



**HAL**  
open science

# ON A FEKETE-SZEG Ö THEOREM

Thérèse Fallière

► **To cite this version:**

| Thérèse Fallière. ON A FEKETE-SZEG Ö THEOREM. 2024. hal-04462386

**HAL Id: hal-04462386**

**<https://univ-avignon.hal.science/hal-04462386>**

Preprint submitted on 16 Feb 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# ON A FEKETE-SZEGÖ THEOREM

THÉRÈSE FALLIERO

ABSTRACT. We consider again a classical theorem relating capacities and algebraic integers and the question of the simultaneous approximation of  $n - 1$  different complex numbers by conjugate algebraic integers of degree  $n$ .

*This is a preliminary version of a further more complete paper, which will be co-written by Ahmed Sebbar.*

## 1. INTRODUCTION

A classical theorem of Fekete and Szegö [12] says that, given a compact set  $K$  in the complex plane, then

- (a) If  $\mathcal{C}(E) < 1$ , there is an open set  $U$  containing  $K$  such that there are only finitely many algebraic integers  $\alpha$  having all of their conjugates in  $U$ , and
- (b) if  $\mathcal{C}(E) \geq 1$ , then every open  $U$  containing  $K$  contains infinitely many such  $\alpha$ .

Furthermore Fekete and Szegö proved also that if  $E$  is a compact set in the complex plane, stable under complex conjugation and having logarithmic capacity  $\mathcal{C}(E) \geq 1$ , then every neighborhood of  $E$  contains infinitely many conjugate sets of algebraic integers. In [10] V. Ennola solved a question raised by R. M. Robinson that if  $\Delta$  is any real interval of length greater than 4, then for any sufficiently large  $n$  there exists an irreducible monic polynomial of degree  $n$  with integer coefficients all of whose zeros lie in  $\Delta$ .

We should emphasize that many diophantine inequalities are reduced to the existence of lattice points in some convex body [7, Chapter III]. In this direction, it is remarkable that T. Chinburg [8] reduces the proof of Fekete theorem to an application of Minkowski's Convex Body theorem [7, Chapter III] that we recall for later use.

This problem is actually related to a precise form of the Stone-Weierstrass theorem. This classical theorem states that every continuous function defined on a closed interval  $[a, b]$  can be uniformly approximated by polynomials. A more general statement is if  $X$  is a compact Hausdorff topological space and if  $C(X)$  is the algebra of real-valued continuous functions  $f : X \rightarrow \mathbb{R}$ , then a subalgebra  $\mathcal{A} \subset C(X)$  is dense if and only if it separates points.

The question is for which compact set continuous functions can be approximated by polynomials with integer coefficients? This question is a major one in approximation theory and the literature is very extensive [13]. Let  $f$  be a continuous real-valued function defined on  $[0, 1]$ , then the sequence of polynomials  $(p_n)$  defined by

$$p_n(x) = \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) \binom{n}{\nu} x^\nu (1-x)^{n-\nu}$$

converges uniformly to  $f$ . This is therefore a constructive proof of the Stone-Weierstrass theorem. It is due to Bernštein. We deduce from this result [14, Theorem 5] that for a continuous real-valued function  $f$  on the unit interval  $I = [0, 1]$  to be uniformly approximable by polynomials with integer coefficients it is necessary and sufficient that it be integer-valued at both 0 and 1.

As was pointed out in [14, Lemma 1], If  $q$  is a non constant polynomial with integer coefficients and  $I$  is an interval of length at least four, then  $\|q\| \leq 2$ . Hence, clearly, the approximation by polynomials with integer coefficients on a set  $E$  is related to the capacity of  $E$ . Consequently

[14, Theorem 2 ] if the capacity  $\mathcal{C}(I) \geq 1$  the only functions that are uniformly approximable in  $I$  by polynomials with integer coefficients are these polynomials themselves. On the other hand one can prove the surprising result that any  $f \in L^2([a, b])$ ,  $b - a < 4$  can be approximated on the interval  $[a, b]$  by polynomials with integer coefficients.

**Theorem 1.1** (Pál). *If  $f$  is continuous on  $[-a, a]$ ,  $|a| < 1$ , and  $f(0)$  is an integer then  $f$  may be uniformly approximated thereon by polynomials with integer coefficients.*

These problems are at the core of the approximation theory by polynomials with integer coefficients. It is a classical result, due to Chebyshev that for an interval  $[a, b]$

$$\inf_Q \|Q(z)\|_\infty = 2 \left( \frac{b-a}{4} \right)^n$$

where  $Q(X)$  describes the set of monic polynomials of degree  $n$ . Hilbert showed in [15] that if we restrict to monic polynomials of degree  $n$  with integer coefficients, then

$$\inf_Q \|Q(z)\|_{L^2([a,b])} \leq C\sqrt{n} \left( \frac{b-a}{4} \right)^{n/2}$$

and Fekete showed in [11] the more flexible

$$\inf_Q \|Q(z)\|_\infty \leq 2^{1-\frac{1}{n+1}} (n+1) \left( \frac{b-a}{4} \right)^{n/2}.$$

The main objective of this work is the following question suggested by J.P. Serre in [28]. If  $K$  is a compact of  $\mathbb{C}$ , symmetric with respect to the real axis with  $\mathcal{C}(K) > 1$ , and if  $U$  is an open containing  $K$ , there is a sequence  $(P_n)$  of monic polynomials with integer coefficients whose roots are in  $U$  and are such that the associated zeros counting measure  $\delta_{P_n}$  converge weakly to equilibrium  $\mu_K$  of  $K$ .

**Organization of the paper:** Very succinctly, we give the definitions in the first section, then we introduce the counting measures in the second section. Minkowski's theorem will be discussed in section three. The fourth section is devoted to certain approximations results and the fifth section, the longest, is devoted to the introduction of certain Riemann surfaces, in relation to certain Jacobi matrices and the solution of the Serre question.

## 2. DEFINITIONS

**2.1. Approximation on intervals.** If  $\mu$  is a measure on  $\mathbb{C}$  of compact support, its logarithm potential is defined by

$$\Phi_\mu(z) = \int \ln(|z-w|^{-1}) d\mu(w).$$

This integral converges if  $z \notin \text{supp}(d\mu)$ , and since  $d\mu$  has compact support,  $\ln(|z-w|^{-1})$  is uniformly bounded below for  $(z, w) \in \text{supp}(d\mu) \times \text{supp}(d\mu)$ , so the integral for each  $z \in \text{supp}(\mu)$  either converges or diverges to  $+\infty$ , in which case we set  $\Phi_\mu(z) = +\infty$ .

Potentials enter naturally in studying growth of polynomials as  $n \rightarrow \infty$ . For if

$$P_n(x) = \prod_{j=1}^n (x - x_j^{(n)})$$

then

$$\frac{1}{n} \ln |P_n(x)| = -\Phi_{\nu_n}(x)$$

where

$$\nu_n = \frac{1}{n} \sum_{j=1}^n \delta_{x_j^{(n)}}$$

is the counting measure for the zeros  $x_j^{(n)}$ . The function  $\Phi_\mu(z)$  is bounded below on  $\text{supp}(\mu)$ , so

$$I(\mu) = \int \Phi_\mu(z) d\mu(z) = \int \int_{K \times K} \ln(|x-y|^{-1}) \mu(x) \mu(y)$$

is either finite or diverges to  $+\infty$ .  $I(\mu)$  is called the potential energy of  $\mu$  or, for short, the energy of  $\mu$ .

Consider a compact  $K \subset \mathbb{C}$ . We consider all probability measures  $\mathcal{M}_{+,1}(K)$  on  $K$ . We say  $K$  has capacity zero if and only if  $I(\mu) = \infty$  for all  $\mu \in \mathcal{M}_{+,1}(K)$ . We set

$$v(K) = \inf_{\mu} I(\mu)$$

where  $\mu$  runs over all positive probability measures supported in  $K$ . Then the capacity of  $K$  is defined as  $\mathcal{C}(K) = e^{-v(K)}$ . Its logarithm  $v(K)$  is called the logarithm capacity of  $K$ .

For a compact  $K$  with non-zero capacity there exists a unique positive probability measure  $\mu$ , such that  $I(\mu) = \ln \mathcal{C}(K)^{-1}$ . This measure  $\mu = \mu_K$  is called the equilibrium measure of  $K$ .

Let us also recall that if  $(\mu_n)_n$  and  $\mu_\infty$  are probability measures on a compact Hausdorff space  $X$ ,  $(\mu_n)_n$  converges weakly  $\mu_\infty$  if

$$\int f d\mu_n \rightarrow \int f d\mu \quad \text{as } n \rightarrow \infty$$

for every  $f$  continuous on  $X$  and if  $f$  a real valued function defined on a topologic space  $E$ ,  $f$  is lower semicontinuous at  $a$  if

$$f(a) = \liminf_{x \rightarrow a} f(x).$$

We have the following property of the potential energy

$$\mu \rightarrow I(\mu)$$

is weakly lower semicontinuous. The lower semi continuity means

$$\mu_n \rightarrow \mu \Rightarrow \liminf I(\mu_n) \geq I(\mu)$$

equivalently

$$(1) \quad I^{-1}((-\infty, a]) \text{ is closed for all } a$$

$$(2) \quad I^{-1}((a, \infty]) \text{ is open for all } a.$$

Before continuing further, we wish to recall some definitions that will be need. Given a bounded set  $\mathcal{E}$  in the complex plane, we denote by  $\mathcal{E}(r)$  the  $r$ -neighborhood of  $\mathcal{E}$ .

**Definition 2.1.** *If  $\mathcal{E}_1$  and  $\mathcal{E}_2$  are two bounded sets in  $\mathbb{C}$ , the difference between  $\mathcal{E}_1$  and  $\mathcal{E}_2$  is the smallest  $r$  such that  $\mathcal{E}_1(r)$  contains  $\mathcal{E}_2$  and  $\mathcal{E}_2(r)$  contains  $\mathcal{E}_1$ .*

We will denote this difference by  $\delta(\mathcal{E}_1, \mathcal{E}_2)$ , it is small if and only if  $\mathcal{E}_1$  and  $\mathcal{E}_2$  are (almost) super imposible. We will say that  $\mathcal{E}_1$  is near  $\mathcal{E}_2$  (and reciprocally). If  $\mathcal{E}$  is a compact set and  $(\mathcal{E}_\nu)$  a family of sets such that  $\delta(\mathcal{E}, \mathcal{E}_\nu)$  tends to zero as  $\nu$  tends to  $\infty$  we will say simply that  $(\mathcal{E}_\nu)$  tends to  $\mathcal{E}$ .

