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On the use of needle-like perturbations in spatially heterogeneous control systems

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Abstract In this paper we consider a general control system involving a *spatially heterogeneous dynamics*. This means that the state space is partitioned into several disjoint regions and that each region has its own (smooth) control system. As a result, the dynamics discontinuously changes whenever the trajectory crosses an interface between two regions. In that spatially heterogeneous setting (and in contrast with the usual smooth case), a needle-like perturbation of the control may generate a perturbed trajectory that does not uniformly converge towards the nominal one, and may lead to the absence of a corresponding first-order variation vector. The first contribution of this paper is to illustrate this issue by means of a simple counterexample. Our second and main contribution is to provide a *modified* needle-like perturbation of the control (adapted to the spatially heterogeneous setting) which generates a perturbed trajectory that uniformly converges towards the nominal one, and leads to a corresponding first-order variation vector (which has the particularity of admitting a discontinuity jump at each interface crossing). This is made possible under several assumptions (including transverse crossing conditions), by introducing new tools such as *auxiliary trajectories* and *auxiliary controls* and by using a conic version of the implicit function theorem.

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1 Introduction

One essential tool in control theory (among others) is the concept of *needle-like perturbation* introduced by Pontryagin *et al.* [30] in order to derive first-order approximations of accessible sets, and to establish first-order necessary optimality conditions, for control systems described by nonlinear differential equations. A needle-like perturbation is a particular perturbation of the control (roughly speaking, large for the L^∞ -norm but small for the L^1 -norm) which generates a perturbed trajectory that uniformly converges towards the nominal one and, above all, leads to a first-order variation quantified thanks to the so-called *variation vector*. We refer to Section 2.1 for basic reminders. The subject of the present work is precisely in line with this perturbation technique, but in the context of spatially heterogeneous dynamics (see below).

Hybrid control systems have recently attracted a significant attention in the literature in order to describe control systems exhibiting both continuous and discrete behaviors (see, *e.g.*, [10, 12, 18, 32, 33] and references therein). A particular case is given by control systems described by differential equations involving *heterogeneous dynamics* (that is, involving several different dynamics) where the changes of dynamics are seen as discrete events. These heterogeneous settings are varied. For example, one can consider *temporally* heterogeneous control systems where the changes of dynamics are determined by the time variable (see, *e.g.*, [6, 15, 17, 27–29]). In this paper we restrict our attention to *spatially* heterogeneous control systems where the changes of dynamics are determined by the state variable. Precisely, in the spirit of [2–7, 19, 22–24], we consider that the state space is partitioned into several disjoint regions and that each region has its own (smooth) control system. As a result, the dynamics discontinuously changes whenever the trajectory crosses an interface between two regions. The instants at which the trajectory goes from one region to another are called *crossing times*.

Our aim in this paper is to conduct a sensitivity analysis of a general spatially heterogeneous control system under needle-like perturbations. This question has received little attention in the literature, except in [19] where an expression of a variation vector corresponding to a classical needle-like perturbation can be found. This variation vector has furthermore the particularity of admitting a discontinuity jump at each crossing time. However, in the spatially heterogeneous setting, and in contrast to what is claimed in [19], a classical needle-like perturbation may generate a perturbed trajectory that does not uniformly converge towards the nominal one, and may lead to the absence of a corresponding variation vector. The first contribution of this paper is to illustrate this issue by means of a simple counterexample (see Example 2.1). We precise here that the error in [19] does not affect the validity of the main part

of [19] dedicated to convergence results on regularized versions of spatially heterogeneous optimal control problems. Nevertheless the above counterexample shows that the use of needle-like perturbations in the context of spatially heterogeneous control systems requires a careful attention and this is the subject of the present paper.

Hence the second and main contribution of the present work is to provide a rigorous derivation of the same variation vector obtained in [19], but by using a *modified* needle-like perturbation of the control (adapted to the spatially heterogeneous setting). For this purpose, new tools are introduced, referred to as *auxiliary trajectories* and *auxiliary controls*, which correspond, roughly speaking, to (smooth) extensions of the restrictions of the nominal trajectory and of the nominal control to each interval between two consecutive crossing times. Next, we apply a classical needle-like perturbation to a given auxiliary control, generating that way a so-called *perturbed auxiliary trajectory*. Under appropriate assumptions (including transverse crossing conditions on the nominal trajectory, similar to the ones considered in [4, 19], see Definition 1), and by using a conic version of the implicit function theorem (see Lemma 1), we prove the existence of a *perturbed crossing time* for this perturbed auxiliary trajectory. This results into a perturbation of both the initial time and the initial condition of the next auxiliary trajectory, and thus into a next perturbed auxiliary trajectory. Hence we construct by induction a series of perturbed auxiliary trajectories, leading by concatenation to a global perturbed trajectory (corresponding to a perturbation of the nominal one). This approach addresses the issues raised by Example 2.1 (see Remark 1) and allows us to state and prove our main result (Theorem 1). This latter asserts that the perturbed trajectory uniformly converges towards the nominal one, and establishes the existence of a corresponding variation vector that coincides with the one announced in [19].

This paper is organized as follows. It contains only one section (Section 2). In Section 2.1, basic reminders about classical needle-like perturbations and their use in smooth control systems are provided. In Section 2.2, a general spatially heterogeneous control system is introduced, together with some terminology and assumptions. The counterexample (Example 2.1), showing that an adaptation of the needle-like perturbations in that context is required, is provided. Our main result (Theorem 1) is stated and, since its proof is quite long and technical, this latter is postponed to the end of the paper (see Appendices A and B). Nevertheless, for the reader's convenience, an overview of this proof is given in Section 2.3 (in order to highlight the new tools and the main ideas of our approach). Finally a list of comments and perspectives is provided in Section 2.4.

2 Main result and overview of its proof

We start with some notations and functional framework required all along the paper. In this work, when $(\mathcal{Z}, d_{\mathcal{Z}})$ is a metric set, we denote by $\bar{B}_{\mathcal{Z}}(z, \varepsilon)$ the

closed ball of \mathcal{Z} centered at some $z \in \mathcal{Z}$ and of some radius $\varepsilon > 0$. For any positive integer $d \in \mathbb{N}^*$, we denote by $\langle \cdot, \cdot \rangle_{\mathbb{R}^d}$ (resp. $\|\cdot\|_{\mathbb{R}^d}$) the standard inner product (resp. Euclidean norm) of \mathbb{R}^d . For any subset $S \subset \mathbb{R}^d$, we denote by ∂S the boundary of S defined by $\partial S := \overline{S} \setminus \text{Int}(S)$, where \overline{S} and $\text{Int}(S)$ stand for the closure and interior of S respectively. Furthermore, for any extended-real number $r \in [1, +\infty]$ and any real interval $I \subset \mathbb{R}$ with nonempty interior, we denote by:

- $L^r(I, \mathbb{R}^d)$ the standard Lebesgue space of r -integrable functions defined on I with values in \mathbb{R}^d , endowed with its usual norm $\|\cdot\|_{L^r}$.
- $C(I, \mathbb{R}^d)$ the standard space of continuous functions defined on I with values in \mathbb{R}^d , endowed with its standard uniform norm $\|\cdot\|_C$.
- $\text{AC}(I, \mathbb{R}^d)$ the subspace of $C(I, \mathbb{R}^d)$ of absolutely continuous functions.

If a function $\gamma : I \rightarrow \mathbb{R}^d$ admits left and right limits at some $\tau \in \text{Int}(I)$, we denote by

$$\gamma^-(\tau) := \lim_{t \uparrow \tau} \gamma(t) \quad \text{and} \quad \gamma^+(\tau) := \lim_{t \downarrow \tau} \gamma(t).$$

Finally, when $I = [0, T]$ for some $T > 0$, recall that a partition of the interval $[0, T]$ is a finite set $\mathbb{T} = \{t_k\}_{k=0, \dots, N}$ such that $0 = t_0 < t_1 < \dots < t_{N-1} < t_N = T$ for some positive integer $N \in \mathbb{N}^*$.

2.1 Reminders about classical needle-like perturbations and their use in smooth control systems

For notions and results presented in this section, we refer to standard monographs such as [1, 11, 13, 16, 30, 31] and references therein. Let $n, m \in \mathbb{N}^*$ be two positive integers and $T > 0$ be a fixed positive real number. Consider the smooth control system given by

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)), & \text{a.e. } t \in [0, T], \\ x(0) = x_{\text{init}}, \end{cases} \quad (\text{CS})$$

where the initial condition $x_{\text{init}} \in \mathbb{R}^n$ is fixed and $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a dynamics of class C^1 . As usual in the literature, x is called the *state* (or *trajectory*) and u is called the *control*. Now, let $(x, u) \in \text{AC}([0, T], \mathbb{R}^n) \times L^\infty([0, T], \mathbb{R}^m)$ be a solution to (CS). Roughly speaking, a first-order sensitivity analysis of the final value $x(T)$ with respect to control perturbations consists in obtaining an expression of the limit

$$\lim_{\alpha \downarrow 0} \frac{x^\alpha(T) - x(T)}{\alpha}, \quad (2.1)$$

where $x^\alpha \in \text{AC}([0, T], \mathbb{R}^n)$ is a *perturbed trajectory* associated with some *perturbed control* $u^\alpha \in L^\infty([0, T], \mathbb{R}^m)$. It plays a fundamental role in control theory in order to obtain first-order approximations of the accessible set and to establish first-order necessary optimality conditions. One of the best-known

(if not the best-known) control perturbation in the literature is called *needle-like perturbation* which consists in taking

$$u^\alpha(t) := \begin{cases} \omega & \text{for all } t \in [\tau - \alpha, \tau), \\ u(t) & \text{otherwise,} \end{cases} \quad (2.2)$$

where $\omega \in \mathbb{R}^m$ and $\tau \in (0, T)$ is a Lebesgue point of the map $t \mapsto f(x(t), u(t))$. It is well known that the corresponding perturbed trajectory x^α uniformly converges to x as $\alpha \downarrow 0$ and that the limit (2.1) is equal to $w(T)$, where $w \in \text{AC}([\tau, T], \mathbb{R}^n)$, commonly called *variation vector*, is the unique solution to the linear Cauchy problem

$$\begin{cases} \dot{w}(t) = \nabla_x f(x(t), u(t)) w(t), & \text{a.e. } t \in [\tau, T], \\ w(\tau) = f(x(\tau), \omega) - f(x(\tau), u(\tau)). \end{cases}$$

Figure 1 illustrates the concept of needle-like perturbation.

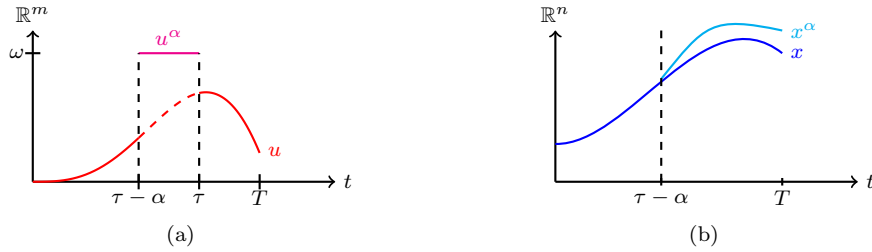


Fig. 1: Illustration of a classical needle-like perturbation of the control (left) and of the corresponding perturbed trajectory (right).

2.2 Counterexample and main result in a spatially heterogeneous context

Now, in the spirit of [2–7, 19, 22–24], consider a partition $\mathbb{R}^n = \cup_{j \in \mathcal{J}} \overline{X_j}$ of the state space, where \mathcal{J} is a family of indexes and where the nonempty open subsets X_j , called *regions*, are pairwise disjoint. In this paper we focus on the *spatially heterogeneous* control system given by

$$\begin{cases} \dot{x}(t) = h(x(t), u(t)), & \text{a.e. } t \in [0, T], \\ x(0) = x_{\text{init}}, \end{cases} \quad (\text{HS})$$

where the fixed initial condition x_{init} belongs to some X_{j_1} (with $j_1 \in \mathcal{J}$) and where the spatially heterogeneous dynamics $h : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is defined regionally by

$$\forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^m, \quad h(x, u) := h_j(x, u) \quad \text{if } x \in X_j,$$

where all the subdynamics $h_j : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ are of class C^1 . Note that $h(x, u)$ is not defined when $x \notin \cup_{j \in \mathcal{J}} X_j$ but this fact will have no impact on the rest of this paper since we will focus on regular solutions to (HS) only (see Definition 1 below and Remark 5 in Section 2.4).

In the above spatially heterogeneous setting, the main difficulty in performing a first-order sensitivity analysis of the final value $x(T)$ lies in handling the changes of dynamics occurring at the interfaces $\partial X_j \cap \partial X_{j'}$. For this purpose, as in [19], we will focus only on particular solutions to (HS), called *regular*, which satisfy several appropriate properties, including transverse crossing conditions.

Definition 1 *A solution $(x, u) \in AC([0, T], \mathbb{R}^n) \times L^\infty([0, T], \mathbb{R}^m)$ to (HS) is said to be regular if the following conditions are satisfied:*

(C1) *There exists a partition $\mathbb{T} = \{t_k^c\}_{k=0, \dots, N}$ of the interval $[0, T]$ such that*

$$\forall k \in \{1, \dots, N\}, \exists j(k) \in \mathcal{J} \setminus \{j(k-1)\}, \forall t \in (t_{k-1}^c, t_k^c), x(t) \in X_{j(k)},$$

with $x(0) \in X_{j(1)}$ (and thus $j(1) = j_1$) and $x(T) \in X_{j(N)}$.

