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# ON A FEKETE-SZEGÖ THEOREM

THÉRÈSE FALLIERO

ABSTRACT. We consider again a classical theorem relating capacities and algebraic integers and the question of the simultaneous approximation of  $n - 1$  different complex numbers by conjugate algebraic integers of degree  $n$ .

*This is a preliminary version of a further more complete paper, which will be co-written by Ahmed Sebbar.*

## 1. INTRODUCTION

A classical theorem of Fekete and Szegö [13] says that, given a compact set  $K$  in the complex plane having logarithmic capacity  $\mathcal{C}(K)$ , then

- (a) If  $\mathcal{C}(K) < 1$ , there is an open set  $U$  containing  $K$  such that there are only finitely many algebraic integers  $\alpha$  having all of their conjugates in  $U$ , and
- (b) if  $\mathcal{C}(K) \geq 1$ , then every open  $U$  containing  $K$  contains infinitely many such  $\alpha$ .

Furthermore Fekete and Szegö proved also that if  $K$  is a compact set in the complex plane, stable under complex conjugation and having logarithmic capacity  $\mathcal{C}(K) \geq 1$ , then every neighborhood of  $K$  contains infinitely many conjugate sets of algebraic integers. In [10] V. Ennola solved a question raised by R. M. Robinson that if  $\Delta$  is any real interval of length greater than 4, then for any sufficiently large  $n$  there exists an irreducible monic polynomial of degree  $n$  with integer coefficients all of whose zeros lie in  $\Delta$ .

We should emphasize that many diophantine inequalities are reduced to the existence of lattice points in some convex body [7, Chapter III]. In this direction, it is remarkable that T. Chinburg [8] reduces the proof of Fekete theorem to an application of Minkowski's Convex Body theorem [7, Chapter III] that we recall for later use.

This problem is actually related to a precise form of the Stone-Weierstrass theorem. This classical theorem states that every continuous function defined on a closed interval  $[a, b]$  can be uniformly approximated by polynomials. A more general statement is if  $X$  is a compact Hausdorff topological space and if  $C(X)$  is the algebra of real-valued continuous functions  $f : X \rightarrow \mathbb{R}$ , then a subalgebra  $\mathcal{A} \subset C(X)$  is dense if and only if it separates points.

The question is for which compact set continuous functions can be approximated by polynomials with integer coefficients? This question is a major one in approximation theory and the literature is very extensive [14]. Let  $f$  be a continuous real-valued function defined on  $[0, 1]$ , then the sequence of polynomials  $(p_n)$  defined by

$$p_n(x) = \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) \binom{n}{\nu} x^\nu (1-x)^{n-\nu}$$

converges uniformly to  $f$ . This is therefore a constructive proof of the Stone-Weierstrass theorem. It is due to Bernštein. We deduce from this result [15, Theorem 5] that for a continuous real-valued function  $f$  on the unit interval  $I = [0, 1]$  to be uniformly approximable by polynomials with integer coefficients it is necessary and sufficient that it be integer-valued at both 0 and 1.

As was pointed out in [15, Lemma 1], If  $q$  is a non constant polynomial with integer coefficients and  $I$  is an interval of length at least four, then  $\|q\| \leq 2$ , where  $\|\cdot\|$  is the supremum norm. Hence, clearly, the approximation by polynomials with integer coefficients on a set  $E$  is related to

to the capacity of  $E$ . Consequently [15, Theorem 2] if the capacity  $\mathcal{C}(I) \geq 1$  the only functions that are uniformly approximable in  $I$  by polynomials with integer coefficients are these polynomials themselves. On the other hand one can prove the surprising result that any  $f \in L^2([a, b])$ ,  $b - a < 4$  can be approximated on the interval  $[a, b]$  by polynomials with integer coefficients.

**Theorem 1.1** (Pál). *If  $f$  is continuous on  $[-a, a]$ ,  $|a| < 1$ , and  $f(0)$  is an integer then  $f$  may be uniformly approximated thereon by polynomials with integer coefficients.*

These problems are at the core of the approximation theory by polynomials with integer coefficients. It is a classical result, due to Chebyshev that for an interval  $[a, b]$

$$\inf_Q \|Q(z)\|_\infty = 2 \left( \frac{b-a}{4} \right)^n$$

where  $Q(X)$  describes the set of monic polynomials of degree  $n$ . Hilbert showed in [16] that if we restrict to monic polynomials of degree  $n$  with integer coefficients, then

$$\inf_Q \|Q(z)\|_{L^2([a,b])} \leq C\sqrt{n} \left( \frac{b-a}{4} \right)^{n/2}$$

and Fekete showed in [12] the more flexible

$$\inf_Q \|Q(z)\|_\infty \leq 2^{1-\frac{1}{n+1}} (n+1) \left( \frac{b-a}{4} \right)^{n/2}.$$

For a set  $V \subset \mathbb{C}$  let  $\text{Pol}_V$  be the set of monic polynomials in  $\mathbb{Z}[X]$  of degree at least 1 such that all their roots lie in  $V$ . Let  $E \subset \mathbb{R}$  be a finite union of segments with  $\mathcal{C}(E) > 1$  and  $\mu_E$  be its equilibrium measure. We have the following theorem ([30] Theorem 1.6.2).

**Theorem 1.2** (Serre). *There exists a sequence of polynomials  $P_n \in \text{Pol}_E$  such that  $\mu_{P_n} \rightarrow \mu_E$ .*

The main objective of this work is the following natural question from the Fekete-Szegő theorem, suggested by J.P. Serre in [30].

**Question 1.1.** *Let  $K$  be a compact of  $\mathbb{C}$  stable under complex conjugation, of capacity  $\mathcal{C}(K) \geq 1$  and  $U$  an open set containing  $K$ . Is there a sequence of polynomials  $P_n \in \text{Pol}_U$  such that  $\mu_{P_n} \rightarrow \mu_K$ ?*

In this paper we obtain the following result.

**Theorem 1.3.** *If  $K$  is a compact of  $\mathbb{C}$ , symmetric with respect to the real axis with  $\mathcal{C}(K) > 1$  and such that  $\Omega \cap H$  is simply connected, where  $\Omega$  is the unbounded component of  $\mathbb{C} \setminus K$  and  $H$  the upper half plane. If  $U$  is an open set containing  $K$ , there is a sequence  $(P_n)$  of monic polynomials with integer coefficients whose roots are in  $U$  and are such that the associated zeros counting measure  $\mu_{P_n}$  converge weakly to the equilibrium measure  $\mu_K$  of  $K$ .*

**Organization of the paper:** Very succinctly, we give the definitions in the first section, then we introduce the counting measures in the second section. Minkowski's theorem will be discussed in section three. The fourth section is devoted to certain approximations results and the fifth section, the longest, is devoted to the introduction of certain Riemann surfaces, in relation to certain Jacobi matrices and the solution of the Serre question.

## 2. DEFINITIONS

**2.1. Approximation on intervals.** If  $\mu$  is a finite Borel measure on  $\mathbb{C}$  with compact support, its logarithm potential is the function  $\Phi_\mu : \mathbb{C} \rightarrow (-\infty, +\infty]$  defined by

$$\Phi_\mu(z) = \int \ln(|z-w|^{-1}) d\mu(w).$$

This integral converges if  $z \notin \text{supp}(d\mu)$ , and since  $d\mu$  has compact support,  $\ln(|z-w|^{-1})$  is uniformly bounded below for  $(z, w) \in \text{supp}(d\mu) \times \text{supp}(d\mu)$ , so the integral for each  $z \in \text{supp}(\mu)$  either converges or diverges to  $+\infty$ , in which case we set  $\Phi_\mu(z) = +\infty$ .

Potentials enter naturally in studying growth of polynomials as  $n \rightarrow \infty$ . For if

$$P_n(x) = \prod_{j=1}^n (x - x_j^{(n)})$$

then

$$\frac{1}{n} \ln |P_n(x)| = -\Phi_{\nu_n}(x)$$

where

$$\mu_{P_n} = \frac{1}{n} \sum_{j=1}^n \delta_{x_j^{(n)}}$$

is the counting measure for the zeros  $x_j^{(n)}$ . The function  $\Phi_\mu(z)$  is bounded below on  $\text{supp}(\mu)$ , so

$$I(\mu) = \int \Phi_\mu(z) d\mu(z) = \int \int_{K \times K} \ln(|z - w|^{-1}) d\mu(z) d\mu(w)$$

is either finite or diverges to  $+\infty$ .  $I(\mu)$  is called the potential energy of  $\mu$  or, for short, the energy of  $\mu$ .

