



**HAL**  
open science

The hybrid maximum principle for optimal control problems with spatially heterogeneous dynamics is a consequence of a Pontryagin maximum principle for “L1 square” local solutions

Terence Bayen, Anas Bouali, Loic Bourdin

► To cite this version:

Terence Bayen, Anas Bouali, Loic Bourdin. The hybrid maximum principle for optimal control problems with spatially heterogeneous dynamics is a consequence of a Pontryagin maximum principle for “L1 square” local solutions. 2023. hal-03985420

HAL Id: hal-03985420

<https://univ-avignon.hal.science/hal-03985420v1>

Preprint submitted on 13 Feb 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

1    **THE HYBRID MAXIMUM PRINCIPLE FOR OPTIMAL CONTROL**  
2    **PROBLEMS WITH SPATIALLY HETEROGENEOUS DYNAMICS IS A**  
3    **CONSEQUENCE OF A PONTRYAGIN MAXIMUM PRINCIPLE FOR**  
4     **$L^1$ -LOCAL SOLUTIONS**

5    TÉRENCE BAYEN\*, ANAS BOUALI†, AND LOÏC BOURDIN‡

6    **Abstract.** The title of the present work is a nod to the paper *The hybrid maximum principle is*  
7    *a consequence of Pontryagin maximum principle* by Dmitruk and Kaganovich (Systems and Control  
8    Letters, 2008). Here we investigate a similar but different framework of hybrid optimal control  
9    problems. Precisely we consider a general control system that is described by a differential equation  
10    involving a spatially heterogeneous dynamics. In that context the sequence of dynamics followed  
11    by the trajectory and the corresponding switching times are fully constrained by the state position.  
12    We prove with an explicit counterexample that the augmentation technique proposed by Dmitruk  
13    and Kaganovich cannot be fully applied to our setting, but we show that it can be adapted by  
14    introducing a new notion of local solution to classical optimal control problems and by establishing a  
15    corresponding Pontryagin maximum principle. Thanks to this method we derive a hybrid maximum  
16    principle adapted to our setting, with a simple proof that does not require any technical tool (such as  
17    implicit function arguments) to handle the dynamical discontinuities.

18    **Key words.** Optimal control, heterogeneous dynamics, hybrid maximum principle.

19    **MSC codes.** 34A38, 49K15.

20    **1. Introduction.**

21    **1.1. General context.** The *Pontryagin Maximum Principle* (in short, PMP),  
22    established at the end of the 1950s (see [27]), has originally been developed for optimal  
23    control problems where the control system is described by an ordinary differential  
24    equation (in short, ODE). It states the corresponding first-order necessary optimality  
25    conditions, in terms of an (absolutely continuous) costate function. As usual in  
26    optimization, the PMP remains valid for *local solutions* only (typically in uniform  
27    norm for the state and in  $L^1$ -norm for the control). Since then, the PMP has been  
28    adapted to many situations, in particular for control systems of different natures.

29    On the other hand, *hybrid systems* are, in a broad sense, dynamical systems  
30    that exhibit both continuous and discrete behaviors. They are particularly used in  
31    automation and robotics to describe complex systems in which, for example, logic  
32    decisions are combined with physical processes. We refer to [32] for an elementary  
33    introduction to hybrid systems. This theory is very large and it is commonly accepted  
34    that it includes ODEs with *heterogeneous dynamics*, that is, ODEs involving a family  
35    of different dynamics (used for example to describe evolutions in heterogeneous media)  
36    where the transitions from one dynamics to another are seen as discrete events.

37    The PMP has been extended to hybrid control systems, especially in the context  
38    of ODEs with heterogeneous dynamics (see, *e.g.*, [19, 22, 26, 29, 30, 31]), resulting  
39    in theorems often referred to as *Hybrid Maximum Principle* (in short, HMP). We  
40    emphasize that the frameworks are very varied. Indeed the rule that supervises the  
41    transitions between the different dynamics is usually described by additional variables

---

\*Avignon Université, Laboratoire de Mathématiques d'Avignon (EA 2151) F-84018, France.  
(terence.bayen@univ-avignon.fr)

†Avignon Université, Laboratoire de Mathématiques d'Avignon (EA 2151) F-84018, France.  
(anas.bouali@univ-avignon.fr)

‡Université de Limoges, Institut de recherche XLIM. UMR CNRS 7252, France.  
(loic.bourdin@unilim.fr)

42 that can be free or constrained and, in that second case, the constraints can be of  
 43 different natures. For example the *switching times* (*i.e.* the instants at which the  
 44 control system moves from one dynamics to another) can be the resultant of a control  
 45 decision or can be (fully or partially) determined by the time variable, the state position  
 46 or both of them. Hence different versions of the HMP can be found in the literature,  
 47 corresponding to different hybrid control systems that are presented under various  
 48 names according to their nature (such as *multi-processes* [18], *switched systems* [28],  
 49 *regional systems* [1], *systems on stratified domains* [14], *variable structure systems* [8]).  
 50 In contrary to the classical PMP, the HMP is usually expressed in terms of an (only)  
 51 piecewise absolutely continuous costate function that admits discontinuity jumps at  
 52 the switching times. A common feature of most of the above references is that the  
 53 mathematical framework somehow guarantees that local perturbations (typically in  
 54 uniform norm for the state and in  $L^1$ -norm for the control) preserve the same hybrid  
 55 structure (that is, the same sequence of dynamics) as the nominal one.

56 **1.2. The augmentation technique of Dmitruk and Kaganovich.** In the  
 57 context of ODEs with heterogeneous dynamics, the difficult part of deriving a HMP  
 58 lies in handling the dynamical discontinuities. To this end, an excellent strategy has  
 59 been proposed in [19], in which the switching times are additional variables satisfying  
 60 equality/inequality constraints involving the corresponding intermediate state values.

61 Roughly speaking, considering an optimal solution (associated with switching times  
 62 denoted by  $\tau_k^*$ ), this technique consists in affine changes of time variable, mapping  
 63 the intervals  $(\tau_{k-1}^*, \tau_k^*)$  into a common interval  $(0, 1)$ . This procedure augments the  
 64 dimensions of the variables and thus is categorized in the set of *augmentation techniques*.  
 65 The authors prove that the augmented solution is a local solution to the augmented  
 66 problem which is classical (that is, non-hybrid) by construction (since the discontinuities  
 67 have been positioned at the endpoints of the interval  $[0, 1]$ ). Therefore the classical  
 68 PMP can be applied to the augmented solution (expressed in terms of an augmented  
 69 absolutely continuous costate function satisfying endpoint transversality conditions).  
 70 Hence, by inverting the above affine changes of time variable, first-order necessary  
 71 optimality conditions are derived for the original nonaugmented solution, expressed in  
 72 terms of a nonaugmented (piecewise absolutely continuous) costate function satisfying  
 73 discontinuity jumps at the switching times  $\tau_k^*$  (whose expressions follow from the  
 74 endpoint transversality conditions at 0 and 1 of the augmented costate function).

75 Hence Dmitruk and Kaganovich have entitled their paper [19] as *The hybrid*  
 76 *maximum principle is a consequence of Pontryagin maximum principle*. The augmen-  
 77 tation technique is particularly satisfactory because it allows to reduce the hybrid  
 78 problem into a classical (non-hybrid) augmented problem, avoiding the use of technical  
 79 arguments (such as implicit function theorems) to handle the dynamical discontinuities.