The times t_k^c , for $k \in \{1, \dots, N-1\}$, are called crossing times since they correspond to the instants at which x goes from the region $X_{j(k)}$ to the region $X_{j(k+1)}$, and thus $x(t_k^c) \in \partial X_{j(k)} \cap \partial X_{j(k+1)}$.

(C2) *There exist $\delta > 0$ and $\nu > 0$ such that at each crossing time t_k^c :*

(i) *The control u is continuous over $[t_k^c - \delta, t_k^c]$ and over $(t_k^c, t_k^c + \delta]$, and admits left and right limits at t_k^c , denoted by $u^-(t_k^c)$ and $u^+(t_k^c)$ respectively.*

(ii) *There exists a C^1 function $F_k : \mathbb{R}^n \rightarrow \mathbb{R}$ such that*

$$\forall y \in \bar{B}_{\mathbb{R}^n}(x(t_k^c), \nu), \begin{cases} y \in X_{j(k)} \Leftrightarrow F_k(y) < 0, \\ y \in \partial X_{j(k)} \cap \partial X_{j(k+1)} \Leftrightarrow F_k(y) = 0, \\ y \in X_{j(k+1)} \Leftrightarrow F_k(y) > 0. \end{cases}$$

In particular it holds that $F_k(x(t_k^c)) = 0$.

(iii) *The transverse crossing conditions*

$$\begin{aligned} \langle \nabla F_k(x(t_k^c)), h_{j(k)}(x(t_k^c), u^-(t_k^c)) \rangle_{\mathbb{R}^n} &> 0, \\ \langle \nabla F_k(x(t_k^c)), h_{j(k+1)}(x(t_k^c), u^+(t_k^c)) \rangle_{\mathbb{R}^n} &> 0, \end{aligned}$$

are both satisfied.

Under the conditions introduced in Definition 1 and using implicit function arguments to handle the interface crossings, the authors of [19] assert (in some way) the following theorem, providing a first-order sensitivity analysis of the final value $x(T)$ which involves a variation vector w admitting a discontinuity jump at each crossing time. We refer to Section 2.4 for comments on Theorem 1 and on the restrictions imposed by Definition 1.

Theorem 1 *Let $(x, u) \in \text{AC}([0, T], \mathbb{R}^n) \times \text{L}^\infty([0, T], \mathbb{R}^m)$ be a regular solution to (HS) and consider the notations introduced in Definition 1. Let $\omega \in \mathbb{R}^m$ and $\tau \in (t_{q-1}^c, t_q^c)$, for some $q \in \{1, \dots, N\}$, be a Lebesgue point of the map $t \mapsto h_{j(q)}(x(t), u(t))$. Then, there exists a solution $(x^\alpha, u^\alpha) \in \text{AC}([0, T], \mathbb{R}^n) \times \text{L}^\infty([0, T], \mathbb{R}^m)$ to (HS) such that x^α uniformly converges to x as $\alpha \downarrow 0$ and*

$$\lim_{\alpha \downarrow 0} \frac{x^\alpha(T) - x(T)}{\alpha} = w(T),$$

where $w : [\tau, T] \rightarrow \mathbb{R}^n$ is the variation vector defined by

$$w(t) := \begin{cases} w_q(t) & \text{for all } t \in [\tau, t_q^c], \\ w_k(t) & \text{for all } t \in [t_{k-1}^c, t_k^c] \text{ and all } k \in \{q+1, \dots, N-1\}, \\ w_N(t) & \text{for all } t \in [t_{N-1}^c, T], \end{cases}$$

where $w_q \in \text{AC}([\tau, t_q^c], \mathbb{R}^n)$ is the unique solution to the linear Cauchy problem

$$\begin{cases} \dot{w}_q(t) = \nabla_x h_{j(q)}(x(t), u(t)) w_q(t), & \text{a.e. } t \in [\tau, t_q^c], \\ w_q(\tau) = h_{j(q)}(x(\tau), \omega) - h_{j(q)}(x(\tau), u(\tau)), \end{cases}$$

and $w_k \in \text{AC}([t_{k-1}^c, t_k^c], \mathbb{R}^n)$ is defined by induction as the unique solution to the linear Cauchy problem

$$\begin{cases} \dot{w}_k(t) = \nabla_x h_{j(k)}(x(t), u(t)) w_k(t), & \text{a.e. } t \in [t_{k-1}^c, t_k^c], \\ w_k(t_{k-1}^c) = w_{k-1}(t_{k-1}^c) + \xi_{k-1}, \end{cases}$$

for all $k \in \{q+1, \dots, N\}$, where $\xi_k \in \mathbb{R}^n$ is the jump vector defined by

$$\xi_k := \sigma_k(h_{j(k+1)}(x(t_k^c), u^+(t_k^c)) - h_{j(k)}(x(t_k^c), u^-(t_k^c))),$$

with $\sigma_k \in \mathbb{R}$ defined by

$$\sigma_k := \frac{\langle \nabla F_k(x(t_k^c)), w_k(t_k^c) \rangle_{\mathbb{R}^n}}{\langle \nabla F_k(x(t_k^c)), h_{j(k)}(x(t_k^c), u^-(t_k^c)) \rangle_{\mathbb{R}^n}},$$

for all $k \in \{q, \dots, N-1\}$.

In the (sketched) proof of Theorem 1 proposed in [19, Section 2.1.1], the perturbed control u^α used by the authors is a classical needle-like perturbation of the form (2.2). Our first contribution in the present work is to provide the next counterexample showing that this approach is not mathematically correct. Nevertheless we precise here that the error in [19] does not affect the validity of Theorem 1, but only of its proof in [19]. It does not affect either the main part of [19] dedicated to convergence results on regularized versions of spatially heterogeneous optimal control problems.

Example 2.1 Consider (HS) with $n = m = 1$, $T = 2$, the state space partition $\mathbb{R} = \overline{X_1} \cup \overline{X_2}$ with $X_1 := \{x \in \mathbb{R} \mid x < 1\}$ and $X_2 := \{x \in \mathbb{R} \mid x > 1\}$, $x_{\text{init}} = 0$ and $h_1(x, u) := +u$ and $h_2(x, u) := -u$ for all $(x, u) \in \mathbb{R} \times \mathbb{R}$. Taking the control $u \in L^\infty([0, 2], \mathbb{R})$ given by $u(t) = +1$ over $[0, 1)$ and $u(t) = -1$ over $[1, 2]$, we get the trajectory $x \in AC([0, 2], \mathbb{R})$ given by $x(t) = t$ over $[0, 2]$, with $t_1^c = 1$ as unique crossing time. Note that all conditions of Definition 1 are satisfied. Now consider a classical needle-like perturbation u^α of the form (2.2) associated with $\omega = -1$ and some $\tau \in (0, 1)$. We get a perturbed trajectory $x^\alpha \in AC([0, 2], \mathbb{R})$ satisfying $x^\alpha(t) \in X_1$ over the whole interval $[0, 2]$ (see Figure 2). We deduce that x^α does not uniformly converge to x as $\alpha \downarrow 0$ and that the limit $\lim_{\alpha \downarrow 0} \frac{x^\alpha(T) - x(T)}{\alpha}$ does not exist.

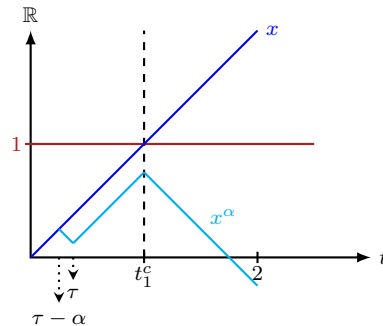


Fig. 2: Illustration of Example 2.1.

The issue highlighted by Example 2.1 is that classical needle-like perturbations do not take into account the perturbation of the crossing times. Hence, in the spatially heterogeneous context, a careful modification of the needle-like perturbations is required in order to ensure the validity of Theorem 1. This is the second and main contribution of the present work. As in [19], our proof of Theorem 1 is based on implicit function arguments to handle the interface crossings, but also on new tools, such as *auxiliary trajectories* and *auxiliary controls*, allowing a technical construction by concatenation of a satisfactory perturbed trajectory x^α , associated with a *modified* needle-like perturbation u^α . To ease the reading, we postpone the detailed proof of Theorem 1 (with all technicalities) to the end of the paper (see Appendices A and B). Nevertheless we propose in the next section an overview of this proof, presenting in a concise way the new tools and the main ideas of our approach. In particular, the definitions of the perturbed trajectory x^α (constructed by concatenation) and of the modified needle-like perturbation u^α are given in Item 7. Their explicit application to the context of Example 2.1 is provided in Remark 1 in Section 2.4.

2.3 Overview of the proof of Theorem 1

This overview is divided into seven major items in which we take care to highlight at which points the conditions introduced in Definition 1 are used.

1. Let (x, u) be a regular solution to (HS) and consider the notations introduced in Definition 1. For all $k \in \{1, \dots, N\}$, introduce the *auxiliary control*, denoted by \tilde{u}_k , that coincides with u over (t_{k-1}^c, t_k^c) and that is continuously extended to a constant function outside (t_{k-1}^c, t_k^c) thanks to Condition (C2)(i) (see Figure 3). We refer to the beginning of Appendix B for the exact definition of \tilde{u}_k .

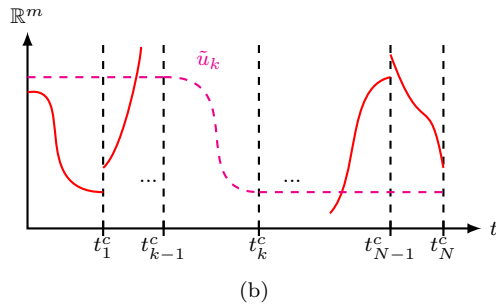
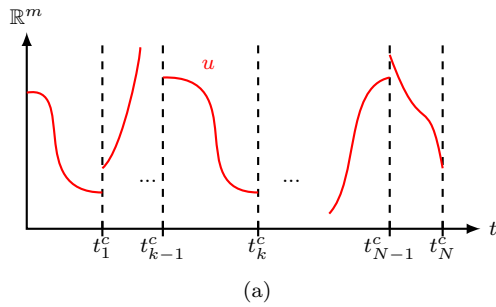


Fig. 3: Illustration of an auxiliary control \tilde{u}_k . In this illustration, for simplicity, we have chosen a control u that is continuous over each interval between two consecutive crossing times, but this is not mandatory. We only know that u satisfies the continuity properties given in Condition (C2)(i).

2. Then, for all $k \in \{1, \dots, N\}$, introduce the *auxiliary trajectory*, denoted by \tilde{z}_k , as the unique solution to the smooth state equation associated with the dynamics $h_{j(k)}$ only (that is, with the dynamics $h_{j(k)}$ all over \mathbb{R}^n , even outside $X_{j(k)}$), with the auxiliary control \tilde{u}_k and with the initial condition $\tilde{z}_k(t_{k-1}^c) = x(t_{k-1}^c)$. Note that \tilde{z}_k corresponds to an extension

of x as illustrated in Figure 4. We refer to the beginning of Appendix B for the exact definition of \tilde{z}_k .

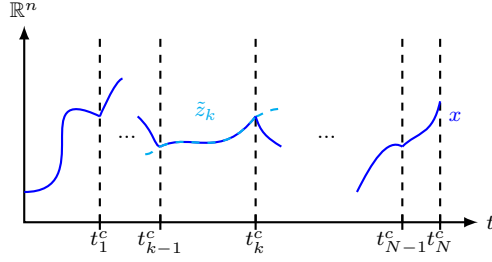


Fig. 4: Illustration of an auxiliary trajectory \tilde{z}_k .

- Now let $\omega \in \mathbb{R}^m$ and $\tau \in (t_{q-1}^c, t_q^c)$, for some $q \in \{1, \dots, N\}$, be a Lebesgue point of the map $t \mapsto h_{j(q)}(x(t), u(t))$. Consider a classical needle-like perturbation of the auxiliary control \tilde{u}_q associated with ω and τ (see Figure 5). This gives us a *perturbed auxiliary trajectory* denoted by \tilde{z}_q^α . Since we deal here with a classical smooth setting (with the dynamics $h_{j(q)}$ only), we can use standard results from the literature such as the uniform convergence of \tilde{z}_q^α to \tilde{z}_q as $\alpha \downarrow 0$, and the existence of a variation vector, denoted by w_q , solution to a linear Cauchy problem with an initial condition at τ given by to $h_{j(q)}(\tilde{z}_q(\tau), \omega) - h_{j(q)}(\tilde{z}_q(\tau), \tilde{u}_q(\tau)) = h_{j(q)}(x(\tau), \omega) - h_{j(q)}(x(\tau), u(\tau))$.

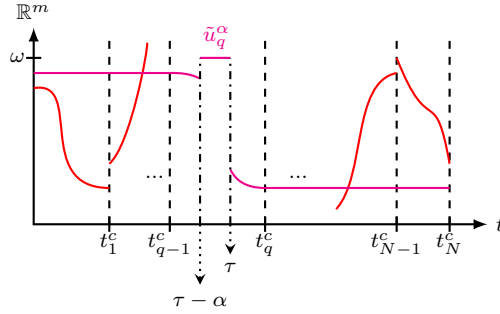


Fig. 5: Illustration of a classical needle-like perturbation of the auxiliary control \tilde{u}_q .