Consider a compact  $K \subset \mathbb{C}$ . We consider all probability measures  $\mathcal{M}_{+,1}(K)$  on  $K$ . We say  $K$  has capacity zero if and only if  $I(\mu) = \infty$  for all  $\mu \in \mathcal{M}_{+,1}(K)$ . We set

$$v(K) = \inf_{\mu} I(\mu)$$

where  $\mu$  runs over all positive probability measures supported in  $K$ . Then the capacity of  $K$  is defined as  $\mathcal{C}(K) = e^{-v(K)}$ . Its logarithm  $v(K)$  is called the logarithm capacity of  $K$ .

For a compact  $K$  with non-zero capacity there exists a unique positive probability measure  $\mu$ , such that  $I(\mu) = \ln \mathcal{C}(K)^{-1}$ . This measure  $\mu = \mu_K$  is called the equilibrium measure of  $K$ .

Before continuing further, we wish to recall some definitions that will be need.

Let us also recall that if  $(\mu_n)_n$  and  $\mu_\infty$  are probability measures on a compact Hausdorff space  $X$ ,  $(\mu_n)_n$  converges weakly  $\mu_\infty$  if

$$\int f d\mu_n \rightarrow \int f d\mu \quad \text{as } n \rightarrow \infty$$

for every  $f \in \mathcal{C}^0(X)$ , function continuous on  $X$ .

Denote by  $\mathcal{M}(K) = \mathcal{C}^0(K)^*$  the set of all measure on  $K$ . The  $\sigma(\mathcal{M}(K), \mathcal{C}^0(X))$ -topology is the weakest topology on  $\mathcal{M}(K)$  in which the maps  $x \mapsto \langle y, x \rangle$  of  $\mathcal{M}(K)$  to  $\mathbb{C}$  are continuous for all  $y \in \mathcal{C}^0(K)$ . By the Banach-Alaoglu theorem, the unit ball in  $\mathcal{M}(K)$  is compact in the  $\sigma(\mathcal{M}(K), \mathcal{C}^0(X))$ -topology.  $\mathcal{M}_{+,1}(K)$  is closed in the unit ball, so it too is compact.

Moreover if  $f$  a real valued function defined on a topological space  $E$ ,  $f$  is lower semi continuous at  $a$  if

$$f(a) = \liminf_{x \rightarrow a} f(x).$$

We have the following property of the potential energy

$$\mu \rightarrow I(\mu)$$

is weakly lower semi continuous. The lower semi continuity means

$$\mu_n \rightarrow \mu \Rightarrow \liminf I(\mu_n) \geq I(\mu)$$

equivalently

$$(1) I^{-1}((-\infty, a]) \text{ is closed for all } a$$

$$(2) I^{-1}((a, \infty]) \text{ is open for all } a.$$

Finally, given a bounded set  $\mathcal{E}$  in the complex plane, we denote by  $\mathcal{E}(r)$  the  $r$ -neighborhood of  $\mathcal{E}$ .

**Definition 2.1.** If  $\mathcal{E}_1$  and  $\mathcal{E}_2$  are two bounded sets in  $\mathbb{C}$ , the difference between  $\mathcal{E}_1$  and  $\mathcal{E}_2$  is the smallest  $r$  such that  $\mathcal{E}_1(r)$  contains  $\mathcal{E}_2$  and  $\mathcal{E}_2(r)$  contains  $\mathcal{E}_1$ .

We will denote this difference by  $\delta(\mathcal{E}_1, \mathcal{E}_2)$ , it is small if and only if  $\mathcal{E}_1$  and  $\mathcal{E}_2$  are (almost) super imposable. We will say that  $\mathcal{E}_1$  is near  $\mathcal{E}_2$  (and reciprocally). If  $\mathcal{E}$  is a compact set and  $(\mathcal{E}_\nu)$  a family of sets such that  $\delta(\mathcal{E}, \mathcal{E}_\nu)$  tends to zero as  $\nu$  tends to  $\infty$  we will say simply that  $(\mathcal{E}_\nu)$  tends to  $\mathcal{E}$ .

We then deduce the lemma

**Lemma 2.1.** *If  $\delta(K_n, K) \rightarrow 0$  and  $\mathcal{C}(K_n) \rightarrow \mathcal{C}(K)$  then  $\mu_{K_n} \rightarrow \mu_K$ .*

*Proof.* As  $\mathcal{M}_{+,1}(K)$  is compact let  $\eta$  be a weak limit point of  $\mu_{K_n}$ . By lower semi continuity of the energy  $I$ ,

$$\begin{aligned} I(\eta) &\leq \liminf I(\mu_{K_n}) \\ &= \lim \ln(\mathcal{C}(K_n)^{-1}) \\ &= \ln(\mathcal{C}(K)^{-1}), \end{aligned}$$

so  $\eta = \mu_K$ , that is,  $\mu_{K_n} \rightarrow \mu_K$ . □

Note that the inverse is not true  $\mu_{P_n} \rightarrow \mu_K$  and  $\mathcal{C}(\{\text{zeros of } P_n\}) = 0$ .

### 3. ALGEBRAIC INTEGERS WITH ALL CONJUGATES IN A GIVEN COMPACT.

For a set  $V \subset \mathbb{C}$  let  $\text{Pol}_V$  be the set of monic polynomials in  $\mathbb{Z}[X]$  of degree at least 1 such that all their roots lie in  $V$ . If  $z$  is an algebraic integer, a root of a polynomial  $P(X) \in \text{Pol}_V$ , then all the conjugate of  $z$  are in  $V$  and we say that  $z$  is totally in  $V$ .

Let  $\text{Irr}_V$  be the set of irreducible monic polynomials in  $\mathbb{Z}[X]$  of degree at least 1 such that all their roots lie in  $V$ . for such a polynomial  $P(X)$  of degree  $g$  let  $\mu_P$  be the corresponding probability measure supported in its roots,  $\mu_P = \frac{1}{g} \sum_{i=1}^g \delta_{x_i}$ . Now let  $K \subset \mathbb{C}$  be compact. There are two quite

different cases [30], [28], depending on the capacity of  $K$ .

(1) If  $\mathcal{C}(K) < 1$ , then  $\text{Irr}_K$  is finite.

(2) If  $K \subset \mathbb{C}$  is  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ -stable and  $\mathcal{C}(K) \geq 1$ , then for any open  $U, K \subset U$ , the set  $\text{Irr}_U$  is infinite.

In particular if  $E \subset \mathbb{R}$  is a union of finite number of segments and  $\mathcal{C}(E) > 1$ , then  $\text{Irr}_E$  is infinite.

**3.1. Precisions on the properties of  $K$ .** Let  $U \subset \mathbb{C}$  be a set and let  $U^*$  be the so called ‘‘symmetric kernel’’ of  $U$  consisting of those points of  $U$  which belong, together with their conjugates to  $U$ . So  $U^*$  is symmetric with respect to the real axis and naturally  $\mathcal{C}(U^*) \leq \mathcal{C}(U)$ .

M.Fekete [12], [8] proved that if  $K$  is a compact of  $\mathbb{C}$  such that  $\mathcal{C}(K^*) < 1$ , then there is only a finite number of irreducible algebraic equations with integer coefficients of the form

$$z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n = 0$$

whose roots lie all in  $K^*$ .

**Theorem 3.1** (Minkowski’s theorem). *Suppose  $K$  to be a symmetric, convex, bounded subset  $\mathbb{R}^d$ . If  $\text{vol}(K) > 2^d$ , then  $K$  contains at least one lattice point other than 0.*

There is an extension to general lattices  $\Lambda = \mathbb{Z}u_1 \oplus \dots \oplus \mathbb{Z}u_d$ , where  $\{u_1, \dots, u_d\}$  is a basis of  $\mathbb{R}^d$ . We define  $\text{vol}(\Lambda)$  as the volume of the parallelotope

$$\left\{ \sum_{i=1}^d \alpha_i u_i, \quad 0 \leq \alpha_i \leq 1 \right\}$$

**Theorem 3.2** (Minkowski’s theorem for general lattices). *Suppose  $\Lambda$  to be a lattice and  $K$  to be a bounded symmetric convex subset in  $\mathbb{R}^d$ . If  $\text{vol}(K) > 2^d \det \Lambda$ , then  $K$  contains at least a point of  $\Lambda$  different from 0.*

We must perhaps insist that Minkowski’s theorem as well as Motzkin’s theorem on simultaneous approximation (which in turn depends on two theorems of Kronecker. The first one [21, p.159]) is at the heart of the diophantine approximation and then at the heart of the approximation by

polynomials with integer coefficients. The second one [18] state that if an algebraic integer  $\alpha$  and all of its conjugates are in the closed unit disk  $\mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$ , then it is either  $\alpha = 0$  or it is root of unity. This is apparent at [7, Chap III], Feruguson [14, Theorem 1.1] and Chinburg [8]. For the sake of completeness and in order to see how the different idea articulate we give an idea of the proof of the first part of Fekete-Szegö theorem.

