



# On the reduction of a spatially hybrid optimal control problem into a temporally hybrid optimal control problem

Térence Bayen, Anas Bouali, Loïc Bourdin, Olivier Cots

## ► To cite this version:

Térence Bayen, Anas Bouali, Loïc Bourdin, Olivier Cots. On the reduction of a spatially hybrid optimal control problem into a temporally hybrid optimal control problem. 2023. hal-04319045

**HAL Id: hal-04319045**

**<https://univ-avignon.hal.science/hal-04319045>**

Preprint submitted on 2 Dec 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# On the reduction of a spatially hybrid optimal control problem into a temporally hybrid optimal control problem

T  rence Bayen<sup>1</sup>, Anas Bouali<sup>2</sup>, Lo  c Bourdin<sup>3</sup>, and Olivier Cots<sup>4</sup>

<sup>1</sup> Avignon Universit  , Laboratoire de Math  matiques d'Avignon (EA 2151)  
F-84018 [terence.bayen@univ-avignon.fr](mailto:terence.bayen@univ-avignon.fr)

<sup>2</sup> Avignon Universit  , Laboratoire de Math  matiques d'Avignon (EA 2151)  
F-84018 [anas.bouali@univ-avignon.fr](mailto:anas.bouali@univ-avignon.fr)

<sup>3</sup> Institut de recherche XLIM. UMR CNRS 7252. Universit   de Limoges, France  
[loic.bourdin@unilim.fr](mailto:loic.bourdin@unilim.fr)

<sup>4</sup> INP-ENSEEIH-IRIT. UMR CNRS 5505. Toulouse Universit  , France  
[olivier.cots@irit.fr](mailto:olivier.cots@irit.fr)

**Summary.** This paper studies a reformulation of a general spatially hybrid optimal control problem for which the dynamics is usually defined over a given partition of the state space into strata so that switchings of the system only occur regionally (i.e., at the time of a change of stratum). Given a global solution to such a problem, we associate a temporally hybrid optimal control problem for which the change of dynamics now occurs at free switching instants that can be optimized (without considering a partition). We prove that under a strong transverse condition on the dynamics at the interfaces between strata, the global solution is a  $L^1$ -local solution to this new problem. Thanks to an explicit example, we also prove that this reformulation fails to hold in general. In fact, the analysis of this example makes it possible to demonstrate that the structure of the solution to the spatially and temporally hybrid optimal control problems are different. The study carried out in this work highlights several ways for obtaining the spatially hybrid maximum principle.

## 1.1 Introduction

### 1.1.1 General context

Optimal control theory experienced major advances in the late 1950s following in particular the proof of the maximum principle by Pontryagin *et al.*, see, [23]. Being an extension of the theory of calculus of variation to any smooth control system, the Pontryagin Maximum Principle (PMP in short) was a breakthrough in the theory of optimal control. Since then, its application has enabled major advances in many fields of science such as in the field of aerospace. Let us emphasize that the PMP originally addressed optimal

control problems (OCPs in short) governed by control systems that are differentiable w.r.t. the state. It has now been extended in more complex settings such as when the dynamics is governed by a hybrid control system which is the main topic of this work. Such systems are extensions of the classical control systems to the case where the dynamics can be discontinuous w.r.t. the state (typically, in the sense of Fillipov, see [17], but the origin of these discontinuities can be varied). They arise in many domains such as in non-smooth mechanics [9], in electricity [10], in biology [2], or in viability theory [8]. In the hybrid framework, the necessary conditions of optimality are named after the hybrid maximum principle (HMP in short) which is the counter part of the Pontryagin Maximum Principle for optimal control problems governed by hybrid control systems and it was been developed in various hybrid settings (see, e.g. [3, 5, 12, 13, 14, 15, 16, 18, 19, 21, 22, 24, 25]). In order to present the contribution in this paper, we would like now to give an overview of the main hybrid frameworks as well the corresponding approaches that have been developed in the literature in order to derive the HMP.

### 1.1.2 Temporally and spatially hybrid optimal control problems

The theory of hybrid optimal control is very broad due to the diversity of hybrid controls systems encountered. As was said previously, the origin of the jump of the dynamics is varied. For instance, it can be controlled by an automaton [18, 24, 25] and in that case, we would rather speak of switched control systems (involving a discrete control responsible of the jump of the dynamics in addition to the continuous control function). Note that the use of an automaton is also encountered in [21, 22] and that in [12, 13], another very general framework called *theory of multiprocesses* is introduced in which the change of dynamics occurs at free instants. In this work, we will focus essentially on to main families of hybrid OCPs:

- *Temporally hybrid optimal control problems.* These are OCPs governed by a dynamics that is subject to changes at some free instants (that can be optimized). The number of these instants is fixed in advance as well as each (smooth) dynamics between two such instants.
- *Spatially hybrid optimal control problems.* These are OCPs that presuppose the existence of a fixed partition of the state space into strata (or regions). Associated with this partition, we are also given a family of smooth dynamics and then, the hybrid system under consideration is defined in such a way that in each stratum, it coincides with the corresponding smooth dynamics. Hence, the control system is discontinuous over the state space and changes of the dynamics occur regionally (i.e., when the trajectory goes from a stratum into another one).

The HMP that is adapted to temporally hybrid OCPs was developed in particular in [14, 15, 16] thanks to an augmentation technique. This approach consists in transforming the problem into a smooth OCP (i.e., involving smooth

data w.r.t. the state) for which the PMP can be applied. Through this transformation, one may prove that a global solution to the original OCP becomes a  $L^1$ -local solution to the augmented one so that the HMP (that we also call temporally HMP) can be obtained by inverting this transformation.

For spatially hybrid OCPs, the situation is more intricate because of the presence of a partition of the state space that requires a priori the concept of Filippov's solutions [17] in order to properly define a solution to the hybrid system. For this reason, the derivation of the corresponding HMP has required the use of transverse conditions that can be classified in two ways:

- by *weak transverse hypothesis*, we mean that the nominal trajectory is supposed to go from a region to another one transversally (i.e., not tangentially). This hypothesis involves only the nominal trajectory and the interface locally at the crossing point (see, e.g., [5, 7, 19]).
- by *strong transverse hypothesis*, we mean that locally at the interface between two regions, the scalar product between the unit outward normal vector and the dynamics is bounded below by a positive constant independently of the control (see e.g., [1, 6]). This implies that every admissible trajectory should cross locally the interface transversally.

Under each one of these hypotheses, sliding modes are thus excluded. Next, the corresponding spatially HMP has been proved following essentially two approaches. The first one is based on a sensitivity analysis in this hybrid setting following the classical proof of the PMP (see [4] dealing with spatially hybrid OCPs without terminal constraints). A second methodology consists in adapting the augmentation technique of [16] to this framework. Indeed, an optimal solution to the spatially hybrid OCP may not necessarily be a solution to the augmented problem (see e.g., [5]). Doing so, a new notion of minimum was introduced in [5] in order in particular to bypass possible switches of trajectories at an interface (which cannot happen in the temporally hybrid setting). This approach already pointed out a possible misconception in the literature and that the proposed reformulation (from a spatially hybrid OCP into a temporally hybrid OCP) in [16] is not completely correct to handle spatially hybrid OCPs. Our work aims precisely to study more into details the question of reformulating a spatially hybrid OCP into a temporally hybrid OCP. This seems to us a relevant question insofar as such a reduction would make it possible to overcome a partition.

### 1.1.3 Contribution and organization of this paper

In this paper, we consider a general spatially hybrid OCP for which we are now given a global solution with a prescribed number of crossing times, and, in addition, we suppose that it satisfies a strong transverse hypothesis. Next, we define a temporally hybrid OCP associated with this global solution. This problem is constructed in such a way that the number of switching instants coincides with the number of crossing points of the global solution. As well,

the dynamics between two consecutive switching instants is the same than the dynamics of the global solution between two consecutive crossing points. The main issue of this paper is Proposition 1 which asserts that the global solution is a  $L^1$ -local solution to the temporally hybrid OCP. Next, we develop an example which shows that the preceding result is no longer valid in absence of a strong transverse condition. Thanks to the temporally and spatially hybrid maximum principles, we compute explicitly a solution to the original spatially hybrid OCP (which has a single crossing point and verifies a weak transverse hypothesis) as well as a solution to the temporally hybrid OCP (which has one switching instant for the dynamics and two switching times for the control). The optimal synthesis as well as numerical simulation show that these two solutions are different which proves our claim.

This paper is organized as follows. In Section 1.2, we recall the PMP for a smooth optimal control problem and we also introduce a general temporally hybrid OCP (in a Mayer form and with mixed initial-terminal state constraints). By using an augmentation technique as in [16], we prove the corresponding temporally HMP by application of the PMP to an augmented OCP. In Section 1.3, we introduce a general spatially hybrid OCP as well as the notions of strong and weak transverse conditions. We prove our main result (Proposition 1) on the reformulation of the general spatially hybrid OCP into a temporally hybrid OCP. We end up this section by stating the corresponding spatially HMP (using the HMP obtained in Section 1.2). Finally, Section 1.4 is devoted to the optimal synthesis of an example in order to highlight that such a transformation fails to hold in absence of a strong transverse hypothesis.

### 1.1.4 Basic notation and functional framework

In this paper, for any positive integer  $d \in \mathbb{N}^*$ , we denote by  $\langle \cdot, \cdot \rangle_{\mathbb{R}^d}$  (resp.  $\|\cdot\|_{\mathbb{R}^d}$ ) the standard inner product (resp. Euclidean norm) of  $\mathbb{R}^d$ . For any subset  $X \subset \mathbb{R}^d$ , we denote by  $\partial X$  the boundary of  $X$  defined by  $\partial X := \overline{X} \setminus \text{Int}(X)$ , where  $\overline{X}$  and  $\text{Int}(X)$  stand respectively for the closure and the interior of  $X$ . Given a closed convex set  $Y \subset \mathbb{R}^d$ , the normal cone to  $Y$  at some point  $y \in Y$  is defined by

$$N_Y[y] := \{y'' \in \mathbb{R}^d \mid \forall y' \in Y, \langle y'', y' - y \rangle_{\mathbb{R}^d} \leq 0\}.$$

For any extended-real number  $r \in [1, \infty]$  and any real interval  $I \subset \mathbb{R}$ , we denote by:

- $L^r(I, \mathbb{R}^d)$  the usual Lebesgue space of  $r$ -integrable functions defined on  $I$  with values in  $\mathbb{R}^d$ , endowed with its usual norm  $\|\cdot\|_{L^r}$ ;
- $C(I, \mathbb{R}^d)$  the standard space of continuous functions defined on  $I$  with values in  $\mathbb{R}^d$ , endowed with the standard uniform norm  $\|\cdot\|_C$ ;
- $AC(I, \mathbb{R}^d)$  the subspace of  $C(I, \mathbb{R}^d)$  of absolutely continuous functions.