- The next step is to prove that the perturbed auxiliary trajectory \tilde{z}_q^α crosses the interface $\partial X_{j(q)} \cap \partial X_{j(q+1)}$ at some perturbed crossing time $\hat{t}_q(\alpha)$ (see Figure 6). For this purpose, we invoke a conic version of the implicit function theorem (see Lemma 1) to the map $G_q : (\alpha, t) \mapsto F_q(\tilde{z}_q^\alpha(t))$ that can be applied thanks to Conditions (C2)(i)(ii)(iii) and the construction of \tilde{u}_q .

In particular, note that $\nabla_t G_q$ is invertible at $(0, t_q^c)$ thanks to the first transverse crossing condition in Condition (C2)(iii).

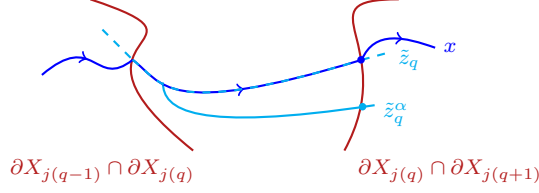


Fig. 6: The perturbed auxiliary trajectory \tilde{z}_q^α crosses the interface $\partial X_{j(q)} \cap X_{j(q+1)}$ at some perturbed crossing time $\tilde{t}_q(\alpha)$.

5. From the construction of the perturbed auxiliary trajectory \tilde{z}_q^α , it can be proved that \tilde{z}_q^α stays inside $X_{j(q)}$ over $(t_{q-1}^c, \tilde{t}_q(\alpha))$. Indeed, thanks to Condition (C2)(iii), one can prove by contradiction that there exist $t_{q-1}^c < s'_q < s_q < \min\{t_q^c, \tilde{t}_q(\alpha)\}$, uniformly with respect to α , such that \tilde{z}_q^α has values in $X_{j(q)}$ over (t_{q-1}^c, s'_q) and over $(s_q, \tilde{t}_q(\alpha))$ (see Lemma 3 for technical details). Then, from the uniform convergence of \tilde{z}_q^α to $\tilde{z}_q = x$ over $[s'_q, s_q]$, we obtain that \tilde{z}_q^α has values in the open set $X_{j(q)}$ over $[s'_q, s_q]$ also.
6. After having analyzed the perturbations in the region $X_{j(q)}$ in Items 3, 4, 5, the next step is to analyze the resulting perturbations in the next regions $X_{j(q+1)}, \dots, X_{j(N)}$. For the reader's convenience, we will develop here only the passage to the region $X_{j(q+1)}$ (the other regions are treated by induction, see Item 7). In contrast with Item 3 (in which we have proceeded to a classical needle-like perturbation of the auxiliary control), we will consider here the perturbation of the initial time t_q^c by $\tilde{t}_q(\alpha)$ (constructed in Item 4) and the perturbation of the initial condition $x(t_q^c)$ by $\tilde{z}_q^\alpha(\tilde{t}_q(\alpha))$. This gives us the perturbed auxiliary trajectory \tilde{z}_{q+1}^α . This construction allows us to proceed to a concatenation of the perturbed auxiliary trajectories \tilde{z}_q^α and \tilde{z}_{q+1}^α (see Figure 7).

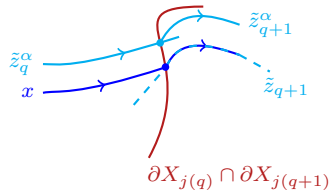


Fig. 7: Perturbed auxiliary trajectory \tilde{z}_{q+1}^α under perturbations of the initial time and of the initial condition.

Since we deal here with a classical smooth setting (with the dynamics $h_{j(q+1)}$ only), we can use standard results from the literature such as the uniform convergence of \tilde{z}_{q+1}^α to \tilde{z}_{q+1} as $\alpha \downarrow 0$, and the existence of a variation vector, denoted by w_{q+1} , solution to a linear Cauchy problem with an initial condition at t_q^c that is the sum of $w_q(t_q^c)$ and of an extra term ξ_q due to the perturbations of the initial time and of the initial condition. Note also that, similarly to Item 4, we prove that \tilde{z}_{q+1}^α crosses the interface $\partial X_{j(q+1)} \cap \partial X_{j(q+2)}$ at some perturbed crossing time $\tilde{t}_{q+1}(\alpha)$ and, using similar arguments to Item 5, it can be proved that the trajectory \tilde{z}_{q+1}^α stays inside $X_{j(q+1)}$ over $(\tilde{t}_q(\alpha), \tilde{t}_{q+1}(\alpha))$.

7. Finally, we proceed by induction, region after region, in order to construct the perturbed auxiliary trajectories \tilde{z}_k^α and the corresponding variation vectors w_k for all $k \in \{q, \dots, N\}$. Then, we construct by concatenation the perturbed trajectory x^α , associated with the modified needle-like perturbation u^α , as follows:

$$x^\alpha(t) := \begin{cases} x(t) & \text{for all } t \in [t_0^c, t_{q-1}^c], \\ \tilde{z}_q^\alpha(t) & \text{for all } t \in [t_{q-1}^c, \tilde{t}_q(\alpha)], \\ \tilde{z}_k^\alpha(t) & \text{for all } t \in [\tilde{t}_{k-1}(\alpha), \tilde{t}_k(\alpha)] \text{ and } k \in \{q+1, \dots, N-1\}, \\ \tilde{z}_N^\alpha(t) & \text{for all } t \in [\tilde{t}_{N-1}(\alpha), t_N^c], \end{cases}$$

and

$$u^\alpha(t) := \begin{cases} u(t) & \text{for all } t \in [t_0^c, \tau - \alpha), \\ \omega & \text{for all } t \in [\tau - \alpha, \tau), \\ \tilde{u}_q(t) & \text{for all } t \in [\tau, \tilde{t}_q(\alpha)), \\ \tilde{u}_k(t) & \text{for all } t \in [\tilde{t}_{k-1}(\alpha), \tilde{t}_k(\alpha)) \text{ and } k \in \{q+1, \dots, N-1\}, \\ \tilde{u}_N(t) & \text{for all } t \in [\tilde{t}_{N-1}(\alpha), t_N^c]. \end{cases}$$

Then we construct the variation vector $w : [\tau, T] \rightarrow \mathbb{R}^n$ by concatenation as it is defined in Theorem 1. This approach guarantees that x^α uniformly converges to x as $\alpha \downarrow 0$ and that

$$\lim_{\alpha \downarrow 0} \frac{x^\alpha(T) - x(T)}{\alpha} = \lim_{\alpha \downarrow 0} \frac{\tilde{z}_N^\alpha(T) - \tilde{z}_N(T)}{\alpha} = w_N(T) = w(T).$$

2.4 A list of comments and perspectives

Remark 1 *In the context of Example 2.1, the modified needle-like perturbation u^α and the corresponding perturbed trajectory x^α , both presented in Item 7 of Section 2.3, are respectively given by*

$$u^\alpha(t) = \begin{cases} +1 & \text{for all } t \in [0, \tau - \alpha), \\ -1 & \text{for all } t \in [\tau - \alpha, \tau), \\ +1 & \text{for all } t \in [\tau, \tilde{t}_1(\alpha)), \\ -1 & \text{for all } t \in [\tilde{t}_1(\alpha), 2], \end{cases}$$

and

$$x^\alpha(t) = \begin{cases} t & \text{for all } t \in [0, \tau - \alpha], \\ -t + 2(\tau - \alpha) & \text{for all } t \in [\tau - \alpha, \tau], \\ t - 2\alpha & \text{for all } t \in [\tau, 2], \end{cases}$$

where $\tilde{t}_1(\alpha) = 1 + 2\alpha$ is the perturbed crossing time of $t_1^c = 1$ (see Figure 8). Here x^α uniformly converges to x as $\alpha \downarrow 0$ and the limit $\lim_{\alpha \downarrow 0} \frac{x^\alpha(T) - x(T)}{\alpha}$ exists and is equal to -2 , which is in accordance with the conclusions of Theorem 1.

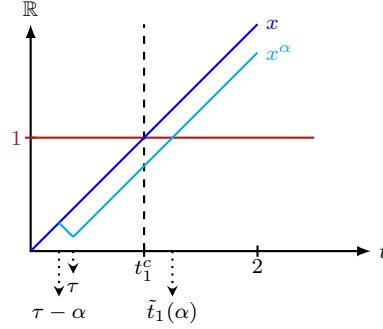


Fig. 8: Illustration of Remark 1 (to be compared with Figure 2).

Remark 2 Example 2.1 shows that using needle-like perturbations of the control in the context of a spatially heterogeneous control system is not trivial. Actually, perturbing the initial condition x_{init} in that context is not trivial either. Indeed, considering a basic perturbation $x_{\text{init}} - \alpha$ in Example 2.1 leads also to a perturbed trajectory x^α satisfying $x^\alpha(t) \in X_1$ over the whole interval $[0, 2]$. Thus x^α does not uniformly converge to x as $\alpha \downarrow 0$ and the limit $\lim_{\alpha \downarrow 0} \frac{x^\alpha(T) - x(T)}{\alpha}$ does not exist. Hence, in the spatially heterogeneous setting, a perturbation of the initial condition should induce also a perturbation of the control in order to take into account the perturbation of the crossing times. Such an extension can be achieved by adapting our approach summarized in Section 2.3, and detailed in Appendices A and B, but this is out of the scope of the present paper and left to the readers.

Remark 3 One main difference between Theorem 1 and the classical smooth context (see Section 2.1 for reminders) is the discontinuity of the variation vector w at each crossing time t_k^c . The jump vector ξ_k arises from the perturbation $\tilde{t}_k(\alpha)$ of the crossing time t_k^c (obtained from the application of a conic version of the implicit function theorem). Nonetheless this jump vector can be zero in some particular cases, such as when $h_{j(k+1)}(x(t_k^c), u^+(t_k^c)) = h_{j(k)}(x(t_k^c), u^-(t_k^c))$ (see Example 2.1 and Remark 1 for an example). Of course, the discontinuity jumps of the variation vector w in Theorem 1 can be directly

correlated with the discontinuity jumps of the costate p arising in Hybrid maximum principles stated for spatially heterogeneous optimal control problems (see, e.g., [4, 7, 19]). We refer to [9, Remark 2.2.4] for a discussion relating the discontinuities of the (forward) variation vector w with those of the (backward) costate p .

Remark 4 *In this paper we have considered a certain framework which is, of course, not the most general possible. We made some choices in order to keep the presentation as simple as possible, while keeping the essence of our work. For example, we have assumed that the subdynamics $h_j : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ are of class C^1 , while it is well known in the literature that this regularity can be relaxed (see, e.g., [14]). Also we have considered autonomous control systems (that is, independent of the time variable t), while it is clear that our approach can easily be extended to time-dependent control systems. Below we list some other possible extensions of our work:*

- *As in [9, 19], a possible extension could be to consider a time-dependent partition of the state space $\mathbb{R}^n = \cup_{j \in \mathcal{J}} \bar{X}_j(t)$ in order to encompass spatially heterogeneous control systems with time-evolutive regions. In that context, one should add continuity conditions on the multifunctions $X_j : [0, T] \rightrightarrows \mathbb{R}^n$ to maintain the applicability of our approach. One could also consider a space-time partition $\mathbb{R}^n \times [0, T] = \cup_{j \in \mathcal{J}} \bar{Y}_j$. Such a generalized context could encompass, as particular cases, temporally heterogeneous control systems (see Introduction), as well as spatially heterogeneous control systems with time-evolutive regions. Note that such an extension could possibly be addressed by using the classical augmentation technique which consists in considering the time variable t as an additional state variable x_{n+1} satisfying $\frac{dx_{n+1}}{dt} = 1$ (see, e.g., [8, Section 1.3.3]).*
- *One may also consider an extended framework that includes a regionally switching parameter (see [9]), meaning that the control system depends on an additional parameter that remains constant in each region but can change its value when the state crosses an interface. This framework is useful when considering control systems with loss control regions (see [3, 5]). One could also consider, for each region X_j , a subdynamics $h_j : \mathbb{R}^n \times \mathbb{R}^{m_j} \rightarrow \mathbb{R}^n$ with possibly different control dimensions $m_j \in \mathbb{N}^*$. This generalized context is interesting to deal with spatially heterogeneous control systems in which the control in one region is fundamentally different from the control in another region.*
- *Finally, this paper focuses on single needle-like perturbations, in the sense that the control is perturbed at only one time $\tau \in (0, T)$ and with only one control value $\omega \in \mathbb{R}^m$. In the literature, it is well known that multiple needle-like perturbations, for which the control is perturbed at several times $\tau_1, \dots, \tau_K \in (0, T)$ and with several control values $\omega_1, \dots, \omega_K \in \mathbb{R}^m$, are very useful in optimal control theory to establish first-order necessary optimality conditions in the presence of mixed initial-final state constraints (see, e.g., [21]). Therefore the extension of our approach to multiple needle-like perturbations would be relevant, but would also require a thor-*

ough and rigorous analysis due to the presence (already) of several crossing times $t_1^c, \dots, t_{N-1}^c \in (0, T)$. This is out of the scope of the present work, but it constitutes an interesting perspective for further researches.