80 **1.3. Framework and contributions of the present work.** In the spirit of [1,  
 81 23], we consider a control system described by an ODE with *spatially* heterogeneous  
 82 dynamics, in the sense that the state space is partitioned into several disjoint regions  
 83 and each region has its own dynamics. In that context the sequence of dynamics  
 84 followed by the trajectory and the corresponding switching times (called *crossing*  
 85 *times* since they correspond to the instants at which the state goes from one region to  
 86 another) are fully constrained by the state position.

87 A HMP corresponding to this setting has already been announced in [23] but with  
 88 a sketch of proof which is, to our best knowledge, erroneous. Indeed the author invoke  
 89 *needle-like perturbations* of the control, while they are not admissible in the present  
 90 setting (see Appendix C for a counterexample). This issue has been corrected in our

91 previous paper [2] by applying needle-like perturbations on *auxiliary controls*. Then,  
 92 to handle the resulting perturbed crossing times, we used an inductive application of  
 93 the implicit function theorem, which results into a technical and extended analysis.  
 94 An attempt to derive a HMP corresponding to our setting, with the simpler approach  
 95 of Dmitruk and Kaganovich, was also presented in [1]. Unfortunately, to our best  
 96 knowledge, this proof is also incorrect. Indeed, in contrary to the framework of  
 97 Dmitruk and Kaganovich in [19], our setting fails to guarantee that the augmented  
 98 solution is a local solution to the classical augmented problem (see Section 3.4 for a  
 99 counterexample) and, therefore, the classical PMP cannot be applied. We emphasize  
 100 that our counterexample shows that, in our setting, a local perturbation (in uniform  
 101 norm for the state and in  $L^1$ -norm for the control) does not preserve the hybrid  
 102 structure of the nominal one in general.

103 Hence the main objective of this paper is to derive a HMP for our setting, with a  
 104 correct proof that adapts the augmentation procedure of Dmitruk and Kaganovich.  
 105 To this aim a new notion of local solution to classical optimal control problems (see  
 106 the definition of  $L^1_{\square}$ -local solution in Definition 2.2) and a corresponding version of  
 107 the PMP (see Theorem 2.1) are required. Indeed we prove in Proposition 3.1 that,  
 108 under appropriate assumptions (such as *transverse conditions* at the crossing times),  
 109 the augmented solution is a  $L^1_{\square}$ -local solution to the classical augmented problem and  
 110 therefore the above new PMP can be applied. Finally, similarly to [19], by inverting the  
 111 affine changes of time variable, a HMP for our setting is obtained (see Theorem 3.1).

112 **1.4. Organization of the paper.** In Section 2, a classical optimal control prob-  
 113 lem is considered (see Problem (P)), the new notion of  $L^1_{\square}$ -local solution is introduced  
 114 (see Definition 2.2) and a corresponding PMP is established (see Theorem 2.1). In  
 115 Section 3, a hybrid optimal control problem with spatially heterogeneous dynamics  
 116 is introduced (see Problem (HP)). Applying the augmentation procedure, Proposi-  
 117 tion 3.1 states that an augmented solution to Problem (HP) is a  $L^1_{\square}$ -local solution to  
 118 the corresponding classical augmented problem of the form of Problem (P). Hence,  
 119 applying the above new PMP and inverting the affine changes of time variable, a HMP  
 120 for Problem (HP) is obtained (see Theorem 3.1). An explicit counterexample showing  
 121 that an augmented solution to Problem (HP) is not a local solution (in the usual sense)  
 122 to the corresponding classical augmented problem is provided in Section 3.4. Finally  
 123 the technical proofs of Proposition 3.1 and Theorem 3.1 are provided in Appendices A  
 124 and B respectively. A counterexample showing that needle-like perturbations of the  
 125 control are not admissible in our setting is provided in Appendix C.

## 126 2. Preliminaries and PMP for the new notion of $L^1_{A_{\square}}$ -local solution.

127 In this paper, for any positive integer  $d \in \mathbb{N}^*$ , we denote by  $\langle \cdot, \cdot \rangle_{\mathbb{R}^d}$  (resp.  $\| \cdot \|_{\mathbb{R}^d}$ )  
 128 the standard inner product (resp. Euclidean norm) of  $\mathbb{R}^d$ . For any subset  $X \subset \mathbb{R}^d$ ,  
 129 we denote by  $\partial X$  the boundary of  $X$  defined by  $\partial X := \overline{X} \setminus \text{Int}(X)$ , where  $\overline{X}$  and  
 130  $\text{Int}(X)$  stand respectively for the closure and the interior of  $X$ . Given a (Lebesgue)  
 131 measurable subset  $A \subset \mathbb{R}$ , we denote by  $\mu(A)$  its (Lebesgue) measure. Furthermore,  
 132 for any extended-real number  $r \in [1, \infty]$  and any real interval  $I \subset \mathbb{R}$ , we denote by:

- 133 •  $L^r(I, \mathbb{R}^d)$  the usual Lebesgue space of  $r$ -integrable functions defined on  $I$  with  
 134 values in  $\mathbb{R}^d$ , endowed with its usual norm  $\| \cdot \|_{L^r}$ ;
- 135 •  $C(I, \mathbb{R}^d)$  the standard space of continuous functions defined on  $I$  with values  
 136 in  $\mathbb{R}^d$ , endowed with the standard uniform norm  $\| \cdot \|_C$ ;
- 137 •  $AC(I, \mathbb{R}^d)$  the subspace of  $C(I, \mathbb{R}^d)$  of absolutely continuous functions.

138 Now take  $I = [0, T]$  for some  $T > 0$ . Recall that a partition of the interval  $[0, T]$  is a  
 139 set  $\mathbb{T} = \{\tau_k\}_{k=0, \dots, N}$  of real numbers such that  $0 = \tau_0 < \tau_1 < \dots < \tau_{N-1} < \tau_N = T$

140 for some  $N \in \mathbb{N}^*$ . In this paper a function  $p : [0, T] \rightarrow \mathbb{R}^d$  is said to be *piecewise*  
 141 *absolutely continuous*, with respect to a partition  $\mathbb{T} = \{\tau_k\}_{k=0, \dots, N}$ , if  $p$  is continuous  
 142 at 0 and  $T$  and the restriction of  $p$  over each open interval  $(\tau_{k-1}, \tau_k)$  admits an  
 143 extension over  $[\tau_{k-1}, \tau_k]$  that is absolutely continuous. If so,  $p$  admits left and right  
 144 limits at each  $\tau_k \in (0, T)$ , denoted respectively by  $p^-(\tau_k)$  and  $p^+(\tau_k)$ . We denote by:  
 145 •  $\text{PAC}_{\mathbb{T}}([0, T], \mathbb{R}^d)$  the space of piecewise absolutely continuous functions, with  
 146 respect to a partition  $\mathbb{T}$  of the interval  $[0, T]$ , with values in  $\mathbb{R}^d$ .  
 147 Finally, when  $(\mathcal{Z}, d_{\mathcal{Z}})$  is a metric space, we denote by  $B_{\mathcal{Z}}(z, \nu)$  (resp.  $\bar{B}_{\mathcal{Z}}(z, \nu)$ ) the  
 148 standard open (resp. closed) ball of  $\mathcal{Z}$  centered at  $z \in \mathcal{Z}$  and of radius  $\nu > 0$ .