We then deduce the lemma

**Lemma 2.1.** *If  $\delta(K_n, K) \rightarrow 0$  and  $\mathcal{C}(K_n) \rightarrow \mathcal{C}(K)$  then  $\mu_{K_n} \rightarrow \mu_K$ .*

*Proof.* As  $\mathcal{M}_{+,1}(K)$  is compact let  $\eta$  be a weak limit point of  $\mu_{K_n}$ . By lower semicontinuity of the energy  $I$ ,

$$\begin{aligned} I(\eta) &\leq \liminf I(\mu_{K_n}) \\ &= \lim \ln(\mathcal{C}(K_n)^{-1}) \\ &= \ln(\mathcal{C}(K)^{-1}), \end{aligned}$$

so  $\eta = \mu_K$ , that is,  $\mu_{K_n} \rightarrow \mu_K$ . □

Note that the inverse is not true  $\delta_{P_n} \rightarrow \mu_K$  and  $\mathcal{C}(\{\text{zeros of } P_n\}) = 0$ .

### 3. ALGEBRAIC INTEGERS WITH ALL CONJUGATES IN A GIVEN COMPACT.

For a set  $V \subset \mathbb{C}$  let  $\text{Pol}_V$  be the set of monic polynomials in  $\mathbb{Z}[X]$  of degree at least 1 such that all their roots lie in  $V$ . If  $z$  is an algebraic integer, a root of a polynomial  $P(X) \in \text{Pol}_V$ , then all the conjugate of  $z$  are in  $V$  and we say that  $z$  is totally in  $V$ .

Let  $\text{Irr}_V$  be the set of irreducible monic polynomials in  $\mathbb{Z}[X]$  of degree at least 1 such that all their roots lie in  $V$ . for such a polynomial  $P(X)$  of degree  $g$  let  $\mu_P$  be the corresponding probability measure supported in its roots,  $\mu_P = \frac{1}{g} \sum_{i=1}^g \delta_{x_i}$ . Now let  $K \subset \mathbb{C}$  be compact. There are two quite different cases [28], [26], depending on the capacity of  $K$ .

- (1) If  $\mathcal{C}(K) < 1$ , then  $\text{Irr}_K$  is finite.
- (2) If  $K \subset \mathbb{R}$  is a union of finite number of segments and  $\mathcal{C}(K) > 1$ , then  $\text{Irr}_K$  is infinite.
- (3) If  $K \subset \mathbb{C}$  is  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ -stable and  $\mathcal{C}(K) \geq 1$ , then for any open  $U, K \subset U$ , the set  $\text{Irr}_U$  is infinite.

**3.1. The case of a finite union of intervals.** Let  $E \subset \mathbb{R}$  be a finite union of segments with  $\mathcal{C}(E) > 1$  and  $\mu_E$  be its equilibrium measure. We have the following theorem ([28] Theorem 1.6.2).

**Theorem 3.1** (Serre). *There exists a sequence of polynomials  $P_n \in \text{Pol}_E$  such that  $\mu_{P_n} \rightarrow \mu_E$ .*

Then from the Fekete-Szego theorem a natural question, suggested by J.P Serre, arises.

**Question 3.1.** *Let  $K$  be a compact of  $\mathbb{C}$  stable under complex conjugation, of capacity  $\mathcal{C}(K) \geq 1$  and  $U$  an open set containing  $K$ . Is there a sequence of polynomials  $P_n \in \text{Pol}_U$  such that  $\mu_{P_n} \rightarrow \mu_K$ ?*

**3.2. Precisions on the properties of  $K$ .** Let  $U \subset \mathbb{C}$  be a set and let  $U^*$  be the so called “symmetric kernel” of  $U$  consisting of those points of  $U$  which belong, together with their conjugates to  $U$ . So  $U^*$  is symmetric with respect to the real axis and naturally  $\mathcal{C}(U^*) \leq \mathcal{C}(U)$ .

M.Fekete [11], [8] proved that if  $K$  is a compact of  $\mathbb{C}$  such that  $\mathcal{C}(K^*) < 1$ , then there is only a finite number of irreducible algebraic equations with integer coefficients of the form

$$z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n = 0$$

whose roots lie all in  $K^*$ .

**Theorem 3.2** (Minkowski’s theorem). *Suppose  $K$  to be a symmetric, convex, bounded subset  $\mathbb{R}^d$ . If  $\text{vol}(K) > 2^d$ , then  $K$  contains at least one lattice point other than 0.*

There is an extension to general lattices  $\Lambda = \mathbb{Z}u_1 \oplus \dots \oplus \mathbb{Z}u_d$ , where  $\{u_1, \dots, u_d\}$  is a basis of  $\mathbb{R}^d$ . We define  $\text{vol}(\Lambda)$  as the volume of the parallelotope

$$\left\{ \sum_{i=1}^d \alpha_i u_i, \quad 0 \leq \alpha_i \leq 1 \right\}$$

**Theorem 3.3** (Minkowski’s theorem for general lattices). *Suppose  $\Lambda$  to be a lattice and  $K$  to be a bounded symmetric convex subset in  $\mathbb{R}^d$ . If  $\text{vol}(K) > 2^d \det \Lambda$ , then  $K$  contains at least a point of  $\Lambda$  different from 0.*

We must perhaps insist that Minkowski’s theorem as well as Motzkin’s theorem on simultaneous approximation (which in turn depends on two theorems of Kronecker. The first one [20, p.159]) is at the heart of the diophantine approximation and then at the heart of the approximation by polynomials with integer coefficients. The second one [17] state that if an algebraic integer  $\alpha$  and all of its conjugates are in the closed unit disk  $\mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$ , then it is either  $\alpha = 0$  or it is root of unity. This is apparent at [7, Chap III], Ferguson [13, Theorem 1.1] and Chinburg [8]. For the sake of completeness and in order to see how the different idea articulate we give an idea of the proof of the first part of Fekete-Szegö theorem. For  $a \in \mathbb{R}^{n+1}$  we define the polynomial  $f_a(z) = a_0 + \dots + a_n z^n$ . Let  $K$  be a compact such that  $\mathcal{C}(K) < 1$ , then [8] then

$$f_n(E) = \{a = (a_0, a_1, \dots, a_n) \in \mathbb{R}^{n+1} \setminus \{0\}, \quad |f_a(E)| < 1\}$$

is an open symmetric convex set, with large volume for a large  $n$ . By Minkowski's theorem,  $f_n(E)$  contains some  $a = (a_0, a_1, \dots, a_n) \in \mathbb{Z}^{n+1} \setminus \{0\}$ . We fix a such  $a$  and we consider the open set

$$U = \{z \in \mathbb{C}, |f_a(z)| < 1\}.$$

If  $\alpha$  and its conjugates are contained in  $U$ , then  $f_a(\alpha)$  and its conjugates are contained in the unit disk  $\mathbb{D}$ . By the second Kronecker theorem  $\alpha$  is one of the many roots of the polynomial  $f_a(z)$ .

**Remark 3.1.** *The idea of using Minkowski's convex body theorem in this context goes back to Hilbert [15]. We can rephrase, in a classical way [7], what we said on the proof of the first part of Fekete-Szegö theorem: Any convex body of volume at least  $2^n$  contains at least  $2^n + 1$  integral points. In particular the system of linear inequalities*

$$\left| \sum_{k=1}^n a_{k,m} x^k \right| \leq b_m, \quad 1 \leq m \leq n$$

with

$$\det (a_{k,m})_{1 \leq k, m \leq n} \neq 0, \quad \prod_{m=1}^n b_m \geq \left| \det (a_{k,m})_{1 \leq k, m \leq n} \right|$$

has a nonzero integral solution.

Now let  $K$  be a compact of  $\mathbb{C}$  of capacity  $\mathcal{C}(K) \geq 1$ . Let  $U$  a neighborhood of  $K$  then  $U^*$  is a neighborhood of  $K^*$  (with the convention that the empty set is the neighborhood of the empty set). If  $\mathcal{C}(K^*) < 1$  then by continuity of the capacity, there exists  $V$  a neighborhood of  $K$  such that  $\mathcal{C}(V^*) < 1$ . From what precedes, there exists only a finite number of algebraic integers with all its conjugate in  $V$ . Then to show that there exists an infinity of algebraic integers totally in  $U$ , we can assume  $\mathcal{C}(K^*) \geq 1$  and finally the hypothesis of  $K$  symmetric with respect to the real axis is natural.

$K$  being a metric compact set, many assumptions can be made on  $\partial K$ .

**Definition 3.1.** *We say that  $K$  has a continuous boundary when  $K$  is a non empty union of connected components non reduced to a point.*

Recovering  $K$  with a finite number of small enough closed balls, we can assume that the boundary  $\partial K$  is continuous. From [4, Proposition p. 18] we can suppose  $K$  with regular boundary of class  $C^\infty$ . In fact let  $U$  be an open set containing  $K$ , there exists a  $C^\infty$  function  $\varphi$  in  $\mathbb{R}^2$  such that

- (1)  $\varphi = 1$  on  $K$ ,
- (2)  $\text{supp}(\varphi) \subset U$ .

Then  $K \subset \text{supp}(\varphi) \subset U$  with a compact support and with regular  $C^\infty$  boundary. Let us recall that a Jordan analytic curve is a closed curve  $\Gamma$  in  $\mathbb{C}$  which possesses a neighborhood  $V$  and a conformal map  $\xi$  from  $V$  on  $\{\alpha < |z| < \beta\}$ , such that the image of  $\Gamma$  by  $\xi$  is the circle  $\{|z| = r\}$ ,  $\alpha < r < \beta$ . We can also suppose that  $\partial K$  is a set of analytic Jordan curves. To see this a first method is to use [12, Theorem G]. Let  $K(\rho)$  be the  $\rho$ -neighborhood of  $K$  ( $\rho > 0$ ), there exists  $\rho$  sufficiently small so that  $K(\rho) \subset U$ . From the preceding theorem, there exists a domain defined by a lemniscate containing  $K$  and contained in  $K(\rho)$ : for  $n \geq n_1(\rho)$ ,  $\{z, |w_n(z)| \leq \nu_n\}$  where

$$w_n(z) = \prod_{k=1}^n (z - \zeta_k^{(n)})(z - \overline{\zeta_k^{(n)}}), \quad \nu_n = \max_K |w_n(z)|.$$

In conclusion this compact set is invariant under complex conjugation and its boundary consists in analytic Jordan curves.