We detail an application of a Chinburg's theorem. See also [2], p.24. For  $a \in \mathbb{R}^{n+1}$  we define the polynomial  $f_a(z) = a_0 + \dots + a_n z^n$ . Let  $K$  be a compact such that  $\mathcal{C}(K) < 1$ , then (see [8]) we consider

$$f_n(K) = \{a = (a_0, a_1, \dots, a_n) \in \mathbb{R}^{n+1} \setminus \{0\}, \quad |f_a(K)| < 1\}.$$

Let  $\psi_n$  the eucliden measure on  $\mathbb{R}^{n+1}$ , theorem 1.2 in [8] says

$$(1) \quad \lim n^{-2} 2 \ln \psi_n(f_n(K)) = -\ln \mathcal{C}(K).$$

We note that, setting  $\tilde{f}_a(z) = \frac{f_a(z)}{a_n}$ , we have  $K \subset \tilde{f}_a^{-1}(D(0, \frac{1}{|a_n|}))$  and we deduce that  $\mathcal{C}(K) \leq \frac{1}{|a_n|^{1/n}}$ , that is

$$|a_n| \leq \frac{1}{\mathcal{C}(K)^n}.$$

Then in the case  $\mathcal{C}(K) > 1$ ,  $\lim a_n = 0$  and  $\forall \epsilon > 0, \exists N, \forall n \geq N, f_n(K) \subset \mathbb{R}^n \times [-\epsilon, \epsilon]$  and  $\lim \psi_n(f_n(K)) = 0$ , we can compare with 1.

In the case  $\mathcal{C}(K) < 1$ , 1 allows to show Fekete's theorem. For this we apply Minkowski's theorem: let  $C$  open symmetric convex set of  $\mathbb{R}^{n+1}$ , if  $\psi_n(C) > 2^{n+1}$ , then  $C$  contains a point with integer coordinates, different from 0.

Here  $f(K) \subset \mathbb{R}^{n+1}$ , as  $f_{-a}(z) = f_a(z)$ ,  $f_n(K)$  is the symmetric.

As  $f_{ta+(1-t)b} = tf_a + (1-t)f_b$ , for  $t \in [0, 1]$ ,  $f_n(K)$  is a convex set.

Let's verify that  $f_n(K)$  is an open bounded set.

Let  $a \in f_n(K)$ , as  $K$  is a compact set and  $f_a$  continue,  $m_a = \max_{z \in K} |f_a(z)| < 1$ . Let  $\rho$  be the radius of the smallest disc contained in  $K$  and centered at 0. Let  $M = \sum_{k=0}^n \rho^k$  and  $\epsilon \in \mathbb{R}^{n+1}$  such that  $\|\epsilon\| = (\sum_0^n |\epsilon_i|^2)^{1/2}$ ,  $\|\epsilon\| < \frac{1-m_a}{M}$ . Then, with  $f_{a+b} = f_a + f_b$ , we deduce that  $D(a, \epsilon) \subset f_n(K)$  and  $f_n(K)$  is open.

We can add that  $f_n(K)$  is bounded obtaining different estimates on  $|a_n|$ . Several methods allow to justify that, for  $0 \leq k \leq n$ ,  $|a_k| \leq (n+1)! \rho^{\varphi(n)} \mathcal{C}(K)^{-\frac{n(n+1)}{2}}$ , where  $\varphi(n)$  is a function depending on  $n$  only.

Consider  $n+1$  points of  $K$ ,  $z_0, z_1, \dots, z_n$  such that  $|\Pi_{0 \leq i < j \leq n} (z_j - z_i)| = \sup_{x_k \in K} |\Pi_{0 \leq i < j \leq n} (x_j - x_i)| := V_{n+1}$ ; such a set is called, Fekete's set. With the preceding notations, we denote by  $w_i = f_a(z_i)$ . Then, we have a linear system in the  $a_i$ :

$$\begin{cases} a_0 + a_1 z_0 + \dots + a_n z_0^n & = w_0 \\ \dots & = \dots \\ a_0 + a_1 z_n + \dots + a_n z_n^n & = w_n \end{cases}$$

Let's denote the determinant of the system ,  $\begin{vmatrix} 1 & z_0 & \dots & z_0^n \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ 1 & z_n & \dots & z_n^n \end{vmatrix}$  by  $\det V = V(z_0, \dots, z_n) =$

$\Pi_{0 \leq i < j \leq n} (z_j - z_i)$ , so that  $|V(z_0, \dots, z_n)| = V_{n+1}$ .

We have, for example,

$$a_0 = \begin{vmatrix} w_0 & z_0 & \dots & z_0^n \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ w_n & z_n & \dots & z_n^n \end{vmatrix} / V(z_0, \dots, z_n).$$

More generally,  $(V^{-1})_{ij} = \frac{\tilde{V}_{ij}(V)}{V(z_0, \dots, z_n)}$ , where  $\tilde{V}_{ij}(V)$  is the "classical mineur"  $(-1)^{i+j} |V_i^j|$ .

If  $V_i^j = (\alpha_{kl})_{1 \leq k, l \leq n}$  then  $|V_i^j| = \sum_{\sigma \in S_n} \epsilon(\sigma) \alpha_{1\sigma(1)} \dots \alpha_{n\sigma(n)}$  and  $|\alpha_{l\sigma(l)}| \leq \rho^{\sigma(l)}$ .

Then  $||V_i^j|| \leq n! \rho^{\sum_{k \neq j} k}$  and as  $a_i = \frac{1}{\det V} \sum_{l=0}^n \tilde{V}_{ij}(V) w_l$ , we have  $|a_i| \leq \frac{n!}{\det V} \sum_{j=0}^n \rho^{\frac{n(n+1)}{2} - j}$ .

Denoting  $d_{n+1} = V_{n+1}^{\frac{2}{n(n+1)}}$ ,  $(d_n)$  tends decreasing to  $\mathcal{C}(K)$ , then  $\det V = V_{n+1} \geq \mathcal{C}(K)^{\frac{2}{n(n+1)}}$  and  $\forall i, 0 \leq i \leq n, |a_i| \leq n! \rho^{\frac{n(n+1)}{2}} \frac{\rho^{n+1} - 1}{\rho^{n+1} - \rho} \mathcal{C}(K)^{-\frac{n(n+1)}{2}}$ .

Another inequality can be obtained noting that

$$\begin{aligned} V(z_0, \dots, z_{i-1}, X, z_{i+1}, \dots, z_n) &= \sum_{l=0}^n \tilde{V}_{il}(V) X^l \\ &= \prod_{0 \leq l < k < i} (z_k - z_l) \prod_{l=0}^{i-1} (X - z_l) \prod_{k=i+1}^n (z_k - X) \prod_{i < l < k \leq n} (z_k - z_l) \prod_{l=0}^{i-1} (X - z_l) \prod_{k=i+1}^n (X - z_k) \\ &= X^n - s_1 X^{n-1} + \dots + (-1)^{(n-j)} s_{n-j} X^j + \dots + (-1)^n s_n. \end{aligned}$$

We have  $s_{n-j} = \sum_{1 \leq i_1 < i_2 < \dots < i_{n-j} \leq k} z_{i_1} z_{i_2} \dots z_{i_{n-j}}$ . As  $\forall l, |z_l| \leq \rho \quad |s_{n-j}| \leq \binom{n}{j} \rho^{n-j}$  and  $|\tilde{V}_{ij}(V)| \leq V_n \binom{n}{j} \rho^{n-j}$ .

$$\text{Finally } \left| \frac{\tilde{V}_{ij}(V)}{\det(V)} \right| \leq \frac{V_n}{V_{n+1}} \binom{n}{j} \rho^{n-j}$$

In conclusion  $f_n(K)$  is an open symmetric convex set, with large volume for a large  $n$ . By Minkowski's theorem,  $f_n(K)$  contains some  $a = (a_0, a_1, \dots, a_n) \in \mathbb{Z}^{n+1} \setminus \{0\}$ . We fix a such  $a$  and we consider the open set

$$U = \{z \in \mathbb{C}, |f_a(z)| < 1\}.$$

If  $\alpha$  and its conjugates are contained in  $U$ , then  $f_a(\alpha)$  and its conjugates are contained in the unit disk  $\mathbb{D}$ . By the second Kronecker theorem  $\alpha$  is one of the many roots of the polynomial  $f_a(z)$ .

**Remark 3.1.** *The idea of using Minkowski's convex body theorem in this context goes back to Hilbert [16]. We can rephrase, in a classical way [7], what we said on the proof of the first part of Fekete-Szegő theorem: Any convex body of volume at least  $2^n$  contains at least  $2^n + 1$  integral points. In particular the system of linear inequalities*

$$\left| \sum_{k=1}^n a_{k,m} x^k \right| \leq b_m, \quad 1 \leq m \leq n$$

with

$$\det (a_{k,m})_{1 \leq k, m \leq n} \neq 0, \quad \prod_{m=1}^n b_m \geq \left| \det (a_{k,m})_{1 \leq k, m \leq n} \right|$$

has a nonzero integral solution.