149 **2.1. A classical optimal control problem and  $L^1_{A\Box}$ -local solution.** Let  $n$ ,  
 150  $m$ ,  $d$  and  $\ell \in \mathbb{N}^*$  be four fixed positive integers and  $T > 0$  be a fixed positive real  
 151 number. In the present section we consider a classical Mayer optimal control problem  
 152 with parameter and mixed terminal state constraints given by

$$\begin{aligned}
 & \text{minimize} && \phi(x(0), x(T), \lambda), \\
 & \text{subject to} && (x, u, \lambda) \in \text{AC}([0, T], \mathbb{R}^n) \times L^\infty([0, T], \mathbb{R}^m) \times \mathbb{R}^d, \\
 153 \text{ (P)} & && \dot{x}(t) = f(x(t), u(t), \lambda), \quad \text{a.e. } t \in [0, T], \\
 & && g(x(0), x(T), \lambda) \in S, \\
 & && u(t) \in U, \quad \text{a.e. } t \in [0, T],
 \end{aligned}$$

154 where the Mayer cost function  $\phi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d \rightarrow \mathbb{R}$ , the dynamics  $f : \mathbb{R}^n \times$   
 155  $\mathbb{R}^m \times \mathbb{R}^d \rightarrow \mathbb{R}^n$  and the constraint function  $g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d \rightarrow \mathbb{R}^\ell$  are of class  $C^1$ ,  
 156 and where  $S \subset \mathbb{R}^\ell$  is a nonempty closed convex subset and  $U \subset \mathbb{R}^m$  is a nonempty  
 157 subset. As usual in the literature,  $x \in \text{AC}([0, T], \mathbb{R}^n)$  is called the *state* (or the  
 158 *trajectory*),  $u \in L^\infty([0, T], \mathbb{R}^m)$  is called the *control* and  $\lambda \in \mathbb{R}^d$  is called the *parameter*.  
 159 A triplet  $(x, u, \lambda) \in \text{AC}([0, T], \mathbb{R}^n) \times L^\infty([0, T], \mathbb{R}^m) \times \mathbb{R}^d$  is said to be *admissible*  
 160 for Problem (P) if it satisfies all the constraints of Problem (P). Finally, such an  
 161 admissible triplet is said to be a *global solution* to Problem (P) if it minimizes the  
 162 Mayer cost  $\phi(x(0), x(T), \lambda)$  among all admissible triplets.

163 *Remark 2.1.* (i) All along this paper (not only for Problem (P)), we have chosen  
 164 to deal with optimal control problems with (only) Mayer cost, fixed final time and  
 165 autonomous dynamics. It is well known in the literature (see, *e.g.*, [9, 15, 16]) that  
 166 standard techniques (such as augmentation or changes of variables) allow to deal with  
 167 more general Bolza cost, free final time and time-dependent dynamics. Similarly, in  
 168 Problem (P), we assume for simplicity that  $\phi$ ,  $f$  and  $g$  are of class  $C^1$  and also some  
 169 topological properties for  $S$ . However the results that we will present in this section  
 170 can be extended to weaker assumptions (see, *e.g.*, [17, 33]). Overall, our aim in this  
 171 paper is not to address the most general framework possible. We keep our setting as  
 172 simple as possible to stay focused on the novel aspects of our work.

173 (ii) The presence of a parameter  $\lambda \in \mathbb{R}^d$  in Problem (P) can also be treated thanks  
 174 to an augmentation (see, *e.g.*, [9]). It is noteworthy that the main problem considered  
 175 in the present work (see Problem (HP) in the next Section 3) is a hybrid optimal  
 176 control problem which does not involve any parameter. However the proof of our main  
 177 result (Theorem 3.1) is based on a reduction of Problem (HP) into a classical optimal  
 178 control problem of the form of Problem (P) that involves parameters. This is the only  
 179 reason why we need to consider the presence of a parameter  $\lambda \in \mathbb{R}^d$  in Problem (P).

180 The classical PMP [27] has originally been developed for global solutions but, as  
 181 usual in optimization, it remains valid for *local solutions*. As a consequence, several  
 182 notions of local solution to classical optimal control problems, and the corresponding

183 versions of the PMP, have been developed in the literature (see, *e.g.*, [11, 25]). Let us  
184 introduce two new notions of local solution which will play central roles in our work.

DEFINITION 2.1 ( $L_A^1$ -local solution). *An admissible triplet  $(x^*, u^*, \lambda^*)$  is said to be a  $L_A^1$ -local solution to Problem (P), for a measurable subset  $A \subset [0, T]$ , if, for all  $R \geq \|u^*\|_{L^\infty}$ , there exists  $\eta > 0$  such that  $\phi(x^*(0), x^*(T), \lambda^*) \leq \phi(x(0), x(T), \lambda)$  for all admissible triplets  $(x, u, \lambda)$  satisfying*

$$\begin{cases} \|x - x^*\|_C + \|u - u^*\|_{L^1} + \|\lambda - \lambda^*\|_{\mathbb{R}^d} \leq \eta, \\ \|u\|_{L^\infty} \leq R, \\ u(t) = u^*(t) \text{ a.e. } t \in [0, T] \setminus A. \end{cases}$$

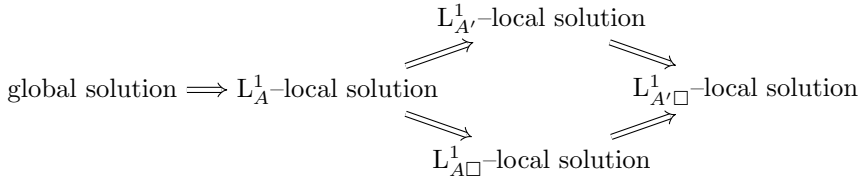
185 DEFINITION 2.2 ( $L_{A^\square}^1$ -local solution). *An admissible triplet  $(x^*, u^*, \lambda^*)$  is said  
186 to be a  $L_{A^\square}^1$ -local solution to Problem (P), for a measurable subset  $A \subset [0, T]$ , if there  
187 exists an increasing family  $(A_\varepsilon)_{\varepsilon>0}$  of measurable subsets of  $A$ , satisfying  $\mu(A_\varepsilon) \rightarrow \mu(A)$   
188 as  $\varepsilon \rightarrow 0$ , such that  $(x^*, u^*, \lambda^*)$  is a  $L_{A_\varepsilon}^1$ -local solution to Problem (P) for all  $\varepsilon > 0$ .*

189 *Remark 2.2.* (i) The notations  $L_A^1$  and  $L_{A^\square}^1$  are very close, while the corresponding  
190 definitions are (slightly) different. Therefore the reader needs to be careful with these  
191 two different concepts, for which we will give each one a version of the PMP (see  
192 Lemma 2.1 for  $L_A^1$ -local solutions and Theorem 2.1 for  $L_{A^\square}^1$ -local solutions).

193 (ii) The concept of  $L_{[0,T]}^1$ -local solution coincides with the classical notion of  $L^1$ -  
194 local solution well established in the literature (see, *e.g.*, [11, 25]). Therefore, in  
195 the sequel, we simply write  $L^1$ -local solution instead of  $L_{[0,T]}^1$ -local solution. To be  
196 consistent we simply write  $L_{\square}^1$ -local solution instead of  $L_{[0,T]\square}^1$ -local solution.

197 (iii) With respect to the classical concept of  $L^1$ -local solution, the refined notion  
198 of  $L_A^1$ -local solution imposes on admissible controls to match the nominal one almost  
199 everywhere outside the measurable subset  $A \subset [0, T]$ . This feature is crucial to reduce  
200 the hybrid optimal control problem considered in the next Section 3 into a classical  
201 optimal control problem. This is not possible with the classical concept of  $L^1$ -local  
202 solution, as shown by a counterexample in Section 3.4.

