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Loss control regions in optimal control problems

T erence Bayen* Anas Bouali† Lo ic Bourdin‡ Olivier Cots§

June 22, 2023

Abstract

This paper addresses optimal control problems with *loss control regions*. In that context the state space is partitioned into disjoint sets, referred to as *regions*, which are classified into two types: *control regions* and *loss control regions*. When the state belongs to a control region, the control is permanent (i.e. the control value is authorized to be modified at any time). On the contrary, when the state belongs to a loss control region, the control must remain constant, equal to the last assigned value before the state enters into the loss control region, and this value is kept until the state exits this region. The objective of this paper is twofold. First, we reformulate the above setting into a hybrid optimal control problem (with spatially heterogeneous dynamics) involving moreover a regionally switching parameter, and we prove a corresponding hybrid maximum principle: hence first-order necessary optimality conditions in a Pontryagin form are obtained. Second, this paper proposes a two-steps numerical scheme to solve optimal control problems with loss control regions. It is based on a direct numerical method (applied to a regularized problem) which initializes an indirect numerical method (applied to the original problem and based on the aforementioned optimality conditions). This numerical approach is applied to several illustrative examples.

1 Introduction

General context. *Optimal control theory* is the mathematical field concerned with the analysis of controlled dynamical systems, where one aims at steering such a system from a given configuration to some desired target by minimizing a given cost. The *Pontryagin maximum principle* (in short, PMP), established at the end of the 50's for general finite-dimensional nonlinear continuous-time control systems (see [34]), is certainly the milestone of optimal control theory, providing first-order necessary optimality conditions. Roughly speaking, the PMP ensures the existence of an (absolutely continuous) adjoint vector such that the optimal control satisfies the so-called *Hamiltonian maximization condition*.

In the literature, optimal control theory and, in particular, the PMP are usually concerned with a *permanent control*, in the sense that the control value is authorized to be modified at any instant of time. However, in practice, when acting on concrete processes, some constraints may prevent changing the control value in a full free way: we speak then of a *nonpermanent control* and the situations are diverse. For instance, in automatic, when the control is digital, the control value can be modified only in a discrete way in time, resulting into a piecewise constant control (also called *sampled-data control*). A version of the PMP has been obtained for sampled-data controls, in which the Hamiltonian maximization condition is replaced by the so-called *averaged Hamiltonian gradient condition* (see [1, 10, 16, 17, 18]). In aerospace, the control is not permanent in the presence of an eclipse constraint (see [29, 30]). Such a constraint applies to satellites that use solar power and that cannot be active when they are in a shadow region (i.e. an area that is not directly exposed to sunlight). In such a shadow region, the control input can only be set to zero due to the absence of power. Additionally, in viability problems or in epidemiology, we often encounter constraint sets or environment sets attached to the controlled dynamics (see, e.g., [14, 33] in the context of an optimal

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control analysis of covid-19). Such sets can be related to thresholds not to exceed and they can in general be described by a set of inequalities involving state variables. Depending on the application model, it is then desirable to maintain the system as much as possible outside this set. This is typically the case in time crisis problems [8] when the state belongs to the environment set. In this context, to reduce operating costs, it can be convenient to make use of constant controls. In the same way, constant controls may also be convenient whenever the system fulfills the constraints.

Objective and methodology. The objective of this paper is to address optimal control problems with *loss control regions*. In that context the state space is partitioned into disjoint sets, referred to as *regions*, which are classified into two types: *control regions* and *loss control regions*. When the state belongs to a control region, the control is permanent (i.e. the control value is authorized to be modified at any time). On the contrary, when the state belongs to a loss control region, the control must remain constant, equal to the last assigned value before the state enters into the loss control region, and this value is kept until the state exits this region. As it was previously mentioned, the consideration of such problems is motivated by various applications.

To address optimal control problems with loss control regions, we pursue the approach initiated in our previous works [4, 5], by considering the control function as a permanent control in control regions and as a constant parameter in loss control regions. With this point of view, our framework falls into the domain of *hybrid* optimal control theory which extends the classical theory to discontinuous dynamics. In that field, the so-called *hybrid maximum principle* (in short, HMP) extends the PMP to various hybrid settings (i.e. to discontinuities of various natures, such as temporal or spatial, see below). In the HMP, the adjoint vector is usually piecewise absolutely continuous, admitting a discontinuity jump at each time the dynamics changes discontinuously. We refer to [21, 26, 28, 32, 36, 37, 38] and references therein.

To be specific, our methodology leads to a hybrid setting where the dynamics changes discontinuously (only) according to the state position (spatial) in a given partition of the state space: we speak of a *spatially heterogeneous dynamics* and this setting corresponds to the spirit of previous works such as [3, 30] (in which transversal crossing assumptions are made to handle the boundary crossings of the optimal trajectory). In contrast, the change of dynamics may depend on time only, i.e. we fix in advance a certain number of instants (fixed or free) at which the dynamics changes. This discontinuity may be controlled or not (see, e.g. [21, 26]). In that case, we would rather speak of a temporal discontinuity. Note that this case is not under consideration throughout this paper which is devoted only to spatially heterogeneous dynamics. Finally we emphasize that our strategy, not only leads to a hybrid optimal control problem (with spatially heterogeneous dynamics), but moreover involving a *regionally switching parameter* (i.e. a parameter that can change its value when the trajectory moves from one region to another). To the best of our knowledge, this last concept has never been considered in the literature until our previous work [4].

Main results.

- (a) **Theoretical contributions.** The main theoretical contribution of this paper is the derivation of a PMP for optimal control problems with loss control regions (see Theorem 3.1). The necessary optimality condition is expressed as the usual Hamiltonian maximization condition whenever the state belongs to a control region, and as the averaged Hamiltonian gradient condition whenever the state belongs to a loss control region. Theorem 3.1 is actually a direct consequence of a more general result that we establish in Theorem 2.1: a HMP for hybrid optimal control problems (with spatially heterogeneous dynamics) involving a regionally switching parameter.

Theorem 2.1 is an extension of a HMP that can be found in our previous work [4] in the simpler context of a fixed initial condition and no final state constraint. Furthermore our methodology in [4] consists in a thorough sensitivity analysis of the hybrid control system, involving an inductive reasoning to handle the consecutive crossing times thanks to implicit function theorems. In the present work, Theorem 2.1, not only handles mixed terminal state constraints, but moreover is proved with a simpler method based on an *augmentation technique*. This approach consists in reducing the hybrid optimal control problem into a classical (nonhybrid) optimal control problem to which the classical PMP can be applied. It was initiated in [26] to deal with *hybrid* optimal control problems in which the word *hybrid* refers to the situation where the dynamics changes discontinuously but at fixed or free instants of time (above all,

independently of the state position). As highlighted in our previous paper [6] with counterexamples, the augmentation technique must be carefully adapted to the present framework of spatially heterogeneous dynamics. In particular it requires the introduction of a new notion of local solution (called L_{\square}^1 -local solution) to classical (nonhybrid) optimal control problems and the derivation of a corresponding PMP (see [6, Theorem 2.1]).

We point out that, similarly to Theorem 2.1 that is an extension of the HMP that can be found in our previous paper [4], Theorem 3.1 is an extension of the PMP that can be found in our previous work [5] in the simpler context of a fixed initial condition and no final state constraint.

Finally, since our framework is related to the hybrid setting (with spatially heterogeneous dynamics), Theorem 2.1 and Theorem 3.1 are both results obtained under appropriate transversal crossing assumptions and both ensure the existence of a piecewise absolutely continuous adjoint vector admitting a discontinuity jump at each crossing time.

- (b) **Numerical contributions.** This paper proposes a two-steps numerical approach to solve optimal control problems with loss control regions. First, we use a direct numerical method applied to a *regularized* problem: a regularization is required to overcome the discontinuities appearing when rewriting the optimal control problem with loss control regions as a hybrid optimal control problem with regionally switching parameter. This first step is useful to determine the *structure* of the optimal trajectory (i.e. the ordered sequence of regions that the optimal trajectory visits) and, second, to initialize an indirect numerical method applied to the original problem and based on the PMP stated in Theorem 3.1. The originality here is to incorporate the averaged Hamiltonian gradient condition, as well as the discontinuity jumps of the adjoint vector, to define an appropriate shooting function (in addition to the classical terms defining the shooting function for non-hybrid control problems, see [20, 25]).

For the sake of brevity, in the rest of this paper, the word *hybrid* will refer (only) to the situation of spatially heterogeneous dynamics.

Organization of the paper. This paper is organized as follows. In Section 2, a general hybrid optimal control problem with regionally switching parameter is introduced (see Problem (HP)). Then Proposition 2.1 asserts that the augmentation of a global solution to Problem (HP) leads to a L_{\square}^1 -local solution to a classical (nonhybrid) augmented optimal control problem. Hence, applying the PMP for L_{\square}^1 -local solutions (see [6, Theorem 2.1]) and reversing the augmentation procedure, a HMP for hybrid optimal control problems with regionally switching parameter is obtained (see Theorem 2.1). In Section 3, a general optimal control problem with loss control regions is introduced (see Problem (P)). By rewriting this problem as a hybrid optimal control problem with regionally switching parameter and applying the previous Theorem 2.1, a PMP for optimal control problems with loss control regions is obtained (see Theorem 3.1). In Section 4, a two-steps numerical scheme is proposed to solve optimal control problems with loss control regions. Afterwards it is applied to numerically solve some illustrative examples, precisely a Zermelo-type problem [2] and a version of the minimal time problem for the harmonic oscillator [35] both including loss control regions. Finally the technical proofs of Proposition 2.1 and Theorem 2.1 are provided in Appendices A and B respectively.

2 Derivation of a HMP with regionally switching parameter

This section addresses hybrid optimal control problems with regionally switching parameter (which will allow us in the next Section 3 to address optimal control problems with loss control regions as a particular case). To this aim, Section 2.1 is devoted to required notations and functional framework. In Section 2.2, a general hybrid optimal control problem with regionally switching parameter is introduced, together with terminology and assumptions. In Section 2.3, we discuss the notion of *regular* solution to the corresponding hybrid control system. In Section 2.4, thanks to an augmentation technique, we establish in Proposition 2.1 the correspondence between a solution to the hybrid optimal control problem with regionally switching parameter, that is regular, and a L_{\square}^1 -local solution (notion that was previously introduced in [6]) to a classical optimal control problem with (constant) parameter. In Section 2.5, applying a version of the PMP that is adapted to L_{\square}^1 -local solutions (extracted from [6, Theorem 2.1]), we derive the main result of this section (Theorem 2.1) which is a HMP with regionally switching parameter.

2.1 Notation and functional framework

Let us start with some basic notation and functional framework. In this paper, for any positive integer $d \in \mathbb{N}^*$, we denote by $\langle \cdot, \cdot \rangle_{\mathbb{R}^d}$ (resp. $\| \cdot \|_{\mathbb{R}^d}$) the standard inner product (resp. Euclidean norm) of \mathbb{R}^d . For any subset $X \subset \mathbb{R}^d$, we denote by ∂X the boundary of X defined by $\partial X := \overline{X} \setminus \text{Int}(X)$, where \overline{X} and $\text{Int}(X)$ stand respectively for the closure and the interior of X . Given a closed convex set $Y \subset \mathbb{R}^d$, the normal cone to Y at some point $y \in Y$ is defined by

$$N_Y[y] := \{y'' \in \mathbb{R}^d \mid \forall y' \in Y, \langle y'', y' - y \rangle_{\mathbb{R}^d} \leq 0\}.$$

Given a (Lebesgue) measurable subset $A \subset \mathbb{R}$, we denote by $\mu(A)$ its (Lebesgue) measure. For any extended-real number $r \in [1, \infty]$ and any real interval $I \subset \mathbb{R}$, we denote by:

- $L^r(I, \mathbb{R}^d)$ the usual Lebesgue space of r -integrable functions defined on I with values in \mathbb{R}^d , endowed with its usual norm $\| \cdot \|_{L^r}$;
- $C(I, \mathbb{R}^d)$ the standard space of continuous functions defined on I with values in \mathbb{R}^d , endowed with the standard uniform norm $\| \cdot \|_C$;
- $AC(I, \mathbb{R}^d)$ the subspace of $C(I, \mathbb{R}^d)$ of absolutely continuous functions.

Now take $I = [0, T]$ for some $T > 0$. Recall that a partition of the interval $[0, T]$ is a finite set $\mathbb{T} = \{\tau_k\}_{k=0, \dots, N}$, for some positive integer $N \in \mathbb{N}^*$, such that $0 = \tau_0 < \tau_1 < \dots < \tau_{N-1} < \tau_N = T$. In this paper:

- A function $\gamma \in L^\infty([0, T], \mathbb{R}^d)$ is said to be *piecewise constant*, with respect to a partition $\mathbb{T} = \{\tau_k\}_{k=0, \dots, N}$ of the interval $[0, T]$, if the restriction of γ over each open interval (τ_{k-1}, τ_k) is almost everywhere equal to a constant denoted by $\gamma_k \in \mathbb{R}^d$. If so, γ is identified to the function $\gamma : [0, T] \rightarrow \mathbb{R}^d$ given by

$$\gamma(t) := \begin{cases} \gamma_k & \text{if } t \in [\tau_{k-1}, \tau_k) \text{ for all } k \in \{1, \dots, N-1\}, \\ \gamma_N & \text{if } t \in [\tau_{N-1}, \tau_N] \text{ for } k = N, \end{cases}$$

for all $t \in [0, T]$.

- A function $\gamma : [0, T] \rightarrow \mathbb{R}^d$ is said to be *piecewise absolutely continuous*, with respect to a partition $\mathbb{T} = \{\tau_k\}_{k=0, \dots, N}$ of the interval $[0, T]$, if γ is continuous at 0 and T and the restriction of γ over each open interval (τ_{k-1}, τ_k) admits an extension over $[\tau_{k-1}, \tau_k]$ that is absolutely continuous. If so, γ admits left and right limits at each $\tau_k \in (0, T)$, denoted by $\gamma^-(\tau_k)$ and $\gamma^+(\tau_k)$ respectively.

In what follows we denote by $\text{PC}_{\mathbb{T}}([0, T], \mathbb{R}^d)$ (resp. $\text{PAC}_{\mathbb{T}}([0, T], \mathbb{R}^d)$) the space of all piecewise constant functions (resp. piecewise absolutely continuous functions) respecting a given partition \mathbb{T} of $[0, T]$. We denote by $\text{PC}([0, T], \mathbb{R}^d)$ (resp. $\text{PAC}([0, T], \mathbb{R}^d)$) the set of all piecewise constant functions (resp. piecewise absolutely continuous functions), independently of the partition considered.

For a differentiable map $\psi : \mathbb{R}^d \rightarrow \mathbb{R}^{d'}$, with $d' \in \mathbb{N}^*$, we denote by $\nabla \psi(x) := (\nabla \psi_1(x) \dots \nabla \psi_{d'}(x)) \in \mathbb{R}^{d \times d'}$ the gradient of ψ at some $x \in \mathbb{R}^d$. We say that ψ is *submersive* at $x \in \mathbb{R}^d$ if the differential $D\psi(x) = \nabla \psi(x)^\top \in \mathbb{R}^{d' \times d}$ is surjective. Finally, when $(\mathcal{Z}, d_{\mathcal{Z}})$ is a metric set, we denote by $B_{\mathcal{Z}}(z, \nu)$ (resp. $\overline{B}_{\mathcal{Z}}(z, \nu)$) the standard open (resp. closed) ball of \mathcal{Z} centered at $z \in \mathcal{Z}$ and of radius $\nu > 0$.

2.2 A hybrid optimal control problem with regionally switching parameter: terminology and assumptions

Let $n, m, d, \ell \in \mathbb{N}^*$ be four fixed positive integers and $T > 0$ be a fixed positive real number. In this section we consider a partition of the state space given by

$$\mathbb{R}^n = \bigcup_{j \in \mathcal{J}} \overline{X}_j,$$

where \mathcal{J} is a (possibly infinite) family of indexes and the nonempty open subsets $X_j \subset \mathbb{R}^n$, called *regions*, are disjoint. Our aim in this section is to derive first-order necessary optimality conditions in a Pontryagin form for the *hybrid* optimal control problem with *regionally switching parameter* given by

$$\begin{aligned}
& \text{minimize} && \phi(x(0), x(T)), \\
& \text{subject to} && (x, \lambda, u) \in \text{AC}([0, T], \mathbb{R}^n) \times \text{PC}([0, T], \mathbb{R}^d) \times \text{L}^\infty([0, T], \mathbb{R}^m), \\
& && \dot{x}(t) = h(x(t), \lambda(t), u(t)), \quad \text{a.e. } t \in [0, T], \\
& && g(x(0), x(T)) \in \text{S}, \\
& && (\lambda(t), u(t)) \in \Lambda \times \text{U}, \quad \text{a.e. } t \in [0, T], \\
& && \lambda \text{ is a regionally switching parameter associated with } x,
\end{aligned} \tag{HP}$$

where both the Mayer cost function $\phi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ and the constraint function $g : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^\ell$ are of class C^1 , where both subsets $\text{S} \subset \mathbb{R}^\ell$ and $\Lambda \subset \mathbb{R}^d$ are nonempty closed convex subsets, where $\text{U} \subset \mathbb{R}^m$ is a nonempty subset and where the *spatially heterogeneous* dynamics $h : \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is defined *regionally* by

$$\forall (x, \lambda, u) \in \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^m, \quad h(x, \lambda, u) := h_j(x, \lambda, u) \quad \text{if } x \in X_j,$$

where the maps $h_j : \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ are of class C^1 . Note that $h(x, \lambda, u)$ is not defined when $x \notin \cup_{j \in \mathcal{J}} X_j$ but this fact will have no impact on the rest of this paper thanks to transverse crossing assumptions (see Definition 2.2).