Remark 5 *Theorem 1 deals with regular solutions to (HS) only, which allows us to develop our approach based on implicit function arguments (see Section 2.3 and Appendices A and B for details) but also induces some restrictions. For example, Condition (C1) does not include the possibility of an infinite number of crossing times (excluding a Zeno-type phenomenon [34]). Also it does not allow trajectories bouncing against a boundary of a region, or moving along such a boundary (excluding sliding modes as described for instance in [2, 26]). By the way, this last restriction is the reason why the fact that $h(x, u)$ is not defined when $x \notin \cup_{j \in \mathcal{J}} X_j$ has no impact on the present work. Also, note that the transverse crossing conditions in Condition (C2)(iii) have a geometrical interpretation (see Figure 9), excluding the possibility for the trajectory x to cross the interfaces $\partial X_{j(k)} \cap \partial X_{j(k+1)}$ tangentially. All these restrictions constitute nontrivial open challenges for further research works. To conclude this remark, we just mention that the right continuity of the control after each crossing time t_k^c and the left continuity before the last crossing time t_{N-1}^c in Condition (C2)(i) are useless in our proof of Theorem 1. We have adopted these hypotheses for the sake of simplicity of the presentation, however they can be removed.*

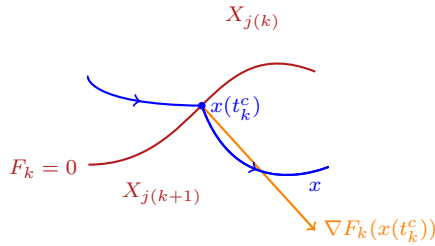


Fig. 9: Geometrical illustration of the transverse crossing conditions (C2)(iii): the trajectory x does not exit $X_{j(k)}$ tangentially and does not enter into $X_{j(k+1)}$ tangentially.

A Preliminaries for the proof of Theorem 1

As explained in Section 2.3, the proof of Theorem 1 is based on a conic version of the implicit function theorem (see Lemma 1 in Appendix A.1). It is also based on results concerning the sensitivity analysis of smooth control systems, not only with respect to needle-like perturbations of the control (as in Section 2.1), but also with respect to perturbations of the initial time and of the initial condition. The aim of Appendix A.2 is to gather these results in a unified framework. The proof of Theorem 1 will be developed with all technical details in the next appendix (Appendix B).

A.1 A conic version of the implicit function theorem

Lemma 1 *Let $\bar{\alpha} > 0$, $t^c \in \mathbb{R}$ and $\delta > 0$. Consider a continuous map*

$$\begin{aligned} \mathcal{G} : [0, \bar{\alpha}] \times [t^c - \delta, t^c + \delta] &\rightarrow \mathbb{R} \\ (\alpha, t) &\mapsto \mathcal{G}(\alpha, t), \end{aligned}$$

satisfying $\mathcal{G}(0, t^c) = 0$, such that $\nabla_{\alpha}\mathcal{G}(0, t^c)$ exists and such that $\nabla_t\mathcal{G}$ exists and is continuous over $[0, \bar{\alpha}] \times [t^c - \delta, t^c + \delta]$ with $\nabla_t\mathcal{G}(0, t^c) \neq 0$. Then there exist $0 < \bar{\beta} \leq \bar{\alpha}$ and an implicit function $\tilde{t} \in C([0, \bar{\beta}], [t^c - \delta, t^c + \delta])$, satisfying $\tilde{t}(0) = t^c$ and $\mathcal{G}(\alpha, \tilde{t}(\alpha)) = 0$ for all $\alpha \in [0, \bar{\beta}]$, that is differentiable at 0 with derivative

$$\tilde{t}'(0) = -\frac{\nabla_{\alpha}\mathcal{G}(0, t^c)}{\nabla_t\mathcal{G}(0, t^c)}.$$

Proof Consider the continuous extension $\mathcal{G}_0 : [-\bar{\alpha}, \bar{\alpha}] \times [t^c - \delta, t^c + \delta] \rightarrow \mathbb{R}$ defined by

$$\forall (\alpha, t) \in [-\bar{\alpha}, \bar{\alpha}] \times [t^c - \delta, t^c + \delta], \quad \mathcal{G}_0(\alpha, t) := \begin{cases} \mathcal{G}(\alpha, t) & \text{if } \alpha \in [0, \bar{\alpha}], \\ 2\mathcal{G}(0, t) - \mathcal{G}(-\alpha, t) & \text{if } \alpha \in [-\bar{\alpha}, 0]. \end{cases}$$

From the assumptions of Lemma 1, one can easily derive that $\mathcal{G}_0(0, t^c) = 0$, $\nabla_{\alpha}\mathcal{G}_0(0, t^c)$ exists and $\nabla_t\mathcal{G}_0$ exists and is continuous over $[-\bar{\alpha}, \bar{\alpha}] \times [t^c - \delta, t^c + \delta]$ with $\nabla_t\mathcal{G}_0(0, t^c) \neq 0$. Using a classical version of the implicit function theorem (see, e.g., [25, Theorem 9.3] and [20, Theorem E]), there exist $0 < \bar{\beta} \leq \bar{\alpha}$ and an implicit function $\tilde{t} \in C([-\bar{\beta}, \bar{\beta}], [t^c - \delta, t^c + \delta])$, satisfying $\tilde{t}(0) = t^c$ and $\mathcal{G}_0(\alpha, \tilde{t}(\alpha)) = 0$ for all $\alpha \in [-\bar{\beta}, \bar{\beta}]$, that is differentiable at 0 with derivative $\tilde{t}'(0) = -\frac{\nabla_{\alpha}\mathcal{G}_0(0, t^c)}{\nabla_t\mathcal{G}_0(0, t^c)}$. To conclude the proof, one has just to consider the restriction of the function \tilde{t} to the interval $[0, \bar{\beta}]$ and to use the facts that $\nabla_{\alpha}\mathcal{G}_0(0, t^c) = \nabla_{\alpha}\mathcal{G}(0, t^c)$ and $\nabla_t\mathcal{G}_0(0, t^c) = \nabla_t\mathcal{G}(0, t^c)$.

A.2 Sensitivity analysis of smooth control systems

Let $g : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a dynamics of class C^1 . For any triplet $\theta = (u, r, y_r) \in L^{\infty}([0, T], \mathbb{R}^m) \times [0, T] \times \mathbb{R}^n$, the classical Cauchy-Lipschitz theorem (or Picard-Lindel of theorem) ensures the existence and the uniqueness of the maximal solution to the Cauchy problem

$$\begin{cases} \dot{y}(t) = g(y(t), u(t)), & \text{a.e. } t \in [0, T], \\ y(r) = y_r. \end{cases}$$

This maximal solution is denoted by $y(\cdot, g, \theta)$ and is defined over the maximal interval denoted by $I(g, \theta) \subset [0, T]$. Recall that the classical blow-up theorem ensures that, either $I(g, \theta) = [0, T]$ (in that case we speak of a *global* solution), either $y(\cdot, g, \theta)$ is unbounded over $I(g, \theta)$. In the sequel we denote by $\text{Glob}(g)$ the set of all triplets θ such that $I(g, \theta) = [0, T]$.

For the technical needs of this section, for any triplet $\theta = (u, r, y_r) \in \text{Glob}(g)$ and any $R \geq \|u\|_{L^{\infty}}$, we denote by $M(g, \theta, R) \geq 0$ a common bound of $\|g\|_{\mathbb{R}^n}$, $\|\nabla_x g\|_{\mathbb{R}^n \times \mathbb{R}^m}$ and $\|\nabla_u g\|_{\mathbb{R}^n \times \mathbb{R}^m}$ over the compact set $K(g, \theta, R) := \cup_{t \in [0, T]} K_t(g, \theta, R)$ where

$$K_t(g, \theta, R) := \left\{ (x, v) \in \mathbb{R}^n \times \mathbb{R}^m \mid \|x - y(t, g, \theta)\|_{\mathbb{R}^n} \leq 1 \text{ and } \|v\|_{\mathbb{R}^m} \leq R \right\}.$$

Note that $(y(t, g, \theta), u(t)) \in K_t(g, \theta, R)$ for almost every $t \in [0, T]$. Since $K_t(g, \theta, R)$ is convex, one can easily get that

$$\|g(y_2, v_2) - g(y_1, v_1)\|_{\mathbb{R}^n} \leq M(g, \theta, R)(\|y_2 - y_1\|_{\mathbb{R}^n} + \|v_2 - v_1\|_{\mathbb{R}^m}),$$

for all $(y_2, v_2), (y_1, v_1) \in K_t(g, \theta, R)$, for all $t \in [0, T]$.

We are now in a position to state and prove the next continuous dependence result for the trajectory $y(\cdot, g, \theta)$ with respect to the triplet θ .

Lemma 2 For any triplet $\theta = (u, r, y_r) \in \text{Glob}(g)$ and any $R \geq \|u\|_{L^\infty}$, there exists $\varepsilon > 0$ such that the neighborhood of θ defined by

$$\mathcal{N}(g, \theta, R, \varepsilon) := \left(\overline{B}_{L^1}(u, \varepsilon) \cap \overline{B}_{L^\infty}(0_{L^\infty}, R) \right) \times \left([r - \varepsilon, r + \varepsilon] \cap [0, T] \right) \times \overline{B}_{\mathbb{R}^n}(y_r, \varepsilon),$$

is included in $\text{Glob}(g)$. Furthermore, for all triplets $\theta' = (u', r', y'_r) \in \mathcal{N}(g, \theta, R, \varepsilon)$ and almost every $t \in [0, T]$, it holds that $(y(t, g, \theta'), u'(t)) \in K_t(g, \theta, R)$. Finally the map

$$\begin{aligned} \mathcal{F} : \mathcal{N}(g, \theta, R, \varepsilon) &\rightarrow C([0, T], \mathbb{R}^n) \\ \theta' &\mapsto y(\cdot, g, \theta'), \end{aligned}$$

is Lipschitz continuous, in the sense that there exists $L(g, \theta, R) \geq 0$ such that

$$\|y(\cdot, g, \theta'') - y(\cdot, g, \theta')\|_C \leq L(g, \theta, R) (\|u'' - u'\|_{L^1} + |r'' - r'| + \|y''_r - y'_r\|_{\mathbb{R}^n}),$$

for all $\theta' = (u', r', y'_r)$, $\theta'' = (u'', r'', y''_r) \in \mathcal{N}(g, \theta, R, \varepsilon)$.

Proof Let $\theta = (u, r, y_r) \in \text{Glob}(g)$ and $R \geq \|u\|_{L^\infty}$. In this proof, for the ease of notations, we denote by $M := M(g, \theta, R)$. Let us fix $\varepsilon > 0$ such that $\varepsilon(1 + 2M)e^{MT} < 1$ and let us prove that $\mathcal{N}(g, \theta, R, \varepsilon) \subset \text{Glob}(g)$. To this aim let $\theta' = (u', r', y'_r) \in \mathcal{N}(g, \theta, R, \varepsilon)$ and introduce the sets

$$\begin{aligned} \mathcal{I}_1 &:= \{t \in I(g, \theta') \cap [0, r'] \mid \|y(t, g, \theta') - y(t, g, \theta)\|_{\mathbb{R}^n} > 1\} \\ \text{and } \mathcal{I}_2 &:= \{t \in I(g, \theta') \cap [r', T] \mid \|y(t, g, \theta') - y(t, g, \theta)\|_{\mathbb{R}^n} > 1\}. \end{aligned}$$

If $\mathcal{I}_1 \cup \mathcal{I}_2 = \emptyset$, then the solution $y(\cdot, g, \theta')$ is bounded over $I(g, \theta')$, and thus $\theta' \in \text{Glob}(g)$ from the blow-up theorem. Therefore, by contradiction, let us assume that $\mathcal{I}_1 \cup \mathcal{I}_2 \neq \emptyset$. In the sequel we only deal with the case $\mathcal{I}_2 \neq \emptyset$ (the case where $\mathcal{I}_2 = \emptyset$, and thus $\mathcal{I}_1 \neq \emptyset$, is similar). From integral representations it holds that

$$\begin{aligned} y(t, g, \theta') - y(t, g, \theta) &= (y'_r - y_r) + \int_{r'}^t g(y(s, g, \theta'), u'(s)) - g(y(s, g, \theta), u(s)) \, ds \\ &\quad - \int_r^{r'} g(y(s, g, \theta), u(s)) \, ds, \end{aligned}$$

for all $t \in I(g, \theta')$. Now let $t_2 := \inf \mathcal{I}_2 \geq r'$. From continuity and definition of t_2 , we know that $\|y(t_2, g, \theta') - y(t_2, g, \theta)\|_{\mathbb{R}^n} \geq 1$ and thus $t_2 > r'$ since

$$\begin{aligned} \|y(r', g, \theta') - y(r', g, \theta)\|_{\mathbb{R}^n} &\leq \|y'_r - y_r\|_{\mathbb{R}^n} + \left| \int_r^{r'} \|g(y(s, g, \theta), u(s))\|_{\mathbb{R}^n} \, ds \right| \\ &\leq \|y'_r - y_r\|_{\mathbb{R}^n} + M|r' - r| \leq \varepsilon(1 + M) < 1. \end{aligned}$$