203 (iv) For a measurable subset  $A \subset [0, T]$ , it is clear that a  $L_A^1$ -local solution is  
204 automatically a  $L_{A^\square}^1$ -local solution. However the converse is not true in general (see  
205 the counterexample in Section 3.4). From a general point of view, the implications



206

207 hold true for any measurable subsets  $A' \subset A \subset [0, T]$ , but not the converses in general.

208 **2.2. PMP for  $L_{A^\square}^1$ -local solutions and comments.** Recall first that the  
209 *normal cone* to  $S$  at some point  $z \in S$  is defined by

$$210 \quad N_S[z] := \{z'' \in \mathbb{R}^\ell \mid \forall z' \in S, \langle z'', z' - z \rangle_{\mathbb{R}^\ell} \leq 0\},$$

211 and that  $g$  is said to be *submersive* at a point of  $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d$  if the differential of  $g$  at  
212 this point is surjective. Finally recall that the *Hamiltonian*  $\mathcal{H} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R}^n \rightarrow$   
213  $\mathbb{R}$  associated with Problem (P) is defined by  $\mathcal{H}(x, u, \lambda, p) := \langle p, f(x, u, \lambda) \rangle_{\mathbb{R}^n}$  for  
214 all  $(x, u, \lambda, p) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R}^n$ . We are now in a position to establish a new  
215 version of the PMP that is dedicated to  $L_{A^\square}^1$ -local solutions to Problem (P).

216 THEOREM 2.1 (PMP for  $L_{A\Box}^1$ -local solutions). *If  $(x^*, u^*, \lambda^*)$  is a  $L_{A\Box}^1$ -local*  
 217 *solution to Problem (P), for a measurable subset  $A \subset [0, T]$ , such that  $g$  is submersive*  
 218 *at  $(x^*(0), x^*(T), \lambda^*)$ , then there exists a nontrivial pair  $(p, p^0) \in \text{AC}([0, T], \mathbb{R}^n) \times \mathbb{R}_+$*   
 219 *satisfying:*

- 220 (i) *the Hamiltonian system  $\dot{x}^*(t) = \nabla_p \mathcal{H}(x^*(t), u^*(t), \lambda^*, p(t))$  and  $-\dot{p}(t) =$*   
 221  *$\nabla_x \mathcal{H}(x^*(t), u^*(t), \lambda^*, p(t))$  for almost every  $t \in [0, T]$ ;*  
 222 (ii) *the endpoint transversality condition*

$$223 \quad \begin{pmatrix} p(0) \\ -p(T) \\ \int_0^T \nabla_\lambda \mathcal{H}(x^*(s), u^*(s), \lambda^*, p(s)) ds \end{pmatrix} = p^0 \nabla \phi(x^*(0), x^*(T), \lambda^*) + \nabla g(x^*(0), x^*(T), \lambda^*) \xi,$$

- 224 *for some  $\xi \in \text{N}_S[g(x^*(0), x^*(T), \lambda^*)]$ ;*  
 225 (iii) *the Hamiltonian maximization condition  $u^*(t) \in \arg \max_{\omega \in U} \mathcal{H}(x^*(t), \omega, \lambda^*, p(t))$*   
 226 *for almost every  $t \in A$ .*

227 The proof of Theorem 2.1 is quite simple and will be developed in a few lines. It  
 228 is based on the next preliminary PMP for  $L_A^1$ -local solutions to Problem (P).

229 LEMMA 2.1 (PMP for  $L_A^1$ -local solutions). *If  $(x^*, u^*, \lambda^*)$  is a  $L_A^1$ -local solu-*  
 230 *tion to Problem (P), for a measurable subset  $A \subset [0, T]$ , such that  $g$  is submersive*  
 231 *at  $(x^*(0), x^*(T), \lambda^*)$ , then the conclusion of Theorem 2.1 holds true.*

232 *About the proof of Lemma 2.1.* A PMP for  $L_A^1$ -local solutions to classical optimal  
 233 control problems can be established via many different methods known in the literature.  
 234 In our context, since the measurable subset  $A$  can be of complex nature (such as a  
 235 Cantor set of positive measure), the classical needle-like perturbations of the control  
 236 (see, *e.g.*, [15, 27]) may not be suitable for the sensitivity analysis of the control system  
 237 and, therefore, one may prefer to use *implicit spike variations* (see, *e.g.*, [10, 12, 24]).  
 238 To deal with the parameter  $\lambda \in \mathbb{R}^d$  in Problem (P), one can simply augment the  
 239 state variable from  $x$  to  $(x, \lambda)$  by adding the state equation  $\dot{\lambda}(t) = 0_{\mathbb{R}^d}$  (see, *e.g.*, [9]).  
 240 Finally, to deal with the general mixed terminal state constraints  $g(x(0), x(T), \lambda) \in S$   
 241 in Problem (P), one may use the Ekeland variational principle on a penalized functional  
 242 involving the square of the distance function to  $S$  (see, *e.g.*, [12, 20]). Since all these  
 243 techniques are very well known in the literature, the proof of Lemma 2.1 is omitted.  $\square$

*Proof of Theorem 2.1.* Consider an increasing family  $(A_\varepsilon)_{\varepsilon > 0}$  of measurable sub-  
 sets of  $A$  associated with  $(x^*, u^*, \lambda^*)$  and a decreasing positive sequence  $(\varepsilon_k)_{k \in \mathbb{N}}$  such  
 that  $\varepsilon_k \rightarrow 0$ . In the sequel we denote by  $A_k := A_{\varepsilon_k}$  and by  $(p_k, p_k^0) \in \text{AC}([0, T], \mathbb{R}^n) \times$   
 $\mathbb{R}_+$  the nontrivial pair provided by Lemma 2.1 (with  $\xi_k \in \text{N}_S[g(x^*(0), x^*(T), \lambda^*)]$ ) for  
 all  $k \in \mathbb{N}$ . From linearity and submersiveness, the pair  $(\xi_k, p_k^0)$  is nontrivial and can  
 be renormalized so that  $\|(\xi_k, p_k^0)\|_{\mathbb{R}^d \times \mathbb{R}} = 1$  for all  $k \in \mathbb{N}$ . Therefore, up to a subse-  
 quence that we do not relabel, the sequence  $(\xi_k, p_k^0)_{k \in \mathbb{N}}$  converges to some nontrivial  
 pair  $(\xi, p^0)$  satisfying  $(\xi, p^0) \in \text{N}_S[g(x^*(0), x^*(T), \lambda^*)] \times \mathbb{R}_+$  from closure of the normal  
 cone. Define  $p \in \text{AC}([0, T], \mathbb{R}^n)$  as the unique global solution to

$$\begin{cases} \dot{p}(t) = -\nabla_x f(x^*(t), u^*(t), \lambda^*)^\top p(t), & \text{a.e. } t \in [0, T], \\ p(T) = -p^0 \nabla_2 \phi(x^*(0), x^*(T), \lambda^*) - \nabla_2 g(x^*(0), x^*(T), \lambda^*) \xi. \end{cases}$$

244 The Hamiltonian system and the second component of the endpoint transversality  
 245 condition are satisfied. Since  $p$  and  $p_k$  satisfy the same linear differential equation  
 246 and  $p_k(T) \rightarrow p(T)$ , the sequence  $(p_k)_{k \in \mathbb{N}}$  uniformly converges to  $p$  over  $[0, T]$ . We  
 247 deduce the first and third components of the endpoint transversality condition and,