Now let  $K$  be a compact of  $\mathbb{C}$  of capacity  $\mathcal{C}(K) \geq 1$ . Let  $U$  a neighborhood of  $K$  then  $U^*$  is a neighborhood of  $K^*$  (with the convention that the empty set is the neighborhood of the empty set). If  $\mathcal{C}(K^*) < 1$  then by continuity of the capacity, there exists  $V$  a neighborhood of  $K$  such that  $\mathcal{C}(V^*) < 1$ . From what precedes, there exists only a finite number of algebraic integers with all its conjugate in  $V$ . Then to show that there exists an infinity of algebraic integers totally in  $U$ , we can assume  $\mathcal{C}(K^*) \geq 1$  and finally the hypothesis of  $K$  symmetric with respect to the real axis is natural.

#### 4. ON A COMPACT SYMMETRIC WITH RESPECT TO THE REAL AXIS

**4.1. The boundary.** First of all,  $K$  being a metric compact set, many assumptions can be made on  $\partial K$ .

**Definition 4.1.** *We say that  $K$  has a continuous boundary when  $K$  is a non empty union of connected components non reduced to a point.*

Recovering  $K$  with a finite number of small enough closed balls, we can assume that the boundary  $\partial K$  is continuous.

From [4, Proposition p. 18] we can suppose  $K$  with regular boundary of class  $C^\infty$ . In fact let  $U$  be an open set containing  $K$ , there exists a  $C^\infty$  function  $\varphi$  in  $\mathbb{R}^2$  such that

$$(1) \quad \varphi = 1 \text{ on } K,$$

(2)  $\text{supp}(\varphi) \subset U$ .

Then  $K \subset \text{supp}(\varphi) \subset U$  with  $\text{supp}(\varphi)$  a compact set with regular  $C^\infty$  boundary.

Let us recall that a analytic Jordan curve is a closed curve  $\Gamma$  in  $\mathbb{C}$  which possesses a neighborhood  $V$  and a conformal map  $\xi$  from  $V$  on  $\{\alpha < |z| < \beta\}$ , such that the image of  $\Gamma$  by  $\xi$  is the circle  $\{|z| = r\}$ ,  $\alpha < r < \beta$ . We can also suppose that  $\partial K$  is a set of analytic Jordan curves. To see this, a first method is to use [13, Theorem G]. Let  $K(\rho)$  be the  $\rho$ -neighborhood of  $K$  ( $\rho > 0$ ), there exists  $\rho$  sufficiently small so that  $K(\rho) \subset U$ . From the preceding theorem, there exists a domain defined by a lemniscate containing  $K$  and contained in  $K(\rho)$ : for  $n \geq n_1(\rho)$ ,  $\{z, |w_n(z)| \leq \nu_n\}$  where

$$w_n(z) = \prod_{k=1}^n (z - \zeta_k^{(n)})(z - \overline{\zeta_k^{(n)}}), \quad \nu_n = \max_K |w_n(z)|.$$

In conclusion this compact set is invariant under complex conjugation and its boundary consists in analytic Jordan curves.

A second method can be found in [1, p.144]. As an open set of  $\mathbb{C}$ ,  $U$  is a natural Riemann open surface, there exists a sequence of regular subregions  $(U_n)$ , such that  $\overline{U_n} \subset U_{n+1}$  and  $U = \bigcup_{i=1}^{\infty} U_n$ .

We recall that  $\Omega$  is regularly imbedded if  $\Omega$  and its exterior have a common boundary which is a 1-dimensional submanifold. A regularly imbedded subregion of a Riemann surface is thus bounded by analytic curves, then  $\forall n$   $\partial U_n$  consists in analytic Jordan curves.

$K$  being a compact in  $U$ , it can be recovered by a finite number of  $U_n$ . Then there exists  $N$  such that  $K \subset U_N$ . Then  $K \subset \overline{U_N} \subset U_{N+1} \subset U$  and  $\overline{U_N}$  is a compact with analytic boundary.

Finally we recall the following definition (see for example [4, p.22]),

**Definition 4.2.** Let  $\Omega$  be an open subset of  $\mathbb{R}^2$ . We say that  $\Omega$  has a regular boundary of class  $C^k$  ( $k \geq 1$ ) if for every  $p \in \partial\Omega$  there is a neighborhood  $U_p$  of  $p$  and a diffeomorphism  $\varphi_p$  of class  $C^k$  from  $U_p$  onto a neighborhood  $V_p$  of 0 in  $\mathbb{R}^2$  such that  $\varphi_p(p) = 0$ ,

$$\varphi_p(U_p \cap \overline{\Omega}) = V_p \cap \{(x, y) \in \mathbb{R}^2 : x \leq 0\}$$

and the Jacobian determinant  $J(\varphi_p)$  is  $> 0$  in  $U_p$ .

We have (see for example [4, Proposition p. 27]) that for  $\Omega$  a relatively compact, open subset of  $\mathbb{C}$  with piecewise regular boundary (of class  $C^k$ ,  $k \geq 1$ ), there is only a finite number of connected components of  $\partial\Omega$  and each of them is a Jordan curve (piecewise  $C^k$ ).

In conclusion we can always suppose that  $K$  is a compact with  $C^\infty$  boundary, then the number of connected components of  $\partial K$  is finite.

In the following the regularity of  $K$  is understood. Note first that which is important in  $K$  is the boundary of the outer component of  $\mathbb{C} \setminus \partial K$ . Then  $K$  can be a compact whose boundaries of the bounded connected components are what ever you want.

Denoting by  $\Omega$  the outer component of  $\mathbb{C} \setminus \partial K$ , we know that the equilibrium measure of  $K$  is supported on  $\partial\Omega$  (see for example, [33], [31] Theorem A.10). We denote by  $\partial\Omega = (\Gamma_1, \dots, \Gamma_r) = \Gamma$ .

For the sake of completeness, we recall the following results.

**4.2. The double of a plane domain.** Let  $\Omega$  be a plane domain. We have seen that we may assume that each boundary component of the boundary of  $\Omega$ , denoted by  $\partial\Omega = \Gamma = (\Gamma_1, \Gamma_2, \dots, \Gamma_r)$ ,  $\Gamma_j$  is a smooth analytic curve. Alternatively, one may think of  $\Omega$  as a plane bordered Riemann surface. More precisely

**Definition 4.3.** For each  $r = 1, 2, \dots$  we shall denote by  $\mathcal{U}_r$  the class of plane domains whose boundary consists of  $r$  disjoint Jordan curves  $\Gamma_1, \Gamma_2, \dots, \Gamma_r$  which satisfy the following smoothness condition: with each  $\Gamma_j$  there is associated a function  $z_j(t)$  analytic and univalent in a neighborhood of  $\Gamma_j$  which maps this neighborhood onto the circular ring  $1 - \delta < |z| < 1 + \delta$  and the curve  $\Gamma_j$  onto the circle  $|z| = 1$ .

For  $r \in \mathbb{N}^*$ , we shall denote by  $\mathcal{U}_r^*$  the class of plane domains whose boundary consists of  $n$  disjoint continua. It is well known that each domain in the class  $\mathcal{U}_r^*$  is conformally equivalent to a domain in  $\mathcal{U}_r$ .



Much of the functions theory on  $\Omega$  is conveniently described in terms of the Schottky double  $\hat{\Omega}$  of  $\Omega$ . This is the compact Riemann surface of genus  $g = r - 1$  obtained, when  $\Omega$  has analytic boundary, as is henceforth assumed, by welding  $\Omega$  along  $\Gamma$ , with an identical copy  $\tilde{\Omega}$ . Thus, as a point set  $\hat{\Omega} = \Omega \cup \Gamma \cup \tilde{\Omega}$ . The "backside"  $\tilde{\Omega}$  is provided with the opposite conformal structure. This means that if  $\tilde{z} \in \tilde{\Omega}$  denotes the opposite to  $z \in \Omega$  then the map  $\tilde{z} \rightarrow z$  is a holomorphic coordinate on  $\tilde{\Omega}$ . The construction of this double generalizes to any bordered Riemann surface and the result is always a symmetric Riemann surface, i.e., a Riemann surface provided with an antiholomorphic involution. In our construction this is the map  $U : \hat{\Omega} \rightarrow \hat{\Omega}$  which exchanges  $z$  and  $\tilde{z}$  and which keeps  $\Gamma$  pointwise fixed.

