \varepsilon.$$



By the arbitrariness of  $\varepsilon$  we conclude that  $h'_\infty(\tau) \leq M(h_\infty(\tau), \tau)$ . Similarly we show that  $h'_\infty(\tau) \geq m(h_\infty(\tau), \tau)$ .

Therefore, the vector  $v$  tangent to the graph of  $h_\infty$  at  $(h_\infty(\tau), \tau)$ , which belongs to  $\mathbb{R}_+(h'_\infty(\tau), 1)$ , satisfies

$$\frac{p_1(v)}{p_2(v)} = h'_\infty(\tau) \in [m(h_\infty(\tau), \tau), M(h_\infty(\tau), \tau)].$$

Equivalently, the vector  $v$  belongs to  $\mathbb{C}_{(h_\infty(\tau), \tau)}^s$ . So its image  $DR_{j_0}(h_\infty(\tau), \tau)v$ , which is the vector tangent to  $D_\infty^s$  at  $x = R_{j_0}(h_\infty(\tau), \tau)$ , belongs to the stable cone  $C_x^{s, \eta}$  (in particular it belongs to the positive stable half-cone at  $x$ ) and we conclude the proof.  $\square$

#### C.5.4 $D_\infty^s$ is $\mathcal{C}^1$ : proof of Proposition C.4.3

Let  $x \in D_\infty^s$ . Observe in particular that its future orbit is contained in  $U$  and so its local stable manifold is well-defined: we will denote it as  $W_{loc, \varepsilon}^s(x)$ , where  $\varepsilon > 0$  can be chosen uniformly for any  $x$  such that  $f^i(x) \in U$  for any  $i \in \mathbb{N}$ .

*Proof of Proposition C.4.3.* Argue by contradiction and assume that for any  $N \in \mathbb{N}$  it holds

$$D_\infty^s \not\subset f^{-N}(W_{loc, \varepsilon}^s(f^N(x))).$$

That is, for any  $N \in \mathbb{N}$  there exists  $y_N \in D_\infty^s$  such that  $f^N(y_N) \notin W_{loc, \varepsilon}^s(f^N(x))$ . Fix now  $n \in \mathbb{N}$  such that

$$\max_{R \in \mathcal{R}} \max_{x \in [0, 1]^2} \|DR(x)\| \mathcal{P} \sqrt{1 + \mathcal{P}^2} < \frac{\varepsilon}{(\mu + \xi)^n}, \quad (\text{C.9})$$

where  $\mathcal{P} > 0$  is the constant such that  $\|\cdot\| \leq \mathcal{P} \|\cdot\|_e^2$  and

$$\mathcal{P} = \max_{R \in \mathcal{R}} \max_{x \in R} \max_{v \in C_x^{s, \eta} \cap S^1} \frac{Dp_1(R^{-1}(x))DR^{-1}(x)v}{Dp_2(R^{-1}(x))DR^{-1}(x)v}.$$

The constant  $\mathcal{P}$  is finite because, from Lemma C.2.1,  $Dp_2(R^{-1}(x))DR^{-1}(x)v$  is not null and because  $\bigcup_{R \in \mathcal{R}} R$  is compact. Let  $y_{nl} \in D_\infty^s$ . Then there exists  $k \geq 0$  such that <sup>3</sup>

$$d(f^{nl+k}(y_{nl}), f^{nl+k}(x)) \geq \varepsilon. \quad (\text{C.10})$$

Denote as  $\gamma$  the curve contained in  $D_\infty^s((j_i)_{i \in \mathbb{Z}})$  joining  $y_{nl}$  and  $x$ . Consider  $f^k \circ \gamma$ : by Lemma C.5.6 it is contained in  $D_\infty^s((j_{i+k})_{i \in \mathbb{Z}})$  and it joins  $f^k(y_{nl})$  and  $f^k(x)$ . The sequence  $(j_{i+k})_{i \in \mathbb{Z}}$  is also admissible. Denote as  $H_\infty$  the Lipschitz function such that  $R_{j_k}(\text{Graph}(H_\infty)) = D_\infty^s((j_{i+k})_{i \in \mathbb{Z}})$ . From Proposition C.4.1, the function  $H_\infty$  is Lipschitz and so it is almost everywhere differentiable.

Let  $f^k(y_{nl}) = R_{j_k}(H_\infty(t_0), t_0)$ ,  $f^k(x) = R_{j_k}(H_\infty(t_1), t_1)$ . Up to invert the roles, assume that  $t_0 < t_1$ .

Let us calculate the length of  $f^k \circ \gamma$ . That is

$$L_{f^k \circ \gamma}(f^k(y_{nl}), f^k(x)) = \int_{t_0}^{t_1} \left\| \frac{d}{ds} R_{j_k}(H_\infty(s), s) \right\| ds \leq$$

2.  $\|\cdot\|_e$  is the standard Euclidean norm.

3.  $l \in \mathbb{N}$  is the constant given by the cone field property.

$$\begin{aligned}
&\leq \int_{t_0}^{t_1} \|DR_{j_k}(H_\infty(s), s)\| \| (H'_\infty(s), 1) \| ds \leq \max_{x \in [0,1]^2} \|DR_{j_k}(x)\| \int_{t_0}^{t_1} \mathcal{P} \sqrt{1 + (H'_\infty(s))^2} ds \leq \\
&\leq \max_{x \in [0,1]^2} \|DR_{j_k}(x)\| \mathcal{P} \sqrt{1 + \mathcal{J}^2} < +\infty,
\end{aligned} \tag{C.11}$$

where

$$\mathcal{J} = \max_{x \in R_{j_k}} \max_{v \in C_x^{s,\eta} \cap \mathbb{S}^1} \frac{Dp_1(R_{j_k}^{-1}(x)) DR_{j_k}^{-1}(x)v}{Dp_2(R_{j_k}^{-1}(x)) DR_{j_k}^{-1}(x)v} < +\infty.$$

The constant  $\mathcal{J}$  is finite because of Lemma C.2.1 and because of the compactness of  $R_{j_k}$ . In particular, the length of  $f^k \circ \gamma$  is bounded.