A second method can be found in [1, p.144]. As an open set of  $\mathbb{C}$ ,  $U$  is a natural Riemann open surface, there exists a sequence of regular subregions  $(U_n)$ , such that  $\overline{U_n} \subset U_{n+1}$  and  $U = \bigcup_{i=1}^{\infty} U_n$ .

We recall the following definition (see for example [4, p.22]),

**Definition 3.2.** Let  $\Omega$  be an open subset of  $\mathbb{R}^2$ . We say that  $\Omega$  has a regular boundary of class  $C^k$  ( $k \geq 1$ ) if for every  $p \in \partial\Omega$  there is a neighborhood  $U_p$  of  $p$  and a diffeomorphism  $\varphi_p$  of class  $C^k$  from  $U_p$  onto a neighborhood  $V_p$  of 0 in  $\mathbb{R}^2$  such that  $\varphi_p(p) = 0$ ,

$$\varphi_p(U_p \cap \overline{\Omega}) = V_p \cap \{(x, y) \in \mathbb{R}^2 : x \leq 0\}$$

and the Jacobian determinant  $J(\varphi_p)$  is  $> 0$  in  $U_p$ .

We have (see for example [4, Proposition p. 27]) that for  $\Omega$  a relatively compact, open subset of  $\mathbb{C}$  with piecewise regular boundary (of class  $C^k$ ,  $k \geq 1$ ), there is only a finite number of connected components of  $\partial\Omega$  and each of them is a Jordan curve (piecewise  $C^k$ ).

In conclusion we can always suppose that  $K$  is a compact with  $C^\infty$  boundary, then the number of connected components of  $\partial K$  is finite.

**3.3. A generalization of Serre's method, the case of a compact of  $\mathbb{R}$ .** We use [3, Theorem 13] which characterizes such domains. In these domains are those symmetric with respect to the real axis and whose intersection with the open upper half plane is simply connected. We can also find domains such that  $\Gamma = (\Gamma_1, \Gamma_2)$  or  $(\Gamma_1, \Gamma_2, \Gamma_3)$  which are invariants under conjugation.

We have to notice that a compact  $e$  of  $\mathbb{R}$  of capacity  $\mathcal{C}(e) \geq 1$  is an example of such domains. The fact that for all neighborhood  $U$  of  $e$  in  $\mathbb{R}$ , there exists a sequence of monic polynomials of degree  $> 0$ , with coefficients in  $\mathbb{Z}$ , whose all roots are in  $U$ ,  $(P_n)$  such that  $\lim \mu_{P_n} = \mu_e$  will be a consequence of [28, Theorem 1.6.2].

For the convenience of the reader we recall some facts (see for example [29], Chapter 5).

**Proposition 3.1.** Let  $e \subset \mathbb{R}$  be compact. Let

$$\tilde{e}_n = \{x \in \mathbb{R}, \text{dist}(x, e) \leq \frac{1}{n}\}.$$

Then

$$(1) \quad e \subset \dots \subset \tilde{e}_{n+1} \subset \tilde{e}_n \subset \dots \subset \mathbb{R} \quad \text{and} \quad \bigcap_n \tilde{e}_n = e,$$

(2) Each  $\tilde{e}_n$  is a finite union of disjoint closed intervals.

*Proof.* The point (1) is clear. To show (2) we use the fact that every open set in  $\mathbb{R}$  is a countable union of disjoint open intervals. Hence  $\mathbb{R} \setminus e$  is a disjoint union of maximal open intervals. Since  $e$  is compact, two of these intervals are unbounded and the others  $\{J_k\}_{k \in I}$ , where  $I$  is a countable set, are contained in the convex hull of  $e$ . Thus,  $\sum_{k \in I} |J_k| < \infty$ , so for each  $n > 0$ ,  $\#\{k, |J_k| > 2/n\}$  is finite. Thus, all but finitely many  $J_k$  lie in a given  $\tilde{e}_n$ , showing  $\mathbb{R} \setminus \tilde{e}_n$  is finite. Thus, all but finitely many  $J_k$  lie in a given  $\tilde{e}_n$ , showing  $\mathbb{R} \setminus \tilde{e}_n$  has finitely many open intervals. It is easy to see that each of the finite disjoint closed intervals in  $\tilde{e}_n$  must have positive measure. □

Furthermore we have [27, Theorem 3.9] and [29, Theorem 5.6.1, p. 306]

**Theorem 3.4.** Let  $e = \bigcup_{j=1}^{l+1} e_j$  be an  $l$ -gap set with  $e_j = [\alpha_j, \beta_j]$ ,  $\alpha_j < \beta_j < \beta_{j+1}$ . Then for all  $m$

large, there exist  $l$ -gap sets  $e^{(m)} = \bigcup_{j=1}^{l+1} e_j^{(m)}$  with

$$(1) \quad e_j \subset e_j^{(m)},$$

(2) Each  $e_j^{(m)}$  has harmonic measure in  $e^{(m)}$  equal to  $k_j^{(m)}/m$  with  $k_j^{(m)} \in \{1, 2, \dots\}$ ,

(3) For some positive constants  $C_1, C_2$ ,

$$(1) \quad |e_j^{(m)} \setminus e_j| \leq C_1 m^{-1}$$

$$(2) \quad \mathcal{C}(e) \leq \mathcal{C}(e^{(m)}) \leq \mathcal{C}(e) + C_2 m^{-1}$$

Then we have

**Theorem 3.5.** *Let  $e \subset \mathbb{R}$  be compact. Then there exist  $e_n$  so that  $e \subset \dots \subset e_{n+1} \subset e_n \subset \dots \subset \mathbb{R}$  and  $\bigcap_n \tilde{e}_n = e$  holds, and*

$$e_n \subset e_{n-1}^{int}$$

and each  $e_n$  is the spectrum of some two-sided periodic Jacobi matrix. Moreover

- (1)  $\rho_{e_n} \rightarrow \rho_e$ ,
- (2)  $\mathcal{C}(e_n) \rightarrow \mathcal{C}(e)$ .

Now, we know that  $e_n$  is a finite union of calibrated intervals and then by [27], [28], there exists a sequence of monic polynomials with coefficients in  $\mathbb{Z}$ ,  $P_{k,e_n}$ , whose all roots are in  $e_n$ , such that  $\lim_k \mu_{P_{k,e_n}} = \mu_e$ . By the diagonal process  $\mu_e = \lim_j \mu_{P_{k_j,e_{n_j}}}$ .

**Remark 3.2.** *In particular we can release the condition of continuous boundary in the theorem 9.3 (Bilu, Rumely) of [2].*

#### 4. SOME APPROXIMATIONS

We have the followings theorem [5], [25].

**Theorem 4.1** (Rumely). *Suppose a compact set  $E \subset \mathbb{C}$  with continue boundary, has capacity  $\mathcal{C}(E) = 1$  and is stable under complex conjugation. Let  $(\alpha_n)_{n \geq 1}$  a sequence of algebraic integers ( $\alpha_n \neq \alpha_m$  if  $n \neq m$ ): for all open  $U$  containing  $K$ , there exists  $n_0$  such that for all  $n \geq n_0$ ,  $\alpha_n$  with all its conjugate,  $O(\alpha_n)$ , are in  $U$ . Let  $\Delta_n$  the measure*

$$\Delta_n = \frac{1}{d_{\alpha_n}} \sum_{\beta \in O(\alpha_n)} \delta_\beta,$$

then the measures  $\Delta_n$  converge weakly to the equilibrium measure of  $E$ ,  $\mu_E$ .

**Theorem 4.2** (Rumely, Bilu). *Suppose a compact set  $E \subset \mathbb{C}$  has capacity  $\mathcal{C}(E) = 1$  and is stable under complex conjugation. If  $\{\alpha_n\} \subset \bar{\mathbb{Q}}$  is a sequence for which  $\deg(\alpha_n) \rightarrow \infty$  and  $h_E(\alpha_n) \rightarrow 0$ , then the measures  $\Delta_n$  converge weakly to the equilibrium measure of  $E$ .*

The following result is due to Pritsker [21, Theorem 2.3]

**Theorem 4.3.** *Given any positive Borel measure  $\mu$ ,  $0 \leq \mu(\mathbb{C}) \leq 1$ , that is symmetric about real line, there is a sequence of complete sets of conjugate algebraic integers such that their counting measures  $\tau_n$  converge weakly to  $\mu$ .*

**There exist many sequences of polynomials with their zeros in  $E$ , that verify  $\mu_n \rightarrow \mu_E$ .**

Let  $E \subset \mathbb{C}$  be compact and infinite. An  $n$  point Fekete set is a set  $\{z_j, j = 1, \dots, n\} \subset E$  that maximizes

$$q_n(z_1, \dots, z_n) = \prod_{i \neq j} |z_i - z_j|.$$

From the maximum principle for analytic functions, it follows that for all  $n$ , the Fekete sets lie on the outer boundary of  $E$ .

The normalized density of Fekete sets converges to  $d\mu_E$ , the equilibrium measure for  $E$ .

**Remark 4.1.** *Let  $\{P_n\}$  be any sequence of monic polynomials having all their zeros in  $E$  and such that the normalized zero counting measures for  $P_n$  converge weakly to  $\mu_E$ . If  $\partial_\infty E$  (the outer boundary of  $E$ , that is the boundary of the unbounded component of  $\mathbb{C} \setminus E$ ) is regular (e.g., if it is connected), then*

- (1)  $P_n$  are asymptotically optimal for the Chebyshev problem:

$$\lim_{n \rightarrow \infty} \|F_n\|^{1/n} = \mathcal{C}(E).$$

If  $\mathcal{C}(E) > 0$  (so that  $\mu_E$  is defined), then we also have:

- (2) Uniformly on compact subsets of the unbounded component of  $\mathbb{C} \setminus E$ ,

$$\lim_{n \rightarrow \infty} |F_n(z)|^{1/n} = \exp\{-\Phi_{\mu_E}(z)\}.$$

Now we recall two results of approximation. In [20], Theorem 3.2, p. 158 (see also [9] for its effective version), it is proving that for every  $n$  given numbers  $z_1, \dots, z_n$ , and every  $\epsilon > 0$ , there exists an irreducible equation with complex integral coefficients  $\alpha_1, \dots, \alpha_{n+1}$  and with roots  $\zeta_{k,\epsilon}$  such that  $|\zeta_{k,\epsilon} - z_k| < \epsilon$  for  $k = 1, \dots, n$ . The same is true for real integral coefficients provided that the numbers  $z_1, \dots, z_n$  are symmetric to the real axis.