Now take  $I = [0, T]$  for some  $T > 0$ . Recall that a partition of the interval  $[0, T]$  is a finite set  $\mathbb{T} = \{\tau_k\}_{k=0, \dots, N}$ , for some positive integer  $N \in \mathbb{N}^*$ , such that  $0 = \tau_0 < \tau_1 < \dots < \tau_{N-1} < \tau_N = T$ . In this paper a function  $\gamma : [0, T] \rightarrow \mathbb{R}^d$  is said to be *piecewise absolutely continuous*, with respect to a partition  $\mathbb{T} = \{\tau_k\}_{k=0, \dots, N}$  of the interval  $[0, T]$ , if  $\gamma$  is continuous at 0 and  $T$  and the restriction of  $\gamma$  over each open interval  $(\tau_{k-1}, \tau_k)$  admits an extension over  $[\tau_{k-1}, \tau_k]$  that is absolutely continuous. If so,  $\gamma$  admits left and right limits at each  $\tau_k \in (0, T)$ , denoted by  $\gamma^-(\tau_k)$  and  $\gamma^+(\tau_k)$  respectively.

In what follows we denote by  $\text{PAC}_{\mathbb{T}}([0, T], \mathbb{R}^d)$  the space of all piecewise absolutely continuous functions respecting a given partition  $\mathbb{T}$  of  $[0, T]$ . We denote by  $\text{PAC}([0, T], \mathbb{R}^d)$  the set of all piecewise absolutely continuous functions, independently of the partition considered.

For a differentiable map  $\psi : \mathbb{R}^d \rightarrow \mathbb{R}^{d'}$ , with  $d' \in \mathbb{N}^*$ , we denote by  $\nabla\psi(x) := (\nabla\psi_1(x) \dots \nabla\psi_{d'}(x)) \in \mathbb{R}^{d \times d'}$  the gradient of  $\psi$  at some  $x \in \mathbb{R}^d$ . We say that  $\psi$  is *submersive* at  $x \in \mathbb{R}^d$  if the differential  $D\psi(x) = \nabla\psi(x)^\top \in \mathbb{R}^{d' \times d}$  is surjective. Finally, when  $(\mathcal{Z}, d_{\mathcal{Z}})$  is a metric set, we denote by  $B_{\mathcal{Z}}(z, \nu)$  (resp.  $\bar{B}_{\mathcal{Z}}(z, \nu)$ ) the standard open (resp. closed) ball of  $\mathcal{Z}$  centered at  $z \in \mathcal{Z}$  and of radius  $\nu > 0$ .

## 1.2 Temporally hybrid OCP and temporally HMP

### 1.2.1 Recalls on classical OCP: terminology and PMP

Let  $n, m, d$  and  $\ell \in \mathbb{N}^*$  be four fixed positive integers and  $T > 0$  be a fixed positive real number. In the present section we consider a classical Mayer optimal control problem with parameter and mixed initial-terminal state constraints given by

$$\begin{aligned} & \text{minimize} && \phi(x(0), x(T)), \\ & \text{subject to} && (x, u, \lambda) \in \text{AC}([0, T], \mathbb{R}^n) \times L^\infty([0, T], \mathbb{R}^m) \times \mathbb{R}^d, \\ & && \dot{x}(t) = f(x(t), u(t), \lambda), \quad \text{a.e. } t \in [0, T], \\ & && g(x(0), x(T)) \in S, \\ & && u(t) \in U, \quad \text{a.e. } t \in [0, T], \\ & && \lambda \in A, \end{aligned} \tag{CP}$$

where the Mayer cost function  $\phi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d \rightarrow \mathbb{R}$ , the dynamics  $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d \rightarrow \mathbb{R}^n$  and the constraint function  $g : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^\ell$  are of class  $C^1$ . Moreover, we assume throughout this paper that  $g : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^\ell$  is submersive at each point  $(x_0, x_T) \in \mathbb{R}^n \times \mathbb{R}^n$ . The subsets  $S \subset \mathbb{R}^\ell$  and  $A \subset \mathbb{R}^d$  are nonempty closed convex subsets and  $U \subset \mathbb{R}^m$  is a nonempty subset. As usual in the literature,  $x \in \text{AC}([0, T], \mathbb{R}^n)$  is called the *state* (or the *trajectory*),  $u \in L^\infty([0, T], \mathbb{R}^m)$  is called the *control* and  $\lambda \in \mathbb{R}^d$  is called the *parameter*. A triplet  $(x, u, \lambda) \in \text{AC}([0, T], \mathbb{R}^n) \times L^\infty([0, T], \mathbb{R}^m) \times \mathbb{R}^d$  is said to be *admissible* for Problem (CP) if it satisfies all the constraints of Problem (CP). Finally,

such an admissible triplet is said to be a *global solution* to Problem (CP) if it minimizes the Mayer cost  $\phi(x(0), x(T))$  among all admissible triplets.

**Definition 1 (L<sup>1</sup>–local solution).** *An admissible triplet  $(x^*, u^*, \lambda^*)$  is said to be a L<sup>1</sup>–local solution to Problem (CP) if there exists  $\eta > 0$  such that  $\phi(x^*(0), x^*(T)) \leq \phi(x(0), x(T))$  for all admissible triplets  $(x, u, \lambda)$  satisfying*

$$\|x - x^*\|_C + \|u - u^*\|_{L^1} + \|\lambda - \lambda^*\|_{\mathbb{R}^d} \leq \eta.$$

Now recall that the *Hamiltonian*  $\mathcal{H} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R}^n \rightarrow \mathbb{R}$  associated with Problem (CP) is defined by  $\mathcal{H}(x, u, \lambda, p) := \langle p, f(x, u, \lambda) \rangle_{\mathbb{R}^n}$  for all  $(x, u, \lambda, p) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R}^n$ . The PMP (as in [11, 23]) can be stated as follows.

**Theorem 1 (Classical PMP for L<sup>1</sup>–local solutions).** *If  $(x^*, u^*, \lambda^*)$  is a L<sup>1</sup>–local solution to Problem (CP), there exists a nontrivial pair  $(p, p^0) \in \text{AC}([0, T], \mathbb{R}^n) \times \mathbb{R}_+$  satisfying:*

(i) *the Hamiltonian system*

$$\dot{x}^*(t) = \nabla_p \mathcal{H}(x^*(t), u^*(t), \lambda^*, p(t)), \quad -\dot{p}(t) = \nabla_x \mathcal{H}(x^*(t), u^*(t), \lambda^*, p(t)),$$

*for almost every  $t \in [0, T]$ ;*

(ii) *the endpoint transversality condition*

$$\begin{pmatrix} p(0) \\ -p(T) \end{pmatrix} = p^0 \nabla \phi(x^*(0), x^*(T)) + \nabla g(x^*(0), x^*(T)) \xi,$$

*for some  $\xi \in N_S[g(x^*(0), x^*(T))]$ ;*

(iii) *the Hamiltonian maximization condition*

$$u^*(t) \in \arg \max_{\omega \in U} \mathcal{H}(x^*(t), \omega, \lambda^*, p(t)),$$

*for almost every  $t \in [0, T]$ ;*

(iv) *the averaged Hamiltonian gradient condition*

$$\int_0^T \nabla_\lambda \mathcal{H}(x^*(s), u^*(s), \lambda^*, p(s)) \, ds \in N_A[\lambda^*].$$

### 1.2.2 Temporally hybrid OCP: terminology and HMP

Let  $n, m, N$  and  $\ell \in \mathbb{N}^*$  be four fixed positive integers and  $T > 0$  be a fixed positive real number. In this section we consider a temporally hybrid Mayer optimal control problem with mixed initial-terminal state constraints given by

$$\begin{aligned}
& \text{minimize} && \phi(x(0), x(T)), \\
& \text{subject to} && (x, u, \mathbb{T}) \in \text{AC}([0, T], \mathbb{R}^n) \times L^\infty([0, T], \mathbb{R}^m) \times \mathbb{R}^{N+1}, \\
& && \dot{x}(t) = f_k(x(t), u(t)), \quad \text{a.e. } t \in (\tau_{k-1}, \tau_k), \quad \text{for all } k \in \{1, \dots, N\}, \\
& && g(x(0), x(T)) \in S, \\
& && u(t) \in U, \quad \text{a.e. } t \in [0, T], \\
& && \mathbb{T} = \{\tau_k\}_{k=0, \dots, N} \in \Delta, \\
& && F_k(x(\tau_k)) = 0, \quad \text{for all } k \in \{1, \dots, N-1\},
\end{aligned} \tag{THP}$$

where  $\Delta \subset \mathbb{R}^{N+1}$  is the nonempty closed convex set defined by

$$\Delta := \{\mathbb{T} = \{\tau_k\}_{k=0, \dots, N} \in \mathbb{R}^{N+1} \mid 0 = \tau_0 \leq \tau_1 \leq \dots \leq \tau_{N-1} \leq \tau_N = T\}.$$

Here, the data assumptions and the terminology for Problem (THP) are the same as those for Problem (CP), except that each dynamics  $f_k : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  and function  $F_k : \mathbb{R}^n \rightarrow \mathbb{R}$  are of class  $C^1$ , with  $F_k$  being submersive at every point  $z \in \mathbb{R}^n$ . Moreover, in this setting, the instants  $\tau_k \in (0, T)$  are chosen to be free and referred to as *switching times*, they represent the instants when the dynamics change.

**Definition 2 (L<sup>1</sup>–local solution).** *An admissible triplet  $(x^*, u^*, \mathbb{T}^*)$  is said to be a L<sup>1</sup>–local solution to Problem (THP) if there exists  $\eta > 0$  such that  $\phi(x^*(0), x^*(T)) \leq \phi(x(0), x(T))$  for all admissible triplets  $(x, u, \mathbb{T})$  satisfying*

$$\|x - x^*\|_C + \|u - u^*\|_{L^1} + \|\mathbb{T} - \mathbb{T}^*\|_{\mathbb{R}^{N+1}} < \eta.$$

Now, recall that the *Hamiltonian*  $H_1 : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$  associated with Problem (SHP) is defined by  $H_1(x, u, p, t) := \langle p, f_k(x, u) \rangle_{\mathbb{R}^n}$  for all  $t \in (\tau_{k-1}, \tau_k)$  and all  $(x, u, \mathbb{T}, p) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{N+1} \times \mathbb{R}^n$ . We are now in a position to establish a temporally HMP to Problem (THP).