In Problem (HP), as usual in the literature, $x \in \text{AC}([0, T], \mathbb{R}^n)$ is called the *state* (or the *trajectory*) and $u \in \text{L}^\infty([0, T], \mathbb{R}^m)$ is called the *control*. Additionally we consider a regionally switching parameter $\lambda \in \text{PC}([0, T], \mathbb{R}^d)$, meaning that the parameter λ stays constant as long as the trajectory x stays within a region, but it is allowed to change its value (i.e. to switch) when the trajectory x crosses a boundary and moves from one region to another (see Definition 2.1). A triplet $(x, \lambda, u) \in \text{AC}([0, T], \mathbb{R}^n) \times \text{PC}([0, T], \mathbb{R}^d) \times \text{L}^\infty([0, T], \mathbb{R}^m)$ is said to be *admissible* for Problem (HP) if it satisfies all the constraints of Problem (HP). Such an admissible triplet is said to be a *global solution* to Problem (HP) if it minimizes the Mayer cost $\phi(x(0), x(T))$ among all admissible triplets.

Remark 2.1. In the whole paper (not only for Problem (HP)), we will consider problems with (only) Mayer cost, fixed final time and autonomous dynamics (i.e. independent of t). It is well known in the literature (see, e.g., [12, 19, 23]) that standard techniques, such as augmentation techniques or changes of variables, allow to deal with more general problems that include Bolza cost, free final time and time-dependent dynamics. Also we assume for simplicity that the maps ϕ , f and g are of class C^1 and some topological conditions on the sets S and Λ . However the results presented in this work can be extended to weaker assumptions (as in [24, 39]). Overall our aim in this paper is not to consider the most general framework possible. We keep our setting as simple as possible to stay focused on the main novelties of this work.

2.3 Regular solution to the hybrid control system with regionally switching parameter

Consider the hybrid control system with regionally switching parameter associated with Problem (HP) given by

$$\begin{cases} \dot{x}(t) = h(x(t), \lambda(t), u(t)), & \text{for a.e. } t \in [0, T], \\ \lambda \text{ is a regionally switching parameter associated with } x. \end{cases} \tag{HS}$$

Due to the discontinuities of the spatially heterogeneous dynamics h and to the presence of a regionally switching parameter, we need to precise the definition of a solution to (HS).

Definition 2.1 (Solution to (HS)). A triplet $(x, \lambda, u) \in \text{AC}([0, T], \mathbb{R}^n) \times \text{PC}([0, T], \mathbb{R}^d) \times \text{L}^\infty([0, T], \mathbb{R}^m)$ is said to be a *solution* to (HS) if there exists a partition $\mathbb{T} = \{\tau_k\}_{k=0, \dots, N}$ of the interval $[0, T]$ such that:

(i) It holds that

$$\forall k \in \{1, \dots, N\}, \quad \exists j(k) \in \mathcal{J}, \quad \forall t \in (\tau_{k-1}, \tau_k), \quad x(t) \in X_{j(k)},$$

where $j(k) \neq j(k-1)$ for all $k \in \{2, \dots, N\}$. The sequence $\{j(1), \dots, j(N)\}$ is called the *switching sequence*.

(ii) It holds that $x(0) \in X_{j(1)}$ and $x(T) \in X_{j(N)}$.

(iii) λ is a regionally switching parameter associated with x , that is, $\lambda \in \text{PC}_{\mathbb{T}}([0, T], \mathbb{R}^d)$.

(iv) It holds that $\dot{x}(t) = h_{j(k)}(x(t), \lambda_k, u(t))$ for almost every $t \in (\tau_{k-1}, \tau_k)$ and all $k \in \{1, \dots, N\}$.

In that case, to ease notation, we set $f_k := h_{j(k)}$ and $E_k := X_{j(k)}$ for all $k \in \{1, \dots, N\}$. With this system of notations, we have

$$\begin{cases} x(t) \in E_1, & \forall t \in [\tau_0, \tau_1), \\ x(t) \in E_k, & \forall t \in (\tau_{k-1}, \tau_k), \quad \forall k \in \{2, \dots, N-1\}, \\ x(t) \in E_N, & \forall t \in (\tau_{N-1}, \tau_N], \\ \dot{x}(t) = f_k(x(t), \lambda_k, u(t)), & \text{a.e. } t \in (\tau_{k-1}, \tau_k), \quad \forall k \in \{1, \dots, N\}. \end{cases}$$

The times τ_k for $k \in \{1, \dots, N-1\}$, called *crossing times*, correspond to the instants at which the trajectory x goes from the region E_k to the region E_{k+1} , and thus $x(\tau_k) \in \partial E_k \cap \partial E_{k+1}$.

The main result of this section (Theorem 2.1 stated in Section 2.5) is based on some regularity assumptions made on the optimal triplet of Problem (HP) at each of its crossing times. These hypotheses are precised in the next definition.

Definition 2.2 (Regular solution to (HS)). Following the notations introduced in Definition 2.1, a solution $(x, \lambda, u) \in \text{AC}([0, T], \mathbb{R}^n) \times \text{PC}([0, T], \mathbb{R}^d) \times \text{L}^\infty([0, T], \mathbb{R}^m)$ to (HS), associated with a partition $\mathbb{T} = \{\tau_k\}_{k=0, \dots, N}$, is said to be *regular* if the following conditions are both satisfied:

(i) At each crossing time τ_k , there exists a C^1 function $F_k : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\exists \nu_k > 0, \quad \forall z \in \bar{B}_{\mathbb{R}^n}(x(\tau_k), \nu_k), \quad \begin{cases} z \in E_k & \Leftrightarrow F_k(z) < 0, \\ z \in \partial E_k \cap \partial E_{k+1} & \Leftrightarrow F_k(z) = 0, \\ z \in E_{k+1} & \Leftrightarrow F_k(z) > 0. \end{cases}$$

In particular it holds that $F_k(x(\tau_k)) = 0$.

(ii) At each crossing time τ_k , there exists $\alpha_k > 0$ and $\beta_k > 0$ such that the *transverse conditions*

$$\begin{cases} \langle \nabla F_k(x(\tau_k)), f_k(x(\tau_k), \lambda_k, u(t)) \rangle_{\mathbb{R}^n} \geq \beta_k, & \text{a.e. } t \in [\tau_k - \alpha_k, \tau_k), \\ \langle \nabla F_k(x(\tau_k)), f_{k+1}(x(\tau_k), \lambda_{k+1}, u(t)) \rangle_{\mathbb{R}^n} \geq \beta_k, & \text{a.e. } t \in (\tau_k, \tau_k + \alpha_k], \end{cases} \quad (\text{HTC})$$

are both satisfied. We refer to Figure 1 for a geometrical illustration.

Remark 2.2. In this remark we comment some aspects related to Definitions 2.1 and 2.2.

(i) In Definition 2.1, we exclude a few possibilities, such as having an infinite number of crossing times (no Zeno phenomenon [22]), bouncing against a boundary or moving along a boundary (as considered in [3]). Also we (only) consider trajectories that start and finish inside the regions (and not on a boundary). Nevertheless we discuss in Remark 2.8 possible issues to address the case where trajectories start or finish on a boundary of a stratum.

(ii) The transverse condition introduced in Definition 2.2 has a geometrical interpretation. It means that the trajectory does not cross the boundary $\partial E_k \cap E_{k+1}$ tangentially (see Figure 1). Note that, in the present work, we consider a transverse condition that is (slightly) weaker than the one examined in previous works such as [4, 30] and given by

$$\begin{cases} u \text{ admits left and right limits at } \tau_k \text{ denoted by } u^-(\tau_k) \text{ and } u^+(\tau_k), \\ \langle \nabla F_k(x(\tau_k)), f_k(x(\tau_k), \lambda_k, u^-(\tau_k)) \rangle_{\mathbb{R}^n} > 0, \\ \langle \nabla F_k(x(\tau_k)), f_{k+1}(x(\tau_k), \lambda_{k+1}, u^+(\tau_k)) \rangle_{\mathbb{R}^n} > 0. \end{cases} \quad (\text{HTC}')$$

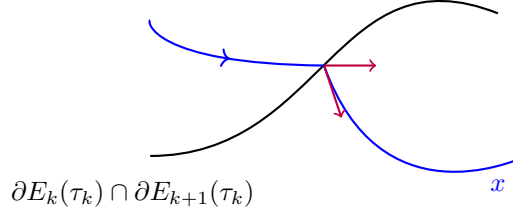


Figure 1: Geometrical illustration of a transversal boundary crossing.

2.4 Reduction to a classical optimal control problem with (constant) parameter

To establish a correspondence from the hybrid optimal control problem with regionally switching parameter (HP) to a classical optimal control problem with (constant) parameter, we proceed to simple affine changes of time variable. Precisely let $(x^*, \lambda^*, u^*) \in AC([0, T], \mathbb{R}^n) \times PC([0, T], \mathbb{R}^d) \times L^\infty([0, T], \mathbb{R}^m)$ be a solution to (HS), associated with a partition $\mathbb{T}^* = \{\tau_k^*\}_{k=0, \dots, N}$, and let us denote by E_k^* , f_k^* and λ_k^* the corresponding regions, functions and constant values of λ^* whenever x^* belongs to E_k^* (see Definition 2.1). We introduce $(y^*, v^*) \in AC([0, 1], \mathbb{R}^{(n+d)N}) \times L^\infty([0, 1], \mathbb{R}^{mN})$ defined by

$$y_k^*(s) := x^*(\tau_{k-1}^* + (\tau_k^* - \tau_{k-1}^*)s), \quad y_{N+k}^*(s) := \lambda^*(\tau_{k-1}^* + (\tau_k^* - \tau_{k-1}^*)s) \quad \text{and} \quad v_k^*(s) := u^*(\tau_{k-1}^* + (\tau_k^* - \tau_{k-1}^*)s), \quad (2.1)$$

for all $s \in [0, 1]$ and all $k \in \{1, \dots, N\}$. Note that the above affine changes of time variable allow mapping each time interval $[\tau_{k-1}^*, \tau_k^*]$ to the common time interval $[0, 1]$ and, therefore, augment the state dimension to $(n+d)N$ and the control dimension to mN , respectively. To reverse the above changes of the time variable, one simply has

$$x^*(t) = y_k^*\left(\frac{t - \tau_{k-1}^*}{\tau_k^* - \tau_{k-1}^*}\right), \quad \lambda^*(t) = y_{N+k}^*\left(\frac{t - \tau_{k-1}^*}{\tau_k^* - \tau_{k-1}^*}\right) \quad \text{and} \quad u^*(t) = v_k^*\left(\frac{t - \tau_{k-1}^*}{\tau_k^* - \tau_{k-1}^*}\right), \quad (2.2)$$

for all $t \in [\tau_{k-1}^*, \tau_k^*]$ and all $k \in \{1, \dots, N\}$. In particular note that $(x^*(0), x^*(T)) = (y_1^*(0), y_N^*(1))$. From a more general point of view, it holds that $x^*(\tau_k^*) = y_{k+1}^*(0)$ for all $k \in \{0, \dots, N-1\}$, and $x^*(\tau_k^*) = y_k^*(1)$ and $\lambda_k^* = y_{N+k}^*(s)$ for all $s \in [0, 1]$ and all $k \in \{1, \dots, N\}$.

With the above context note that the triplet (y^*, v^*, \mathbb{T}^*) satisfies

$$\dot{y}^*(s) = f^*(y^*(s), v^*(s), \mathbb{T}^*), \quad \text{a.e. } s \in [0, 1],$$

where $f^* : \mathbb{R}^{(n+d)N} \times \mathbb{R}^{mN} \times \mathbb{R}^{N+1} \rightarrow \mathbb{R}^{(n+d)N}$ is the C^1 function defined by

$$f^*(y, v, \mathbb{T}) := \left((\tau_1 - \tau_0) f_1^*(y_1, y_{N+1}, v_1), \dots, (\tau_N - \tau_{N-1}) f_N^*(y_N, y_{2N}, v_N), 0_{\mathbb{R}^d}, \dots, 0_{\mathbb{R}^d} \right),$$

for all $y = (y_1, \dots, y_N, y_{N+1}, \dots, y_{2N}) \in \mathbb{R}^{(n+d)N}$, $v = (v_1, \dots, v_N) \in \mathbb{R}^{mN}$ and $\mathbb{T} = \{\tau_0, \dots, \tau_N\} \in \mathbb{R}^{N+1}$. Furthermore, it holds that

$$\begin{cases} y_1^*(s) \in E_1^*, & \forall s \in [0, 1], \\ y_k^*(s) \in E_k^*, & \forall s \in (0, 1), \quad \forall k \in \{2, \dots, N-1\}, \\ y_N^*(s) \in E_N^*, & \forall s \in (0, 1), \end{cases} \quad (2.3)$$

and $y_{k+1}^*(0) = y_k^*(1) \in \partial E_k^* \cap \partial E_{k+1}^*$ for all $k \in \{1, \dots, N-1\}$. Also note that $\mathbb{T}^* \in \Delta$ where $\Delta \subset \mathbb{R}^{N+1}$ is the nonempty closed convex set defined by

$$\Delta := \{\mathbb{T} = \{\tau_k\}_{k=0, \dots, N} \in \mathbb{R}^{N+1} \mid 0 = \tau_0 \leq \tau_1 \leq \dots \leq \tau_{N-1} \leq \tau_N = T\}.$$

Now assume that the triplet (x^*, λ^*, u^*) is moreover a regular solution to (HS) and denote by F_k^* and $\nu_k^* > 0$ the corresponding functions and positive radii (see Definition 2.2). In that context note that $F_k^*(x(\tau_k^*)) = F_k^*(y_k^*(1)) = 0$ for all $k \in \{1, \dots, N-1\}$.

Finally it is clear that, if the triplet (x^*, λ^*, u^*) is furthermore admissible for Problem (HP), then the triplet (y^*, v^*, \mathbb{T}^*) is admissible for the classical optimal control problem with (constant) parameter given by

$$\begin{aligned}
& \text{minimize} && \phi^*(y(0), y(1)), \\
& \text{subject to} && (y, v, \mathbb{T}) \in \text{AC}([0, 1], \mathbb{R}^{(n+d)N}) \times \text{L}^\infty([0, 1], \mathbb{R}^{mN}) \times \mathbb{R}^{N+1}, \\
& && \dot{y}(s) = f^*(y(s), v(s), \mathbb{T}), \quad \text{a.e. } s \in [0, 1], \\
& && g^*(y(0), y(1), \mathbb{T}) \in \text{S}^*, \\
& && v(s) \in \text{U}^N, \quad \text{a.e. } s \in [0, 1],
\end{aligned} \tag{CP*}$$

where $\phi^* : \mathbb{R}^{(n+d)N} \times \mathbb{R}^{(n+d)N} \rightarrow \mathbb{R}$ and $g^* : \mathbb{R}^{(n+d)N} \times \mathbb{R}^{(n+d)N} \times \mathbb{R}^{N+1} \rightarrow \mathbb{R}^{\ell^*}$ are the C^1 functions defined by $\phi^*(y^0, y^1) := \phi(y_1^0, y_N^1)$ and

$$g^*(y^0, y^1, \mathbb{T}) := (g(y_1^0, y_N^1), y_2^0 - y_1^1, \dots, y_N^0 - y_{N-1}^1, F_1^*(y_1^1), \dots, F_{N-1}^*(y_{N-1}^1), y_{N+1}^1, \dots, y_{2N}^1, \mathbb{T}),$$

for all $y^0 = (y_1^0, \dots, y_N^0, y_{N+1}^0, \dots, y_{2N}^0)$, $y^1 = (y_1^1, \dots, y_N^1, y_{N+1}^1, \dots, y_{2N}^1) \in \mathbb{R}^{(n+d)N}$ and $\mathbb{T} = \{\tau_0, \dots, \tau_N\} \in \mathbb{R}^{N+1}$, where $\ell^* := \ell + n(N-1) + (N-1) + dN + (N+1)$, and where $\text{S}^* \subset \mathbb{R}^{\ell^*}$ stands for the nonempty closed convex set defined by

$$\text{S}^* := \text{S} \times \{0_{\mathbb{R}^n}\}^{N-1} \times \{0\}^{N-1} \times \Lambda^N \times \Delta.$$

Before stating the main result of this section, we first need to recall some notions of local solution, extracted from our previous paper [6], which will play central roles in the present work.