248 from submersiveness, that the pair  $(p, p^0)$  is nontrivial. Still from Lemma 2.1, there  
 249 exists a null set  $N_k \subset A_k$  such that  $\mathcal{H}(x^*(t), u^*(t), \lambda^*, p_k(t)) \geq \mathcal{H}(x^*(t), \omega, \lambda^*, p_k(t))$   
 250 for all  $\omega \in U$  and all  $t \in A_k \setminus N_k$ , for all  $k \in \mathbb{N}$ . Now let us prove that the Hamiltonian  
 251 maximization condition holds true at any  $t \in \tilde{A} := (\cup_{k \in \mathbb{N}} A_k) \setminus (\cup_{k \in \mathbb{N}} N_k)$  which is  
 252 a measurable subset of  $A$  with full measure. Let  $t \in \tilde{A}$  and take  $k_0 \in \mathbb{N}$  such  
 253 that  $t \in A_k \setminus N_k$  for all  $k \geq k_0$ . Therefore the previous inequality holds true at  $t$   
 254 for all  $\omega \in U$  and all  $k \geq k_0$ . From convergence of  $p_k(t)$  to  $p(t)$ , we get that  
 255  $\mathcal{H}(x^*(t), u^*(t), \lambda^*, p(t)) \geq \mathcal{H}(x^*(t), \omega, \lambda^*, p(t))$  for all  $\omega \in U$ , which ends the proof.  $\square$

256 *Remark 2.3.* (i) First of all we bring the reader's attention to the fact that the  
 257 Hamiltonian maximization condition in Lemma 2.1 and Theorem 2.1 holds true only  
 258 almost everywhere over  $A$  (and not over the entire interval  $[0, T]$ ). This is the only  
 259 difference with the classical PMP and this is due, of course, to the restrictions to  $L^1_{A\Box}$ -  
 260 and  $L^1_{A\Box}$ -local solutions (see Definitions 2.1 and 2.2 and Item (iii) of Remark 2.2).

261 (ii) Even if the conclusions of Lemma 2.1 and Theorem 2.1 are exactly the same,  
 262 we recall that a  $L^1_{A\Box}$ -local solution is not a  $L^1_A$ -local solution in general (see Item (iv)  
 263 of Remark 2.2). Therefore Theorem 2.1 is not only a consequence of Lemma 2.1 but  
 264 also a strict extension. From the diagram in Remark 2.2, it is also clear that the  
 265 classical PMP (for global solutions or for  $L^1$ -local solutions) is a particular case of  
 266 both Lemma 2.1 and Theorem 2.1 (by taking  $A = [0, T]$ ).

267 (iii) As explained in [6, 7], the submersiveness hypothesis can be removed but, in  
 268 that case, all items of Lemma 2.1 and Theorem 2.1 remain valid, except Item (ii).

269 (iv) Consider the framework of Theorem 2.1 for a  $L^1_{\Box}$ -local solution  $(x^*, u^*, \lambda^*)$ .  
 270 Using the Hamiltonian system and the Hamiltonian maximization condition over  $[0, T]$   
 271 and applying [21, Theorem 2.6.1], we obtain the *Hamiltonian constancy condition*  
 272  $\mathcal{H}(x^*(t), u^*(t), \lambda^*, p(t)) = c$  for almost every  $t \in [0, T]$ , for some  $c \in \mathbb{R}$ .

273 **3. Derivation of a HMP for spatially heterogeneous dynamics.** In this  
 274 section we consider a partition of the state space  $\mathbb{R}^n = \cup_{j \in \mathcal{J}} \overline{X_j}$ , where  $\mathcal{J}$  is a family  
 275 of indexes and the nonempty open subsets  $X_j \subset \mathbb{R}^n$ , called *regions*, are disjoint. Our  
 276 aim is to derive first-order necessary optimality conditions in a Pontryagin form for  
 277 the *hybrid* optimal control problem with mixed terminal state constraints given by

$$\begin{aligned}
 & \text{minimize} && \phi(x(0), x(T)), \\
 & \text{subject to} && (x, u) \in \text{AC}([0, T], \mathbb{R}^n) \times L^\infty([0, T], \mathbb{R}^m), \\
 278 \text{ (HP)} &&& \dot{x}(t) = h(x(t), u(t)), \quad \text{a.e. } t \in [0, T], \\
 &&& g(x(0), x(T)) \in S, \\
 &&& u(t) \in U, \quad \text{a.e. } t \in [0, T],
 \end{aligned}$$

where the data assumptions and the terminology for Problem (HP) are the same  
 as those for Problem (P), except that the dynamics  $h : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  is *spatially*  
*heterogeneous*, in the sense that it is defined *regionally* by

$$\forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^m, \quad h(x, u) := h_j(x, u) \quad \text{when } x \in X_j,$$

279 where the *subdynamics*  $h_j : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  are of class  $C^1$ . Note that  $h(x, u)$  is not  
 280 defined when  $x \notin \cup_{j \in \mathcal{J}} X_j$  but this fact will have no impact on the rest of this work  
 281 (see Item (i) in Remark 3.1). Finally, in contrary to Problem (P) and as explained in  
 282 Item (ii) of Remark 2.1, note that Problem (HP) does not involve any parameter.

283 **3.1. Regular solutions to the hybrid control system.** Due to the disconti-  
 284 nuousities of the spatially heterogeneous dynamics  $h$ , we need to precise the definition of



285 a solution to the hybrid control system

286 (HS)  $\dot{x}(t) = h(x(t), u(t)), \quad \text{for a.e. } t \in [0, T],$

287 associated with Problem (HP).

288 **DEFINITION 3.1** (Solution to (HS)). *A pair  $(x, u) \in \text{AC}([0, T], \mathbb{R}^n) \times \text{L}^\infty([0, T], \mathbb{R}^m)$*   
 289 *is said to be a solution to (HS) if there exists a partition  $\mathbb{T} = \{\tau_k\}_{k=0, \dots, N}$  such that:*

290 (i) *For all  $k \in \{1, \dots, N\}$ , there exists  $j(k) \in \mathcal{J}$  (with  $j(k) \neq j(k-1)$ ) such*  
 291 *that  $x(t) \in X_{j(k)}$  for almost every  $t \in (\tau_{k-1}, \tau_k)$ .*

292 (ii)  *$x(0) \in X_{j(1)}$  and  $x(T) \in X_{j(N)}$ .*

293 (iii)  *$\dot{x}(t) = h_{j(k)}(x(t), u(t))$  for almost every  $t \in (\tau_{k-1}, \tau_k)$  and all  $k \in \{1, \dots, N\}$ .*  
*In that case, to ease notation, we set  $f_k := h_{j(k)}$  and  $E_k := X_{j(k)}$  for all  $k \in \{1, \dots, N\}$ .*

*With this system of notations, we have*

$$\begin{cases} x(t) \in E_1, & \forall t \in [\tau_0, \tau_1), \\ x(t) \in E_k, & \forall t \in (\tau_{k-1}, \tau_k), \quad \forall k \in \{2, \dots, N-1\}, \\ x(t) \in E_N, & \forall t \in (\tau_{N-1}, \tau_N], \\ \dot{x}(t) = f_k(x(t), u(t)), & \text{a.e. } t \in (\tau_{k-1}, \tau_k), \quad \forall k \in \{1, \dots, N\}. \end{cases}$$

294 *Finally the times  $\tau_k$ , for  $k \in \{1, \dots, N-1\}$ , are called crossing times since they*  
 295 *correspond to the instants at which the trajectory  $x$  goes from the region  $E_k$  to the*  
 296 *region  $E_{k+1}$ , and thus  $x(\tau_k) \in \partial E_k \cap \partial E_{k+1}$ .*

297 Our main result (Theorem 3.1 stated in Section 3.3) is based on some regularity  
 298 assumptions made on the behavior of the optimal pair of Problem (HP) at each  
 299 crossing time. These hypotheses are precised in the next definition.