In [13] (Theorem A.1., p. 147) we have that if  $\alpha_1, \dots, \alpha_n$  are a complete set of conjugate algebraic integers over  $\mathbb{Q}[i]$ ,  $\epsilon$  any positive number, and  $z_2, \dots, z_n$  any complex numbers. Then there is a polynomial  $q \in \mathbb{Z}[i][z]$  such that

$$|q(\alpha_j) - z_j| < \epsilon, \quad 2 \leq j \leq n.$$

We have then the following result ([21] p. 16). Let a sequence of polynomials  $(P_n)$  such that the corresponding normalized zero counting measure for  $P_n$ ,  $\mu_n$  converge to  $\mu_E$ . We can suppose  $\deg(P_n) = n$  and denote its zeros by  $z_1, \dots, z_n$ . Therefore, we can approximate this measure by a sequence of the counting measures  $\tau_{n+1}$  for the complete set of conjugate algebraic integers  $\zeta_k = \zeta_{k,1/n}$ ,  $k = 1, \dots, n+1$  by using the theorem of Motzkin. For any  $n \in \mathbb{N}$ , we approximate each point  $z_k$  as close as we wish by one of the conjugate algebraic integers  $\zeta_k$ ,  $1 \leq k \leq n$ , obtained from Motzkin's theorem, while let the remaining  $(n+1)$ th conjugate algebraic integer  $\zeta_{n+1} \rightarrow \infty$  as  $n \rightarrow \infty$  (see [9] p.160-161 for details). It follows that the resulting measures

$$\tau_{n+1} = \frac{1}{n+1} \sum_{k=1}^{n+1} \delta_{\zeta_k}$$

converge to  $\mu_E$  as  $n \rightarrow \infty$ .

In fact  $\mu_n = \frac{1}{n} \sum_{k=1}^n \delta_{z_k} \rightarrow \mu_E$ , then  $\nu_n = \frac{1}{n} \sum_{k=1}^n \delta_{\zeta_k} \rightarrow \mu_E$ . For  $f$  continuous on  $E$  and  $n$  sufficiently large we obtain

$$\int f \tau_{n+1} = \int f \frac{n}{n+1} \nu_n = \frac{n}{n+1} \int f \nu_n \rightarrow \mu_E.$$

But as already seen, the polynomial  $Q_n$  with roots the complete set of conjugate algebraic integers  $\zeta_k$ , has  $\zeta_{n+1}$  not in  $E$  (for  $n$  sufficiently large). So the sequence  $(Q_n)$  doesn't answer Serre's suggestion (Question 3.1).

## 5. DEMONSTRATION OF THE THEOREM FOLLOWING SERRE.

Let  $\Omega$  be a plane domain. The boundary of  $\Omega$  is denoted by  $\partial\Omega = \Gamma = \Gamma_1 + \Gamma_2 + \dots + \Gamma_r$ . In the following the regularity of  $K$  is understood. Note first that which is important in  $K$  is the boundary of the outer component of  $\mathbb{C} \setminus \partial K$ . Then  $K$  can be a compact whose boundaries of the bounded connected components are what ever you want.

Denoting by  $\Omega$  the outer component of  $\mathbb{C} \setminus \partial K$ , we know that the equilibrium measure of  $K$  is supported on  $\partial\Omega$ . We denote by  $\partial\Omega = (\Gamma_1, \dots, \Gamma_r) = \Gamma$ .

**5.1. The double of a plane domain.** Let  $\Omega$  be a plane domain. We have seen that we may assume that each boundary component of the boundary of  $\Omega$ , denoted by  $\partial\Omega = \Gamma = \Gamma_1 + \Gamma_2 + \dots + \Gamma_r$ ,  $\Gamma_j$  is a smooth analytic curve. Alternatively, one may think of  $\Omega$  as a plane bordered Riemann surface. More precisely

**Definition 5.1.** For each  $r = 1, 2, \dots$  we shall denote by  $\mathcal{U}_r$  the class of plane domains whose boundary consists of  $r$  disjoint Jordan curves  $\Gamma_1, \Gamma_2, \dots, \Gamma_r$  which satisfy the following smoothness condition: with each  $\Gamma_j$  there is associated a function  $z_j(t)$  analytic and univalent in a neighborhood of  $\Gamma_j$  which maps this neighborhood onto the circular ring  $1 - \delta < |z| < 1 + \delta$  and the curve  $\Gamma_j$  onto the circle  $|z| = 1$ .

Much of the functions theory on  $\Omega$  is conveniently described in terms of the Schottky double  $\hat{\Omega}$  of  $\Omega$ . This is the compact Riemann surface of genus  $g = r - 1$  obtained, when  $\Omega$  has analytic boundary, as is henceforth assumed, by welding  $\Omega$  along  $\Gamma$ , with an identical copy  $\tilde{\Omega}$ . Thus, as a point set  $\hat{\Omega} = \Omega \cup \Gamma \cup \tilde{\Omega}$ . The "backside"  $\tilde{\Omega}$  is provided with the opposite conformal structure.

This means that if  $\tilde{z} \in \tilde{\Omega}$  denotes the opposite to  $z \in \Omega$  then the map  $\tilde{z} \rightarrow \bar{z}$  is a holomorphic coordinate on  $\tilde{\Omega}$ . The construction of this double generalizes to any bordered Riemann surface and the result is always a symmetric Riemann surface, i.e., a Riemann surface provided with an antiholomorphic involution. In our construction this is the map  $U : \hat{\Omega} \rightarrow \hat{\Omega}$  which exchanges  $z$  and  $\tilde{z}$  and which keeps  $\Gamma$  pointwise fixed.

The double of a plane domain  $\Omega$  has a holomorphic atlas consisting of only two charts: the corresponding coordinate functions are the identity map  $\varphi_1 : z \rightarrow z$  on  $\Omega$  and the map  $\varphi_2 : \tilde{z} \rightarrow \bar{z}$  on  $\tilde{\Omega}$ . When  $\Gamma$  is analytic, both these maps extend analytically across  $\Gamma$  in  $\hat{\Omega}$ , hence their domains of definitions overlap and the union covers all  $\hat{\Omega}$ . Let

$$S = \varphi_2 \circ \varphi_1^{-1}$$

be the coordinate transition function. It is analytic and defined in a neighbourhood of  $\Gamma$  in  $\mathbb{C}$ , and on  $\Gamma$  it satisfies

$$S(z) = \bar{z} \quad (z \in \Gamma).$$

A function  $f$  on  $\hat{\Omega}$  is most conveniently described as a pair of functions  $f_1, f_2$  on  $\Omega$ , continuously extendable to  $\Gamma$ , such that

$$f_1(z) = \overline{f_2(z)} \quad (z \in \Gamma).$$

The formal relations to  $f$  in terms of the coordinate functions  $\varphi_1$  and  $\varphi_2$  above are

$$\begin{cases} f_1 &= f \circ \varphi_1^{-1} \\ f_2 &= c \circ f \circ \varphi_2^{-1} \circ c, \end{cases}$$

where  $c$  denotes complex conjugation. It follows, for example, that  $f$  is meromorphic if and only if  $f_1$  and  $f_2$  are meromorphic.

It is useful to note that if two domains in the class  $\mathcal{U}_r$  are conformally equivalent, then so are their doubles conformally equivalent. Indeed, the Schwarz reflection principle permits an extension of a conformal homeomorphism between plane domains of the class  $\mathcal{U}_r$  to a conformal homeomorphism of their doubles.

We will deal with domain  $\Omega$ , in the class  $\mathcal{U}_r$  whose double is a hyperelliptic Riemann surface. This is equivalent to the fact that  $\Omega$  can be mapped one-to-one conformally onto the exterior of a system of slits taken from the real axis ([3]). At first, notably to put on the notations, we recall some results on the double of the exterior of a system of slits taken from the real axis. Let  $E = \bigcup_{j=1}^r E_j$ ,  $E_j = [e_{2j-1}, e_{2j}] \subset \mathbb{R}$ ,  $e_1 < e_2 < e_3 < \dots < e_{2r-1} < e_{2r}$ . The double of  $\mathbb{C} \setminus E$ ,  $\widehat{\mathbb{C} \setminus E}$ , is the hyperelliptic curve of equation

$$w^2 = \prod_{i=1}^{2r} (z - e_i) = q(z),$$

where we add two points at infinity  $\infty_+$  and  $\infty_-$ , characterised by the fact that  $\frac{w}{z^{r-1}} = 1$  at  $\infty_+$  and  $-1$  at  $\infty_-$ .

It's a two-sheeted branched covering space of the sphere  $P_1(\mathbb{C})$ , branched at the  $2r$  points  $e_j$ . It's a topological covering space of  $\mathbb{C} \setminus E$ .

The covering map  $\pi_E : \widehat{\mathbb{C} \setminus E} \rightarrow P_1(\mathbb{C})$ ,  $\pi : (z, w) \rightarrow z$  is a meromorphic function of order two on  $\widehat{\mathbb{C} \setminus E}$  whose only multiple points, each of multiplicity two, are located at the points  $(e_j, q(e_j))$ ,  $j = 1, \dots, 2r$ .

Let us denote by  $E_j^*$  the inverse image of  $E_j$  in  $\widehat{\mathbb{C} \setminus E}$  and  $E^* = \bigcup_{j=1}^r E_j^*$ .

**5.2. The notion of ‘‘calibrated’’.** In this section we refer to [27]. We will develop the case where  $\widehat{\mathbb{C} \setminus E}$  corresponds to a two-sided  $r$ -periodic real symmetric Jacobi matrix.

First, we recall the following terminology. The meromorphic 1-forms on a compact Riemann surface are called abelian differentials. The abelian differentials which are holomorphic will be

called of the first kind; while the meromorphic abelian differentials with zero residues will be called of the second kind. Finally, a general abelian differential (which may have residues) will be called of the third kind.