Definition 2.3 (L_A^1 -local solution). The triplet (y^*, v^*, \mathbb{T}^*) is said to be a L_A^1 -local solution to Problem (CP*), for a measurable subset $A \subset [0, 1]$, if, for all $R \geq \|v^*\|_{\text{L}^\infty}$, there exists $\eta > 0$ such that $\phi^*(y^*(0), y^*(1)) \leq \phi^*(y(0), y(1))$ for all admissible triplets (y, v, \mathbb{T}) satisfying

$$\begin{cases}
\|y - y^*\|_{\text{C}} + \|v - v^*\|_{\text{L}^1} + \|\mathbb{T} - \mathbb{T}^*\|_{\mathbb{R}^{N+1}} \leq \eta, \\
\|v\|_{\text{L}^\infty} \leq R, \\
v(t) = v^*(t) \text{ a.e. } t \in [0, T] \setminus A.
\end{cases}$$

With $A = [0, T]$, we recover the standard notion of L^1 -local solution that can be found in [13, 31].

Definition 2.4 (L_{\square}^1 -local solution). The triplet (y^*, v^*, \mathbb{T}^*) is said to be a L_{\square}^1 -local solution to Problem (CP*), if there exists an increasing family $(A_\varepsilon)_{\varepsilon>0}$ of measurable subsets of $[0, 1]$, satisfying $\lim_{\varepsilon \rightarrow 0} \mu(A_\varepsilon) = 1$, such that (y^*, v^*, \mathbb{T}^*) is a $L_{A_\varepsilon}^1$ -local solution to Problem (CP*) for all $\varepsilon > 0$.

Proposition 2.1. If $(x^*, \lambda^*, u^*) \in \text{AC}([0, T], \mathbb{R}^n) \times \text{PC}([0, T], \mathbb{R}^d) \times \text{L}^\infty([0, T], \mathbb{R}^m)$ is a global solution to Problem (HP), that is moreover a regular solution to (HS), associated with a partition $\mathbb{T}^* = \{\tau_k^*\}_{k=0, \dots, N}$, then the triplet (y^*, v^*, \mathbb{T}^*) constructed above is a L_{\square}^1 -local solution to Problem (CP*).

Proof. The detailed proof of Proposition 2.1 can be found in Appendix A. Precisely we prove that the triplet (y^*, v^*, \mathbb{T}^*) is a $L_{[\varepsilon, 1-\varepsilon]}^1$ -local solution to Problem (CP*) for any $0 < \varepsilon < \frac{1}{2}$. \square

Remark 2.3. Consider the setting of Proposition 2.1.

- (i) First of all, it is worth mentioning that the triplet (y^*, v^*, \mathbb{T}^*) is not a L^1 -local solution to Problem (CP*) in general. A counterexample is provided in our previous paper [6]. As a consequence one cannot use the classical PMP on the triplet (y^*, v^*, \mathbb{T}^*) . Nevertheless, thanks to Proposition 2.1, we can apply a version of the PMP that is adapted to L_{\square}^1 -local solutions (see [6, Theorem 2.1]).
- (ii) Now let us briefly comment on why the standard notion of L^1 -local solution fails, and why the notion of L_{\square}^1 -local solution is well-suited, when using an augmentation technique in our hybrid setting. Given an admissible triplet (y, v, \mathbb{T}) for Problem (CP*), when we invert the augmentation procedure, we obtain a pair (x, u) that satisfies all the constraints of Problem (HP), except one. Specifically, even if x follows the same dynamics sequence $(f_k^*)_{k=1, \dots, N}$ as the nominal state x^* , it may not visit the

same regions $(E_k^*)_{k=1,\dots,N}$. As a result (x, u) may be not admissible for Problem (HP). To overcome this issue, we rely on the notion of L^1_{\square} -local solution. This approach allows L^1 -perturbations of the control u^* , but only outside the neighborhoods of the crossing times τ_k^* . This strategy ensures that the perturbed pair (x, u) satisfies a transverse condition at each perturbed crossing time τ_k , making it admissible for Problem (HP) since x visits the same regions as the nominal trajectory x^* . We refer to [6] for details.

2.5 HMP with regionally switching parameter and comments

The Hamiltonian $\mathcal{H} : \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$ associated with Problem (HP) is defined by

$$\mathcal{H}(x, \lambda, u, p) := \langle p, h(x, \lambda, u) \rangle_{\mathbb{R}^n},$$

for all $(x, \lambda, u, p) \in \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^n$. We are now in a position to state the main result of this section.

Theorem 2.1 (HMP with regionally switching parameter). Suppose that $(x^*, \lambda^*, u^*) \in AC([0, T], \mathbb{R}^n) \times PC([0, T], \mathbb{R}^d) \times L^\infty([0, T], \mathbb{R}^m)$ is a global solution to Problem (HP), that it is moreover a regular solution to (HS), associated with a partition $\mathbb{T}^* = \{\tau_k^*\}_{k=0,\dots,N}$, and that it is such that g is submersive at $(x^*(0), x^*(T))$. Then there exists a nontrivial pair $(p, p^0) \in PAC_{\mathbb{T}^*}([0, T], \mathbb{R}^n) \times \mathbb{R}_+$ satisfying:

(i) the *Hamiltonian system*

$$\dot{x}^*(t) = \nabla_p \mathcal{H}(x^*(t), \lambda^*(t), u^*(t), p(t)) \quad \text{and} \quad -\dot{p}(t) = \nabla_x \mathcal{H}(x^*(t), \lambda^*(t), u^*(t), p(t)),$$

for almost every $t \in [0, T]$;

(ii) the *transversality condition*

$$\begin{pmatrix} p(0) \\ -p(T) \end{pmatrix} = p^0 \nabla \phi(x^*(0), x^*(T)) + \nabla g(x^*(0), x^*(T)) \xi,$$

for some $\xi \in N_S[g(x^*(0), x^*(T))]$;

(iii) the *discontinuity condition*

$$p^+(\tau_k^*) - p^-(\tau_k^*) = \sigma_k \nabla F_k^*(x^*(\tau_k^*)),$$

for some $\sigma_k \in \mathbb{R}$, for all $k \in \{1, \dots, N-1\}$;

(iv) the *Hamiltonian maximization condition*

$$u^*(t) \in \arg \max_{\omega \in U} \mathcal{H}(x^*(t), \lambda^*(t), \omega, p(t)),$$

for almost every $t \in [0, T]$;

(v) the *averaged Hamiltonian gradient condition*

$$\int_{\tau_{k-1}^*}^{\tau_k^*} \nabla_\lambda \mathcal{H}(x^*(t), \lambda_k^*, u^*(t), p(t)) dt \in N_\Lambda[\lambda_k^*],$$

for all $k \in \{1, \dots, N\}$;

(vi) the *Hamiltonian constancy condition*

$$\mathcal{H}(x^*(t), \lambda^*(t), u^*(t), p(t)) = c,$$

for almost every $t \in [0, T]$, for some $c \in \mathbb{R}$.

Proof. The detailed proof of Theorem 2.1 can be found in Appendix B. It is based on Proposition 2.1 and on the application of a PMP for L^1_{\square} -local solutions [6, Theorem 2.1]. \square

Remark 2.4. As usual in the literature on optimal control theory, the nontrivial pair (p, p^0) provided in Theorem 2.1 is defined up to a positive multiplicative constant. It is said to be *normal* whenever $p^0 > 0$, and *abnormal* whenever $p^0 = 0$. In the normal case $p^0 > 0$, it is usual to renormalize it so that $p^0 = 1$.

Remark 2.5. As explained in [9, 10], the submersiveness hypothesis made in Theorem 2.1 can be removed. In that case, all items of Theorem 2.1 remain valid except Item (ii).

Remark 2.6. In Theorem 2.1, the costate p admits a discontinuity jump at each crossing time τ_k^* , satisfying Item (iii). Under the (slightly) stronger transverse condition (HTC') and using the Hamiltonian constancy condition, one can easily prove that

$$\begin{aligned} \sigma_k &= - \frac{\left\langle p^-(\tau_k^*), f_{k+1}^*(x^*(\tau_k^*), \lambda_{k+1}^*, (u^*)^+(\tau_k^*)) - f_k^*(x^*(\tau_k^*), \lambda_k^*, (u^*)^-(\tau_k^*)) \right\rangle_{\mathbb{R}^n}}{\left\langle \nabla F_k^*(x^*(\tau_k^*)), f_{k+1}^*(x^*(\tau_k^*), \lambda_{k+1}^*, (u^*)^+(\tau_k^*)) \right\rangle_{\mathbb{R}^n}} \\ &= - \frac{\left\langle p^+(\tau_k^*), f_{k+1}^*(x^*(\tau_k^*), \lambda_{k+1}^*, (u^*)^+(\tau_k^*)) - f_k^*(x^*(\tau_k^*), \lambda_k^*, (u^*)^-(\tau_k^*)) \right\rangle_{\mathbb{R}^n}}{\left\langle \nabla F_k^*(x^*(\tau_k^*)), f_k^*(x^*(\tau_k^*), \lambda_k^*, (u^*)^-(\tau_k^*)) \right\rangle_{\mathbb{R}^n}}, \end{aligned}$$

for all $k \in \{1, \dots, N-1\}$. Therefore, in that context, the discontinuity conditions of Theorem 2.1 can be expressed as forward (or backward) discontinuity jumps. Let us recall that discontinuity jumps of the costate are common in general hybrid optimal control theory (see, *e.g.*, [11, 32]). In particular, the previous papers [3, 30] address these discontinuity jumps in the context of spatially heterogeneous dynamics, but in a setting without regionally switching parameter and, above all, their proofs are unsatisfactory (see [6] for details).

Remark 2.7. Consider the setting of Proposition 2.1. From Remark 2.3, we know that (y^*, v^*, \mathbb{T}^*) is not a L^1 -local solution to Problem (CP*) in general. Nevertheless it is possible to avoid the use of the notion of L^1_{\square} -local solution. However, to our best knowledge, this would not be possible without obtaining a weaker result and/or without restricting the framework. We refer to [6, Remark 3.4 Item (i)] for details. Roughly speaking the choice of the transverse conditions (more or less strong) influences the *local quality* (L^1 , L^∞ or L^1_{\square}) of the solution (y^*, v^*, \mathbb{T}^*) to Problem (CP*) and thus the version of the PMP that can be applied to it, and finally the version of the HMP obtained on the original triplet (x^*, λ^*, u^*) .

Remark 2.8. In this remark we would like to emphasize a few relaxations and extensions that can be considered to the present work. They have already been discussed in [6, Remark 3.4] (in a setting without regionally switching parameter).

- (i) To handle trajectories possibly starting or finishing on boundaries, adjustments need to be made in Definition 2.2. For instance, in the context of Theorem 2.1 with $x^*(0) \in E_1^*$ and $x^*(T) \in \partial E_N^*$, additional conditions involving a local C^1 description of ∂E_N^* near $x^*(T)$ and an adapted transverse condition should be included in Definition 2.2. Moreover Problem (CP*) requires careful adjustment to maintain the validity of Proposition 2.1. Finally it is necessary to adapt the submersiveness hypothesis to apply [6, Theorem 2.1] before inverting the augmentation procedure to obtain the corresponding HMP.
- (ii) One may consider a setting where the subdynamics $h_j : \mathbb{R}^n \times \mathbb{R}^{m_j} \rightarrow \mathbb{R}^n$ have possibly different control dimensions $m_j \in \mathbb{N}^*$ and with possibly different control constraint sets $U_j \subset \mathbb{R}^{m_j}$. This generalized context can be useful to set specific control values in particular regions (for example, by taking $U_j = \{0_{\mathbb{R}^{m_j}}\}$ for some $j \in \mathcal{J}$). We are confident that our methodology can be easily adapted to this framework without significant difficulties.

3 Derivation of a PMP with loss control regions

This section is organized as follows. In Section 3.1, we introduce a general optimal control problem with loss control regions, along with terminology and assumptions. In Section 3.2, we discuss the notion of *regular*

solution to the corresponding control system. In Section 3.3, we state and prove the main theoretical result of this paper (Theorem 3.1) which is a PMP with loss control regions. Its proof is based on Theorem 2.1 that was established in the previous section.

3.1 An optimal control problem with loss control regions: terminology and assumptions

Let $n, m, \ell \in \mathbb{N}^*$ be three fixed positive integers and $T > 0$ be a fixed positive real number. In this section we consider a partition of the state space given by

$$\mathbb{R}^n = \bigcup_{j \in \mathcal{J}} \overline{X_j},$$

where \mathcal{J} is a (possibly infinite) family of indexes and the nonempty open subsets $X_j \subset \mathbb{R}^n$, called *regions*, are disjoint. We introduce an indexation $q_j \in \{0, 1\}$ allowing us to separate *control regions* and *loss control regions* (see Introduction for details) as follows

$$q_j := \begin{cases} 1 & \text{if } X_j \text{ is a control region,} \\ 0 & \text{if } X_j \text{ is a loss control region.} \end{cases}$$

Our aim in this section is to derive first-order necessary optimality conditions in a Pontryagin form for the optimal control problem with *loss control regions* given by

$$\begin{aligned} & \text{minimize} && \phi(x(0), x(T)), \\ & \text{subject to} && (x, u) \in \text{AC}([0, T], \mathbb{R}^n) \times \text{L}^\infty([0, T], \mathbb{R}^m), \\ & && \dot{x}(t) = f(x(t), u(t)), \quad \text{a.e. } t \in [0, T], \\ & && g(x(0), x(T)) \in \text{S}, \\ & && u(t) \in \text{U}, \quad \text{a.e. } t \in [0, T], \\ & && u \text{ is constant when } x \text{ is in a loss control region,} \end{aligned} \tag{P}$$

where the Mayer cost function $\phi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$, the dynamics $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ and the constraint function $g : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^\ell$ are of class C^1 , and where both subsets $\text{S} \subset \mathbb{R}^\ell$ and $\text{U} \subset \mathbb{R}^m$ are nonempty closed convex subsets.

A pair $(x, u) \in \text{AC}([0, T], \mathbb{R}^n) \times \text{L}^\infty([0, T], \mathbb{R}^m)$ is said to be *admissible* for Problem (P) if it satisfies all the constraints of Problem (P). Such an admissible pair is said to be a *global solution* to Problem (P) if it minimizes the Mayer cost $\phi(x(0), x(T))$ among all admissible pairs.

3.2 Regular solution to the control system with loss control regions

Consider the control system with loss control regions associated with Problem (P) given by

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)), & \text{for a.e. } t \in [0, T], \\ u \text{ is constant when } x \text{ is in a loss control region.} \end{cases} \tag{CS}$$

Let us precise the definition of a solution to (CS).

Definition 3.1 (Solution to (CS)). A pair $(x, u) \in \text{AC}([0, T], \mathbb{R}^n) \times \text{L}^\infty([0, T], \mathbb{R}^m)$ is said to be a *solution* to (CS) if there exists a partition $\mathbb{T} = \{\tau_k\}_{k=0, \dots, N}$ of the interval $[0, T]$ such that:

(i) It holds that

$$\forall k \in \{1, \dots, N\}, \quad \exists j(k) \in \mathcal{J}, \quad \forall t \in (\tau_{k-1}, \tau_k), \quad x(t) \in X_{j(k)},$$

where $j(k) \neq j(k-1)$ for all $k \in \{2, \dots, N\}$. The sequence $\{j(1), \dots, j(N)\}$ is called the *switching sequence*.

- (ii) It holds that $x(0) \in X_{j(1)}$ and $x(T) \in X_{j(N)}$.
- (iii) For all $k \in \{1, \dots, N\}$ such that $q_{j(k)} = 0$, the control u is constant over (τ_{k-1}, τ_k) (the constant value being denoted by u_k in the sequel).
- (iv) It holds that $\dot{x}(t) = f(x(t), u(t))$ for almost every $t \in [0, T]$.

The times τ_k for $k \in \{1, \dots, N-1\}$, called *crossing times*, correspond to the instants at which the trajectory x goes from the region $X_{j(k)}$ to the region $X_{j(k+1)}$, and thus $x(\tau_k) \in \partial X_{j(k)} \cap \partial X_{j(k+1)}$.

Similarly to Section 2.3, the main result of this section (Theorem 3.1 stated in Section 3.3) is based on some regularity assumptions made on the optimal pair of Problem (P) at each of its crossing times. These hypotheses are made more precise in the next definition.