**Theorem 2 (Temporally HMP for L<sup>1</sup>–local solutions).** *If  $(x^*, u^*, \mathbb{T}^*)$  is a L<sup>1</sup>–local solution to Problem (THP), such that  $\mathbb{T}^* \in \text{Int}(\Delta)$ , there exists a nontrivial pair  $(p, p^0) \in \text{PAC}_{\mathbb{T}^*}([0, T], \mathbb{R}^n) \times \mathbb{R}_+$  satisfying:*

(i) *the Hamiltonian system*

$$\dot{x}^*(t) = \nabla_p H_1(x^*(t), u^*(t), p(t)), \quad -\dot{p}(t) = \nabla_x H_1(x^*(t), u^*(t), p(t)),$$

*for almost every  $t \in [0, T]$ ;*

(ii) *the endpoint transversality condition*

$$\begin{pmatrix} p(0) \\ -p(T) \end{pmatrix} = p^0 \nabla \phi(x^*(0), x^*(T)) + \nabla g(x^*(0), x^*(T)) \xi,$$

*for some  $\xi \in N_S[g(x^*(0), x^*(T))]$ ;*



(iii) the Hamiltonian maximization condition

$$u^*(t) \in \arg \max_{\omega \in U} H_1(x^*(t), \omega, p(t)),$$

for almost every  $t \in [0, T]$ ;

(iv) the discontinuity condition

$$p^+(\tau_k^*) - p^-(\tau_k^*) = \sigma_k \nabla F_k(x^*(\tau_k^*)),$$

for some  $\sigma_k \in \mathbb{R}$ , for all  $k \in \{1, \dots, N-1\}$ .

*Proof.* The proof of Theorem 2 is postponed to Appendix A. It is based on the augmentation procedure and the application of Theorem 1 (similar to the one developed in [16]).

### 1.3 Spatially hybrid OCP and spatially HMP

#### 1.3.1 Spatially hybrid OCP: terminology and HMP

Let  $n, m$  and  $\ell \in \mathbb{N}^*$  be three fixed positive integers and  $T > 0$  be a fixed positive real number. In this section we consider a partition of the state space given by

$$\mathbb{R}^n = \bigcup_{j \in \mathcal{J}} \overline{X_j},$$

where  $\mathcal{J}$  is a (possibly infinite) family of indexes and the nonempty open subsets  $X_j \subset \mathbb{R}^n$ , called *regions*, are disjoint. In this section, we consider a *spatially* hybrid optimal control problem given by

$$\begin{aligned} & \text{minimize} && \phi(x(0), x(T)), \\ & \text{subject to} && (x, u) \in \text{AC}([0, T], \mathbb{R}^n) \times L^\infty([0, T], \mathbb{R}^m), \\ & && \dot{x}(t) = h(x(t), u(t)), \quad \text{a.e. } t \in [0, T], \\ & && g(x(0), x(T)) \in S, \\ & && u(t) \in U, \quad \text{a.e. } t \in [0, T], \end{aligned} \tag{SHP}$$

where both the Mayer cost function  $\phi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  and the constraint function  $g : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^\ell$  are of class  $C^1$ , where  $S \subset \mathbb{R}^\ell$  is a nonempty closed convex subset and  $U \subset \mathbb{R}^m$  is a nonempty compact subset and where the dynamics  $h : \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  is defined *regionally* by

$$\forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^m, \quad h(x, u) := h_j(x, u) \quad \text{if } x \in X_j,$$

where the maps  $h_j : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  are of class  $C^1$ . In order to deal with (SHP), we will recall in the next part the concept of spatially hybrid control system.

### 1.3.2 Regular solutions

Consider the spatially hybrid control system associated with Problem (SHP) given by

$$\dot{x}(t) = h(x(t), u(t)), \quad \text{for a.e. } t \in [0, T]. \quad (\text{HS})$$

Due to the discontinuities of the dynamics  $h$ , we need to precise the definition of a solution to (HS).

**Definition 3 (Solution to (HS)).** *We say that a pair  $(x, u) \in \text{AC}([0, T], \mathbb{R}^n) \times \text{L}^\infty([0, T], \mathbb{R}^m)$  is a solution to (HS) if there exists a partition  $\mathbb{T} = \{\tau_k\}_{k=0, \dots, N}$  of the interval  $[0, T]$  and a switching sequence  $j : \{1, \dots, N\} \rightarrow \mathcal{J}$  such that:*

- (i) *It holds that  $x(t) \in X_{j(k)}$  for all  $t \in (\tau_{k-1}, \tau_k)$  and all  $k \in \{1, \dots, N\}$ , where  $j(k) \neq j(k-1)$  for all  $k \in \{2, \dots, N\}$  ;*
- (ii) *It holds that  $x(0) \in X_{j(1)}$  and  $x(T) \in X_{j(N)}$  ;*
- (iii) *It holds that  $\dot{x}(t) = h_{j(k)}(x(t), u(t))$  for almost every  $t \in (\tau_{k-1}, \tau_k)$  and all  $k \in \{1, \dots, N\}$ .*

*In that case, to ease notation, we set  $f_k := h_{j(k)}$  and  $E_k := X_{j(k)}$  for all  $k \in \{1, \dots, N\}$ . The times  $\tau_k$  for  $k \in \{1, \dots, N-1\}$ , called crossing times, correspond to the instants at which the trajectory  $x$  goes from the region  $E_k$  to the region  $E_{k+1}$ , and thus  $x(\tau_k) \in \partial E_k \cap \partial E_{k+1}$ .*

Note that we do not consider sliding modes in this paper, so, in order to handle solutions to the hybrid control system, there is no need to introduce the concept of Filippov's solutions (see [17]).

**Definition 4 (Regular solution to (HS)).** *Following the notations introduced in Definition 3, a solution  $(x, u) \in \text{AC}([0, T], \mathbb{R}^n) \times \text{L}^\infty([0, T], \mathbb{R}^m)$  to (HS), associated with a partition  $\mathbb{T} = \{\tau_k\}_{k=0, \dots, N}$ , is said to be regular if the following conditions are both satisfied:*

- (i) *At each crossing time  $\tau_k$ , there exists a  $\text{C}^1$  function  $F_k : \mathbb{R}^n \rightarrow \mathbb{R}$  such that*

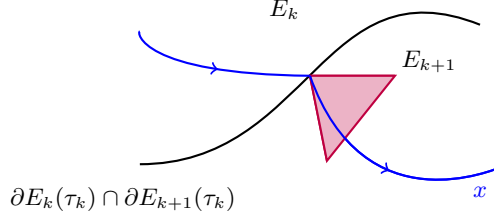
$$\exists \nu_k > 0, \quad \forall z \in \overline{\text{B}}_{\mathbb{R}^n}(x(\tau_k), \nu_k), \quad \begin{cases} z \in E_k \Leftrightarrow F_k(z) < 0, \\ z \in \partial E_k \cap \partial E_{k+1} \Leftrightarrow F_k(z) = 0, \\ z \in E_{k+1} \Leftrightarrow F_k(z) > 0. \end{cases}$$

*In particular it holds that  $F_k(x(\tau_k)) = 0$ .*

- (ii) *At each crossing time  $\tau_k$ , there exists  $\beta_k > 0$  such that the (strong) transverse conditions*

$$\forall \omega \in \text{U}, \quad \begin{cases} \langle \nabla F_k(x(\tau_k)), f_k(x(\tau_k), \omega) \rangle_{\mathbb{R}^n} \geq \beta_k, \\ \langle \nabla F_k(x(\tau_k)), f_{k+1}(x(\tau_k), \omega) \rangle_{\mathbb{R}^n} \geq \beta_k, \end{cases} \quad (\text{TC})$$

*are both satisfied. We refer to Figure 1.1 for a geometrical illustration.*



**Fig. 1.1.** Geometrical illustration of a (strong) transversal boundary crossing.

*Remark 1.* Hereafter, we provide remarks on the transverse condition (TC) and its various relaxations existing in the literature.

- (i) The geometrical interpretation of condition (TC) is that  $x$  does not cross the boundary  $\partial E_k \cap E_{k+1}$  *tangentially* for every admissible control value  $\omega \in U$ .
- (ii) It is noteworthy that such a similar (strong) transverse condition as (TC) can be found in [1, 6].
- (iii) Weaker transverse conditions have been also considered in the literature (see, e.g. [3, 5, 19]). For instance, the transverse condition employed in [5] amounts to assume that at each crossing time  $\tau_k$ , there are  $\beta_k > 0$  and  $\eta_k > 0$  such that

$$\begin{cases} \langle \nabla F_k(x(\tau_k)), f_k(x(\tau_k), u(t)) \rangle_{\mathbb{R}^n} \geq \beta_k, & \text{a.e. } t \in (\tau_k - \eta_k, \tau_k) \\ \langle \nabla F_k(x(\tau_k)), f_{k+1}(x(\tau_k), u(t)) \rangle_{\mathbb{R}^n} \geq \beta_k, & \text{a.e. } t \in (\tau_k, \tau_k + \eta_k). \end{cases} \quad (\text{TC}')$$

whereas the one in [19] requires left and right continuity of the control at each crossing time  $\tau_k$  and that, in addition,

$$\begin{cases} \langle \nabla F_k(x(\tau_k)), f_k(x(\tau_k), u(\tau_k^+)) \rangle_{\mathbb{R}^n} \geq \beta_k, \\ \langle \nabla F_k(x(\tau_k)), f_{k+1}(x(\tau_k), u(\tau_k^-)) \rangle_{\mathbb{R}^n} \geq \beta_k. \end{cases} \quad (\text{TC}'')$$

It is easy to see that  $(\text{TC}) \Rightarrow (\text{TC}')$ , since (TC) implies a transversal crossing in all nearby trajectories, whereas (TC') involves only the nominal trajectory. Additionally,  $(\text{TC}'') \Rightarrow (\text{TC}')$  namely because (TC') does not require a regularity hypothesis on the nominal control.

### 1.3.3 Reduction to a temporally hybrid OCP

In this section, we will establish a correspondence from the spatially hybrid optimal control problem (SHP) to a temporally hybrid optimal control problem. First, let  $(x^*, u^*) \in \text{AC}([0, T], \mathbb{R}^n) \times L^\infty([0, T], \mathbb{R}^m)$  be a global solution to Problem (SHP), associated with a partition  $\mathbb{T}^* = \{\tau_k\}_{k=0, \dots, N}$ , that is a regular solution to (HS). Let us denote by  $E_k^*$ ,  $f_k^*$  and  $F_k^*$  the corresponding

regions, dynamics and local description of  $E_k^* \cap E_{k+1}^*$  (as in Definition 4 Item (i)). Hence, we get that the triplet  $(x^*, u^*, \mathbb{T}^*)$  is admissible for the temporally hybrid optimal control problem given by

$$\begin{aligned}
& \text{minimize} \quad \phi(x(0), x(T)), \\
& \text{subject to} \quad (x, u, \mathbb{T}) \in \text{AC}([0, T], \mathbb{R}^n) \times L^\infty([0, T], \mathbb{R}^m) \times \mathbb{R}^{N+1}, \\
& \quad \dot{x}(t) = f_k^*(x(t), u(t)), \quad \text{a.e. } t \in (\tau_{k-1}, \tau_k), \quad \text{for all } k \in \{1, \dots, N\}, \\
& \quad g(x(0), x(T)) \in S, \\
& \quad u(t) \in U, \quad \text{a.e. } t \in [0, T], \\
& \quad \mathbb{T} = \{\tau_k\}_{k=0, \dots, N} \in \Delta, \\
& \quad F_k^*(x(\tau_k)) = 0, \quad \text{for all } k \in \{1, \dots, N-1\},
\end{aligned} \tag{THP'}$$

where  $\Delta \subset \mathbb{R}^{N+1}$  is the nonempty closed convex set defined by

$$\Delta := \{\mathbb{T} = \{\tau_k\}_{k=0, \dots, N} \in \mathbb{R}^{N+1} \mid 0 = \tau_0 \leq \tau_1 \leq \dots \leq \tau_{N-1} \leq \tau_N = T\}.$$

Now, we are in position to give the correspondence between Problems (SHP) and (THP').