From definition of t_2 , we deduce that $\|y(t, g, \theta') - y(t, g, \theta)\|_{\mathbb{R}^n} \leq 1$ for all $t \in [r', t_2]$. Therefore, since moreover $\|u'\|_{L^\infty} \leq R$, we deduce that $(y(t, g, \theta'), u'(t)) \in K_t(g, \theta, R)$ for almost every $t \in [r', t_2]$. Hence, from integral representations, we get that

$$\begin{aligned} &\|y(t, g, \theta') - y(t, g, \theta)\|_{\mathbb{R}^n} \\ &\leq \|y'_r - y_r\|_{\mathbb{R}^n} + M|r' - r| + M \int_{r'}^t \|y(s, g, \theta') - y(s, g, \theta)\|_{\mathbb{R}^n} + \|u'(s) - u(s)\|_{\mathbb{R}^m} \, ds \\ &\leq \|y'_r - y_r\|_{\mathbb{R}^n} + M|r' - r| + M \int_{r'}^t \|y(s, g, \theta') - y(s, g, \theta)\|_{\mathbb{R}^n} \, ds + M\|u' - u\|_{L^1}, \end{aligned}$$

for all $t \in [r', t_2]$. From the Grönwall lemma we obtain that

$$\|y(t, g, \theta') - y(t, g, \theta)\|_{\mathbb{R}^n} \leq (\|y'_r - y_r\|_{\mathbb{R}^n} + M|r' - r| + M\|u' - u\|_{L^1})e^{MT} \leq \varepsilon(1 + 2M)e^{MT} < 1,$$

for all $t \in [r', t_2]$, which raises a contradiction at $t = t_2$. Thus we have proved that $\mathcal{I}_1 \cup \mathcal{I}_2 = \emptyset$ which gives $\theta' \in \text{Glob}(g)$ but also $(y(t, g, \theta'), u'(t)) \in K_t(g, \theta, R)$ for almost every $t \in [0, T]$. Hence the proofs of the first two claims of Lemma 2 are complete. Now let us prove the last claim. For this purpose, let $\theta' = (u', r', y'_r)$, $\theta'' = (u'', r'', y''_r) \in \mathcal{N}(g, \theta, R, \varepsilon)$. From integral representations it holds that

$$y(t, g, \theta'') - y(t, g, \theta') = (y''_r - y'_r) + \int_{r''}^t g(y(s, g, \theta''), u''(s)) - g(y(s, g, \theta'), u'(s)) ds \\ - \int_{r'}^{r''} g(y(s, g, \theta'), u'(s)) ds,$$

for all $t \in [0, T]$. Using similar arguments than before (in particular using the Gr onwall lemma), we get that

$$\|y(t, g, \theta'') - y(t, g, \theta')\|_{\mathbb{R}^n} \leq (\|y_{r''} - y_{r'}\|_{\mathbb{R}^n} + M|r'' - r'| + M\|u'' - u'\|_{L^1})e^{MT},$$

for all $t \in [0, T]$, which concludes the proof of the last claim of Lemma 2.

Proposition 1 Consider the perturbation $\tilde{\theta}(\alpha) := (\tilde{u}(\alpha), \tilde{r}(\alpha), \tilde{y}_r(\alpha))$ of a triplet $\theta = (u, r, y_r) \in \text{Glob}(g)$ where:

- either $\tilde{u}(\alpha) := u$ (no perturbation of the control), either $\tilde{u}(\alpha)$ is a classical needle-like perturbation of u given by

$$\tilde{u}(\alpha)(t) := \begin{cases} \omega & \text{for all } t \in [\tau - \alpha, \tau), \\ u(t) & \text{otherwise,} \end{cases}$$

where $\omega \in \mathbb{R}^m$ and $\tau \in (0, T)$ is a Lebesgue point of the map $t \mapsto g(y(t, g, \theta), u(t))$.

- either $\tilde{r}(\alpha) := r$ (no perturbation of the initial time), either $\tilde{r}(\alpha)$ satisfies $\tilde{r}(0) = r$ and is differentiable at 0 with derivative denoted by $\tilde{r}'(0)$ (in that second context, assume that $r \in [0, T)$ is a Lebesgue point of the map $t \mapsto g(y(t, g, \theta), u(t))$ and, in case of needle-like perturbation of the control, assume furthermore that $r \neq \tau$).
- $\tilde{y}_r(\alpha)$ satisfies $\tilde{y}_r(0) = y_r$ and is differentiable at 0 with derivative denoted by $\tilde{y}'_r(0)$.

Then:

- There exists $\bar{\alpha} > 0$ such that $\tilde{\theta}(\alpha) \in \text{Glob}(g)$ for all $\alpha \in [0, \bar{\alpha}]$.
- The perturbed trajectory $y(\cdot, g, \tilde{\theta}(\alpha))$ uniformly converges to $y(\cdot, g, \theta)$ as $\alpha \downarrow 0$.
- The map

$$\mathcal{P} : [0, \bar{\alpha}] \rightarrow C([\varsigma, T], \mathbb{R}^n) \\ \alpha \mapsto y(\cdot, g, \tilde{\theta}(\alpha)),$$

with $\varsigma := \tau$ in case of needle-like perturbation of the control and $\varsigma := 0$ otherwise, is differentiable at 0 and its derivative is equal to $w_{(\tilde{r}, \tilde{y}_r)} + w_{\tilde{u}}$, where $w_{(\tilde{r}, \tilde{y}_r)}$ is the unique solution to the linear Cauchy problem given by

$$\begin{cases} \dot{w}(t) = \nabla_x g(y(t, g, \theta), u(t)) w(t), & \text{a.e. } t \in [0, T], \\ w(r) = \tilde{y}'_r(0) - \tilde{r}'(0)g(y(r, g, \theta), u(r)), \end{cases}$$

with $\tilde{r}'(0)g(y(r, g, \theta), u(r)) = 0_{\mathbb{R}^n}$ in case of no perturbation of the initial time, and where $w_{\tilde{u}}$ is the unique solution to the linear Cauchy problem given by

$$\begin{cases} \dot{w}(t) = \nabla_x g(y(t, g, \theta), u(t)) w(t), & \text{a.e. } t \in [0, T], \\ w(\tau) = g(y(\tau, g, \theta), \omega) - g(y(\tau, g, \theta), u(\tau)), \end{cases}$$

in case of needle-like perturbation of the control and $w_{\tilde{u}}$ is the zero function otherwise.

- If furthermore $\tilde{y}_r(\alpha)$ and $\tilde{r}(\alpha)$ are assumed to be continuous with respect to α , then the map $(\alpha, t) \in [0, \bar{\alpha}] \times [0, T] \mapsto y(t, g, \tilde{\theta}(\alpha)) \in \mathbb{R}^n$ is continuous.

Proof This proof is dedicated to the case of a needle-like perturbation of the control, together with a perturbation of the initial time (the other cases are similar and simpler). Let $R \geq \|u\|_{L^\infty} + \|\omega\|_{\mathbb{R}^m}$. As in the proof of Lemma 2, we denote by $M := M(g, \theta, R)$. Consider $\varepsilon > 0$ provided in Lemma 2. It is clear that $\tilde{\theta}(\alpha) \in \mathcal{N}(g, \theta, R, \varepsilon)$ for sufficiently small $\alpha > 0$. As a consequence, from Lemma 2, there exists $\bar{\alpha} > 0$ such that $\tilde{\theta}(\alpha) \in \text{Glob}(g)$ for all $\alpha \in [0, \bar{\alpha}]$ which concludes the proof of the first item. The second and fourth items are trivial consequences of the Lipschitz continuity provided in Lemma 2. Now our aim is to prove the third item. For this purpose we introduce

$$\chi^\alpha(t) := \frac{y(t, g, \tilde{\theta}(\alpha)) - y(t, g, \theta)}{\alpha} - w_{(\tilde{r}, \tilde{y}_r)}(t) - w_{\tilde{u}}(t),$$

for all $t \in [0, T]$ and all $\alpha \in (0, \bar{\alpha}]$. Our aim is to prove that χ^α uniformly converges to zero over $[\tau, T]$ as $\alpha \downarrow 0$. For this purpose, we write $\chi^\alpha = \chi_1^\alpha + \chi_2^\alpha$ where

$$\chi_1^\alpha(t) := \frac{\tilde{y}_1^\alpha(t) - y(t)}{\alpha} - w_{\tilde{u}}(t), \quad \chi_2^\alpha(t) := \frac{\tilde{y}^\alpha(t) - \tilde{y}_1^\alpha(t)}{\alpha} - w_{(\tilde{r}, \tilde{y}_r)}(t),$$

for all $t \in [0, T]$ and all $\alpha \in (0, \bar{\alpha}]$, where we use the notations

$$\tilde{y}^\alpha(t) := y(t, g, \tilde{\theta}(\alpha)), \quad \tilde{y}_1^\alpha(t) := y(t, g, \tilde{\theta}_1(\alpha)), \quad y(t) := y(t, g, \theta),$$

and $\tilde{\theta}_1(\alpha) := (\tilde{u}(\alpha), r, y_r) \in \mathcal{N}(g, \theta, R, \varepsilon)$, for all $t \in [0, T]$ and all $\alpha \in [0, \bar{\alpha}]$. Note that, for almost every $t \in [0, T]$ and all $\alpha \in [0, \bar{\alpha}]$, the five elements

$$(y(t), u(t)), \quad (\tilde{y}^\alpha(t), u(t)), \quad (\tilde{y}_1^\alpha(t), u(t)), \quad (\tilde{y}^\alpha(t), \tilde{u}(\alpha)(t)), \quad (\tilde{y}_1^\alpha(t), \tilde{u}(\alpha)(t)),$$

belong to $K_t(g, \theta, R)$, as well as their convex combinations. Also note that \tilde{y}^α and \tilde{y}_1^α uniformly converge to y over $[0, T]$ as $\alpha \downarrow 0$ from the Lipschitz continuity provided in Lemma 2. In what follows, as in the proof of Lemma 2, we will use integral representations and the Grönwall lemma to prove that χ_1^α and χ_2^α uniformly converge to zero over $[\tau, T]$ as $\alpha \downarrow 0$. To reduce the notations in integrands, we will use the notation $\rho(s) := (y(s), u(s))$.

Step 1: Let us prove that χ_1^α uniformly converges to zero over $[\tau, T]$ as $\alpha \downarrow 0$. From integral representations it holds that

$$\chi_1^\alpha(t) = \chi_1^\alpha(\tau) + \int_\tau^t \frac{g(\tilde{y}_1^\alpha(s), u(s)) - g(y(s), u(s))}{\alpha} - \nabla_x g(\rho(s)) w_{\tilde{u}}(s) ds,$$

for all $t \in [\tau, T]$ and all $\alpha \in (0, \bar{\alpha}]$. Using a Taylor expansion with integral rest, we obtain that

$$\begin{aligned} \|\chi_1^\alpha(t)\|_{\mathbb{R}^n} &\leq \|\chi_1^\alpha(\tau)\|_{\mathbb{R}^n} \\ &+ \int_\tau^t \int_0^1 \|\nabla_x g(y(s) + \eta(\tilde{y}_1^\alpha(s) - y(s)), u(s))\|_{\mathbb{R}^n \times n} \left\| \frac{\tilde{y}_1^\alpha(s) - y(s)}{\alpha} - w_{\tilde{u}}(s) \right\|_{\mathbb{R}^n} d\eta ds \\ &+ \underbrace{\int_\tau^T \int_0^1 \|\nabla_x g(y(s) + \eta(\tilde{y}_1^\alpha(s) - y(s)), u(s)) - \nabla_x g(\rho(s))\|_{\mathbb{R}^n \times n} \|w_{\tilde{u}}(s)\|_{\mathbb{R}^n} d\eta ds}_{\Gamma_1(\alpha)}, \end{aligned}$$

for all $t \in [\tau, T]$ and all $\alpha \in (0, \bar{\alpha}]$. The Grönwall lemma leads to

$$\|\chi_1^\alpha(t)\|_{\mathbb{R}^n} \leq (\|\chi_1^\alpha(\tau)\|_{\mathbb{R}^n} + \Gamma_1(\alpha)) e^{MT},$$

for all $t \in [\tau, T]$ and all $\alpha \in (0, \bar{\alpha}]$. From the uniform convergence of \tilde{y}_1^α to y over $[0, T]$ as $\alpha \downarrow 0$ and from the dominated convergence theorem, we prove that $\Gamma_1(\alpha)$ converges to

zero as $\alpha \downarrow 0$. It remains to prove that $\|\chi_1^\alpha(\tau)\|_{\mathbb{R}^n}$ converges to zero as $\alpha \downarrow 0$. From integral representations it holds that

$$\chi_1^\alpha(\tau) = \int_{\tau-\alpha}^\tau \frac{g(\tilde{y}_1^\alpha(s), \omega) - g(y(s), \omega)}{\alpha} ds + \int_{\tau-\alpha}^\tau \frac{g(y(s), \omega) - g(y(s), u(s))}{\alpha} ds - w_{\tilde{u}}(\tau),$$

for all $\alpha \in (0, \bar{\alpha}]$. From the uniform convergence of \tilde{y}_1^α to y over $[0, T]$ as $\alpha \downarrow 0$, one can easily prove that the first term tends to $0_{\mathbb{R}^n}$ as $\alpha \downarrow 0$. Finally, since τ is a Lebesgue point of the map $t \mapsto g(y(t), u(t))$ and from the value of $w_{\tilde{u}}(\tau)$, the sum of the last two terms tend to $0_{\mathbb{R}^n}$ as $\alpha \downarrow 0$. The proof of Step 1 is complete.