Definition 3.2 (Regular solution to (CS)). Following the notations introduced in Definition 3.1, a solution $(x, u) \in \text{AC}([0, T], \mathbb{R}^n) \times \text{L}^\infty([0, T], \mathbb{R}^m)$ to (CS), associated with a partition $\mathbb{T} = \{\tau_k\}_{k=0, \dots, N}$ and a switching sequence $\{j(1), \dots, j(N)\}$, is said to be *regular* if the following conditions are both satisfied:

- (i) At each crossing time τ_k , there exists a C^1 function $F_k : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\exists \nu_k > 0, \quad \forall z \in \bar{B}_{\mathbb{R}^n}(x(\tau_k), \nu_k), \quad \begin{cases} z \in X_{j(k)} & \Leftrightarrow F_k(z) < 0, \\ z \in \partial X_{j(k)} \cap \partial X_{j(k+1)} & \Leftrightarrow F_k(z) = 0, \\ z \in X_{j(k+1)} & \Leftrightarrow F_k(z) > 0. \end{cases}$$

In particular it holds that $F_k(x(\tau_k)) = 0$.

- (ii) At each crossing time τ_k , there exists $\alpha_k > 0$ and $\beta_k > 0$ such that the *transverse condition*

$$\langle \nabla F_k(x(\tau_k)), f(x(\tau_k), u(t)) \rangle_{\mathbb{R}^n} \geq \beta_k, \quad \text{a.e. } t \in [\tau_k - \alpha_k, \tau_k + \alpha_k], \quad (\text{TC})$$

is satisfied.

Remark 3.1. Similar comments than the ones developed in Remark 2.2 remain true for Definitions 3.1 and 3.2.

3.3 PMP with loss control regions and comments

The Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$ associated with Problem (P) is defined by

$$H(x, u, p) := \langle p, f(x, u) \rangle_{\mathbb{R}^n},$$

for all $(x, u, p) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$. We are now in a position to state the main result of this section.

Theorem 3.1 (PMP with loss control regions). Suppose that $(x^*, u^*) \in \text{AC}([0, T], \mathbb{R}^n) \times \text{L}^\infty([0, T], \mathbb{R}^m)$ is a global solution to Problem (P), that it is moreover a regular solution to (CS), associated with a partition $\mathbb{T}^* = \{\tau_k^*\}_{k=0, \dots, N}$ and a switching sequence $\{j(1), \dots, j(N)\}$, and that it is such that g is submersive at $(x^*(0), x^*(T))$. Then, there exists a nontrivial pair $(p, p^0) \in \text{PAC}_{\mathbb{T}^*}([0, T], \mathbb{R}^n) \times \mathbb{R}_+$ satisfying:

- (i) the *Hamiltonian system*

$$\dot{x}^*(t) = \nabla_p H(x^*(t), u^*(t), p(t)) \quad \text{and} \quad -\dot{p}(t) = \nabla_x H(x^*(t), u^*(t), p(t)),$$

for almost every $t \in [0, T]$;

- (ii) the *transversality condition*

$$\begin{pmatrix} p(0) \\ -p(T) \end{pmatrix} = p^0 \nabla \phi(x^*(0), x^*(T)) + \nabla g(x^*(0), x^*(T)) \xi,$$

for some $\xi \in \text{N}_S[g(x^*(0), x^*(T))]$;

(iii) the *discontinuity condition*

$$p^+(\tau_k^*) - p^-(\tau_k^*) = \sigma_k \nabla F_k^*(x^*(\tau_k^*)),$$

for some $\sigma_k \in \mathbb{R}$ and for all $k \in \{1, \dots, N-1\}$;

(iv) for all $k \in \{1, \dots, N\}$ such that $q_{j(k)} = 1$, the *Hamiltonian maximization condition*

$$u^*(t) \in \arg \max_{\omega \in U} H(x^*(t), \omega, p(t)),$$

for almost every $t \in (\tau_{k-1}^*, \tau_k^*)$;

(v) for all $k \in \{1, \dots, N\}$ such that $q_{j(k)} = 0$, the *averaged Hamiltonian gradient condition*

$$\int_{\tau_{k-1}^*}^{\tau_k^*} \nabla_u H(x^*(t), u_k^*, p(t)) dt \in N_U[u_k^*],$$

for all $k \in \{1, \dots, N\}$;

(vi) the *Hamiltonian constancy condition*

$$H(x^*(t), u^*(t), p(t)) = c,$$

for almost every $t \in [0, T]$, for some $c \in \mathbb{R}$.

Proof. Consider the framework of Theorem 3.1. The proof of Theorem 3.1 is based on the application of Theorem 2.1 to the hybrid optimal control problem with regionally switching parameter given by

$$\begin{aligned} & \text{minimize} && \phi(x(0), x(T)), \\ & \text{subject to} && (x, \lambda, u) \in \text{AC}([0, T], \mathbb{R}^n) \times \text{PC}([0, T], \mathbb{R}^m) \times L^\infty([0, T], \mathbb{R}^m), \\ & && \dot{x}(t) = h(x(t), \lambda(t), u(t)), \quad \text{a.e. } t \in [0, T], \\ & && g(x(0), x(T)) \in S, \\ & && (\lambda(t), u(t)) \in U^2, \quad \text{a.e. } t \in [0, T], \\ & && \lambda \text{ is a regionally switching parameter associated with } x, \end{aligned} \tag{HP'}$$

where the spatially heterogeneous dynamics $h : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is regionally defined by

$$h(x, \lambda, u) := \begin{cases} f(x, u) & \text{if } x \in X_j \text{ with } q_j = 1, \\ f(x, \lambda) & \text{if } x \in X_j \text{ with } q_j = 0, \end{cases}$$

for all $(x, \lambda, u) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$. The proof will be done in two steps.

Step 1. Consider some $\omega_0 \in U$ and introduce $\lambda^* \in \text{PC}_{\mathbb{T}^*}([0, T], \mathbb{R}^m)$ defined by

$$\lambda^*(t) := \begin{cases} \omega_0 & \text{for all } t \in (\tau_{k-1}^*, \tau_k^*) \text{ such that } q_{j(k)} = 1, \\ u_k^* & \text{for all } t \in (\tau_{k-1}^*, \tau_k^*) \text{ such that } q_{j(k)} = 0. \end{cases}$$

One can easily see that (x^*, λ^*, u^*) is admissible for Problem (HP') associated with the partition \mathbb{T}^* . Now let us prove that the triplet (x^*, λ^*, u^*) is a global solution to Problem (HP'). To this aim let $(x, \lambda, u) \in \text{AC}([0, T], \mathbb{R}^n) \times \text{PC}([0, T], \mathbb{R}^m) \times L^\infty([0, T], \mathbb{R}^m)$ be an admissible triplet for Problem (HP'), associated with a partition $\mathbb{T} = \{\tau_k\}_{k=0, \dots, N'}$ and a switching sequence $\{j'(1), \dots, j'(N')\}$. Let us introduce $v \in L^\infty([0, T], \mathbb{R}^m)$ defined by

$$v(t) := \begin{cases} u(t) & \text{for all } t \in (\tau_{k-1}, \tau_k) \text{ such that } q_{j'(k)} = 1, \\ \lambda_k & \text{for all } t \in (\tau_{k-1}, \tau_k) \text{ such that } q_{j'(k)} = 0. \end{cases}$$

One can easily see that (x, v) is admissible for Problem (P) associated with the partition \mathbb{T} . Therefore, using the optimality of (x^*, u^*) , we obtain that $\phi(x^*(0), x^*(T)) \leq \phi(x(0), x(T))$ which completes our intermediate goal. Furthermore one can easily see that, since (x^*, u^*) is a regular solution to (CS), then the triplet (x^*, λ^*, u^*)

is a regular solution to the hybrid control system associated with Problem (HP'). Finally recall that g is submersive at $(x^*(0), x^*(T))$ from hypotheses of Theorem 3.1.

Step 2. From Step 1, we are in a position to apply Theorem 2.1 on the triplet (x^*, λ^*, u^*) . Therefore there exists a nontrivial pair $(p, p^0) \in \text{PAC}_{\mathbb{T}^*}([0, T], \mathbb{R}^n) \times \mathbb{R}_+$ which satisfies items from (i) to (vi) of Theorem 2.1, where the Hamiltonian $\mathcal{H} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies

$$\mathcal{H}(x, \lambda, u, p) = \begin{cases} H(x, u, p) & \text{if } x \in X_j \text{ with } q_j = 1, \\ H(x, \lambda, p) & \text{if } x \in X_j \text{ with } q_j = 0, \end{cases}$$

for all $(x, \lambda, u, p) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^n$. One can easily deduce that items from (i) to (vi) of Theorem 3.1 are satisfied. The proof is complete. \square

Remark 3.2. Similar comments than the ones developed in Remarks 2.4, 2.5, 2.6 and 2.8 also apply to Theorem 3.1.

4 A numerical approach for optimal control problems with loss control regions and application to illustrative examples

In this section we consider the framework outlined in Section 3 and our objective is to introduce a numerical approach that can compute an optimal control for Problem (P) on illustrative examples¹. In Section 4.1, we present this approach which is composed of two steps: a direct method followed by an indirect method. Additionally we will highlight their pros and cons. In Section 4.2, we numerically solve a Zermelo-type problem with loss control regions (in two cases with different state space partitions). In Section 4.3, we numerically solve a minimal time problem for the harmonic oscillator with a loss control region.

4.1 A numerical approach for optimal control problems with loss control regions

In optimal control theory, there are several ways for solving numerically an optimal control problem. Direct and indirect methods represent an important class of methods that we will use hereafter. Direct methods involve discretizing the state and control variables, simplifying the problem into a nonlinear optimization problem. On the other hand, indirect methods tackle the problem by solving a boundary value problem through the use of a shooting method, which is based on the maximum principle. It is important to note that neither of these methods is better than the other.

Moreover, each of these methods has its pros and cons. For instance, although the direct method is simple to implement, more robust, and less sensitive to the choice of the initial condition, it should be noted that it yields less precise results and can converge to local minima that significantly deviate from the optimal solution. Additionally, this method requires a large amount of memory. On the other hand, the indirect method is known for its extreme precision. However, it is based only on necessary optimality conditions (maximum principle) and often requires knowledge of the structure of the optimal solution. Moreover, it is quite sensitive to the choice of the initial condition, which must be chosen carefully to ensure convergence. Therefore, both methods have their own strengths and limitations, and it is not accurate to say that one is better than the other.

Often in the literature, we proceed in two steps. The first step is to implement a direct method to determine the optimal solution's structure and extract the associated adjoint vectors. The second step involves constructing an indirect (shooting) method, where the initial condition is based on the results obtained from the direct method.

¹The scripts for reproducing the numerical experiments in this paper are published in the repository: <https://github.com/control-toolbox/control-loss>

Description of the direct method. For some $\omega_0 \in \mathbb{U}$ and some $\varepsilon > 0$ small enough, we introduce the regularized problem given by

$$\begin{aligned}
\text{minimize} \quad & \phi(x(0), x(T)) + \varepsilon \int_0^T v^2(t) dt + \int_0^T (1 - \Psi_\varepsilon(x(t))) \|u(t) - \omega_0\|_{\mathbb{R}^m}^2 dt, \\
\text{subject to} \quad & (x, \lambda, u, v) \in \text{AC}([0, T], \mathbb{R}^n) \times \text{AC}([0, T], \mathbb{R}^m) \times \text{L}^\infty([0, T], \mathbb{R}^m) \times \text{L}^\infty([0, T], \mathbb{R}), \\
& \dot{x}(t) = \Psi_\varepsilon(x(t))f(x(t), u(t)) + (1 - \Psi_\varepsilon(x(t)))f(x(t), \lambda(t)), \quad \text{a.e. } t \in [0, T], \\
& \dot{\lambda}(t) = \Psi_\varepsilon(x(t))v(t), \quad \text{a.e. } t \in [0, T], \\
& g(x(0), x(T)) \in S, \\
& \lambda(t) \in \mathbb{U}, \quad \text{a.e. } t \in [0, T], \\
& (u(t), v(t)) \in \mathbb{U} \times \mathbb{R}, \quad \text{a.e. } t \in [0, T],
\end{aligned} \tag{P_\varepsilon}$$

where $\Psi_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}$ is the regularization of the characteristic function of $\cup_{q_j=1} \overline{X_j}$ given by

$$\Psi_\varepsilon(x) := \sum_{q_j=1} e^{-\frac{1}{2\varepsilon} d_j^2(x)},$$

for all $x \in \mathbb{R}^n$, where $d_j : \mathbb{R}^n \rightarrow \mathbb{R}$ stands for the distance function to the set $\overline{X_j}$ defined by $d_j(x) := \inf_{y \in \overline{X_j}} \|x - y\|_{\mathbb{R}^n}$ for all $x \in \mathbb{R}^n$ and every $j \in \mathcal{J}$. Note that the above regularized problem arises from considering the hybrid optimal problem with regionally switching parameter associated with Problem (P), as outlined in the proof of Theorem 3.1. Before presenting the direct method we would like to explain the numerical role of certain quantities that are considered in Problem (P_ε):

- (i) Comment on the dynamics of x and λ . Since Ψ_ε (approximately) equals to one in control regions, it follows that the dynamics of x is described by $\dot{x}(t) = f(x(t), u(t))$ in these regions. On the contrary, since Ψ_ε (approximately) vanishes in loss control regions, it follows that λ remains constant and that the dynamics of x is described by $\dot{x}(t) = f(x(t), \lambda(t))$ in these regions.
- (ii) Comment on the additional control v and the penalization cost $\varepsilon \int_0^T v^2(t) dt$. Since Ψ_ε (approximately) equals to one in control regions, the additional control v can operate in these regions to allow λ to change its constant value between two consecutive loss control regions. On the other hand, the penalization cost is introduced to ensure the convergence of the direct method and to guarantee the uniqueness of the optimal control v . In addition, the multiplicative parameter $\varepsilon > 0$ (small enough) ensures that the penalization cost does not influence (too much) the original cost of Problem (P). Finally, note that, since Ψ_ε (approximately) vanishes in loss control regions, the control system is independent of v in these regions and thus, due to the penalization cost, the optimal control v vanishes in these regions.
- (iii) Comment on the running cost $\int_0^T (1 - \Psi_\varepsilon(x(t))) \|u(t) - \omega_0\|_{\mathbb{R}^m}^2 dt$. Since Ψ_ε (approximately) vanishes in loss control regions, the control system is independent of u in these regions and thus, due to this running cost, the optimal control u is unique (given by $u(t) = \omega_0$) in these regions. Furthermore, since Ψ_ε (approximately) equals to one in control regions, we deduce that this running cost is (approximately) equal to zero at the optimal solution.

We are now in a position to present the direct method for solving Problem (P_ε). After rewriting the problem in a Mayer form, we discretize the time interval and approximate the state and the control at discrete time points, resulting in a finite-dimensional nonlinear optimization problem. This problem is solved using JuMP, a modeling language for mathematical optimization embedded in Julia, with the Mayer cost function as objective function. To handle the dynamics, we employ the *Crank-Nicholson* scheme, and the initial, terminal, and control constraints are formulated as optimization problem constraints. We use the IPOPT solver (whose precision is set to 10⁻⁸) to solve the resulting optimization problem and extract the adjoint vectors using the predefined `dual` function.

Description of the indirect method. From the direct method described above, we extract the structure of the optimal solution (x^*, u^*) to Problem (P). This latter contains an initialization of the adjoint vector p ,

crossing times, switching times, constant values of the control (when the state belongs to loss control regions) and discontinuity jumps of p at each crossing time. These elements allow us to construct a shooting method based on Theorem 3.1 which consists in two parts described below.

In the rest of this section, for simplicity, we will assume that, when x^* belongs to a loss control region, then the constant value u_k^* of the control belongs to the interior of U (see Remark 4.1 for details).

- (i) **Part 1.** Recall that the direct method has captured the structure of the optimal pair (x^*, u^*) . Therefore, in the indirect method we will address each arc (bang-bang, constant (interior value to U), feedback etc.) of the optimal solution separately. Indeed, we begin by defining the flow of the Hamiltonian associated with each arc. To accomplish this, we use the function `Flow`². This latter allows to solve the Hamiltonian system over a given time interval from given initial values of the state and the adjoint vector. This function requires necessary libraries such as `ForwardDiff` for calculating gradients and Jacobians and `OrdinaryDiffEq` for solving ordinary differential equations.

In the setting of loss control regions, we distinguish between two types of Hamiltonian flows:

- **Hamiltonian flows in control regions.** Consider the setting of Theorem 3.1, we recall the Hamiltonian associated with Problem (P) as

$$H(x, u, p) = \langle p, f(x, u) \rangle_{\mathbb{R}^n},$$

for all $(x, u, p) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$. Using Theorem 3.1, specifically the maximization condition, we obtain the expression of the control u^* (which can generate a sequence of arcs). Thus, it remains to define a *pseudo-Hamiltonian*³ associated with each arc. Finally, we define the flow associated with each arc, which allows the resolution of the boundary value problem on a time interval satisfied by the pair (x^*, p) with an initial condition.