300 **DEFINITION 3.2** (Regular solution to (HS)). *Following the notations introduced*  
 301 *in Definition 3.1, a solution  $(x, u) \in \text{AC}([0, T], \mathbb{R}^n) \times \text{L}^\infty([0, T], \mathbb{R}^m)$  to (HS) is said*  
 302 *to be regular if the following conditions are both satisfied:*

303 (i) *At each crossing time  $\tau_k$ , there exists a  $C^1$  function  $F_k : \mathbb{R}^n \rightarrow \mathbb{R}$  such that*

304 
$$\exists \nu_k > 0, \quad \forall z \in \bar{B}_{\mathbb{R}^n}(x(\tau_k), \nu_k), \quad \begin{cases} z \in E_k & \Leftrightarrow F_k(z) < 0, \\ z \in \partial E_k \cap \partial E_{k+1} & \Leftrightarrow F_k(z) = 0, \\ z \in E_{k+1} & \Leftrightarrow F_k(z) > 0. \end{cases}$$

305 *In particular it holds that  $F_k(x(\tau_k)) = 0$ .*

306 (ii) *At each crossing time  $\tau_k$ , there exists  $\alpha_k > 0$  and  $\beta_k > 0$  such that the*  
 307 *transverse conditions*

308 (TC) 
$$\begin{cases} \langle \nabla F_k(x(\tau_k)), f_k(x(\tau_k), u(t)) \rangle_{\mathbb{R}^n} \geq \beta_k, & \text{a.e. } t \in [\tau_k - \alpha_k, \tau_k), \\ \langle \nabla F_k(x(\tau_k)), f_{k+1}(x(\tau_k), u(t)) \rangle_{\mathbb{R}^n} \geq \beta_k, & \text{a.e. } t \in (\tau_k, \tau_k + \alpha_k], \end{cases}$$

309 *are both satisfied.*

310 **Remark 3.1.** (i) Definition 3.1 does not include the possibility of an infinite number  
 311 of crossing times (excluding the Zeno phenomenon [34]). Also it does not allow  
 312 trajectories bouncing against a boundary of a region, or moving along a boundary  
 313 (excluding situations as described in [1]). This last restriction is the reason why the  
 314 fact that  $h(x, u)$  is not defined when  $x \notin \cup_{j \in \mathcal{J}} X_j$  has no impact on the present work.  
 315 Finally Definition 3.1 allows terminal states  $x(0)$  and  $x(T)$  that belong to regions only  
 316 (and not to their boundaries). Possible relaxations are presented in Remark 3.4.

317 (ii) The transverse conditions (TC) have a geometrical interpretation, meaning  
 318 that  $x$  does not cross the boundary  $\partial E_k \cap \partial E_{k+1}$  tangentially. At a crossing time  $\tau_k$ ,

319 the transverse conditions

$$320 \quad (\text{TC}') \quad \begin{cases} u \text{ admits left and right limits at } \tau_k \text{ denoted by } u^-(\tau_k) \text{ and } u^+(\tau_k), \\ \langle \nabla F_k(x(\tau_k)), f_k(x(\tau_k), u^-(\tau_k)) \rangle_{\mathbb{R}^n} > 0 \\ \langle \nabla F_k(x(\tau_k)), f_{k+1}(x(\tau_k), u^+(\tau_k)) \rangle_{\mathbb{R}^n} > 0, \end{cases}$$

321 considered in the papers [1, 23], are (slightly) stronger than (TC).

### 322 3.2. Reduction into a classical optimal control problem with parameter.

323 To establish a correspondence from the hybrid optimal control problem (HP) to a  
 324 classical optimal control problem with parameter of the form of Problem (P), we  
 325 will proceed as in [19] to affine changes of time variable. Precisely let  $(x^*, u^*) \in$   
 326  $\text{AC}([0, T], \mathbb{R}^n) \times \text{L}^\infty([0, T], \mathbb{R}^m)$  be a solution to (HS), associated with a partition  $\mathbb{T}^* =$   
 327  $\{\tau_k^*\}_{k=0, \dots, N}$ , and let  $E_k^*$  and  $f_k^*$  stand for the corresponding regions and functions (see  
 328 Definition 3.1). We introduce  $(y^*, v^*) \in \text{AC}([0, 1], \mathbb{R}^{nN}) \times \text{L}^\infty([0, 1], \mathbb{R}^{mN})$  defined by

$$329 \quad (3.1) \quad y_k^*(s) := x^*(\tau_{k-1}^* + (\tau_k^* - \tau_{k-1}^*)s) \quad \text{and} \quad v_k^*(s) := u^*(\tau_{k-1}^* + (\tau_k^* - \tau_{k-1}^*)s),$$

330 for all  $s \in [0, 1]$  and all  $k \in \{1, \dots, N\}$ . To invert the changes of time variable, it holds

$$331 \quad (3.2) \quad x^*(t) = y_k^*\left(\frac{t - \tau_{k-1}^*}{\tau_k^* - \tau_{k-1}^*}\right) \quad \text{and} \quad u^*(t) = v_k^*\left(\frac{t - \tau_{k-1}^*}{\tau_k^* - \tau_{k-1}^*}\right),$$

for all  $t \in [\tau_{k-1}^*, \tau_k^*]$  and all  $k \in \{1, \dots, N\}$ . In particular note that  $(x^*(0), x^*(T)) =$   
 $(y_1^*(0), y_N^*(1))$ . From a more general point of view, it holds that  $x^*(\tau_k^*) = y_{k+1}^*(0)$   
 for all  $k \in \{0, \dots, N-1\}$  and  $x^*(\tau_k^*) = y_k^*(1)$  for all  $k \in \{1, \dots, N\}$ . Note that the  
 triplet  $(y^*, v^*, \mathbb{T}^*)$  satisfies

$$y^*(s) = f^*(y^*(s), v^*(s), \mathbb{T}^*), \quad \text{a.e. } s \in [0, 1],$$

where  $f^* : \mathbb{R}^{nN} \times \mathbb{R}^{mN} \times \mathbb{R}^{N+1} \rightarrow \mathbb{R}^{nN}$  is the  $C^1$  function defined by

$$f^*(y, v, \mathbb{T}) := \left( (\tau_1 - \tau_0) f_1^*(y_1, v_1), \dots, (\tau_N - \tau_{N-1}) f_N^*(y_N, v_N) \right),$$

332 for all  $y = (y_1, \dots, y_N) \in \mathbb{R}^{nN}$ ,  $v = (v_1, \dots, v_N) \in \mathbb{R}^{mN}$  and  $\mathbb{T} = \{\tau_0, \dots, \tau_N\} \in \mathbb{R}^{N+1}$ .