We denote by  $w_{\infty_+, \infty_-}$  the differential of the third kind having a simple pole at  $\infty_+$  and  $\infty_-$ , with residue  $-1$  and  $+1$  respectively, normalized by

$$\int_{\alpha_j} w_{\infty_+, \infty_-} = 0, \quad j = 1, \dots, 2r.$$

We recall now the link between  $w_{\infty_+, \infty_-}$  and the Green function of  $\mathbb{C} \setminus E$  with pole at  $\infty_+$ ,  $g(z)$ :

- (1)  $g(z)$  is harmonic in  $\mathbb{C} \setminus E$ ,
- (2)  $g(z) - \ln |z|$  is harmonic in a neighborhood of  $\infty^+$
- (3)  $\lim_{z \rightarrow \zeta} g(z) = 0$  for all  $z \in E$ .

We shall denote by  $G(z)$  the multiple-valued function obtained by adding to  $g(z)$  its conjugate.

$$g(z) = \operatorname{Re} G(z) = \frac{G(z) + \overline{G(z)}}{2}.$$

The derivative of this multi-valued function is clearly single-valued. Hence,  $G'(z)$  is a holomorphic function in  $\mathbb{C} \setminus E$ .

The functions  $g$  and  $G$  extend to all of  $\widehat{\mathbb{C} \setminus E}$  by reflecting across  $E$ , since they vanish on  $E$ :  $G(T(z)) = -G(z)$  for  $z \in \mathbb{C} \setminus E$  and  $T$  the canonical anti-conformal involution of  $\widehat{\mathbb{C} \setminus E}$ . In particular  $dG = w_{\infty_+, \infty_-}$ . It is an abelian differential with poles at  $\infty_+$  and  $\infty_-$  and zeros at the  $\lambda_j$  (the zeros of  $G'(z)$ ) and  $\lambda_j^*$  (the points of the copy of  $\mathbb{C} \setminus E$  corresponding to the  $\lambda_j \in \mathbb{C} \setminus E$ ).

$$w_{\infty_+, \infty_-} = \frac{\prod_{j=1}^{r-1} (\lambda - \lambda_j)}{\sqrt{q(\lambda)}} d\lambda = i dp,$$

where  $dp$  is called the quasimomentum in [16]

$$\frac{1}{2\pi} \int_{E_{2j}}^{E_{2j+1}} dp = \frac{m_k}{N}, \quad m_k \in \mathbb{N},$$

**Definition 5.2.** A compact set  $E = [E_1, E_2] \cup [E_3, E_4] \cup \dots \cup [E_{2n-1}, E_{2n}]$  of the real line is calibrated if the complex Green's function  $G(z)$  of  $\widehat{\mathbb{C} \setminus E}$  with pole at infinity satisfies the conditions

$$\int_{E_{2k}}^{E_{2k+1}} \sqrt{Q(t)} dt = 0, \quad k = 1, 2, \dots, n-1$$

and

$$\int_{E_{2k-1}}^{E_{2k}} \sqrt{Q(t)} dt = \pm \frac{r_k \pi i}{N}, \quad k = 1, 2, \dots, n,$$

where

$$Q(t) = \frac{(t - c_1)^2 (t - c_2)^2 \dots (t - c_{n-1})^2}{(t - E_1)(t - E_2) \dots (t - E_{2n-1})(t - E_{2n})} = h(t)^2$$

and  $r_1, \dots, r_n, N \in \mathbb{N}$ ,  $r_1 + \dots + r_n = N$ .

We observe that if a compact  $E$  is calibrated with respect to  $r_1, r_2, \dots, r_n, N$  it is also calibrated with respect to  $kr_1, kr_2, \dots, kr_n, kN$ ,  $k \in \mathbb{N}^*$ .

**Proposition 5.1.** There exists signs  $\epsilon_j \in \{-1, +1\}$  such that

$$\sum_{j=1}^r \epsilon_j \int_{e_{2j-1}}^{e_{2j}} w_{\infty_+, \infty_-} = i\pi.$$

**5.3. Approximation of  $\Gamma$  by  $\Gamma_v$  calibrated.** Let  $K$  be a compact with  $C^\infty$  boundary and  $\Omega$  be the unbounded component of  $\mathbb{C} \setminus K$  (i.e. the component of the complement of  $K$  which contains infinity). The boundary  $\partial\Omega$  consists of  $r$  mutually exterior curves. We know that  $\mu_K$  has support in  $\partial\Omega$ .  $K$  and  $\partial\Omega$  have same capacity, same equilibrium measure and same potential function (see for example [30], p.61).

Let  $f$  be a conformal application from  $\Omega$  to  $\mathbb{C} \setminus E$  where  $E$  is the union of  $r$  segments taken from the real axis  $E = \bigcup_{j=1}^r E_j$ ,  $E_j = [e_{2j-1}, e_{2j}] \subset \mathbb{R}$ ,  $e_1 < e_2 < e_3 < \dots < e_{2r-1} < e_{2r}$ . The map  $f$  is such that

$$f(\infty) = \infty, \quad 1 \leq j \leq r \quad f(\Gamma_j) = E_j$$

We know that  $\mathbb{C} \setminus E = \text{Ker}\{\mathbb{C} \setminus E_n\}$  and  $E_n$  calibrated. We chose  $E_n$  such that for  $1 \leq j \leq r$  we have  $E_j \subset E_{j,n}$ .

We begin by open each interval of  $E$  (see for example the "open up" process in Widom) on the model

$$\psi : [-1, 1] \longrightarrow \{|s| = 1\}, \quad \psi(z) = z + (z^2 - 1)^{1/2}.$$

The complement of  $E$ ,  $\mathbb{C} \setminus E$ , is conformal to a domain  $\Delta$ , obtained by opening all the intervals  $E_j$ ,  $\partial\Delta = (C_1, \dots, C_r)$ . The boundary of  $\Delta$  is  $C^\infty$  and symmetric with respect to the real axis. We denote by  $\Psi$  the corresponding conformal map.

We construct in the same way an application from  $E_{j,n}$  ( $E_j \subset E_{j,n}$ )  $j = 1, \dots, r$  on  $C_{j,n}$  near  $C_j$ , on the model

$$\xi : z \in [\alpha, \beta] \longrightarrow z + [(z - \alpha)(z - \beta)]^{1/2}.$$

Consequently there exists a conformal application from  $\mathbb{C} \setminus E_n$  to  $\Delta_n$  where  $\partial\Delta_n = (C_{1,n}, \dots, C_{r,n})$ . The boundary of  $\Delta_n$  is  $C^\infty$  and symmetric with respect to the real axis.

Then let denote by  $\Phi = \Psi \circ f : \Omega \rightarrow \Delta$ .

In the following section we will denote again  $\Gamma_v$  by  $\Gamma$ .

**5.4. How to obtain a polynomial with rational coefficients.** The aim of this section is to obtain the following theorem

**Theorem 5.1.** *In the case  $\Gamma$  is calibrated and symmetric with respect to the real axis, the Chebyshev polynomial of  $\Gamma \cap \mathcal{H}$  of degree  $r$  achieve its maximum at each extremity of  $\Gamma \cap \mathcal{H}$  and has precisely one zero in each of the  $r$  components of  $\Gamma \cap \mathcal{H}$ .*

*Moreover we can achieve that in each neighborhood of  $\Gamma \cap \mathcal{H}$ , we can find such a polynomial with coefficients in  $\mathbb{Q}[i]$ .*

From [19],  $\widehat{\mathbb{C} \setminus E}$  corresponds to a two-sided  $r$ -periodic real symmetric Jacobi matrix,  $A$ , and an algebraic curve of equation  $\{h + h^{-1} - \Delta(z) = 0\}$  with  $\Delta$  a real polynomial of degree  $r$ , known as the discriminant. It is usually defined as the trace of the one-period transfer matrix. It is also the unique polynomial (with positive leading coefficient) such that

$$\sigma(A) = \{x, \Delta(x) \in [-2, 2]\},$$

where  $\sigma(A)$  is the spectrum of the two-sided operator (acting on  $l^2(\mathbb{Z})$ , it is purely absolutely continuous and of multiplicity two). Let us make more precise the polynomial  $\Delta$ . Let  $A$  be such a Jacobi matrix, it defines a difference operator of period  $r$  with  $a, b \in l^\infty(\mathbb{Z})$ ,  $b_n \neq 0$ ,

$$(3) \quad a_{n+r} = a_n, b_{n+r} = b_n, \quad n \in \mathbb{Z}.$$

To simplify the notations, we write the operator  $A$  as

$$A = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & a_1 & b_1 & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & b_1 & a_2 & b_2 & \cdots & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot b_{r-1} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & a_r & b_r & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

We denote by  $\mathcal{B} = \prod_{n=1}^r b_n$ . We consider the solutions of the eigenvalue problem

$$(4) \quad Au = zu.$$

It defines a second-order difference equation for  $n = 1, 2, 3, \dots$ ,

$$u_{n+1} = b_n^{-1}((z - a_n)u_n - b_{n-1}u_{n-1});$$

we rewrite

$$\begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix} = U_n(z) \begin{pmatrix} u_{n-1} \\ u_n \end{pmatrix},$$

where

$$U_n(z) = \frac{1}{b_n} \begin{pmatrix} 0 & b_n \\ -b_{n-1} & z - a_n \end{pmatrix}.$$

The monodromy matrix  $M_r(z)$ , called also the one-period transfer matrix, is defined by the fundamental matrix

$$M_r(z) = U_r(z) \dots U_1(z).$$

Finally, the Floquet discriminant  $\Delta(z)$  is given by a trace

$$(5) \quad \Delta(z) = \text{Tr}(M_r(z)).$$

Similarly, the Floquet multipliers  $m_{\pm}(z)$  are the eigenvalues of  $M_r(z)$ . They are given by

$$m_{\pm}(z) = \frac{\Delta(z)}{2} \pm \frac{[\Delta(z)^2 - 4]^{1/2}}{2}$$

and are independent of  $n$ . They verify

$$m_+(z)m_-(z) = 1, \quad m_+(z) + m_-(z) = \Delta(z).$$

$\{e_j, j = 1, \dots, 2r\}$  are the points where  $\Delta(\lambda) = \pm 2$ . Indeed,  $\Delta(\lambda) = 2$  at  $e_{2r}, e_{2r-3}, e_{2r-4}, e_{2r-7}, \dots$  and  $-2$  at  $e_{2r-1}, e_{2r-2}, e_{2r-5}, e_{2r-6}, \dots$ . Clearly,  $\Delta$  is determined by the  $p$  points where it is  $+2$  and one of the points where it is  $-2$ , showing the rigidity in possibilities of  $E$ . More precisely the Floquet discriminant  $\Delta(z)$  given by (5) of a real  $r$ -periodic Jacobi matrix is a  $r$ -hyperelliptic real polynomial. More precisely, we have

$$(6) \quad \begin{aligned} \Delta(z)^2 - 4 &= \frac{1}{\mathcal{B}^2} \prod_{l=1}^{2r} (z - e_l) \\ \Delta(z) \mp 2 &= \frac{1}{\mathcal{B}} \prod_{j=1}^r (z - e_j^{\pm}). \end{aligned}$$