**Proposition 1.** *If  $(x^*, u^*) \in \text{AC}([0, T], \mathbb{R}^n) \times L^\infty([0, T], \mathbb{R}^m)$  is a global solution to Problem (SHP), that is moreover a regular solution to (HS), associated with a partition  $\mathbb{T}^* = \{\tau_k^*\}_{k=0, \dots, N}$ , then the triplet  $(x^*, u^*, \mathbb{T}^*)$  is a  $L^1$ -local solution to Problem (THP').*

*Proof.* For the sake of simplicity, we assume that  $N = 2$ . Let us prove that there exists  $\eta > 0$  such that  $\phi(x(0), x(T)) \geq \phi(x^*(0), x^*(T))$  for any triplet  $(x, u, \mathbb{T})$  that is admissible for Problem (THP') and satisfying

$$\|x - x^*\|_C + \|u - u^*\|_{L^1} + |\tau - \tau^*| \leq \eta.$$

First, since we have  $\tau^* \in (0, T)$  and  $x^*(0) \in E_1^*$  then, there exists  $0 < \eta_1 < \frac{1}{2}$  such that  $\tau \in (0, T)$ ,  $x(0) \in E_1^*$  and  $x(T) \in E_2^*$  if  $\|x - x^*\|_{\mathbb{R}^n} + |\tau - \tau^*| \leq \eta_1$ . Moreover, note that all constraints of Problem (SHP) are satisfied except the spatially hybrid control system. Since, one has

$$\dot{x}(t) = f_k^*(x(t), u(t)), \quad \text{a.e. } t \in (\tau_{k-1}, \tau_k),$$

for all  $k \in \{1, 2\}$ . So, it remains to prove that there exists  $0 < \eta \leq \eta_1$  such that  $x(t) \in E_1^*$  for all  $t \in [0, \tau)$  and  $x(t) \in E_2^*$  for all  $t \in (\tau, T]$ . We will only prove the first statement (in two steps) since the second one can be proved in a similar way.

**Step 1.** we prove that there exists  $0 < \delta < m$  and  $0 < \eta_2 \leq \min(\eta_1, \frac{\delta}{2})$  such that for all admissible triplets  $(x, u, \tau)$  for Problem (THP'), we have

$$\|x - x^*\|_C + \|u - u^*\|_{L^1} + |\tau - \tau^*| \leq \eta_2 \implies x(t) \in E_1^* \text{ for all } t \in [\tau - \delta, \tau),$$

where  $m := \frac{\min(\tau^*, T - \tau^*)}{3}$ . By contradiction, let us assume that for all  $\delta > 0$  and all  $\eta_2 > 0$ , there exist a triplet  $(x, u, \tau)$  and  $t' := t(\eta_2, \delta) \in [\tau - \delta, \tau)$  such that

$$\begin{cases} \|x - x^*\|_C + \|u - u^*\|_{L^1} + |\tau - \tau^*| \leq \eta_2, \\ x(t') \notin E_1^*. \end{cases}$$

Hence, up to reducing  $\delta > 0$  and  $\eta_2 > 0$  we can obtain that  $x(t') \in \overline{B}_{\mathbb{R}^n}(x^*(\tau^*), \nu_1^*)$ . This implies that  $F_1^*(x(t')) \geq 0$  and since we have  $F_1^*(x(\tau)) = 0$ , then we get

$$\frac{1}{\tau - t'} \int_{t'}^{\tau} \langle \nabla F_1^*(x(s)), f_1^*(x(s), u(s)) \rangle_{\mathbb{R}^n} ds \leq 0. \quad (1.1)$$

On the other hand, from the continuity of the functions  $\nabla F_1^*$  and  $f_1^*$  and using the transverse condition (TC) we get that there exists  $0 < \nu_1 \leq \nu_1^*$  such that

$$\forall (z, \omega) \in \overline{B}_{\mathbb{R}^n}(x^*(\tau^*), \nu_1) \times U, \langle \nabla F_1^*(z), f_1^*(z, \omega) \rangle_{\mathbb{R}^n} \geq \frac{\beta^*}{2}.$$

Let us prove that up to reducing  $\delta > 0$  and  $\eta_2 > 0$ , we get that  $x(s) \in \overline{B}_{\mathbb{R}^n}(x^*(\tau^*), \nu_1)$  for all  $s \in [t', \tau]$ . Indeed, we have that

$$\begin{aligned} \|x(s) - x^*(\tau^*)\|_{\mathbb{R}^n} &\leq \|x - x^*\|_C + \|x^*(s) - x^*(\tau^*)\|_{\mathbb{R}^n} \\ &\leq \eta_2 + M|s - \tau^*| \\ &\leq \eta_2 + M(|s - \tau| + |\tau - \tau^*|) \\ &\leq (1 + M)\eta_2 + \delta, \end{aligned}$$

for all  $s \in [t', \tau]$  where  $M > 0$  is an upper bound of  $\|\nabla_x f_1^*\|_{\mathbb{R}^n \times n}$  and  $\|\nabla_x f_2^*\|_{\mathbb{R}^n \times n}$  over the compact set  $\{z \in \mathbb{R}^n \mid \|z - x^*(t)\|_{\mathbb{R}^n} \leq 1, \forall t \in [0, T]\} \times U$ . Hence, it suffices to take  $(1 + M)\eta_2 + \delta < \frac{\nu_1}{2}$ . Moreover, we have that  $u(s) \in U$  for almost every  $s \in [0, T]$ . Therefore, we get that

$$\langle \nabla F_1^*(x(s)), f_1^*(x(s), u(s)) \rangle_{\mathbb{R}^n} \geq \frac{\beta^*}{2}, \text{ for a.e. } s \in [t', \tau].$$

Hence, we get that  $\frac{1}{\tau - t'} \int_{t'}^{\tau} \langle \nabla F_1^*(x(s)), f_1^*(x(s), u(s)) \rangle_{\mathbb{R}^n} ds \geq \frac{\beta^*}{2}$ , which contradicts (1.1).

**Step 2.** It remains to prove that there exists  $0 < \eta_3 < \eta_2$  such that for all admissible triplet  $(x, u, \tau)$  for Problem (THP') we have

$$\|x - x^*\|_C + \|u - u^*\|_{L^1} + |\tau - \tau^*| \leq \eta_2 \implies x(t) \in E_1^* \text{ for all } t \in [0, \tau - \delta].$$

Since  $x(t) \in E_1^*$  for all  $t \in [0, \tau^*)$  and  $E_1^*$  is open we get that there exists a (uniform)  $\sigma^* > 0$  such that  $\overline{B}_{\mathbb{R}^n}(x^*(t), \sigma^*) \subset E_1^*$  for all  $t \in [0, \tau^* - \frac{\delta}{2}]$ . Now, by taking  $0 < \eta_3 < \min(\eta_2, \frac{\delta}{2}, \frac{\sigma^*}{2})$ , we get that

$$|\tau - \tau^*| < \eta_3 \implies \tau - \delta < \tau^* - \frac{\delta}{2} < \tau.$$

Moreover, one can easily verify that if  $\|x - x^*\|_C < \eta_3$  then we have  $x(t) \in E_1^*$  for all  $t \in [0, \tau^* - \frac{\delta}{2}]$ . This completes the proof.

### 1.3.4 Spatially HMP and comments

The Hamiltonian  $H : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$  associated with Problem (SHP) is defined by  $H(x, u, p) := \langle p, h(x, u) \rangle$  for all  $(x, u, p) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$ . We are now in a position to state a HMP for Problem (SHP).

**Theorem 3 (Spatially HMP for global solutions).** *If  $(x^*, u^*)$  is a global solution to Problem (SHP), that is moreover regular to (HS), associated with a partition  $\mathbb{T}^* = \{\tau_k^*\}_{k=0, \dots, N}$ , then there exists a nontrivial pair  $(p, p^0) \in \text{PAC}_{\mathbb{T}^*}([0, T], \mathbb{R}^n) \times \mathbb{R}_+$  satisfying:*

(i) *the Hamiltonian system*

$$\dot{x}^*(t) = \nabla_p H(x^*(t), u^*(t), p(t)), \quad -\dot{p}(t) = \nabla_x H(x^*(t), u^*(t), p(t)),$$

*for almost every  $t \in [0, T]$ ;*

(ii) *the endpoint transversality condition*

$$\begin{pmatrix} p(0) \\ -p(T) \end{pmatrix} = p^0 \nabla \phi(x^*(0), x^*(T)) + \nabla g(x^*(0), x^*(T)) \xi,$$

*for some  $\xi \in \mathbb{N}_S[g(x^*(0), x^*(T))]$ ;*

(iii) *the Hamiltonian maximization condition*

$$u^*(t) \in \arg \max_{\omega \in U} H(x^*(t), \omega, p(t)),$$

*for almost every  $t \in [0, T]$ ;*

(iv) *the discontinuity condition*

$$p^+(\tau_k^*) - p^-(\tau_k^*) = \sigma_k \nabla F_k^*(x^*(\tau_k^*)),$$

*for some  $\sigma_k \in \mathbb{R}$ , for all  $k \in \{1, \dots, N-1\}$ .*

*Proof.* Let  $(x^*, u^*) \in \text{AC}([0, T], \mathbb{R}^n) \times L^\infty([0, T], \mathbb{R}^m)$  be a global solution to Problem (SHP), associated with a partition  $\mathbb{T} = \{\tau_k^*\}_{k=0, \dots, N}$ , that is moreover a regular solution to (HS). From Proposition 1, we get that the triplet  $(x^*, u^*, \mathbb{T}^*)$  is a  $L^1$ -local solution to Problem (THP'). Hence, the proof is concluded by applying Theorem 2.

*Remark 2.* Hereafter, we provide some comments about the derivation of a HMP for Problem (SHP).

- (i) Let us emphasize that the transverse condition (TC) plays a crucial role in proving the correspondence between Problems (SHP) and (THP') and therefore the (direct) derivation of a HMP for Problem (SHP) from Theorem 2 (we refer to the proof of Proposition 1 for details).

- (ii) Note that under a weaker transverse condition (see [3, 5, 19]), then a reduction to a temporally hybrid OCP is no longer possible. Indeed, in the following section, we provide a counterexample emphasizing this issue.
- (iii) Whereas  $F_k$  is supposed to be submersive in order to prove Theorem 2, this hypothesis is implicit in Theorem 3, namely because of the transverse conditions.
- (iv) Because the problems considered in this paper are autonomous, one can prove that the maximized Hamiltonians  $\mathcal{H}$ ,  $H_1$ , and  $H$  are constant a.e. over  $[0, T]$ . We refer for instance to [20, 16, 5] for more details on this issue.