Write the curve  $f^k \circ \gamma$  as  $f^{-nl}(f^{nl}(f^k \circ \gamma))$ . Consequently

$$L_{f^k \circ \gamma}(f^k(y_{nl}), f^k(x)) = L_{f^{-nl}(f^{nl+k} \circ \gamma)}(f^k(y_{nl}), f^k(x)).$$

By Lemma C.5.6 the curve  $f^{nl+k} \circ \gamma$  is contained in  $D_\infty^s((j_{i+nl+k})_{i \in \mathbb{Z}})$  and joins  $f^{nl+k}(y_{nl})$  and  $f^{nl+k}(x)$ . In particular,  $f^{nl+k} \circ \gamma$  is the image through  $R_{j_{nl+k}}$  of the graph of a function  $\tilde{H}_\infty$ , from Proposition C.4.1. The function  $\tilde{H}_\infty$  is almost everywhere differentiable and at every point of differentiability  $(\tilde{H}_\infty(t), t)$  it holds  $DR_{j_{nl+k}}(\tilde{H}_\infty(t), t)(\tilde{H}'_\infty(t), 1) \in C_{R_{j_{nl+k}}(\tilde{H}_\infty(t), t)}^{s,\eta}$  by Proposition C.4.2. Denote as

$$f^{nl+k}(y_{nl}) = R_{j_{nl+k}}(\tilde{H}_\infty(s_0), s_0) \quad \text{and} \quad f^{nl+k}(x) = R_{j_{nl+k}}(\tilde{H}_\infty(s_1), s_1).$$

Up to invert the roles, assume that  $s_0 < s_1$ .

From (C.10), we have

$$\begin{aligned}
&L_{f^{nl+k} \circ \gamma}(f^{nl+k}(y_{nl}), f^{nl+k}(x)) = \\
&= \int_{s_0}^{s_1} \left\| DR_{j_{nl+k}}(\tilde{H}_\infty(s), s)(\tilde{H}'_\infty(s), 1) \right\| ds \geq d(f^{nl+k}(y_{nl}), f^{nl+k}(x)) \geq \varepsilon.
\end{aligned} \tag{C.12}$$

We obtain so

$$\begin{aligned}
&L_{f^k \circ \gamma}(f^k(y_{nl}), f^k(x)) = L_{f^{-nl} \circ f^{nl+k} \circ \gamma}(f^{-nl}(f^{nl+k}(y_{nl})), f^{-nl}(f^{nl+k}(x))) = \\
&= \int_{s_0}^{s_1} \left\| \frac{d}{ds} f^{-nl}(R_{j_{nl+k}}(\tilde{H}_\infty(s), s)) \right\| ds = \\
&= \int_{s_0}^{s_1} \left\| \prod_{i=0}^{n-1} Df^{-l}(f^{-il}(R_{j_{nl+k}}(\tilde{H}_\infty(s), s))) DR_{j_{i+nl+k}}(\tilde{H}_\infty(s), s) (\tilde{H}'_\infty(s), 1) \right\| ds.
\end{aligned}$$

Since at every point of differentiability the vector  $DR_{j_{nl+k}}(\tilde{H}_\infty(s), s) (\tilde{H}'_\infty(s), 1)$  belongs to the stable cone field and from the cone field property satisfied by  $U$  (see Definition B.0.1), it holds that

$$L_{f^k \circ \gamma}(f^k(y_{nl}), f^k(x)) \geq \frac{1}{(\mu + \xi)^n} \int_{s_0}^{s_1} \left\| DR_{j_{nl+k}}(\tilde{H}_\infty(s), s)(\tilde{H}'_\infty(s), 1) \right\| ds.$$

That is, from (C.12),

$$L_{f^k \circ \gamma}(f^k(y_{nl}), f^k(x)) \geq \frac{1}{(\mu + \xi)^n} L_{f^{nl+k} \circ \gamma}(f^{nl+k}(y_{nl}), f^{nl+k}(x)) \geq \frac{\varepsilon}{(\mu + \xi)^n}. \tag{C.13}$$

By the choice of  $n \in \mathbb{N}$  in (C.9) and from (C.11) and (C.13) we obtain

$$\begin{aligned} \frac{\varepsilon}{(\mu + \xi)^n} &\leq L_{f^k \circ \gamma}(f^k(y_{nl}), f^k(x)) \leq \max_{x \in [0,1]^2} \|DR_{j_k}(x)\| \mathcal{P} \sqrt{1 + \mathcal{J}^2} \leq \\ &\leq \max_{R \in \mathcal{R}} \max_{x \in [0,1]^2} \|DR(x)\| \mathcal{P} \sqrt{1 + \mathcal{P}^2} < \frac{\varepsilon}{(\mu + \xi)^n}, \end{aligned}$$

which is the required contradiction.

We conclude so that there exists  $N \in \mathbb{N}$  such that  $D_\infty^s \subset f^{-N}(W_{loc,\varepsilon}^s(f^N(x)))$ .

□

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