333 Furthermore it holds that

$$334 \quad (3.3) \quad \begin{cases} y_1^*(s) \in E_1, & \forall s \in [0, 1], \\ y_k^*(s) \in E_k, & \forall s \in (0, 1), \quad \forall k \in \{2, \dots, N-1\}, \\ y_N^*(s) \in E_N, & \forall s \in (0, 1], \end{cases}$$

and  $y_{k+1}^*(0) = y_k^*(1) \in \partial E_k^* \cap \partial E_{k+1}^*$  for all  $k \in \{1, \dots, N-1\}$ . Also note that  $\mathbb{T}^* \in \Delta$   
 where  $\Delta \subset \mathbb{R}^{N+1}$  is the nonempty closed convex set defined by

$$\Delta := \{\mathbb{T} = \{\tau_k\}_{k=0, \dots, N} \in \mathbb{R}^{N+1} \mid 0 = \tau_0 \leq \tau_1 \leq \dots \leq \tau_{N-1} \leq \tau_N = T\}.$$

335 Now assume that the pair  $(x^*, u^*)$  is moreover a regular solution to (HS) and denote  
 336 by  $F_k^*$  and  $\nu_k^* > 0$  the corresponding functions and positive radii (see Definition 3.2).  
 337 In that context note that  $F_k^*(x(\tau_k^*)) = F_k^*(y_k^*(1)) = 0$  for all  $k \in \{1, \dots, N-1\}$ .  
 338 Finally it is clear that, if the pair  $(x^*, u^*)$  is furthermore admissible for Problem (HP),  
 339 then the triplet  $(y^*, v^*, \mathbb{T}^*)$  is admissible for the classical optimal control problem with

340 parameter given by

$$\begin{aligned}
& \text{minimize} && \phi^*(y(0), y(1), \mathbb{T}), \\
& \text{subject to} && (y, v, \mathbb{T}) \in \text{AC}([0, 1], \mathbb{R}^{nN}) \times L^\infty([0, 1], \mathbb{R}^{mN}) \times \mathbb{R}^{N+1}, \\
341 \text{ (CP}^*) &&& \dot{y}(s) = f^*(y(s), v(s), \mathbb{T}), \quad \text{a.e. } s \in [0, 1], \\
&&& g^*(y(0), y(1), \mathbb{T}) \in S^*, \\
&&& v(s) \in U^N, \quad \text{a.e. } s \in [0, 1],
\end{aligned}$$

where  $\phi^* : \mathbb{R}^{nN} \times \mathbb{R}^{mN} \times \mathbb{R}^{N+1} \rightarrow \mathbb{R}$  and  $g^* : \mathbb{R}^{nN} \times \mathbb{R}^{mN} \times \mathbb{R}^{N+1} \rightarrow \mathbb{R}^{\ell^*}$  are the  $C^1$  functions defined by  $\phi^*(y^0, y^1, \mathbb{T}) := \phi(y_1^0, y_N^1, \mathbb{T})$  and

$$g^*(y^0, y^1, \mathbb{T}) := (g(y_1^0, y_N^1), y_2^0 - y_1^1, \dots, y_N^0 - y_{N-1}^1, F_1^*(y_1^1), \dots, F_{N-1}^*(y_{N-1}^1), \mathbb{T}),$$

for all  $y^0 = (y_1^0, \dots, y_N^0)$ ,  $y^1 = (y_1^1, \dots, y_N^1) \in \mathbb{R}^{nN}$  and  $\mathbb{T} = \{\tau_0, \dots, \tau_N\} \in \mathbb{R}^{N+1}$ , where  $\ell^* := \ell + n(N-1) + (N-1) + (N+1)$ , and where  $S^* \subset \mathbb{R}^{\ell^*}$  stands for the nonempty closed convex set defined by

$$S^* := S \times \{0_{\mathbb{R}^n}\}^{N-1} \times \{0\}^{N-1} \times \Delta.$$

342 **PROPOSITION 3.1.** *If  $(x^*, u^*) \in \text{AC}([0, T], \mathbb{R}^n) \times L^\infty([0, T], \mathbb{R}^m)$  is a global solution*  
343 *to Problem (HP), that is moreover a regular solution to (HS), associated with a*  
344 *partition  $\mathbb{T}^* = \{\tau_k^*\}_{k=0, \dots, N}$ , then the triplet  $(y^*, v^*, \mathbb{T}^*)$  constructed above is a  $L^1_{\square}$ -*  
345 *local solution to Problem (CP\*).*

346 *Proof.* The proof of Proposition 3.1 is postponed to Appendix A. We prove that the  
347 triplet  $(y^*, v^*, \mathbb{T}^*)$  is a  $L^1_{[\varepsilon, 1-\varepsilon]}$ -local solution to Problem (CP\*) for any  $0 < \varepsilon < 1/2$ .  $\square$

348 *Remark 3.2.* (i) Consider the framework of Proposition 3.1. In Section 3.4 we  
349 will provide a counterexample showing that the triplet  $(y^*, v^*, \mathbb{T}^*)$  is not a  $L^1$ -local  
350 solution to Problem (CP\*) in general. This highlights the fact that the classical PMP  
351 cannot be applied to the triplet  $(y^*, v^*, \mathbb{T}^*)$ . However, thanks to Proposition 3.1, we  
352 can apply the new PMP for  $L^1_{\square}$ -local solution obtained in Theorem 2.1. This allows us  
353 to derive a HMP for Problem (HP) in the next Section 3.3.

354 (ii) Consider the framework of Proposition 3.1. Given an admissible triplet  $(y, v, \mathbb{T})$   
355 for Problem (CP\*), one can easily invert the augmentation procedure and obtain a  
356 pair  $(x, u)$  which satisfies all the constraints of Problem (HP), except one. Precisely,  
357 even if  $(x, u)$  follows the same sequence  $(f_k^*)_{k=1, \dots, N}$  of dynamics than the pair  $(x^*, u^*)$ ,  
358 it does not necessarily follow the same sequence of regions  $(E_k^*)_{k=1, \dots, N}$  (and thus it is  
359 not necessarily admissible for Problem (HP)). This is the major difficulty of the proof  
360 of Proposition 3.1 and, as we will see with a counterexample in Section 3.4, the notion  
361 of  $L^1$ -local solution (which consists in considering the triplet  $(y, v, \mathbb{T})$  in a standard  
362 neighborhood of  $(y^*, v^*, \mathbb{T}^*)$ ) fails to guarantee this property. This is because, even  
363 if transverse conditions are satisfied by the pair  $(x^*, u^*)$ , allowing  $L^1$ -perturbations  
364 of  $u^*$  (with possibly far values in  $U$  from the ones of  $u^*$ ) in the neighborhoods of  
365 the crossing times  $\tau_k^*$  may lead to a perturbed pair  $(x, u)$  that does not satisfy the  
366 transverse conditions, and thus to a perturbed trajectory  $x$  that may visit a different  
367 sequence of regions than  $x^*$ . On the contrary, the new notion of  $L^1_{[\varepsilon, 1-\varepsilon]}$ -local solution,  
368 for  $0 < \varepsilon < 1/2$ , addresses this issue by allowing  $L^1$ -perturbations of  $u^*$  only outside  
369 neighborhoods of the crossing times  $\tau_k^*$ .

370 **3.3. HMP and comments.** The Hamiltonian  $H : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$  associated  
371 with Problem (HP) is defined by  $H(x, u, p) := \langle p, h(x, u) \rangle$  for all  $(x, u, p) \in \mathbb{R}^n \times \mathbb{R}^m \times$   
372  $\mathbb{R}^n$ . We are now in a position to state the main result of this paper.