## 1.4 Failure of reduction : an example

The aim of this section is to highlight the differences between temporally and spatially hybrid OCPs by means of an explicit example. Doing so, we will calculate optimal solutions of two similar (in the spirit of the reduction discussed previously) hybrid optimal control problems in which the first one involves a partition of the state space, but not the second one.

### 1.4.1 Optimal synthesis of a spatially hybrid OCP

Let us start by introducing the spatially OCP. In what follows, we set  $a_+ := \max(a, 0)$  for  $a \in \mathbb{R}$  and let  $\mathcal{U}$  denotes the set of measurable control functions  $u : [0, 2] \rightarrow [-1, 1]$ . The spatially OCP we consider reads as follows:

$$\text{minimize } -(x_1(2) - 2)^3 - \rho \int_0^2 (1 - x_1(t))_+^2 dt, \quad (1.2)$$

among solutions to the control system

$$\dot{x}_1(t) = \begin{cases} u(t) + 2 & \text{a.e. } t \in [0, 2] \text{ s.t. } x_1(t) < 1, \\ u(t) & \text{a.e. } t \in [0, 2] \text{ s.t. } x_1(t) > 1, \end{cases}$$

with  $x_1(0) = 0$  and  $u \in \mathcal{U}$ . Here  $\rho$  is a positive parameter. Since the system is defined over a partition of  $\mathbb{R}$ , this problem falls into the setting of spatially hybrid OCPs. Hereafter, as usual, a *switching time* of a control  $u \in \mathcal{U}$  denotes an instant  $t_s \in (0, 2)$  such that  $u$  is non-constant in every neighborhood of  $t_s$ .

**Proposition 2.** *If  $\rho \neq 4/3$ , then, an optimal control of (1.2) is  $\bar{u}(t) = -1$  for a.e.  $t \in [0, 1]$  ;  $\bar{u}(t) = +1$  for a.e.  $t \in [1, 2]$  and its associated cost is  $-\frac{\rho}{3}$ .*

*Proof.* By the usual transformation of a Bolza Problem into a Mayer one, it is easily seen that (1.2) is equivalent to the Mayer OCP:

$$\text{minimize } -(x_1(2) - 2)^3 - \rho x_2(2),$$

where  $(x_1, x_2)$  is a solution to the spatially hybrid control system over  $[0, 2]$ :

$$\begin{cases} \dot{x}_1 = u + 2 \\ \dot{x}_2 = (1 - x_1)_+^2 \end{cases} \Leftrightarrow (x_1, x_2) \in X_1 ; \begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = (1 - x_1)_+^2 \end{cases} \Leftrightarrow (x_1, x_2) \in X_2, \quad (1.3)$$

with  $x_1(0) = x_2(0) = 0$ ,  $X_1 := (-\infty, 1) \times \mathbb{R}$ ,  $X_2 := (1, +\infty) \times \mathbb{R}$  and  $u \in \mathcal{U}$ . Let also  $\Sigma := \{1\} \times \mathbb{R}$  denotes the interface between  $X_1$  and  $X_2$ . Observe that  $X_2 \cup \Sigma$  is forward invariant by (1.3). We deduce that every admissible trajectory  $(x_1, x_2)$  of (1.3) necessarily intersects  $\Sigma$  at some time  $t_c \in [\frac{1}{3}, 1]$  and that it satisfies  $(x_1(t), x_2(t)) \in X_2 \cup \Sigma$  for every  $t \in [t_c, 2]$ . Now, we are in a position to apply Filippov's existence theorem (see [11]) which proves the existence of a solution to (1.2) (for brevity we omit this issue, but, we point out that the standard argumentation needs a slight technical adjustment in view of (1.3) which has a discontinuous right hand side). Hereafter, we denote by  $\bar{u}$  an optimal control, by  $(\bar{x}_1, \bar{x}_2)$  the associated solution to (1.3), and let  $t_c := \min\{t > 0 ; \bar{x}_1(t) = 1\}$ .

Let us now show that any sliding mode is non optimal. Suppose first that  $\bar{x}_1$  satisfies  $\bar{x}_1(t) \in \Sigma$  for  $[t_c, t_c + \eta]$  where  $\eta \in (0, 1)$ . Let us define an admissible trajectory of the system  $(x_1, x_2)$  such that  $x_1$  coincides with  $\bar{x}_1$  over  $[0, t_c]$  and  $\dot{x}_1(t) = 1$  for a.e.  $t \in [t_c, 2]$ . We define  $u$  as the admissible control function associated with this trajectory over  $[0, 2]$ . Clearly, since  $t_c < 2$ , we have  $x_1(2) > \bar{x}_1(2)$ . Using the forward invariance of  $X_2 \cup \Sigma$  and the fact that  $1 - \bar{x}_1(t)_+ = 0$  if  $\bar{x}_1(t) \geq 1$ , we deduce that

$$\begin{aligned} -(x_1(2) - 2)^3 - \rho \int_0^2 (1 - (x_1(t))_+^2) dt &= -(x_1(2) - 2)^3 - \rho \int_0^{t_c} (1 - (\bar{x}_1(t))_+^2) dt \\ &< -(\bar{x}_1(2) - 2)^3 - \rho \int_0^{t_c} (1 - (\bar{x}_1(t))_+^2) dt \\ &= -(\bar{x}_1(2) - 2)^3 - \rho \int_0^2 (1 - (\bar{x}_1(t))_+^2) dt, \end{aligned}$$

which contradicts the optimality of  $\bar{u}$ . By the same comparison method, one can prove that  $\bar{u}(t) = 1$  for a.e.  $t \in (t_c, 2]$  which shows in particular that the optimal trajectory does not possesses any sliding mode.

We now wish to apply the HMP in order to compute the optimal control over  $[0, t_c]$ . Using that  $\bar{u}(t) = +1$  for  $t \geq t_c$  and since  $2 + \omega \geq 1$  for every  $\omega \in [-1, 1]$ , we deduce that condition (TC') (see also [5]) is fulfilled<sup>5</sup> so that we can apply the HMP. Let

$$H(x_1, x_2, p_1, p_2, u) := p_1(2 + u) + \rho(1 - x_1)_+^2 = p_1 u + \rho(1 - x_1)_+^2,$$

denotes the Hamiltonian associated with (1.2) (written in the two regions). The adjoint system and terminal condition write respectively

<sup>5</sup> However, the optimal solution is not regular at this point since the dynamics for  $x_1$  in  $X_2$  is  $\dot{x}_1 = u$  with  $u \in [-1, 1]$ .



$$\begin{cases} \dot{p}_1 = 2(1 - x_1)_+, \\ \dot{p}_2 = 0, \end{cases} \quad (1.4)$$

and

$$p_1(2) = 3(x_1(2) - 2)^2 \ ; \ p_2(2) = \rho. \quad (1.5)$$

Finally, the jump condition implies that at a crossing time,  $p_1$  can have a jump whereas  $p_2$  is continuous (the outward unit normal vector to  $X_1$  being the vector  $(1, 0)$ ). Hence,  $p_2$  is constant over  $[0, 2]$  and  $p_2 = \rho$ . In order to exclude non-optimal extremals, we proceed as follows.

*Case 1 :* suppose that  $\bar{x}_1(2) \neq 2$ . We know that  $\bar{u}(t) = +1$  for  $t \in [t_c, 2]$  with  $t_c < 1$ . Also, the optimal trajectory fulfills  $\dot{p}_1 = 2(1 - \bar{x}_1) > 0$  over  $[0, t_c]$ , so,  $p_1$  has at most one switching time  $t_s \in (0, t_c)$  from  $-1$  to  $+1$  over  $[0, t_c]$ . If  $\bar{x}_1$  has a switching time  $t_s \in (0, t_c)$ , then,  $p_1(t_s) = 0$ . Using the constancy of the Hamiltonian along extremals, we get:

$$p_1(t_s)(2 + \bar{u}(t_s^-)) + \rho(1 - x_1(t_s))^2 = 3(\bar{x}_1(2) - 2)^2.$$

Using that  $\bar{x}_1(t_s) = t_s$ ,  $\bar{x}_1(t_c) = 1$ ,  $3t_c - 2t_s = 1$ ,  $\bar{x}_1(t) = t + 1 - t_c$  for  $t > t_c$ .  $\bar{x}_1(2) - 2 = 1 - t_c$  and  $p_1(t_s) = 0$ , the preceding equality becomes

$$\rho \left( \frac{3(1 - t_c)}{2} \right)^2 = 3(1 - t_c)^2,$$

which has no solution for  $\rho \neq 4/3$ . We conclude that this case is not possible.

*Case 2 :*  $\bar{x}_1(2) = 2$ . The only possible control values for  $\bar{u}$  are such that  $\bar{u}(t) = -1$  a.e.  $t \in [0, 1]$  and  $\bar{u}(t) = 1$  for a.e.  $t \in [0, 2]$ . Note that the covector corresponding to this solution satisfies  $p_1 \equiv 0$  over  $[0, 2]$  (combining the jump of the covector  $p_1$  and the fact that  $p_1(2) = 0$  and  $\dot{p}_1 = 0$  for  $t > t_c$ ). The computation of the corresponding cost is straightforward which ends the proof.

#### 1.4.2 Optimal synthesis of a temporally hybrid OCP

Observe that the optimal solution satisfies  $\bar{x}_1(t) = t$  for  $t \in [0, 2]$  (see Fig. 1.2), that  $\bar{u}$  is given by

$$\bar{u}(t) = \begin{cases} -1 & \text{a.e. } t \in (0, \bar{\tau}), \\ +1 & \text{a.e. } t \in (0, \bar{\tau}), \end{cases}$$

where  $\bar{\tau} := 1$ , and that  $\bar{x}_1$  is transverse<sup>6</sup> at the interface  $\Sigma$  between the two regions. In view of problem (1.2) and of its solution  $\bar{u}$ , it is naturel to introduce the closely related hybrid OCP:

$$\text{minimize } -(x_1(2) - 2)^3 - \rho \int_0^2 (1 - x_1(t))_+^2 dt, \quad (1.6)$$

among solutions to the control system

---

<sup>6</sup> In the sense given in [5].