- **Hamiltonian flows in loss control regions.** In these regions, we recall that u^* satisfies an averaged Hamiltonian gradient condition (instead of a maximization condition). Here, the difficulty lies in the fact that this condition is given in an integral and implicit form. Therefore, to overcome this difficulty, we first introduce new states λ and y . The state λ comes from the augmentation technique to handle the constant value of u^* , for example u_k^* , so it satisfies the dynamics $\dot{\lambda}(t) = 0$ and $\lambda(\tau_{k-1}^*) = u_k^*$. Second, the state y satisfies $\dot{y}(t) = 0$ and $y(\tau_{k-1}^*) = 0$ (and thus $y = 0$). Now, we can define the new Hamiltonian as follows:

$$\begin{aligned} \tilde{H}(x, \lambda, y, p) &= H(x, \lambda, p) - y \nabla_{\lambda} H(x, \lambda, p) \\ &= \langle p, f(x, \lambda) \rangle_{\mathbb{R}^n} - y \nabla_{\lambda} H(x, \lambda, p), \end{aligned}$$

for all $(x, \lambda, y, p) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n$. It is important to note that $y = 0$ is necessary to recover the same Hamiltonian H . But, the actual utility of introducing the state y is that it allows us to rewrite the integral expressed in the averaged Hamiltonian gradient condition as a terminal value of an adjoint vector. This makes it easier to take into account in the shooting function. Indeed, we define p_y as the solution to the following system:

$$\begin{cases} \dot{p}_y(t) = -\nabla_y \tilde{H}(x^*(t), u_k^*, y(t), p(t)), & \text{for a.e. } t \in [\tau_{k-1}^*, \tau_k^*], \\ p_y(\tau_{k-1}^*) = 0_{\mathbb{R}^n}. \end{cases}$$

Since u_k^* is assumed to be an interior value to U then we get that

$$\int_{\tau_{k-1}^*}^{\tau_k^*} \nabla_u H(x^*(t), u_k^*, p(t)) dt = p_y(\tau_k^*) = 0,$$

so that there is no need to compute an integral in order to take into account the averaged gradient condition.

²the `Flow` function can be found in the `CTFlows.jl` package: <https://github.com/control-toolbox/CTFlows.jl>

³the pseudo-Hamiltonian stands for the Hamiltonian flow associated with each arc.

- (ii) **Part 2.** It consists in determining the appropriate shooting function which includes novel elements such as crossing times, the discontinuity condition of the adjoint vector at each crossing time and (mainly) the averaged Hamiltonian gradient condition. In addition, it also includes standard elements such as the switching times, terminal state constraints and transversality conditions. Once the shooting function is constructed, we use the predefined function `NLsolve` (whose precision is set to 10^{-8}) with an initialization based on the direct method in order to find a zero of the corresponding shooting function.

Remark 4.1. In this section we assumed that, when x^* belongs to a loss control region, then the constant value u_k^* of the control belongs to the interior of U . This was made for simplicity since the averaged Hamiltonian gradient condition simplifies into an equality. Note that the case where u_k^* belongs to the boundary of U can be treated easily when $m = 1$ (typically if $U = [-1, 1]$ for example), since the boundary is finite and the optimal value can be deduced from the direct method (see Remark 4.2 in Section 4.3). When u_k^* belongs to the boundary of U and $m \geq 2$, the averaged Hamiltonian gradient condition is more involved to handle in the shooting function since it only provides an inclusion. A possible way could be to combine this inclusion together with a description of the set U to get additional equations. The consideration of this case is out of the scope of the paper and could be the matter of future works.

In the remaining sections, we will solve numerically a Zermelo-type problem with two different state partitions that falls within the framework described in Section 3. Next, we will study numerically the harmonic oscillator problem including a loss control region. This is a minimum time problem that can be transformed into the framework of Section 3 using a change of variables and introducing a new state variable $\dot{x}_{n+1} = T$ for almost every $s \in [0, 1]$ with $x_{n+1}(0) = 0$, in such a way that the time interval is fixed. However, we obtain an optimal control problem with loss control regions, but with an additional constant parameter. So, to fall in the framework of Section 3, we rely on the augmentation technique.

4.2 Application to a Zermelo-type problem with loss control regions

Our first example is based on Zermelo's problem, which is well-known in viability theory [2] and also in geometric optimal control theory [15]. One interesting issue related to this problem is that, depending on the parameter values, it highlights the link between non-controllability and abnormal curves. However, we do not investigate this point. Instead, we focus on optimizing one coordinate whenever a loss control region is taken into account (which can model various behaviors of the flow in this navigation problem). Such a problem will be formulated as a Mayer optimal control problem. Specifically, we consider the Zermelo-type optimal control problem with loss control regions given by

$$\begin{aligned}
& \text{minimize} && -x_1(8), \\
& \text{subject to} && (x, u) \in \text{AC}([0, 8], \mathbb{R}^2) \times L^\infty([0, 8], \mathbb{R}), \\
& && \dot{x}_1(t) = x_2(t) + \cos(u(t)), \quad \text{a.e. } t \in [0, 8], \\
& && \dot{x}_2(t) = \sin(u(t)), \quad \text{a.e. } t \in [0, 8], \\
& && x(0) = 0_{\mathbb{R}^2}, \quad x_2(8) = 4, \\
& && u(t) \in [-\frac{\pi}{2}, \frac{\pi}{2}], \quad \text{a.e. } t \in [0, 8], \\
& && u \text{ is constant when } x \text{ is in a loss control region.}
\end{aligned} \tag{4.1}$$

In the sequel, based on the numerical approach developed in Section 4.1, we solve Problem (4.1) on two different situations of space partition of \mathbb{R}^2 (see Example 1 and Example 2). Note that when applying Theorem 3.1 to Problem (4.1) we remain in a normal situation, i.e., $p^0 = 1$.

Example 1. Consider the space partition $\mathbb{R}^2 = \overline{X_1} \cup \overline{X_2} \cup \overline{X_3}$ with

$$X_1 := \left\{ x \in \mathbb{R}^2 \mid x_2 < \frac{1}{2} \right\}, \quad X_2 := \left\{ x \in \mathbb{R}^2 \mid \frac{1}{2} < x_2 < \frac{7}{2} \right\} \quad \text{and} \quad X_3 := \left\{ x \in \mathbb{R}^2 \mid x_2 > \frac{7}{2} \right\},$$

with $q_1 = q_3 = 1$ and $q_2 = 0$ (see Figure 2).

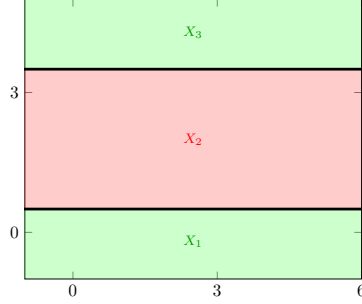


Figure 2: Partition of \mathbb{R}^2 into one loss control region (in red) and two control regions (in green).

Application of the direct method. We obtain an optimal triplet (x, λ, u) that visits the loss control region X_2 once with $\lambda \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Moreover, the adjoint vector p_2 has discontinuity jumps at each crossing time as illustrated in Figure 3.

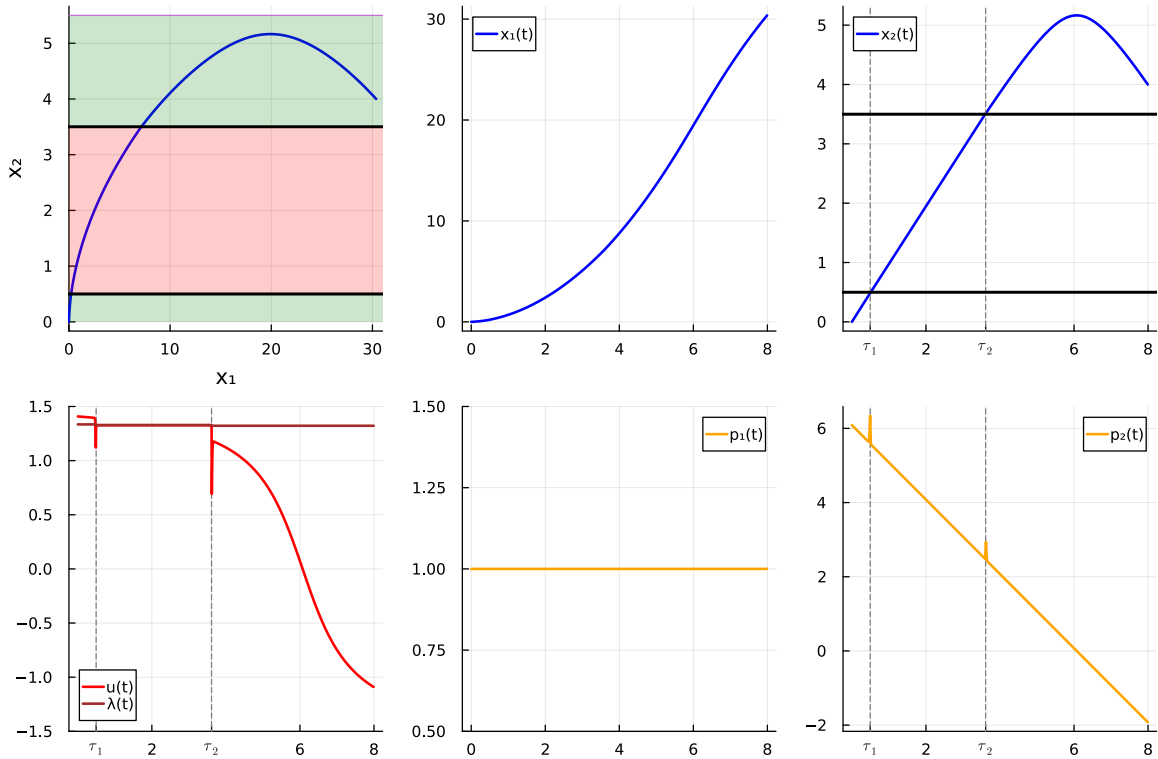


Figure 3: Direct method: optimal triplet (x, λ, u) and adjoint vector p for Problem (4.1) (Example 1).

Application of the indirect method. From the above results, we deduce that the optimal solution to Problem 4.1 (Example 1), denoted by (x^*, u^*) , has three arcs: feedback, then a constant value in $(-\frac{\pi}{2}, \frac{\pi}{2})$, then feedback. From the adjoint equation and the transversality condition we get that $p_1(t) = 1$ for all $t \in [0, 8]$. Therefore, we can express the control u^* in a feedback form using the Hamiltonian maximization condition.

$$u^*(t) \in \arg \max_{\omega \in [-\frac{\pi}{2}, \frac{\pi}{2}]} p_1(t) \cos(\omega) + p_2(t) \sin(\omega) = \arg \max_{\omega \in [-\frac{\pi}{2}, \frac{\pi}{2}]} \left\langle \begin{pmatrix} \cos(\omega) \\ \sin(\omega) \end{pmatrix}, \begin{pmatrix} 1 \\ p_2(t) \end{pmatrix} \right\rangle_{\mathbb{R}^2},$$

which is satisfied for almost every $t \in [0, \tau_1^*] \cup [\tau_2^*, 8]$. From the classical Cauchy-Schwartz inequality, we get

that

$$\begin{pmatrix} \cos(u^*(t)) \\ \sin(u^*(t)) \end{pmatrix} = \frac{1}{\|(1, p_2(t))\|_{\mathbb{R}^2}} \begin{pmatrix} 1 \\ p_2(t) \end{pmatrix},$$

for almost every $[0, \tau_1^*] \cup [\tau_2^*, 8]$, we deduce that $\cos(u^*(t)) \neq 0$ and thus $u^*(t) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ over $[0, \tau_1^*] \cup [\tau_2^*, 8]$. We conclude that $\tan(u^*(t)) = p_2(t)$ and thus $u^*(t) = \arctan(p_2(t))$ for almost every $t \in [0, \tau_1^*] \cup [\tau_2^*, 8]$. Now, we give the shooting function for Problem (4.1) (Example 1).

$$\mathcal{S}_1 : \mathbb{R}^7 \rightarrow \mathbb{R}^7$$

$$(p_1^0, p_2^0, \tau_1, \tau_2, a_1, a_2, u_1) \mapsto \begin{pmatrix} x_2(8) - 4 \\ p_1(8) - 1 \\ x_2(\tau_1) - \frac{1}{2} \\ x_2(\tau_2) - \frac{7}{2} \\ p_y(\tau_2) \\ a_1 - \frac{p_1(\tau_1)(\cos(u_1) - \cos(u^-(\tau_1))) + p_2^-(\tau_1)(\sin(u_1) - \sin(u^-(\tau_1)))}{\sin(u_1)} \\ a_2 - \frac{p_1(\tau_2)(\cos(u^+(\tau_2)) - \cos(u_1)) + p_2^-(\tau_2)(\sin(u^+(\tau_2)) - \sin(u_1))}{\sin(u^+(\tau_2))} \end{pmatrix}$$

The numerical results presented in Figure 4 depict an optimal pair (x^*, u^*) of Problem (4.1) (Example 1)

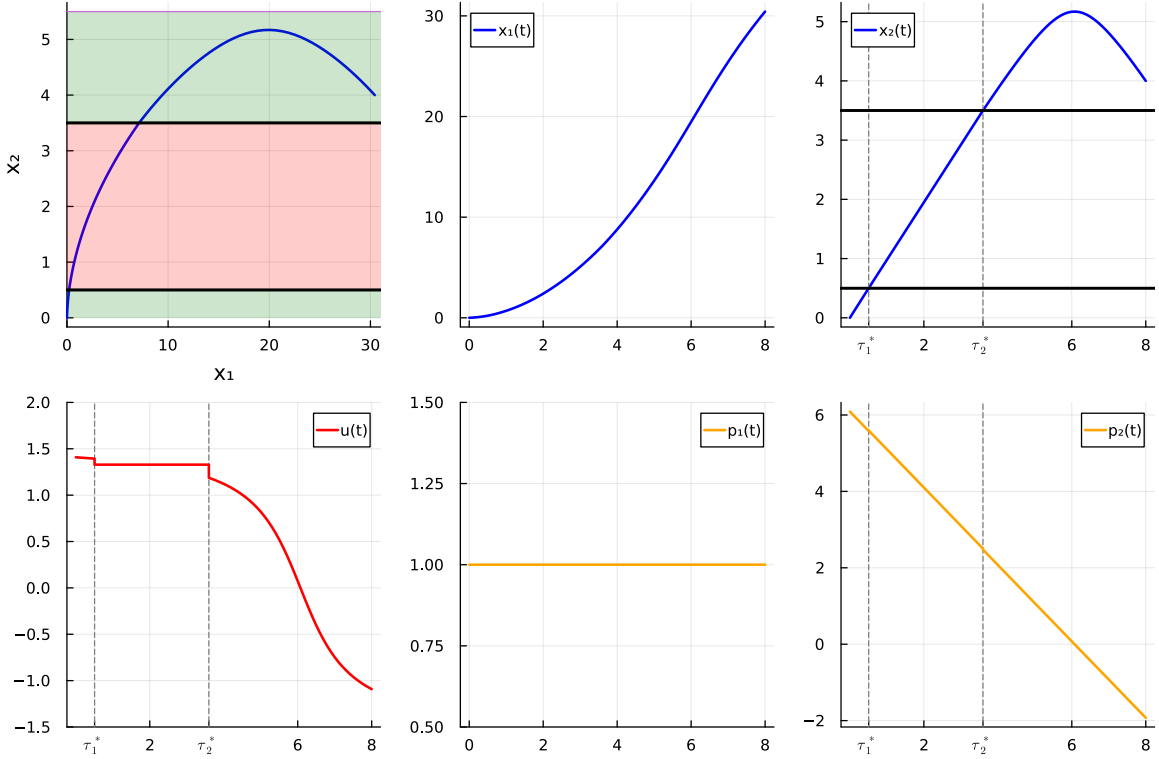


Figure 4: Indirect method: optimal pair (x^*, u^*) and adjoint vector p for Problem (4.1) (Example 1).

together with an adjoint vector p . We observe (small) jumps in the adjoint vector p_2 at both crossing times. In summary, the indirect method confirms the solution obtained using the direct method. Finally, note in the direct method we notice that constancy is satisfied (see Figure 5).

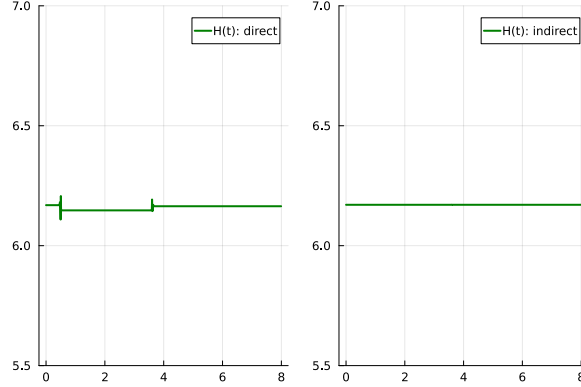


Figure 5: Comparison of Hamiltonian functions: direct method vs. indirect method for Problem (4.1) (Example 1).