Step 2: Let us prove that χ_2^α uniformly converges to zero over $[0, T]$ as $\alpha \downarrow 0$. From integral representations it holds that

$$\chi_2^\alpha(t) = \chi_2^\alpha(r) + \int_r^t \frac{g(\tilde{y}^\alpha(s), \tilde{u}(\alpha)(s)) - g(\tilde{y}_1^\alpha(s), \tilde{u}(\alpha)(s))}{\alpha} - \nabla_x g(\rho(s)) w_{(\tilde{r}, \tilde{y}_r)}(s) ds,$$

for all $t \in [0, T]$ and all $\alpha \in (0, \bar{\alpha}]$. Using a Taylor expansion with integral rest, we obtain that

$$\begin{aligned} \|\chi_2^\alpha(t)\|_{\mathbb{R}^n} &\leq \|\chi_2^\alpha(r)\|_{\mathbb{R}^n} \\ &+ \left| \int_r^t \int_0^1 \|\nabla_x g(\tilde{y}_1^\alpha(s) + \eta(\tilde{y}^\alpha(s) - \tilde{y}_1^\alpha(s)), \tilde{u}(\alpha)(s))\|_{\mathbb{R}^n \times \mathbb{R}^n} \left\| \frac{\tilde{y}^\alpha(s) - \tilde{y}_1^\alpha(s)}{\alpha} - w_{(\tilde{r}, \tilde{y}_r)}(s) \right\|_{\mathbb{R}^n} d\eta ds \right| \\ &+ \underbrace{\int_0^T \int_0^1 \|\nabla_x g(\tilde{y}_1^\alpha(s) + \eta(\tilde{y}^\alpha(s) - \tilde{y}_1^\alpha(s)), \tilde{u}(\alpha)(s)) - \nabla_x g(\rho(s))\|_{\mathbb{R}^n \times \mathbb{R}^n} \|w_{(\tilde{r}, \tilde{y}_r)}(s)\|_{\mathbb{R}^n} d\eta ds}_{\Gamma_2(\alpha)}, \end{aligned}$$

for all $t \in [0, T]$ and all $\alpha \in (0, \bar{\alpha}]$. The Gr onwall lemma leads to

$$\|\chi_2^\alpha(t)\|_{\mathbb{R}^n} \leq (\|\chi_2^\alpha(r)\|_{\mathbb{R}^n} + \Gamma_2(\alpha)) e^{MT},$$

for all $t \in [0, T]$ and all $\alpha \in (0, \bar{\alpha}]$. From the uniform convergences of \tilde{y}^α and \tilde{y}_1^α to y over $[0, T]$ as $\alpha \downarrow 0$ and from the dominated convergence theorem, we prove that $\Gamma_2(\alpha)$ converges to zero as $\alpha \downarrow 0$. It remains to prove that $\|\chi_2^\alpha(r)\|_{\mathbb{R}^n}$ converges to zero as $\alpha \downarrow 0$. From integral representations it holds that

$$\chi_2^\alpha(r) = \left(\frac{\tilde{y}_r(\alpha) - y_r}{\alpha} - \tilde{y}'_r(0) \right) + \left(\tilde{r}'(0)g(y(r), u(r)) + \frac{1}{\alpha} \int_{\tilde{r}(\alpha)}^r g(\tilde{y}^\alpha(s), \tilde{u}(\alpha)(s)) ds \right),$$

for all $\alpha \in (0, \bar{\alpha}]$. From differentiability of $\tilde{y}_r(\cdot)$ at 0, the first term converges to $0_{\mathbb{R}^n}$ as $\alpha \downarrow 0$. Since $r \neq \tau$ and from the continuity of $\tilde{r}(\cdot)$ at 0, we know that the second term can be rewritten as

$$\tilde{r}'(0)g(y(r), u(r)) + \frac{1}{\alpha} \int_{\tilde{r}(\alpha)}^r g(y(s), u(s)) ds - \frac{1}{\alpha} \int_{\tilde{r}(\alpha)}^r g(y(s), u(s)) - g(\tilde{y}^\alpha(s), u(s)) ds,$$

for sufficiently small $\alpha > 0$. Since r is a Lebesgue point of the map $t \mapsto g(y(t), u(t))$ and from the differentiability of $\tilde{r}(\cdot)$ at 0, the sum of the two first terms in the above equation converges to $0_{\mathbb{R}^n}$ as $\alpha \downarrow 0$. Finally the norm of the last term in the above equation can be bounded by

$$\left| \frac{1}{\alpha} \int_{\tilde{r}(\alpha)}^r \|g(y(s), u(s)) - g(\tilde{y}^\alpha(s), u(s))\|_{\mathbb{R}^n} ds \right| \leq M \left| \frac{\tilde{r}(\alpha) - r}{\alpha} \right| \|y - \tilde{y}^\alpha\|_C,$$

which tends to zero as $\alpha \downarrow 0$, thanks to the differentiability of $\tilde{r}(\cdot)$ at 0 and from the uniform convergence of \tilde{y}^α to y over $[0, T]$ as $\alpha \downarrow 0$. The proof of Step 2 is complete, which ends the proof of Proposition 1.

Remark 6 *The proof of Theorem 1 (developed in the next Appendix B) is based on several applications of the results established in the present Appendix A.2, but for smooth control systems considered on compact intervals of the form $[a, b]$ that are not necessarily $[0, T]$. This is, of course, not a difficulty.*

B Proof of Theorem 1

Let $(x, u) \in AC([0, T], \mathbb{R}^n) \times L^\infty([0, T], \mathbb{R}^m)$ be a regular solution to (HS) and consider the notations introduced in Definition 1. For convenience we reduce $\delta > 0$ so that $0 < \delta \leq \frac{1}{3} \min_{k=1, \dots, N} |t_k^c - t_{k-1}^c|$. Then, for all $k \in \{1, \dots, N\}$, we introduce, following the notations introduced in Appendix A.2, the *auxiliary trajectory* $\tilde{z}_k := y(\cdot, h_{j(k)}, \theta_k)$ associated with the triplet $\theta_k := (\tilde{u}_k, t_{k-1}^c, x(t_{k-1}^c))$, where the *auxiliary control* $\tilde{u}_k \in L^\infty([0, T], \mathbb{R}^m)$ is defined by

$$\tilde{u}_k(t) := \begin{cases} u^+(t_{k-1}^c), & \text{for all } t \in [t_0^c, t_{k-1}^c], \\ u(t), & \text{for all } t \in (t_{k-1}^c, t_k^c), \\ u^-(t_k^c), & \text{for all } t \in [t_k^c, t_N^c], \end{cases}$$

where the above definition has to be slightly and trivially adapted for the cases $k = 1$ and $k = N$. Note that $\tilde{z}_k = x$ over $[t_{k-1}^c, t_k^c] \subset I(h_{j(k)}, \theta_k)$ for all $k \in \{1, \dots, N\}$. We refer to Figures 3 and 4 in Section 2.3.

From the blow-up theorem and up to reducing $\delta > 0$ again, we will consider in the sequel that $[t_{k-1}^c - \delta, t_k^c + \delta] \cap [0, T] \subset I(h_{j(k)}, \theta_k)$ for all $k \in \{1, \dots, N\}$. Furthermore, up to reducing $\delta > 0$ again, we will consider that $\tilde{z}_k(t) \in \bar{B}_{\mathbb{R}^n}(x(t_{k-1}^c), \frac{\delta}{2})$ for all $t \in [t_{k-1}^c - \delta, t_{k-1}^c + \delta]$ and all $k \in \{2, \dots, N\}$, and that $\tilde{z}_k(t) \in \bar{B}_{\mathbb{R}^n}(x(t_k^c), \frac{\delta}{2})$ for all $t \in [t_k^c - \delta, t_k^c + \delta]$ and all $k \in \{1, \dots, N-1\}$.

Furthermore, from Condition (C2)(i) and for any $k \in \{1, \dots, N-1\}$, note that \tilde{u}_k is continuous over $[t_k^c - \delta, T]$ and thus \tilde{z}_k is of class C^1 over $[t_k^c - \delta, t_k^c + \delta]$ with $\dot{z}_k(t) = h_{j(k)}(\tilde{z}_k(t), \tilde{u}_k(t))$ for all $t \in [t_k^c - \delta, t_k^c + \delta]$. In particular t_k^c is a Lebesgue point of the map $t \mapsto h_{j(k)}(\tilde{z}_k(t), \tilde{u}_k(t))$ and it holds that $\dot{z}_k(t_k^c) = h_{j(k)}(x(t_k^c), u^-(t_k^c))$. Similarly, from Condition (C2)(i) and for any $k \in \{1, \dots, N-1\}$, note that \tilde{u}_{k+1} is continuous over $[0, t_k^c + \delta]$ and thus \tilde{z}_{k+1} is of class C^1 over $[t_k^c - \delta, t_k^c + \delta]$ with $\dot{z}_{k+1}(t) = h_{j(k+1)}(\tilde{z}_{k+1}(t), \tilde{u}_{k+1}(t))$ for all $t \in [t_k^c - \delta, t_k^c + \delta]$. In particular t_k^c is a Lebesgue point of the map $t \mapsto h_{j(k+1)}(\tilde{z}_{k+1}(t), \tilde{u}_{k+1}(t))$ and it holds that $\dot{z}_{k+1}(t_k^c) = h_{j(k+1)}(x(t_k^c), u^+(t_k^c))$.

We are now in a position to start the proof of Theorem 1. Let $\omega \in \mathbb{R}^m$ and $\tau \in (t_{q-1}^c, t_q^c)$, for some $q \in \{1, \dots, N\}$, be a Lebesgue point of the map $t \mapsto h_{j(q)}(x(t), u(t))$.

B.1 Construction of the perturbed auxiliary trajectories and of the perturbed crossing times

The next proposition uses the notations of Definition 1, the ones introduced in Appendix A.2 and the variation vector w constructed by induction in the statement of Theorem 1.

Proposition 2 *There exists $\bar{\alpha} > 0$ such that, for all $k \in \{q, \dots, N-1\}$, there exists a function $\tilde{t}_k \in C([0, \bar{\alpha}], [t_k^c - \delta, t_k^c + \delta])$ differentiable at 0 with $\tilde{t}_k(0) = t_k^c$ and*

$$\tilde{t}'_k(0) = -\frac{\langle \nabla F_k(x(t_k^c)), w_k(t_k^c) \rangle_{\mathbb{R}^n}}{\langle \nabla F_k(x(t_k^c)), h_{j(k)}(x(t_k^c), u^-(t_k^c)) \rangle_{\mathbb{R}^n}},$$

such that the perturbed auxiliary trajectories $\tilde{z}_k^\alpha := y(\cdot, h_{j(k)}, \theta_k^\alpha)$ associated with the perturbed triplets θ_k^α defined by the induction

$$\theta_k^\alpha := \begin{cases} (\tilde{u}_q^\alpha, t_{q-1}^c, x(t_{q-1}^c)) & \text{if } k = q, \\ (\tilde{u}_k, \tilde{t}_{k-1}^\alpha, \tilde{z}_{k-1}^\alpha(\tilde{t}_{k-1}^\alpha)) & \text{if } k \in \{q+1, \dots, N\}, \end{cases}$$

for all $\alpha \in [0, \bar{\alpha}]$ and all $k \in \{q, \dots, N\}$, where \tilde{u}_q^α is the needle-like perturbation of \tilde{u}_q (see Figure 5 in Section 2.3) given by

$$\tilde{u}_q^\alpha(t) := \begin{cases} \omega & \text{for all } t \in [\tau - \alpha, \tau), \\ \tilde{u}_q(t) & \text{otherwise,} \end{cases}$$

satisfy:

- for all $k \in \{q, \dots, N\}$, it holds that $[t_{k-1}^c - \delta, t_k^c + \delta] \cap [0, T] \subset I(h_{j(k)}, \theta_k^\alpha)$ for all $\alpha \in [0, \bar{\alpha}]$, that \tilde{z}_k^α uniformly converges to \tilde{z}_k over $[t_{k-1}^c - \delta, t_k^c + \delta] \cap [0, T]$ as $\alpha \downarrow 0$, and

$$\lim_{\alpha \downarrow 0} \frac{\tilde{z}_k^\alpha(t_k^c) - \tilde{z}_k(t_k^c)}{\alpha} = w_k(t_k^c).$$

- for all $k \in \{q, \dots, N-1\}$, it holds that $\tilde{z}_k^\alpha(t) \in \bar{B}_{\mathbb{R}^n}(x(t_k^c), \nu)$ for all $(\alpha, t) \in [0, \bar{\alpha}] \times [t_k^c - \delta, t_k^c + \delta]$, that $F_k(\tilde{z}_k^\alpha(\tilde{t}_k(\alpha))) = 0$ for all $\alpha \in [0, \bar{\alpha}]$, and that the map $\alpha \in [0, \bar{\alpha}] \mapsto \tilde{z}_k^\alpha(\tilde{t}_k(\alpha)) \in \mathbb{R}^n$ is continuous over $[0, \bar{\alpha}]$ and differentiable at 0 with

$$\lim_{\alpha \downarrow 0} \frac{\tilde{z}_k^\alpha(\tilde{t}_k(\alpha)) - \tilde{z}_k(t_k^c)}{\alpha} = w_k(t_k^c) + \tilde{t}'_k(0)h_{j(k)}(x(t_k^c), u^-(t_k^c)).$$

Proof The case $q = N$ follows directly from Proposition 1. In the sequel we deal with the case $q \in \{1, \dots, N-1\}$ and we will proceed by induction over $k \in \{q, \dots, N\}$. Note that we will construct $\bar{\alpha} > 0$ in the base case and that it will be reduced a finite number of times at each step of the induction.