$$\dot{x}_1(t) = \begin{cases} u(t) + 2 & \text{a.e. } t \in [0, \tau), \\ u(t) & \text{a.e. } t \in (\tau, 2], \end{cases}$$

such that  $x_1(0) = 0$ ,  $x_1(\tau) = 1$ ,  $u \in \mathcal{U}$ , and  $\tau \in (0, 2)$  is an additional optimization parameter. In contrast with the previous problem, (1.6) falls into the setting of temporally hybrid OCPs since a change of dynamics is imposed at time  $\tau$  (but, no partition is considered here). In view of the solution to (1.2), the question we are interested in is as follows : is there a solution to (1.6) identical to the one found previously? We will see that, depending on the parameter values of  $\rho$ , the answer to the preceding question can be negative (see Proposition 3 and table 1.1). More precisely, our computations show that for  $\rho$  large enough, the optimal solution to (1.6) is not  $\bar{u}$ . An intuitive reason for getting a different solution for this problem is that trajectories can benefit of the Lagrangian cost in (1.6) which is not the case for admissible solutions to (1.2) since  $X_2 \cup \Sigma$  is forward invariant by (1.2). Note that, of course,  $\bar{x}_1$  is also not regular in the sense given above (since one has  $\dot{x}_1 = u$  in  $X_2$  with  $u \in [-1, 1]$ ). Otherwise, we would conclude from the preceding part that  $\bar{u}$  is a solution to (1.6) and that (1.2)-(1.6) have the same solution. Let us now turn to the resolution of (1.6). In contrast with (1.2), we can straightforwardly apply the HMP (for instance from [16]) on (1.6). For conveniency, we rewrite (1.6) as the Mayer OCP:

$$\text{minimize } -(x_1(2) - 2)^3 - \rho x_2(2),$$

where  $(x_1, x_2)$  is a solution to the control system

$$\begin{cases} \dot{x}_1 = u + 2, \\ \dot{x}_2 = (1 - x_1)_+^2, \end{cases}$$

over  $(0, \tau)$  and of

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = (1 - x_1)_+^2, \end{cases}$$

over  $(\tau, 2)$  such that  $x_1(0) = 0$ ,  $x_1(\tau) = 1$ ,  $u \in \mathcal{U}$ , and  $\tau \in (0, 2)$  is an optimization variable. As in (1.2), the existence of an optimal control follows from a slight adaptation of Fillipov's existence Theorem. Let us now apply the HMP (which in this setting does not require verification of a transverse hypothesis). The Hamiltonian associated with (1.6) (written over  $(0, \tau)$  and over  $(\tau, 2)$ ) is:

$$H(x_1, x_2, p_1, p_2, u) := p_1(2 + u) + \rho(1 - x_1)_+^2 = p_1 u + \rho(1 - x_1)_+^2,$$

and the adjoint system and terminal condition are given by (1.4)-(1.5). Finally, the covector  $p$  can have a jump only at time  $t = \tau$  and similarly as for (1.2),  $p_2$  is continuous at  $t = \tau$  so that  $p_2 = \rho$  over  $[0, 2]$ . Even though necessary conditions are analogous in the two problems, let us remind that the problem under investigation is temporally which may change the structure

of a solution with respect to  $\bar{u}$ . Hereafter,  $B_{\pm}$  denotes a bang arc  $u = \pm 1$  over some time interval,  $(u^*, \tau^*)$  denotes an optimal pair, and  $(x_1^*, x_2^*)$  is the associated trajectory.

**Proposition 3.** *The optimal solution is one of the following three types:*

- $\{B_-B_+\}$  with  $(u^*, \tau^*) = (\bar{u}, 1)$  ;
- $\{B_+B_-B_+\}$  with two switching times at  $\tau^* = \frac{1}{3}$  and at some  $t_s \in (\frac{1}{3}, 2)$  ;
- $\{B_-B_+B_-B_+\}$  with three switching times at some instant  $t_s \in (\frac{1}{3}, 1)$ , at the crossing time  $\tau^*$ , and at some instant  $t'_s \in (\tau^*, 2)$  such that  $x_1^*(t_s) = x_1^*(t'_s)$ .

*Proof. First case :* suppose that  $x_1^*(2) > 2$ . From the terminal condition and from the adjoint equation, we get  $p_1 = 0$  as long as  $x_1^* > 1$ . It follows that there is  $\tau^* \in (1/3, 1)$  such that  $u^*$  fulfills  $u^*(t) = 1$  for every  $t \in (\tau^*, 2)$ . By using the constancy of the Hamiltonian, we get that  $p_1(\tau^*)^- > 0$ . Since  $x_1^*$  is increasing over  $[0, \tau^*]$ , we obtain from the adjoint equation  $\dot{p}_1 = 2(1 - x_1^*)_+$  that the trajectory has at most one switching time  $t_s \in (0, \tau^*)$  (for which  $p_1(t_s) = 0$ ). It follows that the cost of  $x_1^*$  writes  $\int_0^{t_c} (1 - x_1^*(t))^2 dt$  where  $t_c := \min\{t > 0 ; x_1^*(t) = 1\}$ , but one has

$$\int_0^{t_c} (1 - x_1^*(t))^2 dt < \int_0^{t_c} (1 - t)^2 dt.$$

This implies that  $\bar{x}_1$  has a strictly lower cost than  $x_1^*$  which shows that an optimal solution cannot satisfy  $x_1^*(2) > 2$ .

*Second case :* suppose that  $x_1^*(2) = 2$ . We necessarily have  $u^*(t) = +1$  a.e.  $t \in [1, 2]$ . Otherwise, by using a similar argumentation as in case 1,  $x_1^*$  would have a crossing point at an instant  $\tau^* \in (1/3, 1)$ , but, then, its cost would be strictly greater than the cost of  $\bar{x}_1$ . Hence, this case leads to  $u^* = \bar{u}$ .

*Third case :* suppose that  $x_1^*(2) < 2$ . Note that  $x_1^*$  satisfies  $x_1^*(0) = 0$ , so, one must have  $\tau^* \in (0, 1/3)$ . Now, we have  $p_1(2) = 3(x_1^*(2) - 2)^2 > 0$ , hence we necessarily have  $u^* = 1$  in a left neighborhood of  $t = 2$ . Now,  $u^*$  necessarily switches at some instant from  $-1$  to the last bang arc  $u^* = 1$  (otherwise, this would be in contradiction with  $x_1^*(0) = 0$ ). Since  $p_1$  is constant whenever  $x_1^* \geq 1$  and  $\dot{p}_1 > 0$  whenever  $x_1^* < 1$ , we deduce that there is a unique  $t'_s \in (2/3, 2)$  such that  $u^*$  switches from  $u = -1$  to  $u = 1$  at  $t = t'_s$ . Hence, one has  $u^*(t) = -1$  for a.e.  $t < t'_s$  sufficiently close to  $t'_s$ . The only possibility for the trajectory to reach the origin at time  $t = 0$  (by reasoning backward in time) is to switch from  $u = +1$  to  $u = -1$  at the crossing time  $\tau^*$  for which  $x_1^*(\tau^*) = 1$  (otherwise, we would have  $u^*(t) = -1$  for a.e.  $t < \tau^*$  contradicting  $x_1^*(0) = 0$ ). Now, by a similar reasoning backward in time from  $t = \tau^*$ , the monotonicity property of the switching function implies that the trajectory possesses at most one switching time  $t_s \in (0, \tau^*)$ . It follows that  $(u^*, \tau^*)$  is of type  $\{B_+B_-B_+\}$  and that  $\tau^* = 1/3$  if  $t_s$  does not exist. Otherwise, if  $t_s$  occurs, then it is of type  $\{B_-B_+B_-B_+\}$  and using the constancy of  $H$  at  $t = t_s$  and  $t = t'_s$ , we get that  $x_1^*(t_s) = x_1^*(t'_s)$ . This ends the proof.

At this step, we see that  $\bar{u}$  is also a candidate for optimality for Problem (1.6), but, additionally, an optimal control can also be a sequence of three or four bang arcs. From the HMP, we can completely characterize extremals of type  $\{B_+B_-B_+\}$  as we show in the next lemma.

**Lemma 1.** *Suppose that  $\rho > 96/25$  and consider an extremal  $(u^*, \tau^*)$  of type  $\{B_+B_-B_+\}$ . Then, one has  $\tau^* = 1/3$  and  $t_s = \frac{5}{3} - \frac{\alpha}{2}$  where*

$$\alpha := \frac{2(2\rho + 36 - \sqrt{25\rho^2 - 96\rho})}{9\rho + 36}.$$

*Proof.* Setting  $\alpha := x_1^*(2)$ ,  $x_1^*$  can be expressed as follows:

$$x_1^*(t) = \begin{cases} 3t & \text{if } t \in [0, \frac{1}{3}], \\ -t + \frac{4}{3} & \text{if } t \in [\frac{1}{3}, t_s], \\ t + \alpha - 2 & \text{if } t \in [t_s, 2], \end{cases}$$

where  $t_s = 5/3 - \alpha/2$ . Using the constancy of the Hamiltonian (at time  $t = 2$  and  $t = t_s$ ), we find that  $\alpha$  is a solution to the algebraic equation

$$3(\alpha - 2)^2 + \rho(1 - \alpha)_+^2 = \rho(1 - x_1(t_s))^2,$$

or equivalently to  $3(\alpha - 2)^2 + \rho(1 - \alpha)_+^2 = \rho(\frac{4}{3} - \frac{\alpha}{2})^2$ . Solving this equation gives us the desired value of  $\alpha$  which ends the proof.

By using a similar argumentation, one can also characterize extremals of type  $\{B_-B_+B_-B_+\}$ .

**Lemma 2.** *Suppose that  $\rho > 48/9$  and consider an extremal  $(u^*, \tau^*)$  of type  $\{B_-B_+B_-B_+\}$ . Then, one has  $t_s = \frac{3\alpha}{4} - \frac{1}{2}$ ,  $\tau = \frac{\alpha}{2}$  and  $t'_s = \frac{3}{2} - \frac{\alpha}{4}$  where*

$$\alpha = \frac{2(-\rho + 48 + 2\sqrt{9\rho^2 - 48\rho})}{7\rho + 48}.$$

*Proof.* For brevity, we omit the proof that is analogous to the proof of the preceding lemma.

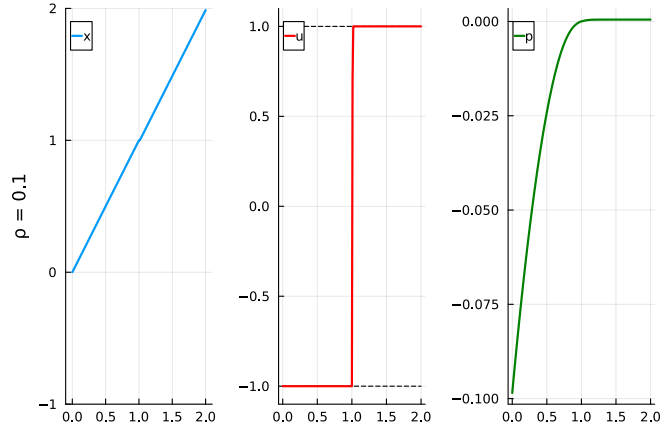
Thanks to the preceding results, we see that for  $\rho > 48/9$ , then there are three types of possible extremals (candidate solution of (1.6)). We end up by a numerical computation of an optimal solution which allows us to determine which extremal is optimal :

- for small values of  $\rho$  (typically,  $\rho = 0.1$ ), then, the solution to (1.6) is  $\bar{u}$  (see Fig. 1.2).
- for larger values of  $\rho$  (typically,  $\rho = 10, 30, 100$ ), then, solutions to (1.6) can benefit of the Lagrangian cost and the solution is the extremal  $B_+B_-B_+$  (see Fig. 1.3 for  $\rho = 10$ , Fig. 1.4 for  $\rho = 30$  and Fig. 1.5 for  $\rho = 100$ ).