Example 2. We study now a variant of the Problem (4.1) including several loss control regions. Consider the space partition $\mathbb{R}^2 = \overline{X_1} \cup \overline{X_2} \cup \overline{X_3} \cup \overline{X_4} \cup \overline{X_5}$ with

$$X_1 := \{x \in \mathbb{R}^2 \mid x_1 < 2\}, \quad X_2 := \{x \in \mathbb{R}^2 \mid 2 < x_1 < 16\}, \quad X_3 := \{x \in \mathbb{R}^2 \mid 16 < x_1 < 20\},$$

$$X_4 := \{x \in \mathbb{R}^2 \mid 20 < x_1 < 25\} \quad \text{and} \quad X_5 := \{x \in \mathbb{R}^2 \mid 25 < x_1\},$$

with $q_1 = q_3 = q_5 = 1$ and $q_2 = q_4 = 0$ (see Figure 6).

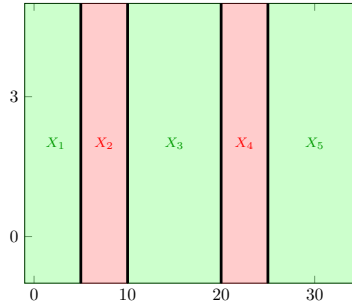


Figure 6: Partition of \mathbb{R}^2 into two loss control regions (in red) and three control regions (in green).

Application of the direct method. We obtain an optimal triplet (x, λ, u) that visits the loss control regions X_2 and X_4 with $\lambda \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Moreover, the adjoint vector p_1 has discontinuity jumps at each crossing time as illustrated in Figure 7.

Application of the indirect method. We use similar arguments as in Example 1, we get that in control regions one has $u^*(t) = \arctan(p_2(t))$ for almost every $t \in [0, \tau_1^*] \cup [\tau_2^*, \tau_3^*] \cup [\tau_4^*, 8]$. Now, we give the

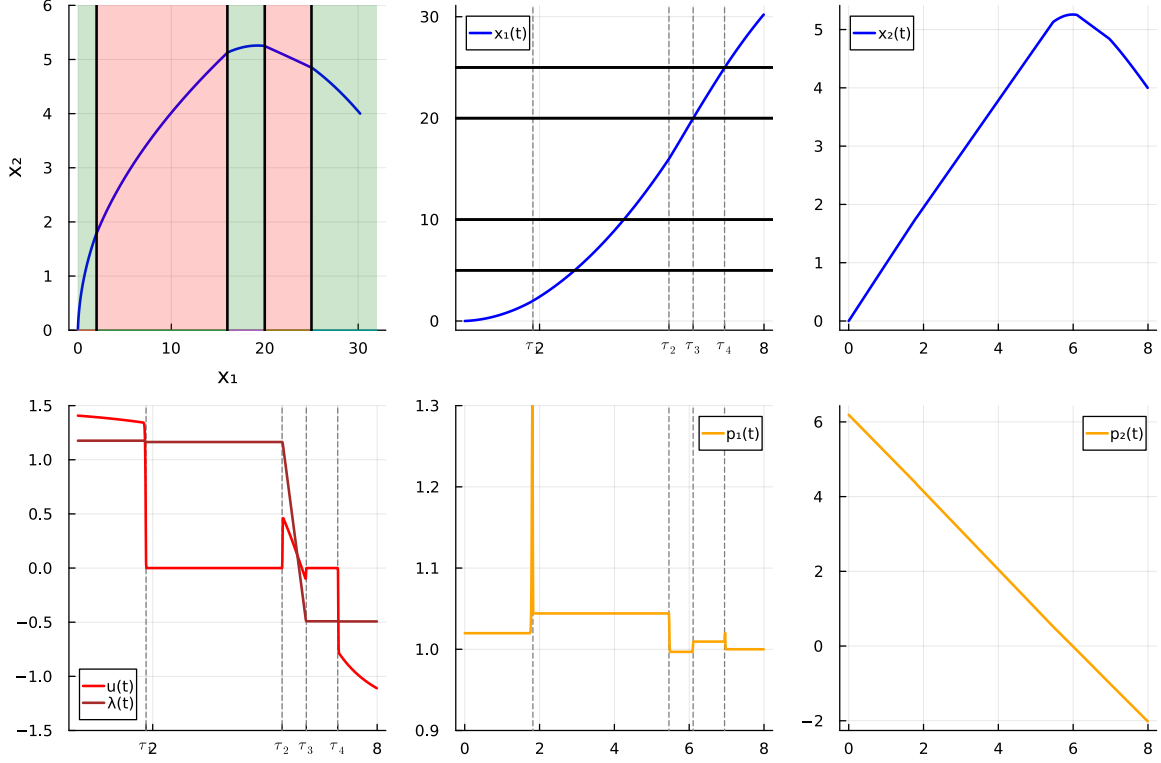


Figure 7: Direct method: optimal triplet (x, λ, u) and adjoint vector p for Problem (4.1) (Example 2).

shooting function for Problem (4.1) (Example 2).

$$\begin{aligned}
 \mathcal{S}_2 : \mathbb{R}^{12} &\rightarrow \mathbb{R}^{12} \\
 (p_1^0, p_2^0, \tau_1, \tau_2, \tau_3, \tau_4, a_1, a_2, a_3, a_4, u_1, u_2) &\mapsto \begin{pmatrix} x_2(8) - 4 \\ p_1(8) - 1 \\ x_1(\tau_1) - 2 \\ x_1(\tau_2) - 16 \\ x_1(\tau_3) - 20 \\ x_1(\tau_4) - 25 \\ p_y(\tau_2) \\ p_y(\tau_4) \\ a_1 - \frac{p_1^-(\tau_1)(\cos(u_1) - \cos(u^-(\tau_1))) + p_2(\tau_1)(\sin(u_1) - \sin(u^-(\tau_1)))}{\sin(u_1)} \\ a_2 - \frac{p_1^-(\tau_2)(\cos(u^+(\tau_2)) - \cos(u_1)) + p_2(\tau_2)(\sin(u^+(\tau_2)) - \sin(u_1))}{\sin(u^+(\tau_2))} \\ a_3 - \frac{p_1^-(\tau_3)(\cos(u_2) - \cos(u^-(\tau_3))) + p_2(\tau_3)(\sin(u_2) - \sin(u^-(\tau_3)))}{\sin(u_2)} \\ a_4 - \frac{p_1^-(\tau_4)(\cos(u^+(\tau_4)) - \cos(u_2)) + p_2(\tau_4)(\sin(u^+(\tau_4)) - \sin(u_2))}{\sin(u^+(\tau_4))} \end{pmatrix}
 \end{aligned}$$

The numerical results presented in Figure 8 depict an optimal pair (x^*, u^*) of Problem (4.1) (Example 2) together with an adjoint vector p . We observe jumps in the adjoint vector p_1 at each crossing time (see Figures 8). Again, our simulations via the indirect method yields a solution (x^*, u^*) that is similar to the one obtained by the direct method. Finally, note in the direct method we notice that constancy is satisfied (see Figure 9).

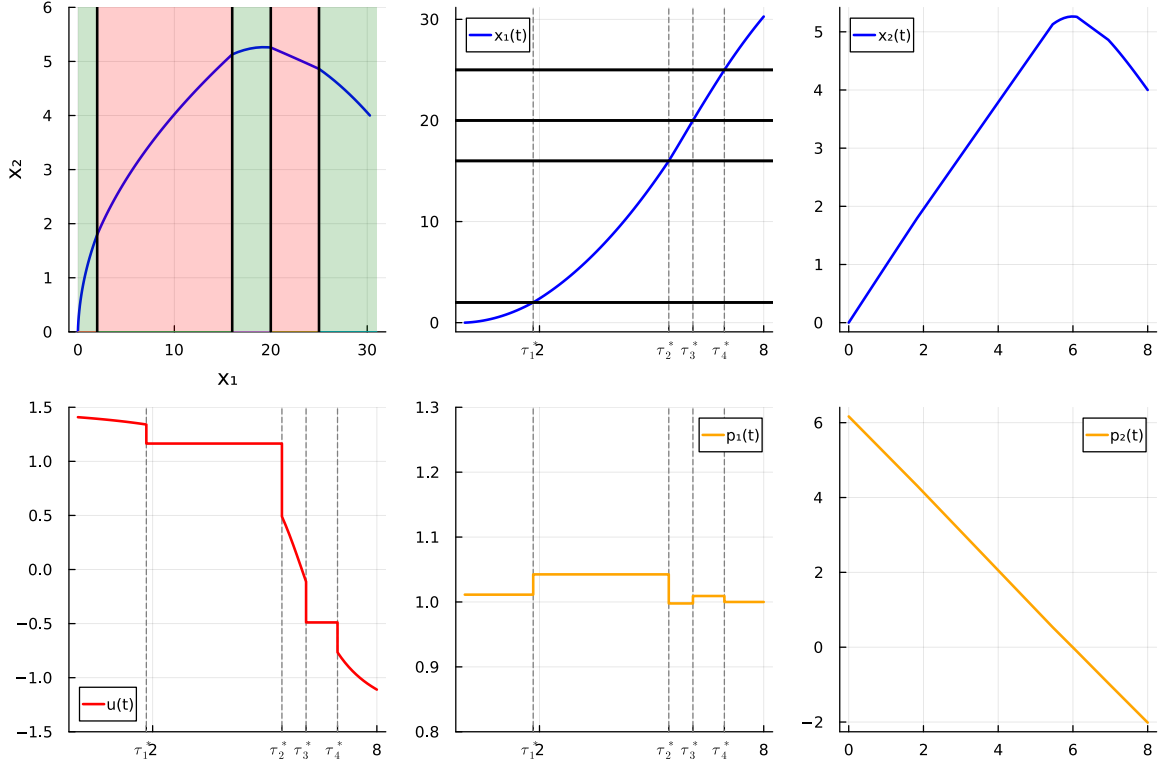


Figure 8: Indirect method: optimal pair (x^*, u^*) and adjoint vector p for Problem (4.1) (Example 2).

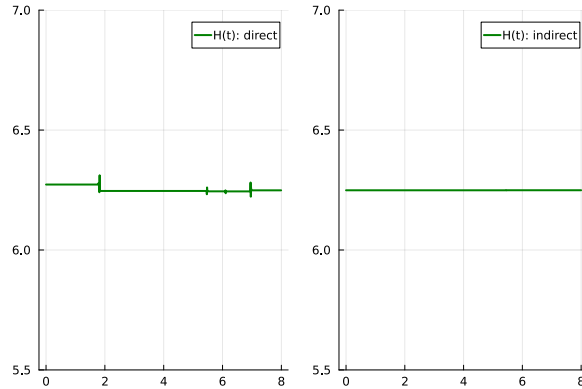


Figure 9: Comparison of Hamiltonian functions: direct method vs. indirect method for Problem (4.1) (Example 2).

4.3 Minimum time problem for the harmonic oscillator with a loss control region

This example is in line with [7] in which we have computed theoretically optimal paths for the classical minimum time control problem governed by the double integrator with a loss control region. Here, we have considered a variant of the minimum time control problem governed by the harmonic oscillator (the classical version is treated in [35]) including a loss control region. One main issue (in contrast with the double integrator, as in [7]) is that trajectories spiral around the target (origin) in a finite number in order to reach this target. Hence, depending on the choice of a control loss region, we expect trajectories to visit this region several times. Thus, the constant value of the control at each visit could be modified. Specifically, we consider the minimum time problem for the harmonic oscillator with a loss control region given by

$$\begin{aligned}
& \text{minimize} && T, \\
& \text{subject to} && (x, u, T) \in \text{AC}([0, T], \mathbb{R}^2) \times \text{L}^\infty([0, T], \mathbb{R}) \times (0, +\infty), \\
& && \dot{x}_1(t) = x_2(t), \quad \text{a.e. } t \in [0, T], \\
& && \dot{x}_2(t) = u(t) - x_1(t), \quad \text{a.e. } t \in [0, T], \\
& && x(0) = \left(\frac{5}{2}, 4\right), \quad x(T) = 0_{\mathbb{R}^2}, \\
& && u(t) \in [-1, 1], \quad \text{a.e. } t \in [0, T], \\
& && u \text{ is constant when } x \text{ is in a loss control region,}
\end{aligned} \tag{4.2}$$

with the space partition $\mathbb{R}^2 = \overline{X_1} \cup \overline{X_2}$ with

$$X_1 := \{x \in \mathbb{R}^2 \mid x_2 > 0\}, \quad \text{and} \quad X_2 := \{x \in \mathbb{R}^2 \mid x_2 < 0\},$$

with $q_1 = 1$ and $q_2 = 0$ (see Figure 10). Note that when applying Theorem 3.1 to Problem (4.2) we remain in a normal situation i.e. $p^0 = 1$.

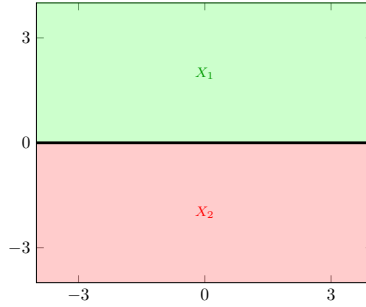


Figure 10: Partition of \mathbb{R}^2 into a “loss control regions” (in red) and a “control region” (in green).

Application of the direct method. We obtain an optimal triplet (x, λ, u) that visits the loss control regions X_2 twice. In the first visit, λ is equal to 1, and in the second visit, λ belongs to $(-1, 1)$. Moreover, we observe that the control is bang-bang during the second visit of the control region as illustrated in in Figure 11.

Application of the indirect method. Based on the above results, we deduce that the optimal solution to Problem (4.2) consists of four arcs. These arcs include sequences of bang arcs and one arc with a constant value in the range of $(-1, 1)$. Now, we give the shooting function for Problem (4.2).

$$\begin{aligned}
& \mathcal{S}_3 : \mathbb{R}^9 \rightarrow \mathbb{R}^{10} \\
& (p_1^0, p_2^0, \tau_1, \tau_2, \tau_3, \sigma, a_1, a_2, u_2) \mapsto \begin{pmatrix} x_1(T) \\ x_2(T) \\ x_2(\tau_1) \\ x_2(\tau_2) \\ x_2(\tau_3) \\ p_y(T) \\ a_1 - \frac{2p_1^-(\tau_1)}{1 - x_1(\tau_1)} \\ a_2 - \frac{p_1^-(\tau_3)(u_2 + 1)}{u_2 - x_1(\tau_3)} \\ p_2(\sigma) \\ p_1(0)x_2(0) - p_2(0)(1 + x_1(0)) - 1 \end{pmatrix}
\end{aligned}$$

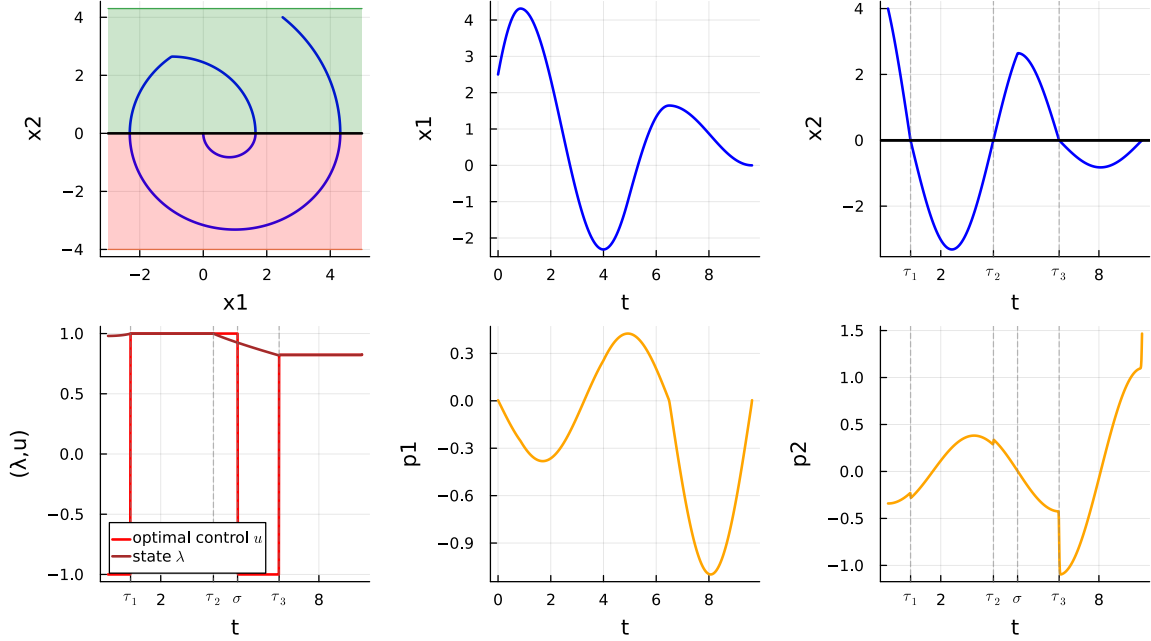


Figure 11: Direct method: optimal triplet (x, λ, u) and adjoint vector p for Problem (4.2).