The double of a plane domain  $\Omega$  has a holomorphic atlas consisting of only two charts: the corresponding coordinate function are the identity map  $\varphi_1 : z \rightarrow z$  on  $\Omega$  and the map  $\varphi_2 : \tilde{z} \rightarrow \bar{z}$  on  $\tilde{\Omega}$ . When  $\Gamma$  is analytic, both these maps extend analytically across  $\Gamma$  in  $\hat{\Omega}$ , hence their domains of definitions overlap and the union covers all  $\hat{\Omega}$ . Let

$$S = \varphi_2 \circ \varphi_1^{-1}$$

be the coordinate transition function. It is analytic and defined in a neighbourhood of  $\Gamma$  in  $\mathbb{C}$ , and on  $\Gamma$  it satisfies

$$S(z) = \bar{z} \quad (z \in \Gamma).$$

A function  $f$  on  $\hat{\Omega}$  is most conveniently described as a pair of functions  $f_1, f_2$  on  $\Omega$ , continuously extendable to  $\Gamma$ , such that

$$f_1(z) = \overline{f_2(z)} \quad (z \in \Gamma).$$

The formal relations to  $f$  in terms of the coordinates functions  $\varphi_1$  and  $\varphi_2$  above are

$$\begin{cases} f_1 &= f \circ \varphi_1^{-1} \\ f_2 &= c \circ f \circ \varphi_2^{-1} \circ c, \end{cases}$$

where  $c$  denotes complex conjugation. It follows, for example, that  $f$  is meromorphic if and only if  $f_1$  and  $f_2$  are meromorphic.

**Remark 4.1.** *A homology basis of  $\hat{\Omega}$  consists of any  $r - 1$  of the  $\Gamma_k$  together with any  $r - 1$  of the curves  $B_k$ , which consists of an arc lying in  $\Omega$  joining  $\Gamma_k$  to  $\Gamma_{k+1}$  (to  $\gamma_1$  if  $k = r$ ) together with its conjugate lying in the copy of  $\Omega$ . In the following, we will take for such a basis  $B_1, \dots, B_{r-1}, \Gamma_1, \dots, \Gamma_{r-1}$ .*

It is useful to note that if two domains in the class  $\mathcal{U}_r$  are conformally equivalent, then so are their doubles conformally equivalent. Indeed, the Schwarz reflection principle permits an extension of a conformal homeomorphism between plane domains of the class  $\mathcal{U}_r$  to a conformal homeomorphism of their doubles.

We will deal with domain  $\Omega$ , in the class  $\mathcal{U}_r$  whose double is a hyperelliptic Riemann surface.

This is equivalent to the fact that  $\Omega$  can be mapped one-to-one conformally onto the exterior of a system of slits taken from the real axis. To be complete, we recall the following results.

**Definition 4.4.** *A closed Riemann surface of genus  $g$  is hyperelliptic if it admits an analytic involution with precisely  $2g+2$  fixed points. Such an analytic involution is called a sheet interchange. All of the Weierstrass points on a hyperelliptic surface are located at the fixed points of the sheet interchange.*

We have the theorem ([3] Theorem 3 p.13).

**Theorem 4.1.** *On a hyperelliptic Riemann surface*

- (1) *there is only one sheet interchange, and*
- (2) *any two meromorphic functions of order two are related by a fractional linear transformation.*

We will also need the following theorem ([3] Theorem 7 p.19).

**Theorem 4.2.** *Let  $\Omega$  be a domain in the class  $\mathcal{U}_r$ ,  $r \geq 3$ . Then the following statements are equivalent:*

- (1) *the double of  $\Omega$  is a hyperelliptic Riemann surface,*
- (2) *the domain  $\Omega$  can be mapped one-to-one conformally onto the exterior of a system of slits taken from the real axis,*
- (3) *the domain  $\Omega$  admits an anticonformal involution possessing precisely  $2r$  fixed points on the boundary of  $\Omega$ .*

Because we will need some notations, we write here the proof of the equivalence between the first and third points.

*Proof.* (1)  $\Leftrightarrow$  3))

1) implies 3). Suppose that the double of  $\Omega$  is the hyperelliptic Riemann surface  $W$  of genus  $g = n - 1$  and with the sheet interchange  $T$ . We let  $U$  denote the canonical conformal involution of  $W$  obtained in the doubling process. If the fixed points of  $T$  are  $p_1, p_2, \dots, p_{2g+2}$ , the mapping  $UTU : W \rightarrow W$  is an analytic involution of  $W$  with precisely  $2g+2$  fixed points  $Up_1, Up_2, \dots, Up_{2g+2}$ . But the uniqueness of the sheet interchange, Theorem 4.1, forces the identity  $T = UTU$ , and so  $U$  commutes with  $T$ .

Observe now that a point  $p$  resides on the boundary of  $\Omega$  if and only if  $Up = p$ . For such a point  $Tp = (UTU)p = UT(Up) = u(Tp)$ , and so  $T$  preserves the boundary of  $\Omega$ . It follows that  $T$  maps  $\Omega$  univalently either onto  $\Omega$  or on  $\tilde{\Omega} = U(\Omega)$ . If  $T(\Omega) = \Omega$ , then  $T$  is an analytic involution of the domain  $\Omega$ . Additionally, since  $T = UTU$ , whenever  $p$  is a fixed point of  $T$ ,  $Up$  is also a fixed point of  $T$ . Thus  $\bar{D}$  contains at least  $\frac{2g+2}{2} = g + 1 \geq 3$  fixed points of  $T$ . But Theorem ?? then yields  $T$  is the identity map, an impossibility in view of the finite number of fixed points of  $T$ . The only possibility left is for  $T$  to map  $\Omega$  univalently onto  $\tilde{\Omega}$ , and we conclude that the  $2g + 2$  fixed points of  $T$  are located on the boundary of  $\Omega$ .

We now set  $V = TU$ ,  $V$  is an anticonformal involution of  $W$  which preserves  $\Omega$  and its boundary. Moreover  $Vp = p \Leftrightarrow Tp = p$ ; then  $T$  obeys 3).

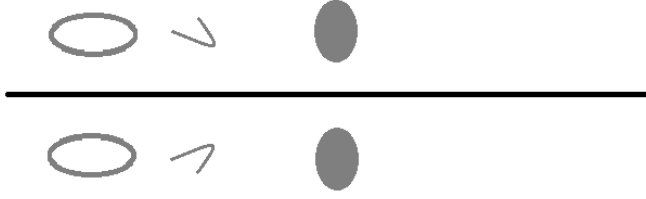
3) implies 1). Suppose a domain  $\Omega \in \mathcal{U}_n$  admits an anticonformal involution  $V : \Omega \rightarrow \Omega$  with precisely  $2n$  fixed points on the boundary of  $\Omega$ . Let  $W$  be the double of  $\Omega$  and let  $U$  be the canonical anticonformal involution on  $W$ ; observe that  $w$  is a closed Riemann surface of genus  $g = n - 1$ . The mapping  $V$  can now be extended to an anticonformal involution of  $W$  by setting  $V(p) = UVU(p)$  for all  $p \in \tilde{\Omega} = U(\Omega)$ . Now since the condition  $U(p) = p$  defines the boundary of  $\Omega$  and since  $U$  interchanges the sets  $\Omega$  and  $\tilde{\Omega}$ , the conformal involution  $T = UV$  of  $W$  has precisely  $2g + 2$  fixed points on  $W$ ; these are of course those fixed points of  $V$  which are located on the boundary of  $\Omega$ . But  $T$  then has the properties of a sheet interchange, then  $W$  is hyperelliptic. □

Finally ([3] Theorem 10 p.26)

**Theorem 4.3.** *If  $\Omega \in \mathcal{U}_r^*$  is a domain which is symmetric with respect to the real axis and for which the intersection of  $\Omega$  with the open upper half plane  $H$ , is simply connected, then  $\Omega$  is conformally equivalent to the exterior of a system of  $r$  slits taken from the real axis.*

We will see that Serre's method generalizes naturally at multiply connected domains of the plane, conformal to the union of the intervals of the real axis. Among these domains are those symmetric with respect to the real axis and whose intersection with the open upper half plane is simply connected. We can also find domains such that  $\Gamma = (\Gamma_1, \Gamma_2)$  or  $(\Gamma_1, \Gamma_2, \Gamma_3)$  which are invariant under conjugation.

On the contrary, we not have yet the answer for a domain such that  $\Omega \cap H$  is not simply connected:



At first, notably to put on the notations, we recall some results on the double of the exterior of a system of slits taken from the real axis. Let  $E = \bigcup_{j=1}^r E_j$ ,  $E_j = [e_{2j-1}, e_{2j}] \subset \mathbb{R}$ ,  $e_1 < e_2 < e_3 < \dots < e_{2r-1} < e_{2r}$ .

We let  $q(z)$  be the polynomial  $q(z) = \prod_{i=1}^{2r} (z - e_i)$ . Such a polynomial will be called the structure polynomial of  $\mathbb{C} \setminus E$ . Consider the subset of  $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$  given by

$$W = \{(z, w), w^2 = q(z)\}.$$

where we add two points at infinity  $\infty_+$  and  $\infty_-$ , characterized by the fact that  $\frac{w}{z^{r-1}} = 1$  at  $\infty_+$  and  $-1$  at  $\infty_-$ .