Base case $k = q$. From Proposition 1, there exists $\bar{\alpha} > 0$ such that $[t_{q-1}^c - \delta, t_q^c + \delta] \cap [0, T] \subset I(h_{j(q)}, \theta_q^\alpha)$ for all $\alpha \in [0, \bar{\alpha}]$, that \tilde{z}_q^α uniformly converges to \tilde{z}_q over $[t_{q-1}^c - \delta, t_q^c + \delta] \cap [0, T]$ as $\alpha \downarrow 0$, and that the map

$$(\alpha, t) \in [0, \bar{\alpha}] \times ([t_{q-1}^c - \delta, t_q^c + \delta] \cap [0, T]) \mapsto \tilde{z}_q^\alpha(t) \in \mathbb{R}^n, \quad (\text{B.3})$$

is continuous. Since moreover $\tilde{z}_q(t) \in \bar{B}_{\mathbb{R}^n}(x(t_q^c), \frac{\nu}{2})$ for all $t \in [t_q^c - \delta, t_q^c + \delta]$, up to reducing $\bar{\alpha} > 0$, we have $\tilde{z}_q^\alpha(t) \in \bar{B}_{\mathbb{R}^n}(x(t_q^c), \nu)$ for all $(\alpha, t) \in [0, \bar{\alpha}] \times [t_q^c - \delta, t_q^c + \delta]$. We define now the map

$$G_q : [0, \bar{\alpha}] \times [t_q^c - \delta_0, t_q^c + \delta_0] \rightarrow \mathbb{R} \\ (\alpha, t) \mapsto F_q(\tilde{z}_q^\alpha(t)),$$

where $0 < \delta_0 < \delta$ is selected such that $\tau < t_q^c - \delta_0$ and where F_q is the C^1 function provided in Definition 1.

Let us check that G_q satisfies all the assumptions of Lemma 1. First, G_q is continuous from the continuity of the map (B.3) and $G_q(0, t_q^c) = F_q(x(t_q^c)) = 0$. Second, for any $\alpha \in [0, \bar{\alpha}]$, we have that $\tilde{u}_q^\alpha = \tilde{u}_q$ is continuous over $[t_q^c - \delta_0, t_q^c + \delta_0]$. This implies that \tilde{z}_q^α is of class C^1 over $[t_q^c - \delta_0, t_q^c + \delta_0]$ and that $\nabla_t G_q(\alpha, t)$ exists with

$$\nabla_t G_q(\alpha, t) = \langle \nabla F_q(\tilde{z}_q^\alpha(t)), h_{j(q)}(\tilde{z}_q^\alpha(t), \tilde{u}_q(t)) \rangle_{\mathbb{R}^n},$$

for all $(\alpha, t) \in [0, \bar{\alpha}] \times [t_q^c - \delta_0, t_q^c + \delta_0]$. Furthermore, from the continuity of the map (B.3), one can see that $\nabla_t G_q$ is continuous over $[0, \bar{\alpha}] \times [t_q^c - \delta_0, t_q^c + \delta_0]$ and, from Condition (C2)(iii), it holds that

$$\nabla_t G_q(0, t_q^c) = \langle \nabla F_q(x(t_q^c)), h_{j(q)}(x(t_q^c), u^-(t_q^c)) \rangle_{\mathbb{R}^n} \neq 0.$$

Finally, from the third item of Proposition 1, we get that

$$\lim_{\alpha \downarrow 0} \frac{\tilde{z}_q^\alpha(t_q^c) - \tilde{z}_q(t_q^c)}{\alpha} = w_q(t_q^c),$$

which implies that $\nabla_\alpha G_q(0, t_q^c)$ exists with $\nabla_\alpha G_q(0, t_q^c) = \langle \nabla F_q(x(t_q^c)), w_q(t_q^c) \rangle_{\mathbb{R}^n}$.

We deduce from Lemma 1 that, up to reducing $\bar{\alpha} > 0$ (precisely, by taking $\bar{\alpha} = \bar{\beta}$), there exists a function $\tilde{t}_q \in C([0, \bar{\alpha}], [t_q^c - \delta_0, t_q^c + \delta_0])$, satisfying $\tilde{t}_q(0) = t_q^c$ and $F_q(\tilde{z}_q^\alpha(\tilde{t}_q(\alpha))) = 0$ for all $\alpha \in [0, \bar{\alpha}]$, that is differentiable at 0 with

$$\tilde{t}'_q(0) = - \frac{\langle \nabla F_q(x(t_q^c)), w_q(t_q^c) \rangle_{\mathbb{R}^n}}{\langle \nabla F_q(x(t_q^c)), h_{j(q)}(x(t_q^c), u^-(t_q^c)) \rangle_{\mathbb{R}^n}}.$$

From the continuities of the function \tilde{t}_q and of the map (B.3), we deduce that the map $\alpha \in [0, \bar{\alpha}] \mapsto \tilde{z}_q^\alpha(\tilde{t}_q(\alpha)) \in \mathbb{R}^n$ is continuous over $[0, \bar{\alpha}]$. It remains to prove that

$$\lim_{\alpha \downarrow 0} \frac{\tilde{z}_q^\alpha(\tilde{t}_q(\alpha)) - \tilde{z}_q(t_q^c)}{\alpha} = w_q(t_q^c) + \tilde{t}'_q(0)h_{j(q)}(x(t_q^c), u^-(t_q^c)).$$

For this purpose, using integral representations, we can write

$$\begin{aligned} & \frac{\tilde{z}_q^\alpha(\tilde{t}_q(\alpha)) - \tilde{z}_q(t_q^c)}{\alpha} \\ &= \frac{\tilde{z}_q^\alpha(t_q^c) - \tilde{z}_q(t_q^c)}{\alpha} + \frac{\tilde{t}_q(\alpha) - t_q^c}{\alpha} \frac{1}{\tilde{t}_q(\alpha) - t_q^c} \int_{t_q^c}^{\tilde{t}_q(\alpha)} h_{j(q)}(\tilde{z}_q(s), \tilde{u}_q(s)) \, ds \\ & \quad + \frac{1}{\alpha} \int_{t_q^c}^{\tilde{t}_q(\alpha)} h_{j(q)}(\tilde{z}_q^\alpha(s), \tilde{u}_q(s)) - h_{j(q)}(\tilde{z}_q(s), \tilde{u}_q(s)) \, ds, \end{aligned}$$

for all $\alpha \in (0, \bar{\alpha}]$. We already proved that the first term tends to $w_q(t_q^c)$ as $\alpha \downarrow 0$. Since t_q^c is a Lebesgue point of the map $t \mapsto h_{j(q)}(\tilde{z}_q(t), \tilde{u}_q(t))$ and since \tilde{t}_q is differentiable at 0, the second term tends to $\tilde{t}'_q(0)h_{j(q)}(x(t_q^c), u^-(t_q^c))$. Finally the third term tends to zero as $\alpha \downarrow 0$, since \tilde{z}_q^α uniformly converges to \tilde{z}_q over $[t_q^c - \delta, t_q^c + \delta]$, $h_{j(q)}$ is of class C^1 and \tilde{t}_q is differentiable at 0. The proof for the base case is complete.

Inductive step. Let $k \in \{q+1, \dots, N\}$ and assume that the induction hypothesis holds true for all $\ell \in \{q, \dots, k-1\}$. The case $k = N$ follows directly from Proposition 1 and from the induction hypothesis (in particular from the differentiability at 0 of the function \tilde{t}_{N-1} and of the map $\alpha \in [0, \bar{\alpha}] \mapsto \tilde{z}_{N-1}^\alpha(\tilde{t}_{N-1}(\alpha)) \in \mathbb{R}^n$). Therefore, in the sequel, we deal with the case $k \in \{q+1, \dots, N-1\}$ and we will proceed similarly to the base case. Therefore some details will be omitted.

Thanks to the induction hypothesis ensuring the continuities of the function \tilde{t}_{k-1} and of the map $\alpha \in [0, \bar{\alpha}] \mapsto \tilde{z}_{k-1}^\alpha(\tilde{t}_{k-1}(\alpha))$, we deduce from Proposition 1 that, up to reducing $\bar{\alpha} > 0$, it holds that $[t_{k-1}^c - \delta, t_k^c + \delta] \cap [0, T] \subset I(h_{j(k)}, \theta_k^\alpha)$ for all $\alpha \in [0, \bar{\alpha}]$, that \tilde{z}_k^α uniformly converges to \tilde{z}_k over $[t_{k-1}^c - \delta, t_k^c + \delta] \cap [0, T]$ as $\alpha \downarrow 0$, and that the map

$$(\alpha, t) \in [0, \bar{\alpha}] \times ([t_{k-1}^c - \delta, t_k^c + \delta] \cap [0, T]) \mapsto \tilde{z}_k^\alpha(t) \in \mathbb{R}^n, \quad (\text{B.4})$$

is continuous. Similarly to the base case, up to reducing $\bar{\alpha} > 0$, we get that $\tilde{z}_k^\alpha(t) \in \overline{B}_{\mathbb{R}^n}(x(t_k^c), \nu)$ for all $(\alpha, t) \in [0, \bar{\alpha}] \times [t_k^c - \delta, t_k^c + \delta]$ and we define the map

$$G_k : [0, \bar{\alpha}] \times [t_k^c - \delta, t_k^c + \delta] \rightarrow \mathbb{R} \\ (\alpha, t) \mapsto F_k(\tilde{z}_k^\alpha(t)).$$

Similarly to the base case, G_k is continuous, $G_k(0, t_k^c) = F_k(x(t_k^c)) = 0$ and $\nabla_t G_k(\alpha, t)$ exists and is continuous over $[0, \bar{\alpha}] \times [t_k^c - \delta, t_k^c + \delta]$ and

$$\nabla_t G_k(0, t_k^c) = \langle \nabla F_k(x(t_k^c)), h_{j(k)}(x(t_k^c), u^-(t_k^c)) \rangle_{\mathbb{R}^n} \neq 0.$$

Finally, from the third item of Proposition 1 and from the induction hypothesis (in particular from the differentiability at 0 of the function \tilde{t}_{k-1} and of the map $\alpha \in [0, \bar{\alpha}] \mapsto \tilde{z}_{k-1}^\alpha(\tilde{t}_{k-1}(\alpha)) \in \mathbb{R}^n$), we get that

$$\lim_{\alpha \downarrow 0} \frac{\tilde{z}_k^\alpha(t_k^c) - \tilde{z}_k(t_k^c)}{\alpha} = w_k(t_k^c),$$

which implies that $\nabla_\alpha G_k(0, t_k^c)$ exists with $\nabla_\alpha G_k(0, t_k^c) = \langle \nabla F_k(x(t_k^c)), w_k(t_k^c) \rangle_{\mathbb{R}^n}$.

From Lemma 1, up to reducing $\bar{\alpha} > 0$, there exists a function $\tilde{t}_k \in C([0, \bar{\alpha}], [t_k^c - \delta, t_k^c + \delta])$, satisfying $\tilde{t}_k(0) = t_k^c$ and $F_k(\tilde{z}_k^\alpha(\tilde{t}_k(\alpha))) = 0$ for all $\alpha \in [0, \bar{\alpha}]$, that is differentiable at 0 with

$$\tilde{t}'_k(0) = - \frac{\langle \nabla F_k(x(t_k^c)), w_k(t_k^c) \rangle_{\mathbb{R}^n}}{\langle \nabla F_k(x(t_k^c)), h_{j(k)}(x(t_k^c), u^-(t_k^c)) \rangle_{\mathbb{R}^n}}.$$

From the continuities of the function \tilde{t}_k and of the map (B.4), we deduce that the map $\alpha \in [0, \bar{\alpha}] \mapsto \tilde{z}_k^\alpha(\tilde{t}_k(\alpha)) \in \mathbb{R}^n$ is continuous over $[0, \bar{\alpha}]$. Similarly to the base case, one can prove that

$$\lim_{\alpha \downarrow 0} \frac{\tilde{z}_k^\alpha(\tilde{t}_k(\alpha)) - \tilde{z}_k(t_k^c)}{\alpha} = w_k(t_k^c) + \tilde{t}'_k(0)h_{j(k)}(x(t_k^c), u^-(t_k^c)),$$

which completes the proof of the inductive step.

B.2 Admissibility of the perturbed auxiliary trajectories

Our objective in this section is to prove that each perturbed auxiliary trajectory \tilde{z}_k^α is *admissible*, in the sense that it is with values in the open set $X_{j(k)}$ over the interval $(\tilde{t}_{k-1}(\alpha), \tilde{t}_k(\alpha))$ between the two consecutive corresponding perturbed crossing times. This result is formalized in Proposition 3 that is based on the next preliminary lemma.