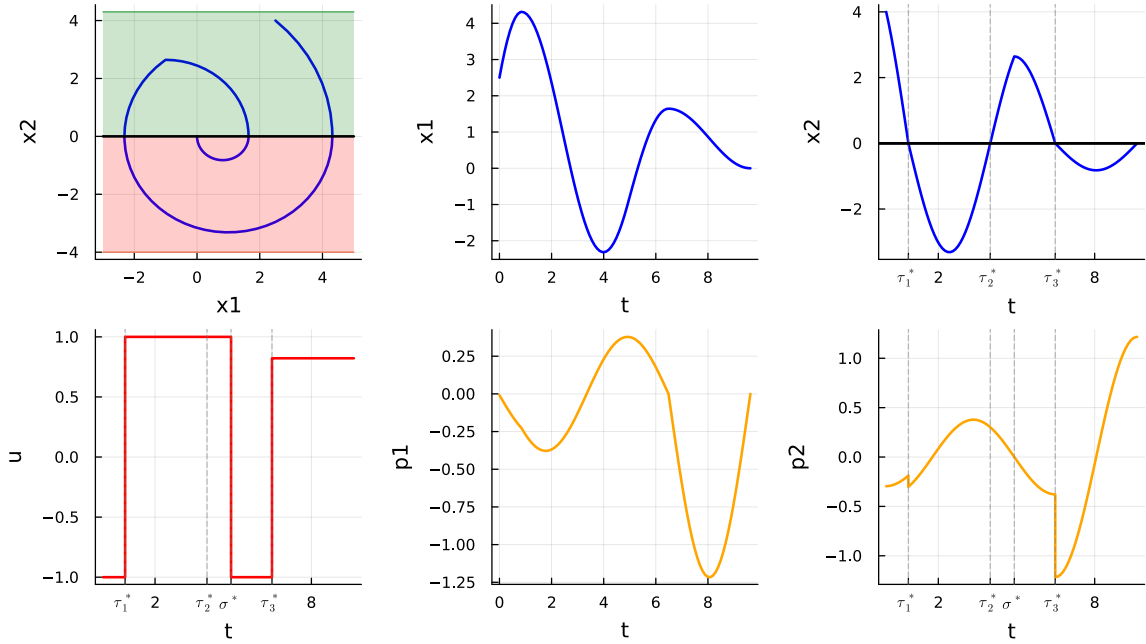


Figure 12: Indirect method: optimal pair (x^*, u^*) and adjoint vector p for Problem (4.2).

The numerical results presented in Figure 12 depict an optimal pair (x^*, u^*) of Problem (4.2) together with an adjoint vector p . Contrary to the direct method, we observe jumps in the adjoint vector p_2 at τ_1^* and τ_3^* . There are no discontinuity jumps at τ_2^* because $(u^*)^+(\tau_2^*) = u_1^*$. Finally, note in the direct method we notice that constancy is satisfied (see Figure 13).

Remark 4.2. As discussed in Remark 4.1, this example shows that an extremal value of the control (namely, $u_1^* = 1$) is possible whenever the trajectory visits the loss control region. This value was expected from the

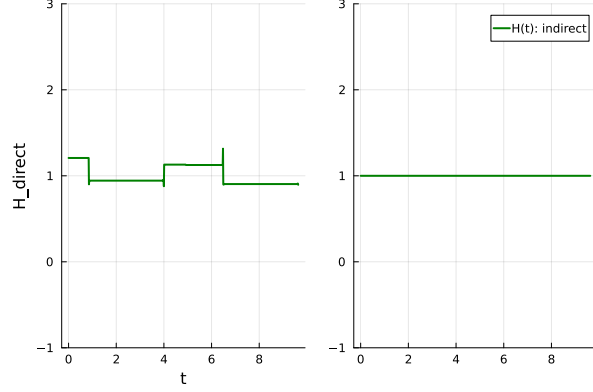


Figure 13: Comparison of Hamiltonian functions: direct method vs. indirect method for Problem (4.2).

numerical solution obtained via the direct method. Observe that it requires the verification of the averaged Hamiltonian gradient condition (here as an inequality). However, whenever the trajectory visits the loss control region for the second time, we obtain that $u_2^* \approx 0.89 \in (-1, 1)$. Thus, finding the accurate value of u_2^* requires the use of an additional equation in the shooting function (see Section 4.1).

A Proof of Proposition 2.1

Consider the framework of Proposition 2.1 and let us prove that the triplet (y^*, v^*, \mathbb{T}^*) is a $L^1_{[\varepsilon, 1-\varepsilon]}$ -local solution to Problem (CP*) for any $0 < \varepsilon < \frac{1}{2}$. Therefore let us fix some $0 < \varepsilon < \frac{1}{2}$ and some $R \geq \|v^*\|_{L^\infty}$. Our aim is to prove that there exists $\eta > 0$ such that $\phi^*(y^*(0), y^*(1)) \leq \phi^*(y(0), y(1))$ for any triplet (y, v, \mathbb{T}) that is admissible for Problem (CP*) and satisfying

$$\begin{cases} \|y - y^*\|_C + \|v - v^*\|_{L^1} + \|\mathbb{T} - \mathbb{T}^*\|_{\mathbb{R}^{N+1}} \leq \eta, \\ \|v\|_{L^\infty} \leq R, \\ v(s) = v^*(s) \text{ a.e. } s \in [0, \varepsilon] \cup [1 - \varepsilon, 1]. \end{cases} \quad (\text{A.1})$$

To this aim we need to introduce several technical positive parameters:

(\mathfrak{P}_1) First we introduce $\underline{\theta} := \min_{k \in \{1, \dots, N\}} |\tau_k^* - \tau_{k-1}^*| > 0$ and $\bar{\theta} := \max_{k \in \{1, \dots, N\}} |\tau_k^* - \tau_{k-1}^*| > 0$.

(\mathfrak{P}_2) From the transverse conditions (see Definition 2.2) and the continuities of the functions ∇F_k^* and f_k^* (and thus the uniform continuities of the functions $\langle \nabla F_k^*, f_k^* \rangle_{\mathbb{R}^n}$ on compact sets), there exist $0 < \nu \leq \min_{k \in \{1, \dots, N-1\}} \nu_k^*$ and $0 < \alpha \leq \min\{\frac{\underline{\theta}}{3}, \min_{k \in \{1, \dots, N-1\}} \alpha_k^*\}$ such that

$$\begin{cases} \langle \nabla F_k^*(z), f_k^*(z, \lambda^-, u^*(t)) \rangle_{\mathbb{R}^n} > 0, & \text{a.e. } t \in [\tau_k^* - \alpha, \tau_k^*], \\ \langle \nabla F_k^*(z), f_{k+1}^*(z, \lambda^+, u^*(t)) \rangle_{\mathbb{R}^n} > 0, & \text{a.e. } t \in (\tau_k^*, \tau_k^* + \alpha], \end{cases}$$

for all $z \in \bar{B}_{\mathbb{R}^n}(x^*(\tau_k^*), \nu)$, all $\lambda^- \in \bar{B}_{\mathbb{R}^d}(\lambda_k^*, \nu)$, all $\lambda^+ \in \bar{B}_{\mathbb{R}^d}(\lambda_{k+1}^*, \nu)$ and all $k \in \{1, \dots, N-1\}$.

(\mathfrak{P}_3) From continuity of y^* over $[0, 1]$, there exists $0 < \chi < \frac{1}{2}$ such that $\|y_k^*(s) - y_k^*(0)\|_{\mathbb{R}^n} \leq \frac{\nu}{2}$ for all $s \in [0, \chi]$ and $\|y_k^*(s) - y_k^*(1)\|_{\mathbb{R}^n} \leq \frac{\nu}{2}$ for all $s \in [1 - \chi, 1]$, for all $k \in \{1, \dots, N\}$.

(\mathfrak{P}_4) Define $\gamma := \frac{\underline{\theta}}{3} \min\{\varepsilon, \chi, \frac{\alpha}{\bar{\theta}}\} > 0$ and $r := \frac{\gamma}{\bar{\theta} + \underline{\theta}} > 0$. Note that $0 < \gamma \leq \alpha \leq \frac{\underline{\theta}}{3}$ and $0 < r < \frac{1}{2}$.

(\mathfrak{P}_5) From continuity of y^* , from (2.3) and the openness of the regions E_k^* , there exists $\delta > 0$ such that

$$\begin{cases} \bar{B}_{\mathbb{R}^n}(y_1^*(s), \delta) \subset E_1^*, & \forall s \in [0, 1 - r], \\ \bar{B}_{\mathbb{R}^n}(y_k^*(s), \delta) \subset E_k^*, & \forall s \in [r, 1 - r], \quad \forall k \in \{2, \dots, N-1\}, \\ \bar{B}_{\mathbb{R}^n}(y_N^*(s), \delta) \subset E_N^*, & \forall s \in [r, 1]. \end{cases}$$

We are now in a position to continue the proof. To this aim let us fix $\eta := \min\{\frac{\theta}{3}, \frac{\nu}{2}, \delta\} > 0$ and let (y, v, \mathbb{T}) be an admissible triplet for Problem (CP*) satisfying (A.1). Our aim is to prove that $\phi^*(y^*(0), y^*(1)) \leq \phi^*(y(0), y(1))$.

Step 1. Since $0 = \tau_0^* < \tau_1^* < \dots < \tau_{N-1}^* < \tau_N^* = T$ and $\mathbb{T} \in \Delta$ with $\|\mathbb{T} - \mathbb{T}^*\|_{\mathbb{R}^{N+1}} \leq \eta \leq \frac{\theta}{3}$, one can easily deduce that $0 = \tau_0 < \tau_1 < \dots < \tau_{N-1} < \tau_N = T$. Therefore we are in a position to define $(x, \lambda, u) \in \text{AC}([0, T], \mathbb{R}^n) \times \text{PC}([0, T], \mathbb{R}^d) \times \text{L}^\infty([0, T], \mathbb{R}^m)$ by

$$x(t) := y_k \left(\frac{t - \tau_{k-1}}{\tau_k - \tau_{k-1}} \right), \quad \lambda(t) = \lambda_k := y_{N+k} \left(\frac{t - \tau_{k-1}}{\tau_k - \tau_{k-1}} \right) \quad \text{and} \quad u(t) := v_k \left(\frac{t - \tau_{k-1}}{\tau_k - \tau_{k-1}} \right), \quad (\text{A.2})$$

for all $t \in [\tau_{k-1}, \tau_k]$ and all $k \in \{1, \dots, N\}$. Note that x is well defined since $y_{k+1}(0) = y_k(1)$ for all $k \in \{1, \dots, N-1\}$ and λ is a piecewise constant function with respect to \mathbb{T} (from admissibility of the triplet (y, v, \mathbb{T})). Observe that $(y_1(0), y_N(1)) = (x(0), x(T))$ and recall that $(y_1^*(0), y_N^*(1)) = (x^*(0), x^*(T))$. Therefore, from the definition of ϕ^* (see Section 2.4) and since (x^*, λ^*, u^*) is a global solution to Problem (HP), to obtain that $\phi^*(y^*(0), y^*(1)) \leq \phi^*(y(0), y(1))$, we only need to prove that the pair (x, λ, u) is admissible for Problem (HP).

From admissibility of the triplet (y, v, \mathbb{T}) , it is clear that $g(x(0), x(T)) \in \text{S}$ and $(\lambda(t), u(t)) \in \Lambda \times \text{U}$ for almost every $t \in [0, T]$. Therefore it only remains to prove that (x, λ, u) is a solution to the hybrid control system with regionally switching parameter (HS) (see Definition 2.1). From (A.2) and the admissibility of the triplet (y, v, \mathbb{T}) , one can easily obtain that

$$\dot{x}(t) = f_k^*(x(t), \lambda(t), u(t)), \quad \text{a.e. } t \in (\tau_{k-1}, \tau_k), \quad (\text{A.3})$$

for all $k \in \{1, \dots, N\}$. Therefore, to conclude the proof, we only need to prove that

$$\begin{cases} x(t) \in E_1^*, & \forall t \in [\tau_0, \tau_1], \\ x(t) \in E_k^*, & \forall t \in (\tau_{k-1}, \tau_k), \quad \forall k \in \{2, \dots, N-1\}, \\ x(t) \in E_N^*, & \forall t \in (\tau_{N-1}, \tau_N]. \end{cases}$$

This is exactly our goal in the next two steps.

Step 2. Since $\|\mathbb{T} - \mathbb{T}^*\|_{\mathbb{R}^{N+1}} \leq \eta \leq \frac{\theta}{3}$, note that $\tau_k - \tau_{k-1} \leq \bar{\theta} + 2\eta \leq \bar{\theta} + \underline{\theta}$ for all $k \in \{1, \dots, N\}$. Hence, since moreover $r := \frac{\gamma}{\bar{\theta} + \underline{\theta}}$, observe that

$$\begin{cases} \frac{t - \tau_0}{\tau_1 - \tau_0} \in [0, 1 - r], & \forall t \in [\tau_0, \tau_1 - \gamma], \\ \frac{t - \tau_{k-1}}{\tau_k - \tau_{k-1}} \in [r, 1 - r], & \forall t \in [\tau_{k-1} + \gamma, \tau_k - \gamma], \quad \forall k \in \{2, \dots, N-1\}, \\ \frac{t - \tau_{N-1}}{\tau_N - \tau_{N-1}} \in [r, 1], & \forall t \in [\tau_{N-1} + \gamma, \tau_N]. \end{cases}$$

As a consequence, from (A.2) and (P5), and since $\|y_k - y_k^*\|_C \leq \|y - y^*\|_C \leq \eta \leq \delta$, one can easily obtain that

$$\begin{cases} x(t) \in E_1^*, & \forall t \in [\tau_0, \tau_1 - \gamma], \\ x(t) \in E_k^*, & \forall t \in [\tau_{k-1} + \gamma, \tau_k - \gamma], \quad \forall k \in \{2, \dots, N-1\}, \\ x(t) \in E_N^*, & \forall t \in [\tau_{N-1} + \gamma, \tau_N]. \end{cases}$$

Therefore, to conclude the proof, it only remains to prove that $x(t) \in E_k^*$ for all $t \in [\tau_k - \gamma, \tau_k]$ and $x(t) \in E_{k+1}^*$ for all $t \in (\tau_k, \tau_k + \gamma]$, for all $k \in \{1, \dots, N-1\}$. This is the objective of the following step.

Step 3. Let us start with two observations. First, since $\|\mathbb{T} - \mathbb{T}^*\|_{\mathbb{R}^{N+1}} \leq \eta \leq \frac{\theta}{3}$, it holds that $|\tau_k - \tau_{k-1}| \geq \frac{\theta}{3}$ for all $k \in \{1, \dots, N\}$. Second, since $\gamma := \frac{\theta}{3} \min\{\varepsilon, \chi, \frac{\alpha}{\theta}\}$, one can get that

$$\frac{t - \tau_{k-1}}{\tau_k - \tau_{k-1}} \in [1 - \varepsilon, 1], \quad \frac{t - \tau_{k-1}}{\tau_k - \tau_{k-1}} \in [1 - \chi, 1] \quad \text{and} \quad \tau_{k-1}^* + (\tau_k^* - \tau_{k-1}^*) \frac{t - \tau_{k-1}}{\tau_k - \tau_{k-1}} \in [\tau_k^* - \alpha, \tau_k^*],$$

for all $t \in [\tau_k - \gamma, \tau_k]$ and all $k \in \{1, \dots, N-1\}$. We deduce the following results:

(i) Since $v_k(s) = v_k^*(s)$ for almost every $s \in [1 - \varepsilon, 1]$, one can easily obtain from (A.2) and (2.1) that

$$u(t) = u^* \left(\tau_{k-1}^* + (\tau_k^* - \tau_{k-1}^*) \frac{t - \tau_{k-1}}{\tau_k - \tau_{k-1}} \right) \quad \text{with} \quad \tau_{k-1}^* + (\tau_k^* - \tau_{k-1}^*) \frac{t - \tau_{k-1}}{\tau_k - \tau_{k-1}} \in [\tau_k^* - \alpha, \tau_k^*],$$

for almost every $t \in [\tau_k - \gamma, \tau_k)$ and all $k \in \{1, \dots, N-1\}$.

(ii) Since $\|y_k - y_k^*\|_C \leq \|y - y^*\|_C \leq \eta \leq \frac{\nu}{2} \leq \nu$ for all $k \in \{1, \dots, 2N\}$, one can easily obtain from (A.2), from the equality $x^*(\tau_k^*) = y_k^*(1)$ and from (\mathfrak{P}_3) that $x(t) \in \overline{B}_{\mathbb{R}^n}(x^*(\tau_k^*), \nu)$ and $\lambda(t) = \lambda_k \in \overline{B}_{\mathbb{R}^d}(\lambda_k^*, \nu)$ for all $t \in [\tau_k - \gamma, \tau_k)$ and all $k \in \{1, \dots, N-1\}$.