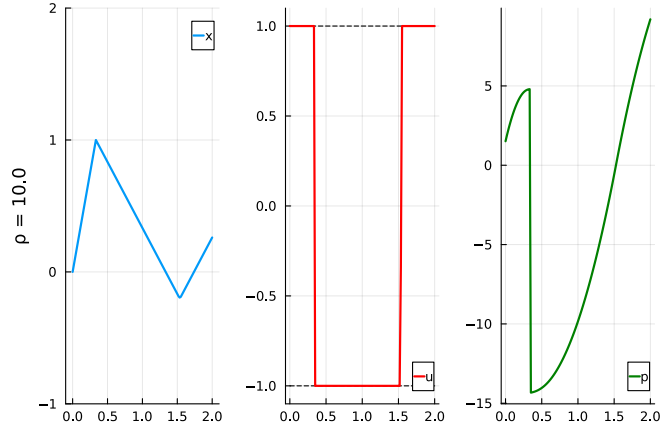
In table 1.1, we indicate the cost  $J^*$  of the optimal solution  $u^*$  and its structure as well as the optimal cost value  $J^\dagger := -\rho/3$  of (1.2). In particular, when  $\rho$  increases, then,  $|J^\dagger - J^*|$  also increases.

$\rho$	0.1	10	30	100
$J^*$	-0.33	-6.13	-34.55	-148.02
$u^*$	$B_+B_-$	$B_+B_-B_+$	$B_+B_-B_+$	$B_+B_-B_+$
$J^\dagger$	-0.33	-3.33	-10	-33.33

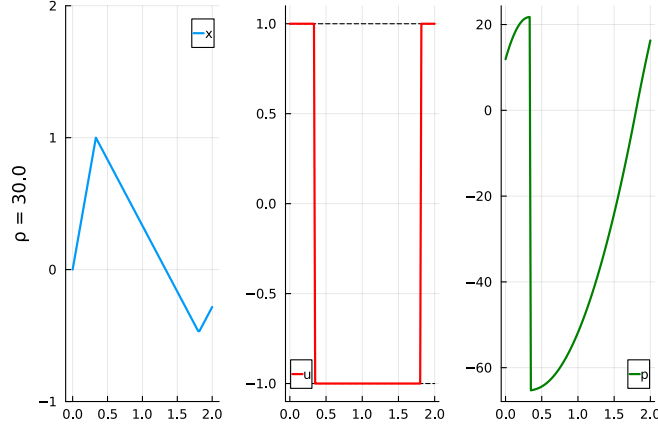
**Table 1.1.** Comparison of the optimal cost value of (1.2) and (1.6). For large values of  $\rho$ , both problems have different solutions.



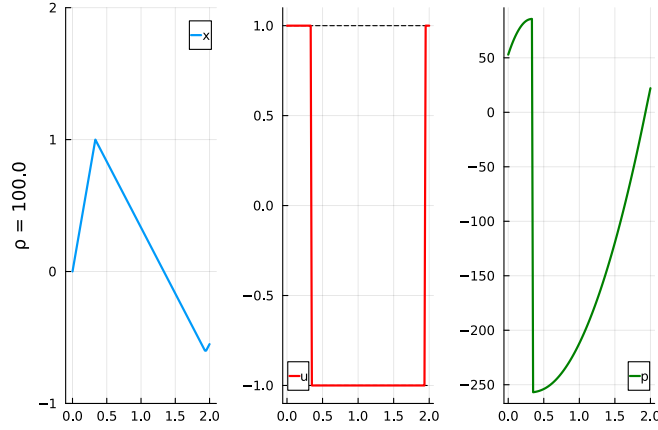
**Fig. 1.2.** Optimal solution of (1.2) and (1.6) for  $\rho = 0.1$  (plot of the trajectory, control, and co-state  $p_1$ ).



**Fig. 1.3.** Optimal solution  $B_+B_-B_+$  of (1.6) for  $\rho = 10$  (plot of the trajectory, control, and co-state  $p_1$  with a jump at  $\tau^*$ ).



**Fig. 1.4.** Optimal solution  $B_+B_-B_+$  of (1.6) for  $\rho = 30$  (plot of the trajectory, control, and co-state  $p_1$  with a jump at  $\tau^*$ ).



**Fig. 1.5.** Optimal solution  $B_+B_-B_+$  of (1.6) for  $\rho = 100$  (plot of the trajectory, control, and co-state  $p_1$  with a jump at  $\tau^*$ ).

### 1.4.3 Further comments

In conclusion, this example made it possible to show a concrete application of the HMP in two different hybrid frameworks. The optimal synthesis has shown that the solution to the spatially hybrid OCP (1.2) has exactly one crossing time which has allowed us to define the auxiliary temporally hybrid OCP (1.6). Next, we have seen that for  $\rho$  large enough, then, an optimal solution of (1.6) is of type  $B_+B_-B_+$  whereas the solution to (1.2) is always  $\bar{u}$ . In this case, the value functions for the two problems widely differ. In contrast,

for small values of the parameter  $\rho$ , solutions to the two problems can be the same (intuitively, there is no point in taking advantage of the Lagrangian cost in (1.6)). Hence, a solution to (1.2) is not necessarily a solution to (1.6) which corroborates the fact that these two frameworks are different. More generally, this example highlights the fact that even under a transverse hypothesis (in the sense given in [5]) on an optimal solution of a spatially hybrid OCP, such a problem cannot in general be reduced to a temporally hybrid OCP (that is, without a stratification, but taking into account the same number of switching times of the dynamics than the solution to the spatially hybrid OCP). It also indicates that the HMP for spatially hybrid OCPs cannot be straightforwardly retrieved from the HMP for temporally hybrid OCPs as in [16] under a (weak) transverse hypothesis on the nominal trajectory at the crossing points. Let us mention that in [5], a similar approach (based on the augmentation technique) could be completely implemented without the concept of regular solution (using only a weak transverse hypothesis on an optimal solution). But, it required the introduction of new tools (such as a new notion of minimum in optimal control) to circumvent the difficulties highlighted by this type of example which shows that a global solution to the spatially hybrid OCP is not necessarily a solution to the temporally hybrid OCP. Thanks to our results, we expect to have clarified the connections between temporally and spatially hybrid OCPs. In particular, necessary optimality conditions in both cases are the same (Hamiltonian maximization condition, transversality condition, jump of the covector), but the frameworks being different, these do not necessarily describe the same set of extremal solutions.

In view of the jumps of the covector arising in the HMP, future works could also investigate if it possible to derive the spatially HMP from the PMP with state constraints (for which the adjoint system involves a measure with a discrete part which has an analogy with the aforementioned jumps).

## A Proof of Theorem 2

Let  $(x^*, u^*, \mathbb{T}^*) \in AC([0, T], \mathbb{R}^n) \times L^\infty([0, T], \mathbb{R}^m) \times \mathbb{R}^{N+1}$  be a  $L^1$ -local solution to Problem (THP). Then, there exists  $\eta^* > 0$  such that  $\phi(x^*(0), x^*(T)) \leq \phi(x(0), x(T))$  for every admissible triplet  $(x, u, \mathbb{T})$  satisfying:

$$\|x - x^*\|_C + \|u - u^*\|_{L^1} + \|\mathbb{T} - \mathbb{T}^*\|_{\mathbb{R}^{N+1}} < \eta^*. \quad (1.7)$$

The proof is done in four steps.

**Step 1: augmentation procedure.** The goal is to reduce Problem (THP) to a classical OCP of the form (CP). To do so, we introduce  $(y^*, v^*)$  as follows

$$y_k^*(s) := x^*(\tau_{k-1}^* + (\tau_k^* - \tau_{k-1}^*)s) \quad \text{and} \quad v_k^*(s) := u^*(\tau_{k-1}^* + (\tau_k^* - \tau_{k-1}^*)s), \quad (1.8)$$

for all  $s \in [0, 1]$  and all  $k \in \{1, \dots, N\}$ . We get that the triplet  $(y^*, v^*, \mathbb{T}^*)$  is admissible for the classical OCP given by

$$\begin{aligned}
& \text{minimize} && \phi^*(y(0), y(1)), \\
& \text{subject to} && (y, v, \mathbb{T}) \in \text{AC}([0, 1], \mathbb{R}^{nN}) \times L^\infty([0, 1], \mathbb{R}^{mN}) \times \mathbb{R}^{N+1}, \\
& && \dot{y}(s) = f^*(y(s), v(s), \mathbb{T}), \quad \text{a.e. } s \in [0, 1], \\
& && g^*(y(0), y(1)) \in S^*, \\
& && v(s) \in U^N, \quad \text{a.e. } s \in [0, 1], \\
& && \mathbb{T} \in \Delta,
\end{aligned} \tag{CP'}$$

where the Mayer cost  $\phi^* : \mathbb{R}^{nN} \times \mathbb{R}^{nN} \times \mathbb{R}^{N+1} \rightarrow \mathbb{R}$ , the dynamics  $f^* : \mathbb{R}^{nN} \times \mathbb{R}^{mN} \times \mathbb{R}^{N+1} \rightarrow \mathbb{R}^{nN}$  and the function  $g^* : \mathbb{R}^{nN} \times \mathbb{R}^{nN} \times \mathbb{R}^{N+1} \rightarrow \mathbb{R}^{\ell^*}$  are of class  $C^1$  and are defined as follows : one has  $\phi^*(y^0, y^1) := \phi(y_1^0, y_N^1)$  and  $g^*$  is given by

$$g^*(y^0, y^1, \mathbb{T}) := (g(y_1^0, y_N^1), y_2^0 - y_1^1, \dots, y_N^0 - y_{N-1}^1, F_1(y_1^1), \dots, F_{N-1}(y_{N-1}^1), \mathbb{T}).$$

As well,  $f^*$  is defined as

$$f^*(y, v, \mathbb{T}) := \left( (\tau_1 - \tau_0)f_1(y_1, v_1), \dots, (\tau_N - \tau_{N-1})f_N(y_N, v_N) \right),$$

for all  $y^0 = (y_1^0, \dots, y_N^0)$ ,  $y^1 = (y_1^1, \dots, y_N^1) \in \mathbb{R}^{nN}$ ,  $y = (y_1, \dots, y_N) \in \mathbb{R}^{nN}$ ,  $v = (v_1, \dots, v_N) \in \mathbb{R}^{mN}$  and  $\mathbb{T} = \{\tau_0, \dots, \tau_N\} \in \mathbb{R}^{N+1}$ , where  $\ell^* := \ell + n(N-1) + (N-1) + (N+1)$ . Finally,  $S^* \subset \mathbb{R}^{\ell^*}$  stands for the nonempty closed convex set defined by

$$S^* := S \times \{0_{\mathbb{R}^n}\}^{N-1} \times \{0\}^{N-1} \times \Delta.$$

**Step 2:  $(y^*, v^*, \mathbb{T}^*)$  is a  $L^1$ -local solution to Problem (CP').** Let us prove that there exists  $\eta > 0$  such that  $\phi^*(y^*(0), y^*(1)) \leq \phi^*(y(0), y(1))$  for any triplets  $(y, v, \mathbb{T})$  admissible for Problem (CP') and satisfying

$$\|y - y^*\|_C + \|v - v^*\|_{L^1} + \|\mathbb{T} - \mathbb{T}^*\|_{\mathbb{R}^{N+1}} < \eta.$$

To this aim let  $0 < \eta < \eta^*$  and  $(y, v, \mathbb{T})$  an admissible triplet for Problem (CP') satisfying the above inequality. Let us explain how we obtain  $\phi^*(y^*(0), y^*(1)) \leq \phi^*(y(0), y(1))$ .