It's a two-sheeted branched covering space of the sphere  $P_1(\mathbb{C})$ , branched at the  $2r$  points  $e_j$ . It's a topological covering space of  $\mathbb{C} \setminus E$ .

The covering map  $\pi_E : W \rightarrow P_1(\mathbb{C})$ ,  $\pi : (z, w) \rightarrow z$  is a meromorphic function of order two on  $W$  whose only multiple points, each of multiplicity two, are located at the points  $(e_j, q(e_j))$ ,  $j = 1, \dots, 2r$ . Then  $W$  is a hyperelliptic surface whose sheet interchange  $T : W \rightarrow W$  is given by  $T : (z, w) \rightarrow (z, -w)$ .

Finally we exhibit  $W$  as the double of the domain  $\Omega$  ([3]).

**Theorem 4.4.** *The double of  $\mathbb{C} \setminus E$ , is conformally equivalent to the Riemann surface  $W$ :*

$$w^2 - q(z) = 0.$$

*Proof.* Let denote by  $D = \mathbb{C} \setminus E$ . First observe that  $D$  admits an analytic square root of its structure polynomial  $\sqrt{q(z)}$ . We want to define  $\sqrt{q}$  as an analytic function on  $D$ , the branch with

$$\sqrt{q(x)} > 0 \quad \text{if } x > e_{2r}.$$

This implies

$$(2) \quad \sqrt{q(x)} < 0 \quad (e_{2r-2}, e_{2r-1}) \cup (e_{2r-6}, e_{2r-5}) \cup \dots$$

$$(3) \quad \sqrt{q(x)} > 0 \quad (e_{2r-4}, e_{2r-3}) \cup (e_{2r-8}, e_{2r-7}) \cup \dots$$

$$(-1)^r \sqrt{q(x)} > 0 \quad \text{if } x < e_1$$

and  $(\sqrt{q(x+i0)})$  means  $\lim_{\epsilon \downarrow 0} \sqrt{q(x+i\epsilon)}$

$$(4) \quad (-i)\sqrt{q(x+i0)} > 0 \quad (e_{2r-1}, e_{2r}) \cup (e_{2r-5}, e_{2r-6}) \cup \dots$$

$$(5) \quad i\sqrt{q(x+i0)} > 0 \quad (e_{2r-3}, e_{2r-4}) \cup (e_{2r-7}, e_{2r-6}) \cup \dots$$

So that it is possible to write  $W$  as the disjoint union  $W = W \cup W^+ \cup W^-$  where

$$\begin{aligned} W &= \{(z, w) \in W, z \in \partial D\} \\ W^+ &= \{(z, w) \in W, w = \sqrt{q(z)}, z \in D\} \\ W^- &= \{(z, w) \in W, w = -\sqrt{q(z)}, z \in D\}. \end{aligned}$$

Additionally the meromorphic function  $\pi$  on  $W$  defined by  $\pi : (z, w) \rightarrow z$  effects a univalent mapping of both  $W^+$  and  $W^-$  onto  $D$ , and if  $(z, w) \in W$ , then  $\pi(z, w) \in \partial D$  and so in particular is real.

To obtain the double of  $D$  we let  $f$  be an analytic and univalent map of  $D$  onto a domain  $D' \in \mathcal{U}_n$ , and we let  $W'$  be the double of  $D'$ . If  $U'$  is the canonical anticonformal involution on  $W'$  obtained by the doubling process, we have then the conformal homeomorphism of  $W$  onto  $W'$  given by

$$(z, w) \rightarrow \begin{cases} f(z), & (z, w) \in W^+ \\ f(z) = U'(f(\bar{z})), & (z, w) \in W \\ U'(f(\bar{z})), & (z, w) \in W^- \end{cases}$$

and the proof is complete.  $\square$

It is easy to check that the sheet interchange corresponds to  $T = UV = VU$ , where  $U = f^{-1} \circ U' \circ f$  and  $V : D \rightarrow D, z \mapsto \bar{z}$ .

We will also denote  $W$  by  $\widehat{\mathbb{C} \setminus E}$ .

Let us denote by  $E_j^*$  the reciprocal image of  $E_j$  in  $\widehat{\mathbb{C} \setminus E}$ ,  $E^* = \bigcup_{j=1}^r E_j^*$  and  $D_j^*$  the reciprocal image of  $[e_{2j}, e_{2j+1}]$  for  $j = 1, \dots, r-1$ . Then (see remark 4.1)  $D_1^*, \dots, D_{r-1}^*, E_1^*, \dots, E_{r-1}^*$  form an homology basis of  $\widehat{\mathbb{C} \setminus E}$ .

The adherence of  $W^+$  is  $W^+ \cup W$  and will denote  $E_+^* = W$  its border. We will define in the same manner  $E_-^*$ .

**4.3. The notion of “calibrated”.** In this section we refer in particular to [25], [29], [30] and the references within, the appellation calibrated corresponds to “Pell-Abel type” in [30].

First, we recall the following terminology. The meromorphic 1-forms on a compact Riemann surface are called abelian differentials. The abelian differentials which are holomorphic will be called of the first kind; while the meromorphic abelian differentials with zero residues will be called of the second kind. Finally, a general abelian differential (which may have residues) will be called of the third kind.

We denote by  $w_{\infty_+, \infty_-}$  the differential of the third kind having a simple pole at  $\infty_+$  and  $\infty_-$ , with residue  $-1$  and  $+1$  respectively, normalized by

$$\int_{e_{2j}}^{e_{2j+1}} w_{\infty_+, \infty_-} = 0, \quad j = 1, \dots, r-1.$$

We recall now the link between  $w_{\infty_+, \infty_-}$  and the Green function of  $\mathbb{C} \setminus E$  with pole at  $\infty_+$ ,  $g(z)$  (see for example [34], [11], and the references within):

- (1)  $g(z)$  is harmonic in  $\mathbb{C} \setminus E$ ,
- (2)  $g(z) - \ln |z|$  is harmonic in a neighborhood of  $\infty^+$
- (3)  $\lim_{z \rightarrow \zeta} g(z) = 0$  for all  $z \in E$ .

Consequently,  $g(z) = \operatorname{Re} G(z)$  where  $G$  is a holomorphic function uniquely determined up to the addition of a purely imaginary constant. A function  $\tilde{g}$ , such that  $\tilde{g}(z) = \operatorname{Im} G(z)$  is called a harmonic conjugate of  $g$ . So  $G$  is the multiple-valued function obtained by adding to  $g$  its conjugate:

$$G(z) = \int_{\gamma_z} 2 \frac{\partial g}{\partial w}(w) dw = g(z) + i\tilde{g}(z), \quad \text{where } \gamma_z \text{ path from a fixed point } z_1 \text{ to } z.$$

$$g(z) = \operatorname{Re} G(z) = \frac{G(z) + \overline{G(z)}}{2}.$$

You can find an expression of  $G$  in terms of theta function in [11].

The derivative of this multi-valued function is clearly single-valued. Hence,  $G'(z)$  is a holomorphic function in  $\mathbb{C} \setminus E$ .

The functions  $g$  and  $G$  extend to all of  $\widehat{\mathbb{C} \setminus E}$  by reflecting across  $E$ , since  $g$  vanishes on  $E$  and  $\operatorname{Re}(G(z)) = 0$  on  $E$ :  $g(U(z)) = -g(z)$  and  $G(U(z)) = -\overline{G(z)}$  for  $z \in \mathbb{C} \setminus E$  and  $U$  the canonical anti-conformal involution of  $\widehat{\mathbb{C} \setminus E}$ . In particular  $dG = w_{\infty_+, \infty_-}$ . From the theory of 1-differential form on  $W$ ,  $w_{\infty_+, \infty_-}$  is of the form

$$w_{\infty_+, \infty_-} = \frac{\lambda^{r-1} + \sum_{k=0}^{r-2} c_k \lambda^k}{\sqrt{q(\lambda)}}.$$

It is an abelian differential with poles at  $\infty_+$  and  $\infty_-$  and zeros at the  $\lambda_j$  (the zeros of  $G'(z)$ ) and  $\lambda_j^*$  (the points of the copy of  $\mathbb{C} \setminus E$  corresponding to the  $\lambda_j \in \mathbb{C} \setminus E$ ). It is clear from the behavior of  $g(x)$  on the real axis that there is exactly one such  $\lambda_j$  in the  $(e_{2j}, e_{2j+1})$ .

$$w_{\infty_+, \infty_-} = \frac{\prod_{j=1}^{r-1} (\lambda - \lambda_j)}{\sqrt{q(\lambda)}} d\lambda = i dp,$$

where  $dp$  is called the quasimomentum in [17].