(iii) We obtain from (A.3), from the previous two items and (\mathfrak{P}_2) that the derivative of $F_k^* \circ x$ satisfies

$$\left\langle \nabla F_k^*(x(t)), f_k^* \left(x(t), \lambda_k, u^* \left(\tau_{k-1}^* + (\tau_k^* - \tau_{k-1}^*) \frac{t - \tau_{k-1}}{\tau_k - \tau_{k-1}} \right) \right) \right\rangle_{\mathbb{R}^n} > 0,$$

for almost every $t \in [\tau_k - \gamma, \tau_k)$ and all $k \in \{1, \dots, N-1\}$. From admissibility of the triplet (y, v, \mathbb{T}) and (A.2), we know that $F_k^*(x(\tau_k)) = F_k^*(y_k(1)) = 0$ for all $k \in \{1, \dots, N-1\}$. As a consequence we obtain that $F_k^*(x(t)) < 0$ for all $t \in [\tau_k - \gamma, \tau_k)$ which implies from Definition 2.2, since $x(t) \in \overline{B}_{\mathbb{R}^n}(x^*(\tau_k^*), \nu)$ and $\nu \leq \nu_k^*$, that $x(t) \in E_k^*$ for all $t \in [\tau_k - \gamma, \tau_k)$ and all $k \in \{1, \dots, N-1\}$.

Following the same strategy one can obtain that $x(t) \in E_{k+1}^*$ for all $t \in (\tau_k, \tau_k + \gamma]$ and all $k \in \{1, \dots, N-1\}$. The proof of Proposition 2.1 is complete.

B Proof of Theorem 2.1

Let $(x^*, \lambda^*, u^*) \in \text{AC}([0, T], \mathbb{R}^n) \times \text{PC}([0, T], \mathbb{R}^d) \times \text{L}^\infty([0, T], \mathbb{R}^m)$ be a global solution to Problem (HP), that is moreover a regular solution to (HS), associated with a partition $\mathbb{T}^* = \{\tau_k^*\}_{k=0, \dots, N}$, such that g is submersive at $(x^*(0), x^*(T))$. From Proposition 2.1, the corresponding triplet (y^*, v^*, \mathbb{T}^*) constructed in Section 2.4 is a L_{\square}^1 -local solution to Problem (CP*). Before applying [6, Theorem 2.1], we need to verify that g^* is submersive at $(y^*(0), y^*(1), \mathbb{T}^*)$. From the definition of the function g^* (see Section 2.4), note that the matrix $\nabla g^*(y^*(0), y^*(1), \mathbb{T}^*) \in \mathbb{R}^{((n+d)N + (n+d)N + (N+1)) \times \ell^*}$ is given by

$\nabla_1 g(y_1^*(0), y_N^*(1))$	$0_{\mathbb{R}^n \times n(N-1)}$	$0_{\mathbb{R}^n \times (N-1)}$	$0_{\mathbb{R}^n \times dN}$	$0_{\mathbb{R}^n \times (N+1)}$
$0_{\mathbb{R}^n(N-1) \times \ell}$	$\text{Id}_{\mathbb{R}^n(N-1) \times n(N-1)}$	$0_{\mathbb{R}^n(N-1) \times (N-1)}$	$0_{\mathbb{R}^n(N-1) \times dN}$	$0_{\mathbb{R}^n(N-1) \times (N+1)}$
$0_{\mathbb{R}^d N \times \ell}$	$0_{\mathbb{R}^d N \times n(N-1)}$	$0_{\mathbb{R}^d N \times (N-1)}$	$0_{\mathbb{R}^d N \times dN}$	$0_{\mathbb{R}^d N \times (N+1)}$
$0_{\mathbb{R}^n(N-1) \times \ell}$	$-\text{Id}_{\mathbb{R}^n(N-1) \times n(N-1)}$	$\nabla F_1^*(y_1^*(1))$	$0_{\mathbb{R}^n(N-1) \times dN}$	$0_{\mathbb{R}^n(N-1) \times (N+1)}$
		\ddots		
		$\nabla F_{N-1}^*(y_{N-1}^*(1))$	$0_{\mathbb{R}^n(N-1) \times dN}$	$0_{\mathbb{R}^n(N-1) \times (N+1)}$
$\nabla_2 g(y_1^*(0), y_N^*(1))$	$0_{\mathbb{R}^n \times n(N-1)}$	$0_{\mathbb{R}^n \times (N-1)}$	$0_{\mathbb{R}^n \times dN}$	$0_{\mathbb{R}^n \times (N+1)}$
$0_{\mathbb{R}^d N \times \ell}$	$0_{\mathbb{R}^d N \times n(N-1)}$	$0_{\mathbb{R}^d N \times (N-1)}$	$\text{Id}_{\mathbb{R}^d N \times dN}$	$0_{\mathbb{R}^d N \times (N+1)}$
$0_{\mathbb{R}^{(N+1)} \times \ell}$	$0_{\mathbb{R}^{(N+1)} \times n(N-1)}$	$0_{\mathbb{R}^{(N+1)} \times (N-1)}$	$0_{\mathbb{R}^{(N+1)} \times dN}$	$\text{Id}_{\mathbb{R}^{(N+1)} \times (N+1)}$

From Definition 2.2, it holds that $\nabla F_k^*(y_k^*(1)) = \nabla F_k^*(x^*(\tau_k^*)) \neq 0_{\mathbb{R}^n}$ for all $k \in \{1, \dots, N-1\}$. Since g is submersive at $(x^*(0), x^*(T)) = (y_1^*(0), y_N^*(1))$, one can easily conclude that g^* is submersive at $(y^*(0), y^*(1), \mathbb{T}^*)$.

Application of [6, Theorem 2.1]. Let us introduce the Hamiltonian $\tilde{H} : \mathbb{R}^{(n+d)N} \times \mathbb{R}^{mN} \times \mathbb{R}^{N+1} \times \mathbb{R}^{(n+d)N} \rightarrow \mathbb{R}$ associated with Problem (CP*) given by

$$\tilde{H}(y, v, \mathbb{T}, q) := \langle q, f^*(y, v, \mathbb{T}) \rangle_{\mathbb{R}^{(n+d)N}} = \sum_{k=1}^N \langle q_k, (\tau_k - \tau_{k-1}) f_k^*(y_k, y_{N+k}, v_k) \rangle_{\mathbb{R}^n},$$

for all $y = (y_1, \dots, y_N, y_{N+1}, \dots, y_{2N}) \in \mathbb{R}^{(n+d)N}$, $v = (v_1, \dots, v_N) \in \mathbb{R}^{mN}$, $\mathbb{T} = \{\tau_0, \dots, \tau_N\} \in \mathbb{R}^{N+1}$ and $q = (q_1, \dots, q_N, q_{N+1}, \dots, q_{2N}) \in \mathbb{R}^{(n+d)N}$. From [6, Theorem 2.1], there exists a nontrivial pair $(q, q^0) \in \text{AC}([0, 1], \mathbb{R}^{(n+d)N}) \times \mathbb{R}_+$ satisfying:

(i) the Hamiltonian system

$$\dot{y}^*(s) = \nabla_q \tilde{H}(y^*(s), v^*(s), \mathbb{T}^*, q(s)) \quad \text{and} \quad -\dot{q}(s) = \nabla_y \tilde{H}(y^*(s), v^*(s), \mathbb{T}^*, q(s)),$$

for almost every $s \in [0, 1]$;

(ii) the transversality condition

$$\begin{pmatrix} q(0) \\ -q(1) \\ \int_0^1 \nabla_{\mathbb{T}} \tilde{H}(y^*(s), v^*(s), \mathbb{T}^*, q(s)) ds \end{pmatrix} = q^0 \nabla \phi^*(y^*(0), y^*(1)) + \nabla g^*(y^*(0), y^*(1), \mathbb{T}^*) \tilde{\xi},$$

for some $\tilde{\xi} \in \mathbb{N}_{S^*}[g^*(y^*(0), y^*(1), \mathbb{T}^*)]$;

(iii) the Hamiltonian maximization condition

$$v^*(s) \in \arg \max_{\tilde{\omega} \in U^N} \tilde{H}(y^*(s), \tilde{\omega}, \mathbb{T}^*, q(s)).$$

for almost every $s \in [0, 1]$.

Introduction of the nontrivial pair (p, p^0) . Since the pair (q, q^0) is not trivial, it is clear that the pair $(p, p^0) \in \text{PAC}_{\mathbb{T}^*}([0, T], \mathbb{R}^n) \times \mathbb{R}_+$ defined by $p^0 := q^0$ and

$$p(t) := \begin{cases} q_1 \left(\frac{t - \tau_0^*}{\tau_1^* - \tau_0^*} \right), & \forall t \in [\tau_0^*, \tau_1^*), \\ q_k \left(\frac{t - \tau_{k-1}^*}{\tau_k^* - \tau_{k-1}^*} \right), & \forall t \in (\tau_{k-1}^*, \tau_k^*), \quad \forall k \in \{2, \dots, N-1\}, \\ q_N \left(\frac{t - \tau_{N-1}^*}{\tau_N^* - \tau_{N-1}^*} \right), & \forall t \in (\tau_{N-1}^*, \tau_N^*], \end{cases}$$

is not trivial.

Hamiltonian system and Hamiltonian maximization condition of Theorem 2.1. From the above Items (i) and (iii) and from (2.2), one can easily obtain that (p, p^0) satisfies the Hamiltonian system and the Hamiltonian maximization condition of Theorem 2.1.

Transversality condition of Theorem 2.1. From the definitions of g^* and S^* (see Section 2.4) and since $\tilde{\xi} \in \mathbb{N}_{S^*}[g^*(y^*(0), y^*(1), \mathbb{T}^*)]$, we can write $\tilde{\xi} := (\xi, \xi^2, \xi^3, \xi^4, \xi^5) \in \mathbb{R}^\ell \times \mathbb{R}^{n(N-1)} \times \mathbb{R}^{N-1} \times \mathbb{R}^{dN} \times \mathbb{R}^{N+1}$ with

$$\xi \in \mathbb{N}_S[g(y_1^*(0), y_N^*(1))] \quad \text{and} \quad \xi^4 \in \mathbb{N}_{\Lambda^N}[(y_{N+1}^*(1), \dots, y_{2N}^*(1))] \quad \text{and} \quad \xi^5 \in \mathbb{N}_\Delta[\mathbb{T}^*].$$

Since $(y_1^*(0), y_N^*(1)) = (x^*(0), x^*(T))$ and $y_{N+k}^*(1) = \lambda_k^*$, note that $\xi \in \mathbb{N}_S[g(x^*(0), x^*(T))]$ and $\xi_k^4 \in \mathbb{N}_\Lambda[\lambda_k^*]$ for all $k \in \{1, \dots, N\}$. Furthermore, from the first two components of the above Item (ii), from the expression of $\nabla g^*(y^*(0), y^*(1), \mathbb{T}^*)$ given at the beginning of Appendix B and from the expression of $\nabla \phi^*(y^*(0), y^*(1))$ (see Section 2.4 for the definition of ϕ^*), we obtain that

$$\begin{aligned} p(0) = q_1(0) &= q^0 \nabla_1 \phi(y_1^*(0), y_N^*(1)) + \nabla_1 g(y_1^*(0), y_N^*(1)) \xi \\ &= p^0 \nabla_1 \phi(x^*(0), x^*(T)) + \nabla_1 g(x^*(0), x^*(T)) \xi, \end{aligned}$$

and

$$\begin{aligned} -p(T) = -q_N(1) &= q^0 \nabla_2 \phi(y_1^*(0), y_N^*(1)) + \nabla_2 g(y_1^*(0), y_N^*(1)) \xi \\ &= p^0 \nabla_2 \phi(x^*(0), x^*(T)) + \nabla_2 g(x^*(0), x^*(T)) \xi. \end{aligned}$$

Therefore the transversality condition of Theorem 2.1 is proved.

Discontinuity condition of Theorem 2.1. From the first two components of the above Item (ii), from the expression of $\nabla g^*(y^*(0), y^*(1), \mathbb{T}^*)$ given at the beginning of Appendix B and from the expression of $\nabla \phi^*(y^*(0), y^*(1))$ (see Section 2.4 for the definition of ϕ^*), we obtain that

$$\forall k \in \{2, \dots, N\}, \quad q_k(0) = \xi_{k-1}^2 \quad \text{and} \quad \forall k \in \{1, \dots, N-1\}, \quad -q_k(1) = -\xi_k^2 + \xi_k^3 \nabla F_k^*(y_k^*(1)).$$

We deduce that

$$p^+(\tau_k^*) - p^-(\tau_k^*) = q_{k+1}(0) - q_k(1) = \xi_k^3 \nabla F_k^*(y_k^*(1)) = \xi_k^3 \nabla F_k^*(x^*(\tau_k^*)), \quad (\text{B.1})$$

for all $k \in \{1, \dots, N-1\}$. Therefore the discontinuity condition of Theorem 2.1 is satisfied with $\sigma_k := \xi_k^3$ for all $k \in \{1, \dots, N-1\}$.

Averaged Hamiltonian gradient condition of Theorem 2.1. From the first two components of Item (ii), from the expression of $\nabla g^*(y^*(0), y^*(1), \mathbb{T}^*)$ given at the beginning of Appendix B and from the expression of $\nabla \phi^*(y^*(0), y^*(1))$ (see Section 2.4 for the definition of ϕ^*), we obtain that

$$\forall k \in \{1, \dots, N\}, \quad q_{N+k}(0) = 0_{\mathbb{R}^d} \quad \text{and} \quad \forall k \in \{1, \dots, N\}, \quad -q_{N+k}(1) = \xi_k^4.$$

From the Hamiltonian system, we deduce that

$$\int_0^1 \nabla_{y_{N+k}} \tilde{H}(y^*(s), v^*(s), \mathbb{T}, q(s)) ds = \xi_k^4 \in N_\Lambda[y_{N+k}^*(1)].$$

for all $k \in \{1, \dots, N\}$. From affine changes of time variable, we obtain that

$$\int_{\tau_{k-1}^*}^{\tau_k^*} \nabla_\lambda \mathcal{H}(x^*(t), \lambda_k^*, u^*(t), p(t)) dt \in N_\Lambda[\lambda_k^*],$$

for all $k \in \{1, \dots, N\}$.

Hamiltonian constancy condition of Theorem 2.1. From the Hamiltonian system and the maximization condition, and applying [27, Theorem 2.6.1] on each interval $[\tau_{k-1}^*, \tau_k^*]$, we obtain that, for all $k \in \{1, \dots, N\}$, there exists a constant $c_k \in \mathbb{R}$ such that

$$\langle p(t), f_k^*(x^*(t), \lambda^*(t), u^*(t)) \rangle_{\mathbb{R}^n} = c_k,$$

for almost every $t \in [\tau_{k-1}^*, \tau_k^*]$. Furthermore, from the definition of Δ (see Section 2.4) and since $0 = \tau_0^* < \tau_1^* < \dots < \tau_{N-1}^* < \tau_N^* = T$, we deduce from $\xi^5 \in N_\Delta[\mathbb{T}^*]$ that all components of ξ^5 are zero, except possibly the first and last components. Thus, from the third component of the above Item (ii), from the expression of $\nabla g^*(y^*(0), y^*(1), \mathbb{T}^*)$ given at the beginning of Appendix B and from the expression of $\nabla \phi^*(y^*(0), y^*(1))$ (see Section 2.4 for the definition of ϕ^*), we obtain that

$$\int_0^1 \langle q_{k+1}(s), f_{k+1}^*(y_{k+1}^*(s), y_{N+k+1}^*(s), v_{k+1}^*(s)) \rangle_{\mathbb{R}^n} ds = \int_0^1 \langle q_k(s), f_k^*(y_k^*(s), y_{N+k}^*(s), v_k^*(s)) \rangle_{\mathbb{R}^n} ds,$$

for all $k \in \{1, \dots, N-1\}$. From affine changes of time variable, we obtain that

$$\frac{1}{\tau_{k+1}^* - \tau_k^*} \int_{\tau_k^*}^{\tau_{k+1}^*} \langle p(t), f_{k+1}^*(x^*(t), \lambda^*(t), u^*(t)) \rangle_{\mathbb{R}^n} dt = \frac{1}{\tau_k^* - \tau_{k-1}^*} \int_{\tau_{k-1}^*}^{\tau_k^*} \langle p(t), f_k^*(x^*(t), \lambda^*(t), u^*(t)) \rangle_{\mathbb{R}^n} dt,$$

for all $k \in \{1, \dots, N-1\}$. From constancy of the above two integrands, we deduce that $c_{k+1} = c_k$ for all $k \in \{1, \dots, N-1\}$. Therefore the Hamiltonian constancy condition is satisfied and the proof of Theorem 2.1 is complete.

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