According to [18], [6], the zeros  $\{e_j^+\}_{1 \leq j \leq r}$  and  $\{e_j^-\}_{1 \leq j \leq r}$  are respectively the eigenvalues of the periodic Jacobi matrix  $\tilde{A}^+$  and the anti-periodic Jacobi matrix  $\tilde{A}^-$  given by

$$\tilde{A}^{\pm} = \begin{pmatrix} -a_1 & b_1 & & & \pm b_r \\ b_1 & -a_2 & & & \\ & & \cdots & & \\ & & & -a_{r-1} & b_{r-1} \\ \pm b_r & & & b_{r-1} & -a_r \end{pmatrix}$$

If  $r$  is even, we have

$$e_1^+ < e_1^- \leq e_2^- < e_2^+ \leq e_3^+ \cdots \leq e_{r-1}^+ < e_{r-1}^- \leq e_r^- < e_r^+ .$$

If  $r$  is odd

$$e_1^- < e_1^+ \leq e_2^+ < e_2^- \leq e_3^- \cdots \leq e_{r-2}^- < e_{r-2}^+ \leq e_{r-1}^+ < e_{r-1}^- \leq e_r^- < e_r^+ .$$

The set  $\{e_l\}_{1 \leq l \leq 2r}$  coincides with the corresponding sequence  $(e_j^\pm)$ . We need also to recall the existence of a regular divisor  $\mathcal{D}$  on  $\widehat{\mathbb{C} \setminus E}$ . Then the divisor  $\mathcal{D} = \sum_{j=1}^{r-1} \lambda_j$  has the property that  $\mathcal{D} + \tilde{\mathcal{D}} - \infty_+ - \infty_-$  is the divisor of the differential  $dp$  on  $\widehat{\mathbb{C} \setminus E}$ , which is real positive on  $E$ . Then  $\mathcal{D}$  is regular ([19], p. 120). Therefore the curve is hyperelliptic of genus  $g = r - 1$  with two points  $\infty_+$  and  $\infty_-$  at infinity; besides  $(h) = -r\infty + r\infty_-$ . Moreover, the hyperelliptic involution coincides with  $(z, h) \rightarrow (z, h^{-1})$ .  $h(z) = e^{rG(z)}$ . In this case, the fixed points for the anti-holomorphic involution are given by  $\{p = (z, h), |h(p)| = 1\} = E^*$  (see [19], p.122), and  $\pi(E^*) = R^{-1}([-2, 2])$ .

The fixed points for the hyperelliptic involution are given by the  $2r$  points where  $h = \pm 1$ , equivalently  $\Delta = \pm 2$ , i.e. the branched points. A sufficient condition to be mapped one-to-one conformally onto the exterior of a system of slits taken from the real axis is that  $\Omega$  is symmetric with respect to the real axis and  $\Omega \cap \mathcal{H}$  is simply connected.

We will adopt this situation and denote by  $f$  a conformal application from  $\Omega$  to  $\mathbb{C} \setminus E$  where  $E$  is the union of  $r$  segments taken from the real axis  $E = \bigcup_{j=1}^r E_j$ ,  $E_j = [e_{2j-1}, e_{2j}] \subset \mathbb{R}$ ,  $e_1 < e_2 < e_3 < \dots < e_{2r-1} < e_{2r}$ . The map  $f$  is such that

$$f(\infty) = \infty, \quad 1 \leq j \leq r, \quad f(\Gamma_j) = E_j.$$

Hence the double of  $\Omega$ ,  $\hat{\Omega}$  is hyperelliptic ([3]). The covering map  $\pi_\Omega : \hat{\Omega} \rightarrow P_1(\mathbb{C})$  is a branched covering of order 2. Let us denote by  $\Gamma_j^*$  the mirror image of  $\Gamma_j$  in  $\hat{\Omega}$  and  $\Gamma^* = \bigcup_{j=1}^r \Gamma_j^*$ .

The conformal application from  $\Omega$  to  $\mathbb{C} \setminus E$   $f$ , can be extended to conformal homeomorphism of their double. Let us denote this conformal application by

$$\hat{f} : \hat{\Omega} \rightarrow \widehat{\mathbb{C} \setminus E}, \quad p \rightarrow \hat{f}(p).$$

The Weierstrass points of  $\hat{\Omega}$  and those of  $\widehat{\mathbb{C} \setminus E}$  are in one to one correspondence by  $\hat{f}$ . We introduce

$$\alpha_j^\pm = (e_j^\pm, h(e_j^\pm)), \quad A_j^\pm = \hat{f}^{-1}(\alpha_j^\pm)$$

and define  $H(p) = h(\hat{f}(p))$ . We will identify, with a small abuse of notations,  $Z$  with  $Z(p)$  and  $z$  with  $z(\hat{f}(p))$ . The maps  $Z$  and  $H$  have the property stated in [19]. In particular we have for the divisor

$$(H) = -r\infty + +r\infty - .$$

We will denote in the same manner  $\mathcal{D}$  and  $\hat{f}^{-1}(\mathcal{D})$ , the regular divisor corresponding to the preceding regular divisor  $\mathcal{D}$  of the zeros of  $\Delta'$ .

To simplify the notations, we write the operator  $C$  as corresponding to the hyperelliptic surface following [19]

$$C = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & A_1 & B'_1 & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & B_1 & A_2 & B'_2 & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & B_2 & A_3 & B'_3 & \cdots & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & B'_{r-1} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & B_{r-1} & A_r & B'_r & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

with  $A_{i+r} = A_i$ ,  $B_{i+r} = B_i$ ,  $B'_{i+r} = B'_i$ ,  $i \in \mathbb{Z}$ . Note that being  $r$ -periodic,  $C$  is bounded.

Following [19],  $\det(C_H - ZI) = (-1)^{r-1} \Phi_0 H + \Phi_1(Z) + (-1)^{r-1} \Phi_2 H^{-1}$ , where  $\Phi_0 = \prod_{i=1}^r B'_i$ ,  $\Phi_2 = \prod_{i=1}^r B_i$  and  $\Phi_1$  is a polynomial of degree  $r$ ,  $\Phi_1(Z) = (-1)^r Z^r + \dots$

$$\det(C_H - ZI) = \begin{vmatrix} A_1 - Z & B'_1 & \cdots & \cdots & \cdots & B_r H^{-1} \\ B_1 & A_2 - Z & B'_2 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & B_{r-2} & A_{r-1} - h & B'_{r-1} \\ B'_r H & \cdots & \cdots & \cdots & B_{r-1} & A_r - Z \end{vmatrix}$$

If

$$\Delta(i, j) = \begin{vmatrix} A_i - Z & B'_i & \cdots & \cdots & \cdots & \cdots \\ B_i & A_{i+1} - Z & B_{i+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & B_{j-2} & A_{j-1} - h & B'_{j-1} \\ \cdots & \cdots & \cdots & \cdots & B_{j-1} & A_j - h \end{vmatrix}$$

some computations yield to the following

$$\Phi_1(Z) = \Delta(2, r+1) - B_1 B'_1 \Delta(3, r).$$

Let  $\alpha$  such that  $\alpha^2 = \frac{\Phi_0}{\Phi_2}$ . Then

$$\det(C_H - ZI) = 0 \iff \alpha H + (\alpha H)^{-1} + \frac{\alpha}{\Phi_2} \Phi_1(Z) = 0.$$

We denote by  $P_1(Z) = -\frac{\alpha}{\Phi_2} \Phi_1(Z)$  and  $H_1 = \alpha H$ . The curve

$$\left\{ (Z_1, H_1); H_1 + (H_1)^{-1} + \frac{\alpha}{\Phi_2} \Phi_1(Z_1) = 0 \right\}$$

is invariant under the transformation  $(Z_1, H_1) \longrightarrow (Z_1, H_1^{-1})$ , which corresponds to the hyperelliptic involution. The fixed points of this hyperelliptic involution satisfy  $H_1 = H_1^{-1}$  and are located on  $\Gamma$  ([?], p. 20). At such a point  $p_0$ ,  $|H(p_0)| = 1$ . As  $H_1(p_0) = \pm 1 = \alpha H(p_0)$ , we conclude  $|\alpha| = 1$ .

Let  $f$  be an infinite column vector  $f = (\dots f_{-1}, f_0, f_1, \dots)$ . Let  $D$  acts on  $f$  as the shift  $Df_k = f_{k+1}$ .  $C$  is assumed to be periodic of period  $r$ , this amounts to the commutation relation  $CS = SC$ , where  $S = D^r$ .

Following [22],  $\alpha S + (\alpha S)^{-1} - P_1(C) =$  and by the analytic functional calculus

$$\sigma(C) = P_1^{-1}(\sigma(\alpha S + (\alpha S)^{-1})).$$

But  $\alpha S + (\alpha S)^{-1}$  is conjugate to  $S + S^{-1}$ ,  $\Delta \alpha S + (\alpha S)^{-1} \Delta^{-1} = S + S^{-1}$  with  $\Delta = \alpha I$ . Then

$$\sigma(\alpha S + (\alpha S)^{-1}) = \sigma(S + S^{-1}) = [-2, 2]$$

and

$$\sigma(C) = P_1^{-1}([-2, 2]).$$

$$H_1(Z_1) = P_1(Z_1) \pm \sqrt{P_1(Z_1)^2 - 4} = \frac{1}{P_1(Z_1) \pm \sqrt{P_1(Z_1)^2 - 4}}$$

and

$$P_1^{-1}([-2, 2]) = \{Z_1, |H_1(Z_1)| = 1\}.$$

Consequently if  $Z_1 \in \sigma(C)$ ,  $|H_1(Z_1)| = |\alpha H(Z_1)| = |\alpha| |H(Z_1)| = 1$ . We deduce that  $|H(Z_1)| = 1$  and then  $Z_1 \in \Gamma$ . In conclusion  $\sigma(C) \subset \Gamma$ .