**Definition 4.5.** *A compact set  $E = [e_1, e_2] \cup [e_3, e_4] \cup \dots \cup [e_{2r-1}, e_{2r}]$  of the real line is  $N$ -calibrated for some  $N \in \mathbb{N}$ ,  $N \geq r$ , if the complex Green's function  $G(z)$  of  $\widehat{\mathbb{C} \setminus E}$  with pole at infinity satisfies the conditions*

$$(6) \quad \int_{e_{2k}}^{e_{2k+1}} \frac{R(t)}{\sqrt{q(t)}} dt = 0, \quad \text{i.e.} \quad \int_{D_k^*} \frac{R(t)}{\sqrt{q(t)}} dt = 0, \quad k = 1, 2, \dots, r-1$$

and

$$(7) \quad \int_{e_{2k-1}}^{e_{2k}} \frac{R(t)}{\sqrt{q(t)}} dt = \pm \frac{n_k \pi i}{N}, \quad \text{i.e.} \quad \int_{E_k^*} \frac{R(t)}{\sqrt{q(t)}} dt = \pm \frac{2n_k \pi i}{N}, \quad k = 1, 2, \dots, r,$$

where

$$R(t) = (t - \lambda_1)(t - \lambda_2) \cdots (t - \lambda_{r-1})$$

and  $n_1, \dots, n_r, N \in \mathbb{N}$ ,  $n_1 + \dots + n_r = N$ .

We observe that if a compact  $E$  is calibrated with respect to  $n_1, n_2, \dots, n_r, N$  it is also calibrated with respect to  $kn_1, kn_2, \dots, kn_r, kN$ ,  $k \in \mathbb{N}^*$ .

We have the following results (see for example, [30], [29])

**Proposition 4.1.** *There exists signs  $\epsilon_j \in \{-1, +1\}$  such that*

$$\sum_{j=1}^r \epsilon_j \int_{e_{2j-1}}^{e_{2j}} w_{\infty_+, \infty_-} = i\pi.$$

**Proposition 4.2.** *Let*

$$f(z) = \cosh \left( N \int_{e_{2r}}^z \frac{R(t)}{\sqrt{q(t)}} dt \right),$$

then  $f$  is a polynomial of degree  $N$ . Moreover, let  $M_n$ , such that  $T_N(z) = M_n f(z)$  is a monic polynomial, then  $T_N$  is the Chebyshev polynomial of degree  $N$  of  $E$ .

For the commodity of the lecture we write the proof of the preceding proposition ( see for example [26], [30], [32], [29]). We use a result of Ostrovskii, Pakovitch and Zaidenberg [24]; we first recall some definitions. Let  $D = D(a, r)$  be a closed disc centered at  $a \in \mathbb{C}$  and of radius  $r$ . We say that a compact  $K \subset D$  supports  $D$  if  $D$  is the (unique) disc of smallest radius which contains  $K$ . For a given compact  $K \subset \mathbb{C}$ , a monic polynomial  $p(z) \in \mathbb{C}[z]$  of degree  $n \geq 0$  is called the  $n$ -th polynomial of least deviation (from zero) or the Chebyshev polynomial of degree  $n$  if  $\|p\|_K \leq \|q\|_K$  for any monic polynomial  $q(z) \in \mathbb{C}[z]$  of degree  $n$ , where  $\|p\|_K = \max_{z \in K} \{|p(z)|\}$ . The classical inequality of Jung asserts that each compact convex  $K$  of diameter  $\delta$  is contained in a closed disc  $D(a, \rho)$ , with  $\frac{\delta}{2} \leq \rho \leq \delta$ . The following theorem is the main result in [24]

**Theorem 4.5.** *Let  $\Delta_r = \Delta(0, r) \subset \mathbb{C}$  be the disc of radius  $r$  centered at the origin,  $K \subset \Delta_r$  be a supporting compact of  $\Delta_r$ , and  $p \in \mathbb{C}[z]$  be a monic polynomial of degree  $n$ . Then  $p$  is the unique  $n$ -th polynomial of least deviation on  $K_p = p^{-1}(K)$ .*

*Proof.* Let  $G(z) = \int_{e_{2r}}^z \frac{R(t)}{\sqrt{q(t)}} dt$  because of (7)  $G$  is a multi valued function on  $\widehat{\mathbb{C} \setminus E}$ , but as  $\int_{\mathcal{D}_j^*} \frac{R(t)}{\sqrt{q(t)}} dt$  is in  $2i\pi\mathbb{Z}$ , then, by  $2i\pi$  multiplicity of  $\cosh$ ,  $f$  is single valued in  $\widehat{\mathbb{C} \setminus E}$ , being meromorphic  $f$  is of the form  $\frac{R_1 + \sqrt{q}}{R_2}$ , where  $R_1$  and  $R_2$  are polynomial functions. Now it is easy to verify that  $G(T(z)) = G(z)$ , where  $T$  is the hyperelliptic involution, then  $f(T(z)) = f(z)$ , consequently  $f$  is a rational function. Since  $G(z) = \int_{e_{2r}}^z w_{\infty_+, \infty_-}$  the only poles of  $G$  are  $\infty_+, \infty_-$  and so it is also the case for  $f$ .

Hence  $f$  is an entire function with a pole of order  $r$  at  $\infty$ , then a polynomial of degree  $r$ . □

**4.4. A generalization of Serre's method, the case of a compact of  $\mathbb{R}$ .** We have to notice that a compact  $E$  of  $\mathbb{R}$  of capacity  $\mathcal{C}(E) \geq 1$  is an example of symmetric domain with respect to the real axis.

The following theorem will be a consequence of [30, Theorem 1.6.2 ].

**Theorem 4.6.** *Let  $E$  be a compact of capacity  $\mathcal{C}(E) \geq 1$ , then for all neighborhood  $U$  of  $E$  in  $\mathbb{R}$ , there exists a sequence of monic polynomials of degree  $> 0$ , with coefficients in  $\mathbb{Z}$ , whose all roots are in  $U$ ,  $(P_n)$  such that  $\lim \mu_{P_n} = \mu_E$ .*

Before proving it, for the convenience of the reader we recall some facts (see for example [32], Chapter 5).

**Proposition 4.3.** *Let  $E \subset \mathbb{R}$  be compact. Let*

$$\tilde{E}_n = \{x \in \mathbb{R}, \text{dist}(x, E) \leq \frac{1}{n}\}.$$

*Then*

$$(1) E \subset \dots \subset \tilde{E}_{n+1} \subset \tilde{E}_n \subset \dots \subset \mathbb{R} \quad \text{and} \quad \bigcap_n \tilde{E}_n = E,$$

$$(2) \text{ Each } \tilde{E}_n \text{ is a finite union of disjoint closed intervals.}$$

*Proof.* The point (1) is clear. To show (2) we use the fact that every open set in  $\mathbb{R}$  is a countable union of disjoint open intervals. Hence  $\mathbb{R} \setminus E$  is a disjoint union of maximal open intervals. Since  $E$  is compact, two of these intervals are unbounded and the others  $\{J_k\}_{k \in I}$ , where  $I$  is a countable set, are contained in the convex hull of  $E$ . Thus,  $\sum_{k \in I} |J_k| < \infty$ , so for each  $n > 0$ ,  $\#\{k, |J_k| > 2/n\}$

is finite. Thus, all but finitely many  $J_k$  lie in a given  $\tilde{E}_n$ , showing  $\mathbb{R} \setminus \tilde{E}_n$  is finite. Thus, all but finitely many  $J_k$  lie in a given  $\tilde{E}_n$ , showing  $\mathbb{R} \setminus \tilde{E}_n$  has finitely many open intervals. It is easy to see that each of the finite disjoint closed intervals in  $\tilde{E}_n$  must have positive measure. □

Furthermore we have [29, Theorem 3.9], [30] and [32, Theorem 5.6.1, p. 306]

**Theorem 4.7.** Let  $E = \bigcup_{j=1}^{l+1} E_j$  be an  $l$ -gap set with  $E_j = [\alpha_j, \beta_j]$ ,  $\alpha_j < \beta_j < \beta_{j+1}$ . Then for all  $m$

large, there exist  $l$ -gap sets  $E^{(m)} = \bigcup_{j=1}^{l+1} E_j^{(m)}$  with

- (1)  $E_j \subset E_j^{(m)}$ ,
- (2) Each  $E_j^{(m)}$  has harmonic measure in  $E^{(m)}$  equal to  $k_j^{(m)}/m$  with  $k_j^{(m)} \in \{1, 2, \dots\}$ ,
- (3) For some positive constants  $C_1, C_2$ ,

$$(8) \quad |E_j^{(m)} \setminus E_j| \leq C_1 m^{-1}$$

$$(9) \quad \mathcal{C}(E) \leq \mathcal{C}(E^{(m)}) \leq \mathcal{C}(E) + C_2 m^{-1}$$

Then we have

**Theorem 4.8.** Let  $E \subset \mathbb{R}$  be compact. Then there exist  $E_n$  so that  $E \subset \dots \subset E_{n+1} \subset E_n \subset \dots \subset \mathbb{R}$  and  $\bigcap_n E_n = E$  holds, and

$$E_n \subset E_{n-1}^{int}$$

and each  $E_n$  is the spectrum of some two-sided periodic Jacobi matrix. Moreover

- (1)  $\mu_{E_n} \rightarrow \mu_E$ ,
- (2)  $\mathcal{C}(E_n) \rightarrow \mathcal{C}(E)$ .