Lemma 3 *Consider the framework of Proposition 2. Then, up to reducing $\bar{\alpha} > 0$, the following properties are satisfied:*

1. *There exists $s'_{q-1} \in (t_{q-1}^c, t_{q-1}^c + \delta]$ such that $\tilde{z}_q^\alpha(t) \in X_{j(q)}$ for all $(\alpha, t) \in [0, \bar{\alpha}] \times (t_{q-1}^c, s'_{q-1}]$ (and for all $(\alpha, t) \in [0, \bar{\alpha}] \times [t_{q-1}^c, s'_{q-1}]$ if $q = 1$).*
2. *For all $k \in \{q, \dots, N-1\}$, there exists $s_k \in [t_k^c - \delta, t_k^c)$ such that $\tilde{z}_k^\alpha(t) \in X_{j(k)}$ for all $(\alpha, t) \in [0, \bar{\alpha}] \times [s_k, \tilde{t}_k(\alpha))$.*
3. *For all $k \in \{q, \dots, N-1\}$, there exists $s'_k \in (t_k^c, t_k^c + \delta]$ such that $\tilde{z}_{k+1}^\alpha(t) \in X_{j(k+1)}$ for all $(\alpha, t) \in [0, \bar{\alpha}] \times (\tilde{t}_k(\alpha), s'_k]$.*
4. *There exists $s_N \in [t_N^c - \delta, t_N^c)$ such that $\tilde{z}_N^\alpha(t) \in X_{j(N)}$ for all $(\alpha, t) \in [0, \bar{\alpha}] \times [s_N, t_N^c]$.*

Proof This proof does not require induction. We will prove each item separately. Note that we will reduce $\bar{\alpha} > 0$ in each item.

Proof of the first item. Note that $\tilde{z}_q^\alpha = \tilde{z}_q = x$ (with values in $X_{j(q)}$) over $(t_{q-1}^c, \tau - \bar{\alpha}]$. Therefore one can easily conclude the first item by taking $s'_{q-1} := \min\{t_{q-1}^c + \delta, \tau - \bar{\alpha}\}$. The case $q = 1$ is similar.

Proof of the fourth item. Since \tilde{z}_N^α uniformly converges to $\tilde{z}_N = x$ (with values in the open set $X_{j(N)}$) over $[t_N^c - \delta, t_N^c]$ as $\alpha \downarrow 0$, one can easily conclude the fourth item by reducing $\bar{\alpha} > 0$ and by taking $s_N = t_N^c - \delta$.

Proof of the second item. Let $k \in \{q, \dots, N-1\}$ and $0 < \delta_0 < \delta$ be fixed such that $\tau < t_q^c - \delta_0$. Recall that $\tilde{z}_k^\alpha(t) \in \bar{B}_{\mathbb{R}^n}(x(t_k^c), \nu)$, and therefore $\tilde{z}_k^\alpha(t) \in X_{j(k)}$ if and only if $F_k(\tilde{z}_k^\alpha(t)) < 0$, for all $(\alpha, t) \in [0, \bar{\alpha}] \times [t_k^c - \delta_0, t_k^c + \delta_0]$. Also recall that $\tilde{t}_k(\alpha)$ tends to t_k^c as $\alpha \downarrow 0$. Therefore, for any $s_k \in [t_k^c - \delta_0, t_k^c)$, there exists $0 < \bar{\beta}(s_k) \leq \bar{\alpha}$ such that $s_k < \tilde{t}_k(\alpha) \leq t_k^c + \delta_0$ for all $\alpha \in [0, \bar{\beta}(s_k)]$. By contradiction assume that

$$\forall s_k \in [t_k^c - \delta_0, t_k^c), \quad \forall 0 < \beta \leq \bar{\beta}(s_k), \quad \exists \alpha \in [0, \beta], \quad \exists t \in [s_k, \tilde{t}_k(\alpha)), \quad F_k(\tilde{z}_k^\alpha(t)) \geq 0. \quad (\text{B.5})$$

Let $s_k \in [t_k^c - \delta_0, t_k^c)$ and $0 < \beta \leq \bar{\beta}(s_k)$ and consider the pair (α, t) given in (B.5). Since $F_k(\tilde{z}_k^\alpha(\tilde{t}_k(\alpha))) = 0$ (see Proposition 2), we obtain that

$$F_k(\tilde{z}_k^\alpha(\tilde{t}_k(\alpha))) - F_k(\tilde{z}_k^\alpha(t)) \leq 0.$$

Since \tilde{z}_k^α is of class C^1 over $[t_k^c - \delta_0, t_k^c + \delta_0]$, note that the above inequality can be rewritten as

$$\frac{1}{\tilde{t}_k(\alpha) - t} \int_t^{\tilde{t}_k(\alpha)} \Psi_k(s) \, ds \leq \frac{1}{\tilde{t}_k(\alpha) - t} \int_t^{\tilde{t}_k(\alpha)} \Psi_k(s) - \Psi_k^\alpha(s) \, ds,$$

where

$$\Psi_k(s) := \langle \nabla F_k(\tilde{z}_k(s)), h_{j(k)}(\tilde{z}_k(s), \tilde{u}_k(s)) \rangle_{\mathbb{R}^n},$$

and

$$\Psi_k^\alpha(s) := \langle \nabla F_k(\tilde{z}_k^\alpha(s)), h_{j(k)}(\tilde{z}_k^\alpha(s), \tilde{u}_k(s)) \rangle_{\mathbb{R}^n},$$

for all $s \in [t_k^c - \delta_0, t_k^c + \delta_0]$. Since \tilde{u}_k is continuous at t_k^c , note that t_k^c is a Lebesgue point of Ψ_k . Therefore, when making tend $s_k \rightarrow t_k^c$ and $\beta \rightarrow 0$, we make tend $\alpha \rightarrow 0$, $\tilde{t}_k(\alpha) \rightarrow t_k^c$ and $t \rightarrow t_k^c$ and thus the left term of (B.2) tends to $\langle \nabla F_k(x(t_k^c)), h_{j(k)}(x(t_k^c), u^-(t_k^c)) \rangle_{\mathbb{R}^n}$. It remains to prove that the right term of (B.2) tends to zero when $\alpha \rightarrow 0$, $\tilde{t}_k(\alpha) \rightarrow t_k^c$ and $t \rightarrow t_k^c$. For this purpose, recall that $\tilde{z}_k^\alpha(t) \in \overline{B}_{\mathbb{R}^n}(x(t_k^c), \nu)$ for all $(\alpha, t) \in [0, \bar{\alpha}] \times [t_k^c - \delta, t_k^c + \delta]$ and that $\nabla_x F_k$ is uniformly continuous over the compact set $\overline{B}_{\mathbb{R}^n}(x(t_k^c), \nu)$ (since F_k is of class C^1). Therefore, since \tilde{z}_k^α uniformly converges to \tilde{z}_k over $[t_k^c - \delta, t_k^c + \delta]$ when $\alpha \rightarrow 0$, one can easily prove that the right term of (B.2) tends to zero when $\alpha \rightarrow 0$, $\tilde{t}_k(\alpha) \rightarrow t_k^c$ and $t \rightarrow t_k^c$. Hence we have obtained that

$$\langle \nabla F_k(x(t_k^c)), h_{j(k)}(x(t_k^c), u^-(t_k^c)) \rangle_{\mathbb{R}^n} \leq 0,$$

which raises a contradiction with Condition (C2)(iii). Therefore we have proved the negation of (B.5) which is given by

$$\exists s_k \in [t_k^c - \delta_0, t_k^c), \quad \exists 0 < \beta \leq \bar{\beta}(s_k), \quad \forall \alpha \in [0, \beta], \quad \forall t \in [s_k, \tilde{t}_k(\alpha)], \quad F_k(\tilde{z}_k^\alpha(t)) < 0.$$

This concludes the proof of the second item by reducing $\bar{\alpha} > 0$ to β .

Proof of the third item. The proof is similar to the proof of the second item.

Proposition 3 *Consider the framework of Proposition 2. Then, up to reducing $\bar{\alpha} > 0$, the following properties are satisfied:*

1. $\tilde{z}_q^\alpha(t) \in X_{j(q)}$ for all $(\alpha, t) \in [0, \bar{\alpha}] \times (t_{q-1}^c, \tilde{t}_q(\alpha))$ (and for all $(\alpha, t) \in [0, \bar{\alpha}] \times [t_{q-1}^c, \tilde{t}_q(\alpha))$ if $q = 1$).
2. $\tilde{z}_k^\alpha(t) \in X_{j(k)}$ for all $(\alpha, t) \in [0, \bar{\alpha}] \times (\tilde{t}_{k-1}(\alpha), \tilde{t}_k(\alpha))$ and all $k \in \{q+1, \dots, N-1\}$.
3. $\tilde{z}_N^\alpha(t) \in X_{j(N)}$ for all $(\alpha, t) \in [0, \bar{\alpha}] \times (\tilde{t}_{N-1}(\alpha), t_N^c]$.

Proof This proof does not require induction. Let us prove the second item only. The other items can be proved similarly (and note that $\bar{\alpha} > 0$ is reduced in each item). Let $k \in \{q+1, \dots, N-1\}$. From Lemma 3, we know that:

- there exists $s'_{k-1} \in (t_{k-1}^c, t_{k-1}^c + \delta]$ such that $\tilde{z}_k^\alpha(t) \in X_{j(k)}$ for all $(\alpha, t) \in [0, \bar{\alpha}] \times (\tilde{t}_{k-1}(\alpha), s'_{k-1}]$.
- there exists $s_k \in [t_k^c - \delta, t_k^c)$ such that $\tilde{z}_k^\alpha(t) \in X_{j(k)}$ for all $(\alpha, t) \in [0, \bar{\alpha}] \times [s_k, \tilde{t}_k(\alpha))$.

Now recall that $\tilde{z}_k = x$ over $[t_{k-1}^c, t_k^c]$ and that $x(t) \in X_{j(k)}$ for all $t \in (t_{k-1}^c, t_k^c)$ and thus for all $t \in [s'_{k-1}, s_k]$. Since $X_{j(k)}$ is an open set and \tilde{z}_k^α uniformly converges to \tilde{z}_k over $[t_{k-1}^c - \delta, t_k^c + \delta]$ as $\alpha \downarrow 0$, one can easily see that, up to reducing $\bar{\alpha} > 0$, one has $\tilde{z}_k^\alpha(t) \in X_{j(k)}$ for all $(\alpha, t) \in [0, \bar{\alpha}] \times [s'_{k-1}, s_k]$. We finally deduce that $\tilde{z}_k^\alpha(t) \in X_{j(k)}$ for all $(\alpha, t) \in [0, \bar{\alpha}] \times (\tilde{t}_{k-1}(\alpha), \tilde{t}_k(\alpha))$. The proof of the second item is complete.

B.3 End of the proof

As explained in Section 2.3, we define by concatenation the perturbed trajectory

$$x^\alpha(t) := \begin{cases} x(t) & \text{for all } t \in [t_0^c, t_{q-1}^c], \\ \tilde{z}_q^\alpha(t) & \text{for all } t \in [t_{q-1}^c, \tilde{t}_q(\alpha)], \\ \tilde{z}_k^\alpha(t) & \text{for all } t \in [\tilde{t}_{k-1}(\alpha), \tilde{t}_k(\alpha)] \text{ and } k \in \{q+1, \dots, N-1\}, \\ \tilde{z}_N^\alpha(t) & \text{for all } t \in [\tilde{t}_{N-1}(\alpha), t_N^c], \end{cases}$$

associated with the modified needle-like perturbation

$$u^\alpha(t) := \begin{cases} u(t) & \text{for all } t \in [t_0^c, \tau - \alpha), \\ \omega & \text{for all } t \in [\tau - \alpha, \tau), \\ \tilde{u}_q(t) & \text{for all } t \in [\tau, \tilde{t}_q(\alpha)), \\ \tilde{u}_k(t) & \text{for all } t \in [\tilde{t}_{k-1}(\alpha), \tilde{t}_k(\alpha)) \text{ and } k \in \{q+1, \dots, N-1\}, \\ \tilde{u}_N(t) & \text{for all } t \in [\tilde{t}_{N-1}(\alpha), t_N^c]. \end{cases}$$

From the construction and the results obtained in Propositions 2 and 3, one can easily see that $(x^\alpha, u^\alpha) \in \text{AC}([0, T], \mathbb{R}^n) \times L^\infty([0, T], \mathbb{R}^m)$ is a (perturbed) solution to (HS), admitting the $\tilde{t}_k(\alpha)$ as (perturbed) crossing times, where we have introduced $\tilde{t}_k(\alpha) := t_k^c$ for all $k \in \{1, \dots, q-1\}$. The uniform convergence of x^α towards x as $\alpha \downarrow 0$ follows from the uniform convergence of \tilde{z}_k^α to \tilde{z}_k over $[t_{k-1}^c - \delta, t_k^c + \delta] \cap [0, T]$ for all $k \in \{q, \dots, N\}$ as $\alpha \downarrow 0$, from the convergence of $\tilde{t}_k(\alpha)$ to t_k^c for all $k \in \{q, \dots, N-1\}$ as $\alpha \downarrow 0$, and from the equality $\tilde{z}_k = x$ over $[t_{k-1}^c, t_k^c]$ for all $k \in \{q, \dots, N\}$. Finally, from Proposition 2, it also holds that

$$\lim_{\alpha \downarrow 0} \frac{x^\alpha(T) - x(T)}{\alpha} = \lim_{\alpha \downarrow 0} \frac{\tilde{z}_N^\alpha(t_N^c) - \tilde{z}_N(t_N^c)}{\alpha} = w_N(t_N^c) = w(T).$$

This concludes the proof of Theorem 1.

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