$Z$  and  $Z_1$  are two meromorphic function of degree 2 then ([?]), then there exists complex numbers  $a, b, c, d$  such that  $ad - bc = 1$  and  $Z_1 = \frac{aZ + b}{cZ + d}$ .

Looking at infinity, we have  $f(\infty) = \infty$ ,  $h(\infty) = H(\infty)$ ;  $H_1 = \alpha H$  then  $H_1(\infty) = \infty$ . Then when  $Z = \infty$ ,  $Z_1 = \infty$  and  $Z_1$  is of the form  $Z_1 = aZ + b$ .

$$P_1^{-1}(\{2\}) = \{aA_j^+ + b\}, P_1^{-1}(\{-2\}) = \{aA_j^- + b\}..$$

By an argument of connectivity, we deduce that, for exemple  $\sigma(C) = P_1^{-1}([-2, 2]) = (a\Gamma + b) \cap (a\mathcal{H} + b)$ .

Consider then the polynomial  $Q(Z) = P_1(aZ + b)$ .  $P_1^{-1}([-2, 2]) = Q^{-1}([-2, 2])$  and  $\frac{1}{a^r}Q$  is the Chebychev polynomial of  $\Gamma \cap \mathcal{H}$ .

Approximating  $C$  by rational coefficients, we can construct  $C_q$  as close of  $C$  as we please and so, corresponding  $P_{1,q}$  as close of  $P_1$  as we please, with the coefficient of  $P_{1,q}$  in  $\mathbb{Q}[i]$ . As  $\sigma(C_q) = P_{1,q}^{-1}([-2, 2]) = \{Z_1 = aZ + b, P_{1,q}(Z_1) \in [-2, 2]\}$ , if  $a_\tau$  and  $b_\tau$ , in  $\mathbb{Q}[i]$ , are closed to  $a$  and  $b$  respectively, then  $\{Z_1 = a_\tau Z + b_\tau, P_{1,q}(Z_1) \in [-2, 2]\}$  are closed to  $\sigma(C_q)$ . Then the polynomial  $Q_\tau(Z) = P_{1,q}(a_\tau Z + b_\tau)$  has its coefficients in  $\mathbb{Q}[i]$ .

On the other hand  $P_{1,q}^{-1}([-2, 2]) = Q_\tau^{-1}([-2, 2])$  and for some convenient  $\beta \in \mathbb{Q}[i]$ , such that  $Q_r(z) = \frac{1}{\beta}z^R + \dots$ , the polynomial  $\beta Q_\tau$  is the Chebychev polynomial of  $\mathcal{K}_\tau = \mathcal{K}_{1,\tau} + \mathcal{K}_{2,\tau} + \dots + \mathcal{K}_{r,\tau}$ , with  $\mathcal{K}_{j,\tau}$  closed to the corresponding component in  $\Gamma \cap \mathcal{H}$ .

Let us denote by  $aA_{j,q}^\pm + b$  the points  $P_{1,q}^{-1}\{\pm 2\}$ , they are closed to the  $aA_j^\pm + b$ . Now  $\beta Q_\tau$  is the Chebychev polynomial of  $\mathcal{K}_\tau = \mathcal{K}_{1,\tau} + \mathcal{K}_{2,\tau} + \dots + \mathcal{K}_{r,\tau}$  and it achieves its extrema at the extremity of  $\mathcal{K}_{j,\tau}$ , denoted by  $a_\tau A_{j,q,\tau}^\pm + b_\tau$ . In sumery it oscillates on  $\mathcal{K}_{j,\tau}$  being extremum at the  $a_\tau A_{j,q,\tau}^\pm + b_\tau$ .

**5.5. How to obtain integer coefficients.** We start with the notations of the preceding section: The polynomial  $\beta Q_\tau$  is the Chebychev polynomial of  $\mathcal{K}_\tau = \mathcal{K}_{1,\tau} + \mathcal{K}_{2,\tau} + \dots + \mathcal{K}_{r,\tau}$  with coefficients in  $\mathbb{Q}[i]$  and it achieves its extrema at the end points of  $\mathcal{K}_{j,\tau}$ , denoted by  $a_\tau A_{j,q,\tau}^\pm + b_\tau$ . Moreover  $Q_r(\mathcal{K}_\tau) = [-2, 2]$  and since it is of degree  $r$ ,  $Q_r$  has an unique zero in each  $\mathcal{K}_{l,\tau}$ , denoted by  $\xi_l$ ,  $l = 1, \dots, r$ . Let  $T_n$  the Chebychev polynomial of  $[-2, 2]$ , then the polynomial  $P_n$  defined by  $P_n = \beta^{-n} T_n(Q_r)$  is the Chebychev polynomial of degree  $nr$  of  $\mathcal{K}_\tau$ . In fact it verifies clearly

$$\mathcal{K}_\tau = \beta^{-n} T_n(Q_r)^{-1}[-2, 2], \quad Q_r(\mathcal{K}_\tau) = [-2, 2], \quad Q_r(a_\tau A_{j,q,\tau}^\pm + b_\tau) = \pm 2.$$

Let  $B_{k,n}$  such that  $Q_r(B_{k,n}) = 2 \cos \frac{k\pi}{n}$ , then  $T_n(B_{k,n}) = (-2)^k$  and  $|P_n|$  achieves its maximum at  $a_\tau A_{j,q,\tau}^\pm + b_\tau$  and the  $B_{k,n}$ . The polynomial  $P_n$  has  $n$  zeros in each  $\mathcal{K}_{j,\tau}$ . It enables us to prove the lemma

**Lemma 5.1.**  $\mu_{P_n}$  converge to the equilibrium measure of  $\mathcal{K}_\tau$ .

*Proof.* The proof follows [29, Theorem B1]. To simplify the notations  $\mathcal{K}_\tau$  is denoted by  $\mathcal{K}$ . By the Bernstein-Walsh lemma we have for  $z \in \mathbb{C} \setminus \mathcal{K}$ ,

$$(7) \quad \frac{1}{nr} \log |P_n(z)| \leq \log \left( \frac{\|P_n\|_{\mathcal{K}}^{\frac{1}{nr}}}{\mathcal{C}(\mathcal{K})} \right) - \Phi_{\mu_{\mathcal{K}}}(z)$$

with the remark that  $\lim_n \|P_n\|_{\mathcal{K}}^{\frac{1}{nr}} = \mathcal{C}(\mathcal{K})$ .

Now let  $d\eta$  be a limit point of the normalized density of zeros of  $P_n(z)$ ,  $d\eta_n$  and let  $f$  be continuous with support contained in  $\mathbb{C} \setminus \mathcal{K}$ , then  $\int f d\eta_n = 0$  because  $P_n$  has its zeros in  $\mathcal{K}$  and if  $d\eta_n \rightarrow d\eta$ , then  $\int f d\eta = 0$  and  $\text{supp}(d\eta) \subset \mathcal{K}$ . By the Upper Envelope Theorem [29, Theorem A7]

$$\liminf_{n \rightarrow \infty} \Phi_{\eta_n}(z) = \Phi_\eta(z)$$

for all  $z \in \mathbb{C} \setminus \mathcal{K}$  except for  $z$  in a polar set. Then letting  $n \rightarrow \infty$  in 7, we get

$$\Phi_\eta(z) \geq \Phi_{\mu_{\mathcal{K}}}$$

for all  $z \in \mathbb{C} \setminus \mathcal{K}$  except for  $z$  in a polar set. By continuity of  $\Phi_\eta$  and  $\Phi_{\mu_{\mathcal{K}}}$  in  $\mathbb{C} \setminus \mathcal{K}$ , we have  $\Phi_\eta \geq \Phi_{\mu_{\mathcal{K}}}$  for all  $z$  in  $\mathbb{C} \setminus \mathcal{K}$ . By using [29, Theorem A.21], we have  $\eta = \mu_{\mathcal{K}}$ . Thus,  $\mu_{\mathcal{K}}$  is the only limit point of the zeros, and so the limit is  $\mu_{\mathcal{K}}$ .  $\square$

With the same proof we deduce that for any  $l \geq 1$ ,  $\mu_{P_n^l} \rightarrow \mu_{\mathcal{K}}$ . Now, following [12, p 164] and its notations we consider

$$P_n(z) = z^k + \frac{\gamma_1 z^{k-1} + \gamma_2 z^{k-2} + \dots + \gamma_k}{m}, \quad m \in \mathbb{N}^*, \gamma_j \in \mathbb{Z}[i].$$

Let us denote by  $b = a^k, a \in \mathbb{N}$  and by  $c = b!m^b, c > b \geq a$ , then  $(P_n(z))^c = z^{kc} + \dots$  and the coefficients of  $z^{kc-1}, z^{kc-2}, \dots, z^{kc-b}$  are in  $\mathbb{Z}[i]$ . We can then determinate complex numbers,  $\lambda_1^{(1)}, \dots, \lambda_k^{(1)}, \lambda_1^{(2)}, \dots, \lambda_k^{(2)}, \dots, \lambda_1^{(c-a)}, \dots, \lambda_k^{(c-a)}$  such that the polynomial

$$(8) \quad \Gamma_a(z) = (P_n(z))^c + \sum_{i=1}^{c-a} \sum_{j=1}^k \lambda_j^{(i)} z^{k-j} (P_n(z))^{c-a-i}$$

$$(9) \quad = (P_n(z))^c + \Delta_a(z)$$

has its coefficients in  $\mathbb{Z}[i]$ . We have

$$\begin{aligned} |\Gamma_a(z) - (P_n(z))^c| &\leq 2 \sum_{i=1}^{c-a} \sum_{j=1}^k |z|^{k-j} |P_n(z)|^{c-a-i} \\ &\leq 2 \sum_{j=1}^k |z|^{k-j} \frac{|P_n(z)|^{c-a} - 1}{|P_n(z)| - 1} \\ \frac{|\Gamma_a(z) - (P_n(z))^c|}{|(P_n(z))^c|} &\leq 2 \left( \sum_{j=1}^k |z|^{k-j} \right) \frac{|P_n(z)|^{-a} - |P_n(z)|^{-c}}{|P_n(z)| - 1}. \end{aligned}$$

On the lemniscate  $|P_n(z)| = R_2$  with  $R_2 > 1$  for example  $R_2 = M_n^{\frac{1}{n}}$ , then with  $M = \max_{|P_n(z)|=R_2} \sum_{j=1}^k |z|^{k-j}$ ,

$$\begin{aligned} |\Gamma_a(z) - (P_n(z))^c| &\leq \frac{2M}{R_2^a(R_2 - 1)} |P_n(z)|^c \\ &\leq \frac{1}{2} |P_n(z)|^c < |P_n(z)|^c \end{aligned}$$

for  $a$  sufficiently large. By Rouché's theorem, in each  $\mathcal{K}_{j,\tau}$ ,  $\Gamma_a$  has  $nc$  zeros near from the  $nc$  zeros of  $(P_n(z))^c$ . After clarifying the earliest "near", we conclude that  $\lim \mu_{\Gamma_a} = \mu_{\mathcal{K}}$ .