Now, we know that  $E_n$  is a finite union of calibrated intervals and then by [29], [30], there exists a sequence of monic polynomials with coefficients in  $\mathbb{Z}$ ,  $P_{k,E_n}$ , whose all roots are in  $E_n$ , such that  $\lim_k \mu_{P_{k,E_n}} = \mu_{E_n}$ . By the diagonal process  $\mu_E = \lim_j \mu_{P_{k_j, E_{n_j}}}$ .

**Remark 4.2.** In particular, in the case of a compact of  $\mathbb{R}$ , we can release the condition of continuous boundary in the theorem 9.1 (Bilu, Rumely) of [2]. We state this theorem in general terms in the following section.

## 5. SOME APPROXIMATIONS

In this section we give some known theorems on approximation of equilibrium measure by counting measure.

First of all, Rumely ([27]), generalizing a Bilu's theorem ([5]) obtained the following equidistribution result.

**Theorem 5.1** (Rumely, Bilu). *Suppose a compact set  $K \subset \mathbb{C}$  with continue boundary, has capacity  $\mathcal{C}(K) = 1$  and is stable under complex conjugation. Let  $(\alpha_n)_{n \geq 1}$  a sequence of algebraic integers ( $\alpha_n \neq \alpha_m$  if  $n \neq m$ ),  $\alpha_n$  of degree  $d_{\alpha_n}$ : for all open  $U$  containing  $K$ , there exists  $n_0$  such that for all  $n \geq n_0$ ,  $\alpha_n$  with all its conjugate,  $O(\alpha_n)$ , are in  $U$ . Let  $\Delta_n$  the measure*

$$\Delta_n = \frac{1}{d_{\alpha_n}} \sum_{\beta \in O(\alpha_n)} \delta_\beta,$$

then the measures  $\Delta_n$  converge weakly to the equilibrium measure of  $K$ ,  $\mu_K$ .

This theorem answers Serre's question in the case of a compact of capacity 1, but in the proof the role of the polynomials  $P_n(z) = \prod_{\beta \in O(\alpha_n)} (z - \delta_\beta)$  is not explicit.

The second result we want to point out is the following theorem due to Pritsker [22, Theorem 2.3].

**Theorem 5.2.** *Given any positive Borel measure  $\mu$ ,  $0 \leq \mu(\mathbb{C}) \leq 1$ , that is symmetric about real line, there is a sequence of complete sets of conjugate algebraic integers such that their counting measures  $\tau_n$  converge weakly to  $\mu$ .*

In this theorem we consider then the particular case of  $\mu_K$ , the equilibrium measure of a compact  $K$  symmetric with respect to the real axis.

The proof needs in particular, some results on approximation of a finite set of points by algebraic integers.

In [21], Theorem 3.2, p.158 (see also [9] for its effective version), it is proving that for every  $n$  given numbers  $z_1, \dots, z_n$ , and every  $\epsilon > 0$ , there exists an irreducible equation with complex integral coefficients  $\alpha_1, \dots, \alpha_{n+1}$  and with roots  $\zeta_{k,\epsilon}$  such that  $|\zeta_{k,\epsilon} - z_k| < \epsilon$  for  $k = 1, \dots, n$ . The same is true for real integral coefficients provided that the numbers  $z_1, \dots, z_n$  are symmetric to the real axis.

For completeness we recall also the more general context of Ferguson's theorem. In [14] (Theorem A.1., p. 147) we have that if  $\alpha_1, \dots, \alpha_n$  are a complete set of conjugate algebraic integers over  $\mathbb{Q}[i]$ ,  $\epsilon$  any positive number, and  $z_2, \dots, z_n$  any complex numbers. Then there is a polynomial  $q \in \mathbb{Z}[i][z]$  such that

$$|q(\alpha_j) - z_j| < \epsilon, \quad 2 \leq j \leq n.$$

A second ingredient of the proof of [22, Theorem 2.3] is to express  $\mu_K$  as the weak limit of counting measures. In fact, there exist sequences of polynomials with their zeros in  $K$ , that verify  $\mu_n \rightarrow \mu_K$ . We give an example which is linked with other interpretations of the capacity of a compact (see for example [33], [30] Appendix, [31] Appendix B).

First of all, let  $K \subset \mathbb{C}$  be compact and infinite. An  $n$  point Fekete set is a set  $\{z_j, j = 1, \dots, n\} \subset K$  that maximizes

$$q_n(z_1, \dots, z_n) = \prod_{i \neq j} |z_i - z_j|.$$

The Fekete constant (or diameter of  $K$ ) is defined by

$$d_n(K) = q_n(z_1, \dots, z_n)^{1/n(n-1)}$$

for the maximizing set.  $d_n(K)$  has a limit called the transfinite diameter of  $K$ , which is in fact, equal to  $\mathcal{C}(K)$ .

Secondly, the Chebyshev polynomials,  $T_n$ , are defined as those monic polynomials of degree  $n$  which minimize  $\|z^n + p(z)\|$ , where  $p \in \mathcal{P}_{n-1}$ , the vectorial space of all polynomials of degree  $\leq n-1$  and  $\|\cdot\|$  is defined by  $\|f\| = \max_{z \in K} |f(z)|$ . We have  $\mathcal{C}(K) = \lim \|T_n\|^{1/n}$ .

From the maximum principle for analytic functions, it follows that for all  $n$ , the Fekete sets lie on the outer boundary of  $K$ . We have the following theorem.

**Theorem 5.3.** *The normalized density of Fekete sets converges to  $\mu_K$ , the equilibrium measure for  $K$ .*

**Remark 5.1.** *Let  $\{P_n\}$  be any sequence of monic polynomials having all their zeros in  $K$  and such that the normalized zero counting measures for  $P_n$  converge weakly to  $\mu_K$ . If  $\partial K$  is regular (e.g., if it is connected), then*

(1)  $P_n$  are asymptotically optimal for the Chebyshev problem:

$$\lim_{n \rightarrow \infty} \|P_n\|^{1/n} = \mathcal{C}(K).$$

If  $\mathcal{C}(K) > 0$  (so that  $\mu_K$  is defined), then we also have:

(2) Uniformly on compact subsets of the unbounded component of  $\mathbb{C} \setminus K$ ,

$$\lim_{n \rightarrow \infty} |P_n(z)|^{1/n} = \exp\{-\Phi_{\mu_K}(z)\}.$$

We come now to the proof that interest us. We have then the following result ([22] p. 16). Let a sequence of polynomials  $(P_n)$  such that the corresponding normalized zero counting measure for  $P_n$ ,  $\mu_n$  converge to  $\mu_K$ . We can suppose  $\deg(P_n) = n$  and denote its zeros by  $z_1, \dots, z_n$ . Therefore, we can approximate this measure by a sequence of the counting measures  $\tau_{n+1}$  for the complete set of conjugate algebraic integers  $\zeta_k = \zeta_{k,1/n}$ ,  $k = 1, \dots, n+1$  by using the theorem of Motzkin. For any  $n \in \mathbb{N}$ , we approximate each point  $z_k$  as close as we wish by one of the conjugate algebraic integers  $\zeta_k$ ,  $1 \leq k \leq n$ , obtained from Motzkin's theorem, while let the remaining  $(n+1)$ th conjugate



algebraic integer  $\zeta_{n+1} \rightarrow \infty$  as  $n \rightarrow \infty$  (see [9] p.160-161 for details). It follows that the resulting measures

$$\tau_{n+1} = \frac{1}{n+1} \sum_{k=1}^{n+1} \delta_{\zeta_k}$$

converge to  $\mu_K$  as  $n \rightarrow \infty$ .

In fact  $\mu_n = \frac{1}{n} \sum_{k=1}^n \delta_{z_k} \rightarrow \mu_K$ , then  $\nu_n = \frac{1}{n} \sum_{k=1}^n \delta_{\zeta_k} \rightarrow \mu_K$ . For  $f$  continuous on  $K$  and  $n$  sufficiently large we obtain

$$\int f \tau_{n+1} = \int f \frac{n}{n+1} \nu_n = \frac{n}{n+1} \int f \nu_n \rightarrow \mu_K.$$

But as already seen, the polynomial  $Q_n$  with roots the complete set of conjugate algebraic integers  $\zeta_k$ , has  $\zeta_{n+1}$  not in  $K$  (for  $n$  sufficiently large). So the sequence  $(Q_n)$  doesn't answer Serre's suggestion (Question 1.1).

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