- (i) First, since  $\mathbb{T}^* \in \text{Int}(\Delta)$  and up to reducing  $\eta > 0$  we get that  $0 = \tau_0 < \tau_1 < \dots < \tau_N = T$  which allows us to define the pair  $(x, u)$  as follows

$$x(t) = y_k \left( \frac{t - \tau_{k-1}}{\tau_k - \tau_{k-1}} \right) \quad \text{and} \quad u(t) = v_k \left( \frac{t - \tau_{k-1}}{\tau_k - \tau_{k-1}} \right),$$

for all  $t \in [\tau_{k-1}, \tau_k]$  and all  $k \in \{1, \dots, N\}$ .

- (ii) Second, in order to get  $\phi^*(y^*(0), y^*(1)) \leq \phi^*(y(0), y(1))$ , it remains to reduce  $0 < \eta < \eta^*$  such that  $(x, u, \mathbb{T})$  is admissible for Problem (THP). Indeed, it is easy to verify that the triplet  $(x, u, \mathbb{T})$  satisfies all constraints of Problem (THP) and from (1.7) we get that

$$\phi(x^*(0), x^*(T)) \leq \phi(x(0), x(T)) \implies \phi^*(y^*(0), y^*(1)) \leq \phi^*(y(0), y(1)).$$



**Step 3: application of Theorem 1.** Consider the Hamiltonian:  $\mathcal{H} : \mathbb{R}^{nN} \times \mathbb{R}^{mN} \times \mathbb{R}^{N+1} \times \mathbb{R}^{nN} \rightarrow \mathbb{R}$  associated with Problem (CP') defined by

$$\mathcal{H}(y, v, \mathbb{T}, q) := \langle q, f^*(y, v, \mathbb{T}) \rangle_{\mathbb{R}^{nN}} = \sum_{k=1}^N (\tau_k - \tau_{k-1}) \langle q_k, f_k(y_k, v_k) \rangle_{\mathbb{R}^n},$$

for all  $(y, v, \mathbb{T}, q) \in \mathbb{R}^{nN} \times \mathbb{R}^{mN} \times \mathbb{R}^{N+1} \times \mathbb{R}^{nN}$ . From Theorem 1 applied to the triplet  $(y^*, v^*, \mathbb{T}^*)$ , there exists a nontrivial pair  $(q, q^0) \in \text{AC}([0, 1], \mathbb{R}^{nN}) \times \mathbb{R}_+$  such that:

(i) it holds that

$$\begin{aligned} \dot{y}^*(t) &= \nabla_q \mathcal{H}(y^*(t), v^*(t), \mathbb{T}^*, q(t)), \\ -\dot{q}(t) &= \nabla_y \mathcal{H}(y^*(t), v^*(t), \mathbb{T}^*, q(t)), \end{aligned}$$

for almost every  $t \in [0, T]$ ;

(ii) it holds that

$$\begin{pmatrix} q(0) \\ -q(T) \end{pmatrix} = q^0 \nabla \phi^*(y^*(0), y^*(1)) + \nabla g^*(y^*(0), y^*(1)) \tilde{\xi},$$

for some  $\tilde{\xi} \in \text{NS}^*[g^*(y^*(0), y^*(1))]$ ;

(iii) it holds that

$$v^*(t) \in \arg \max_{\omega \in \mathbb{U}^N} \mathcal{H}(y^*(t), \omega, \mathbb{T}^*, q(t)),$$

for almost every  $t \in [0, 1]$ .

(iv) it holds that

$$\int_0^1 \nabla_{\mathbb{T}} \mathcal{H}(y^*(s), v^*(s), \mathbb{T}^*, q(s)) ds \in \text{N}_{\Delta}[\mathbb{T}^*].$$

**Step 4: construction of a nontrivial pair  $(p, p^0)$ .** Define  $p^0 := q^0 \in \mathbb{R}_+$  and  $p \in \text{PAC}_{\mathbb{T}^*}([0, T], \mathbb{R}^n)$  by  $p(0) = q_1(0)$ ,  $p(T) = q_N(1)$ , and by

$$p(t) := q_k \left( \frac{t - \tau_{k-1}^*}{\tau_k^* - \tau_{k-1}^*} \right), \text{ for all } t \in (\tau_{k-1}^*, \tau_k^*), \text{ and all } k = 1, \dots, N.$$

From the nontriviality of the pair  $(q, q^0)$  we get the nontriviality of the pair  $(p, p^0)$ . From Items (i), (ii) and (iii) above, we get Items (i), (ii) and (iv) of Theorem 2. Moreover, from the definition of  $g^*$  and  $\text{S}^*$ , we get that  $\xi = (\xi, \xi^2, \xi^3, \xi^4) \in \mathbb{R}^\ell \times \mathbb{R}^{n(N-1)} \times \mathbb{R}^{N-1} \times \mathbb{R}^{N+1}$  satisfies the following

$$\begin{aligned} \forall k \in \{2, \dots, N\}, \quad q_k(0) &= \xi_{k-1}^2 \\ \text{and } \forall k \in \{1, \dots, N-1\}, \quad -q_k(1) &= -\xi_k^2 + \xi_k^3 \nabla F_k(y_k^*(1)). \end{aligned}$$

We deduce that

$$p^+(\tau_k^*) - p^-(\tau_k^*) = q_{k+1}(0) - q_k(1) = \xi_k^3 \nabla F_k(y_k^*(1)) = \xi_k^3 \nabla F_k(x^*(\tau_k^*)),$$

for all  $k \in \{1, \dots, N-1\}$ . Therefore Item (iii) of Theorem 2 is satisfied with  $\sigma_k := \xi_k^3$  for all  $k \in \{1, \dots, N-1\}$  and the proof is complete.

## References

1. L. T. Ashchepkov, D. V. Dolgy, T. Kim and R. P. Agarwal, General optimal control problem, in *Optimal Control*, Springer, 2022, 141–164.
2. N. Augier and A. G. Yabo, Time-optimal control of piecewise affine bistable gene-regulatory networks, *International Journal of Robust and Nonlinear Control*, **33** (2023), 4967–4988.
3. G. Barles, A. Briani and E. Trélat, Value function for regional control problems via dynamic programming and Pontryagin maximum principle, *Math. Control Relat. Fields*, **8** (2018), 509–533.
4. T. Bayen, A. Bouali and L. Bourdin, Hybrid maximum principle with regionally switching parameter, (*in revision*), *hal-03638701*.
5. T. Bayen, A. Bouali and L. Bourdin, The hybrid maximum principle for optimal control problems with spatially heterogeneous dynamics is a consequence of a pontryagin maximum principle for  $L^1_\square$ -local solutions, (*in revision*), *hal-03985420*.
6. T. Bayen, A. Bouali and L. Bourdin, Minimum time problem for the double integrator with a loss control region, *submitted*, *hal-03928967v2*.
7. T. Bayen, K. Boumaza and A. Rapaport, Necessary optimality condition for the minimal time crisis relaxing transverse condition via regularization, *ESAIM Control Optim. Calc. Var.*, **27**, URL <https://doi.org/10.1051/cocv/2021102>.
8. T. Bayen and A. Rapaport, Minimal time crisis versus minimum time to reach a viability kernel: a case study in the prey-predator model, *Optim. Control Appl.*, **40** (2019), 330–350.
9. A. Blumentals, B. Brogliato and F. Bertails-Descoubes, The contact problem in lagrangian systems subject to bilateral and unilateral constraints, with or without sliding coulomb’s friction: a tutorial, *Multibody System Dynamics*, **38** (2016), 43–76.
10. B. Brogliato and L. Thibault, Existence and uniqueness of solutions for non-autonomous complementarity dynamical systems, *Journal of Convex Analysis*, **17** (2010), 961–990.
11. L. Cesari, *Lagrange and Bolza Problems of optimal control and other problems*, 196–205, Springer, New York, 1983.
12. F. H. Clarke and R. B. Vinter, Applications of optimal multiprocesses, *SIAM Journal on Control and Optimization*, **27** (1989), 1048–1071.
13. F. H. Clarke and R. B. Vinter, Optimal multiprocesses, *SIAM Journal on Control and Optimization*, **27** (1989), 1072–1091.
14. A. V. Dmitruk and A. Kaganovich, Maximum principle for optimal control problems with intermediate constraints, *Computational Mathematics and Modeling*, **22** (2011), 180–215.
15. A. Dmitruk, Maximum principle for the general optimal control problem with phase and regular mixed constraints, *Computational Mathematics and Modeling*, **4** (1993), 364–377.
16. A. Dmitruk and A. Kaganovich, The hybrid maximum principle is a consequence of Pontryagin maximum principle, *Systems Control Lett.*, **57** (2008), 964–970.
17. A. F. Filippov, Differential equations with discontinuous right-hand side, *Matematicheskii sbornik*, **93** (1960), 99–128.
18. M. Garavello and B. Piccoli, Hybrid necessary principle, *SIAM J. Control Optim.*, **43** (2005), 1867–1887.

19. T. Haberkorn and E. Tr  lat, Convergence results for smooth regularizations of hybrid nonlinear optimal control problems, *SIAM J. Control Optim.*, **49** (2011), 1498–1522.
20. D. G. Luenberger, *Dynamic Systems*, J. Wiley Sons, 1979.
21. A. Pakniyat and P. E. Caines, On the hybrid minimum principle: the Hamiltonian and adjoint boundary conditions, *IEEE Trans. Automat. Contr.*, **66** (2020), 1246–1253.
22. A. Pakniyat and P. E. Caines, The hybrid minimum principle in the presence of switching costs, in *52nd IEEE Conference on Decision and Control*, IEEE, 2013, 3831–3836.
23. L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze and E. F. Mishchenko, *The mathematical theory of optimal processes*, Translated by D. E. Brown, A Pergamon Press Book. The Macmillan Co., New York, 1964.
24. H. J. Sussmann, A nonsmooth hybrid maximum principle, in *Stability and stabilization of nonlinear systems*, vol. 246, Springer London, 1999, 325–354.
25. H. J. Sussmann, A maximum principle for hybrid optimal control problems, in *Proc. 38th IEEE Conf. Decis. Control*, vol. 1, Ieee, 1999, 425–430.