**5.6. End of the proof.** We will need the following result

**Lemma 5.2.** *Let  $K$  be a compact set of  $\mathbb{C}$ , then*

- (1) *If  $T_n$  is the Chebychev polynomial of  $K$ , then  $\overline{T_n}$  is the Chebychev polynomial of  $\overline{K}$ ,*
- (2)  *$\mathcal{C}(K) = \mathcal{C}(\overline{K})$ .*

*Proof.* It is clear that the second point will follow from the first, but also from the fact that if  $P_n$  a monic polynomial of degree  $n$  then

$$\|P_n\|_{\infty, K} = \|\overline{P_n}\|_{\infty, \overline{K}}.$$

□

Again in this first part we withdraw the  $\nu$  symbol. We are going to prove that

$$\mu_{\mathcal{K}_\tau} \rightarrow \mu_{\Gamma^+},$$

where  $\Gamma^+ = \Gamma \cap H$ . For this we use lemma 2.1. By construction  $\delta(K_\tau, \Gamma^+) \rightarrow 0$ . Now  $\sigma(C) = P_1^{-1}([-2, 2]) = a\Gamma^+ + b$  and then  $\mathcal{C}(\sigma(C)) = |a|\mathcal{C}(\Gamma^+)$ .

We recall that for a  $r$ -periodic Jacobi matrix  $J$ , the capacity of its spectrum is the  $\frac{1}{r}$ th power of the product of the diagonal coefficients. Since the rational coefficients of  $C_q$  approximate those of  $C$ , we have  $\mathcal{C}(\sigma(C_q)) \rightarrow \mathcal{C}(\sigma(C))$ . Since  $\mathcal{C}(\sigma(C_q)) = \mathcal{C}(a\Gamma_q + b) = |a|\mathcal{C}(\Gamma_q)$ , then  $\mathcal{C}(\Gamma_q) \rightarrow \mathcal{C}(\Gamma^+)$ . Moreover

$$\mathcal{C}(\mathcal{K}_\tau) = \mathcal{C}\left(\frac{1}{a_\tau}(a\Gamma_q + b - b_\tau)\right) = \frac{|a|}{|a_\tau|} \mathcal{C}(\Gamma_q),$$

hence  $\mathcal{C}(\mathcal{K}_\tau) \rightarrow \mathcal{C}(\Gamma^+)$ .

Hence we have found a sequence of monic polynomials with integer coefficients  $(\Theta_\lambda)_\lambda$  such that  $\mu_{\Theta_\lambda} \rightarrow \mu_{\mathcal{K}_\tau}$ . By the diagonal process we conclude that  $\mu_{\Theta_\lambda} \rightarrow \mu_{\Gamma^+}$ .

The second step is to verify that  $\lim \mu_{\bar{P}_n} = \mu_{\Gamma^-}$  where  $\Gamma^- = \Gamma \setminus \Gamma^+$ .

The third step is to remark that  $P_n \bar{P}_n$  is a monic polynomial with integer coefficients and that  $\lim_n \delta_{P_n \bar{P}_n} = \frac{1}{2}(\delta_{\Gamma^+} + \delta_{\Gamma^-})$ . Which follows from the definition of  $\delta_{P_n \bar{P}_n}$  and the previous results.

The ultimate step is to verify that  $\frac{1}{2}(\delta_{\Gamma^+} + \delta_{\Gamma^-}) = \delta_{\Gamma}$ . For this we look at the energy. Let  $\mu_1 = \frac{1}{2}\delta_{\Gamma^+}$  and  $\mu_2 = \frac{1}{2}\delta_{\Gamma^-}$

$$\begin{aligned} I\left(\frac{1}{2}(\delta_{\Gamma^+} + \delta_{\Gamma^-})\right) &= I(\mu_1) + I(\mu_2) + 2 \int \Phi_{\mu_1}(x) d\mu_2(x) \\ &= \frac{1}{4} \frac{1}{\ln(\mathcal{C}(\Gamma^+))} + \frac{1}{4} \frac{1}{\ln(\mathcal{C}(\Gamma^-))} + 2 \int \Phi_{\mu_1}(x) d\mu_2(x). \end{aligned}$$

From the beginning we have that  $\mathcal{C}(\Gamma^+) = \mathcal{C}(\Gamma^-) = \mathcal{C}(\Gamma)$ . Moreover as  $\Gamma^+$  and  $\Gamma^-$  are disjoint we have that on  $\Gamma^-$ ,  $\Phi_{\mu_1}(x) \leq \frac{1}{2} \frac{1}{\ln(\mathcal{C}(\Gamma^+))}$ . Finally

$$I\left(\frac{1}{2}(\delta_{\Gamma^+} + \delta_{\Gamma^-})\right) \leq I(\delta_{\Gamma^+ \cup \Gamma^-}) = I(\delta_{\Gamma}),$$

and by definition of the equilibrium measure  $I\left(\frac{1}{2}(\delta_{\Gamma^+} + \delta_{\Gamma^-})\right) = I(\delta_{\Gamma})$ .

## References

- [1] Ahlfors, L. V., Sario, L., (1960) Riemann Surfaces. Princeton University Press, Princeton.
- [2] Autissier, P., Chapter VIII : Autour du théorème de Fekete-Szegö, 329-340, in Arakelov Geometry and Diophantine Applications, E. Peyre, G. Rémond Editors, LNM, volume 2276.
- [3] Baker, W., Plane domains with hyperelliptic double, Phd thesis Stanford University, 1975.
- [4] Berenstein, C.A., Gay, R., Complex variables, An introduction, Springer Verlag, 1991.
- [5] Bilu, Y., Limit distribution of small points on algebraic tori, Duke Math. Journal, 89 (1997), p. 465-476.
- [6] W. Bulla, F. Gesztesy, H. Holden, and G. Teschl. algebro-geometric quasi-periodic finite-gap solutions of the Toda and Kac-van Moerbeke hierarchies. *Memoirs of the AMS*, 135(641):1–79, 1998.
- [7] Cassels, J.W.S., An introduction to the geometry of numbers, Springer (1972)
- [8] Chinburg, T., Capacity theory on varieties, *Compositio Mathematica*, tome 80, no 1 (1991), p. 75-84.
- [9] Dubickas, A., Conjugate algebraic numbers close to a symmetric set, *St. Petersburg Math. J.* 16 (2005), 1013–1016.
- [10] Ennola, V., Conjugate algebraic integers in an interval, *Proc. Amer. Math. Soc.* 53 (1975), 259-261.
- [11] Fekete, M., Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten, *Math Z.* 17, p. 228-249 (1923).
- [12] Fekete, M., Szegö, G., On algebraic equations with integral coefficients whose roots belong to a given set, *Math. Zeitschrift.* 63 (1955), 158–172.
- [13] Ferguson, L., Approximation by polynomials with integral coefficients. *Math. Surveys*, 17, 1980.
- [14] Ferguson, Le Baron O., What can be approximated by polynomials with Integer coefficients. *The American Mathematical Monthly*, 2006, Vol. 113, No. 5, pp. 403-414.
- [15] Hilbert, D., Ein Betrag zur Theorie des Legendreschen Polynoms *Acta Math.* 18, 155-159 (1894).
- [16] Krichever, I.M., Nonlinear equations and elliptic curves. *J. Math. Sci.* 28, 51-90(1985)
- [17] Kronecker, L., Zwei Sätze über Gleichungen mit ganzzahligen Koeffizienten, *J. Reine Angew. Math.* 53(1857), 173-175.
- [18] P.van Moerbeke. The spectrum of Jacobi matrices. *Invent.Math.*, 37:45–81, 1976.
- [19] Van Moerbeke, P., Mumford, D., The spectrum of difference operators and algebraic curves. *Acta Math.* 143 (1979), no. 1-2, 93–154.
- [20] Motzkin, Th., From among  $n$  conjugate algebraic integers,  $n - 1$  can be approximatively given. *Bull. Amer. Math. Soc.* 53 (1947), 156–162.
- [21] Pritsker, I. E., Asymptotic distribution and symmetric means of algebraic numbers
- [22] Naïman, P. B., On the theory of periodic and limit-periodic Jacobi matrices. *Soviet Math.Dokl.*, 3:383-385, 1962.
- [23] Robinson, R. M., Conjugate algebraic integers in real point sets. *Math.Zeitschr.*, 84:415–427, 1964.
- [24] Robinson, R.M. Intervals containing infinitely many sets of conjugate algebraic units. *Annals of Math.* (2), 80:411-428, 1964.
- [25] Rumely, R., On Bilu’s equidistribution theorem. *Contemporary Math.*, 237(1999),159-166
- [26] Rumely, R., Capacity Theory With Local Rationality: The Strong Fekete-szegö Theorem on Curves *Mathematical Surveys and Monographs*, 193.
- [27] Sebbar, A., Falliero, Th. Capacities and Jacobi matrices. *Proc. Edin. Math. Soc.* 46, 719:745 (2003).
- [28] Serre, J. P., Distribution asymptotique des valeurs propres des endomorphismes de Frobenius. *Sém. Bourbaki*, 70:1146 (mars 2018).
- [29] Simon, B., Equilibrium measures and capacities in spectral theory, *Inverse Probl. Imaging* 1 (2007), 713-772.
- [30] Tsuji, M., Potential Theory in Modern Function Theory, Chelsea Publ. Co., New York, 1975.

THÉRÈSE FALLIERO, AVIGNON UNIVERSITÉ, LABORATOIRE DE MATHÉMATIQUES D'AVIGNON (EA 2151), F-84018  
AVIGNON, FRANCE

*Email address:* `therese.falliero@univ-avignon.fr`