373 THEOREM 3.1 (HMP). *If  $(x^*, u^*) \in \text{AC}([0, T], \mathbb{R}^n) \times \text{L}^\infty([0, T], \mathbb{R}^m)$  is a global*  
 374 *solution to Problem (HP), that is moreover a regular solution to (HS), associated with*  
 375 *a partition  $\mathbb{T}^* = \{\tau_k^*\}_{k=0, \dots, N}$ , such that  $g$  is submersive at  $(x^*(0), x^*(T))$ , then there*  
 376 *exists a nontrivial pair  $(p, p^0) \in \text{PAC}_{\mathbb{T}^*}([0, T], \mathbb{R}^n) \times \mathbb{R}_+$  satisfying:*

- 377 (i) *the Hamiltonian system  $\dot{x}^*(t) = \nabla_p H(x^*(t), u^*(t), p(t))$  and  $-\dot{p}(t) = \nabla_x H(x^*(t),$*   
 378  *$u^*(t), p(t))$  for almost every  $t \in [0, T]$ ;*  
 379 (ii) *the endpoint transversality condition*

$$380 \quad \begin{pmatrix} p(0) \\ -p(T) \end{pmatrix} = p^0 \nabla \phi(x^*(0), x^*(T)) + \nabla g(x^*(0), x^*(T)) \xi,$$

- 381 *for some  $\xi \in \text{N}_S[g(x^*(0), x^*(T))]$ ;*  
 382 (iii) *the discontinuity condition  $p^+(\tau_k^*) - p^-(\tau_k^*) = \sigma_k \nabla F_k^*(x^*(\tau_k^*))$  for some  $\sigma_k \in \mathbb{R}$ ,*  
 383 *for all  $k \in \{1, \dots, N-1\}$ ;*  
 384 (iv) *the Hamiltonian maximization condition  $u^*(t) \in \arg \max_{\omega \in \mathcal{U}} H(x^*(t), \omega, p(t))$*   
 385 *for almost every  $t \in [0, T]$ ;*  
 386 (v) *the Hamiltonian constancy condition  $H(x^*(t), u^*(t), p(t)) = c$  for almost ev-*  
 387 *ery  $t \in [0, T]$ , for some  $c \in \mathbb{R}$ .*

388 *Proof.* The proof of Theorem 3.1 is postponed to Appendix B. It is based on  
 389 Proposition 3.1 and on the application of Theorem 2.1 to the triplet  $(y^*, v^*, \mathbb{T}^*)$ .  $\square$

390 *Remark 3.3.* (i) In the classical PMP (that is, when the dynamics is not hetero-  
 391 geneous), the costate  $p$  is absolutely continuous over the entire interval  $[0, T]$  and  
 392 satisfies Items (i), (ii), (iv) and (v) of Theorem 3.1 (see, *e.g.*, [27]). In the present  
 393 setting of heterogeneous dynamics, the costate  $p$  is (only) piecewise absolutely contin-  
 394 uous over  $[0, T]$ , admitting at each crossing time  $\tau_k^*$  a discontinuity jump satisfying  
 395 Item (iii) of Theorem 3.1. Under the (slightly) stronger transverse conditions (TC'),  
 396 the Hamiltonian constancy condition allows to obtain

$$397 \quad \sigma_k = - \frac{\left\langle p^-(\tau_k^*), f_{k+1}^*(x^*(\tau_k^*), (u^*)^+(\tau_k^*)) - f_k^*(x^*(\tau_k^*), (u^*)^-(\tau_k^*)) \right\rangle_{\mathbb{R}^n}}{\left\langle \nabla F_k^*(x^*(\tau_k^*)), f_{k+1}^*(x^*(\tau_k^*), (u^*)^+(\tau_k^*)) \right\rangle_{\mathbb{R}^n}}$$

$$398 \quad = - \frac{\left\langle p^+(\tau_k^*), f_{k+1}^*(x^*(\tau_k^*), (u^*)^+(\tau_k^*)) - f_k^*(x^*(\tau_k^*), (u^*)^-(\tau_k^*)) \right\rangle_{\mathbb{R}^n}}{\left\langle \nabla F_k^*(x^*(\tau_k^*)), f_k^*(x^*(\tau_k^*), (u^*)^-(\tau_k^*)) \right\rangle_{\mathbb{R}^n}},$$

401 for all  $k \in \{1, \dots, N-1\}$ , and thus the discontinuity conditions can be expressed  
 402 as forward (or backward) discontinuity jumps. Such discontinuity jumps are very  
 403 standard in the literature on hybrid optimal control problems (see, *e.g.*, [8, 26]) and  
 404 the discontinuity conditions have even been announced in our setting of spatially  
 405 heterogeneous dynamics in the papers [1, 23]. However, as explained in Introduction,  
 406 we recall that the proofs in [1, 23] are not satisfactory for several and different reasons.

407 (ii) Similarly to Item (iii) of Remark 2.3, and as explained in [6, 7], the submer-  
 408 siveness hypothesis made in Theorem 3.1 can be removed but, in that case, all items  
 409 of Theorem 3.1 remain valid, except Item (ii).

410 *Remark 3.4.* (i) Consider the framework of Proposition 3.1. From Item (i) of  
 411 Remark 3.2, we know that  $(y^*, v^*, \mathbb{T}^*)$  is not a  $\text{L}^1$ -local solution to Problem (CP\*)  
 412 in general. Nevertheless, according to the ideas presented in Item (ii) of Remark 3.2, it  
 413 may be possible to avoid the use of the notion of  $\text{L}^1_{\square}$ -local solution introduced in the

414 present paper. However, to our best knowledge, this would not be possible without  
 415 obtaining a weaker result and/or without restricting the framework. Let us develop  
 416 two options in that direction:

- 417 • First, under the (slightly) stronger transverse conditions (TC'), it can be  
 418 proved that  $(y^*, v^*, \mathbb{T}^*)$  is a  $L^\infty$ -local solution to Problem (CP\*), in the sense  
 419 that there exists  $\eta > 0$  such that  $\phi^*(y^*(0), y^*(1), \mathbb{T}^*) \leq \phi^*(y(0), y(1), \mathbb{T})$  for all  
 420 admissible triplets  $(y, v, \mathbb{T})$  satisfying  $\|y - y^*\|_C + \|v - v^*\|_{L^\infty} + \|\mathbb{T} - \mathbb{T}^*\|_{\mathbb{R}^{N+1}} \leq \eta$ .  
 421 This idea is in-line with the approach developed in [5]. In that context,  
 422 assuming for simplicity that  $U$  is closed and convex and applying a *weak*  
 423 *version* of the classical PMP (that is, a version adapted to  $L^\infty$ -local solutions,  
 424 see [13] and discussion therein), one can derive a weaker version of Theorem 3.1,  
 425 that is, without the Hamiltonian constancy condition and, above all, where the  
 426 Hamiltonian maximization condition is replaced by the weaker *Hamiltonian*  
 427 *gradient condition*  $\nabla_u H(x^*(t), u^*(t), p(t)) \in N_U[u^*(t)]$  for a.e.  $t \in [0, T]$ .  
 428 • Second, under the (very) stronger transverse conditions given by

$$429 \quad (\text{TC}'') \quad \forall \omega \in U, \quad \begin{cases} \langle \nabla F_k^*(x^*(\tau_k^*)), f_k^*(x^*(\tau_k^*), \omega) \rangle_{\mathbb{R}^n} \geq \beta_k, \\ \langle \nabla F_k^*(x^*(\tau_k^*)), f_{k+1}^*(x^*(\tau_k^*), \omega) \rangle_{\mathbb{R}^n} \geq \beta_k, \end{cases}$$

430 for some  $\beta_k > 0$  at each crossing time  $\tau_k^*$ , it can be proved that  $(y^*, v^*, \mathbb{T}^*)$   
 431 is a  $L^1$ -local solution to Problem (CP\*). In that context one can derive  
 432 Theorem 3.1 from the application of the classical PMP. However the strong  
 433 transverse conditions (TC'') are quite restrictive and are not satisfied in  
 434 practice (see the counterexample presented in the next Section 3.4).

435 From a general point of view, it can be observed that the choice of the transverse  
 436 conditions (more or less strong) influences the *local quality* ( $L^1$ ,  $L^\infty$  or  $L^1_{\square}$ ) of the  
 437 solution  $(y^*, v^*, \mathbb{T}^*)$  to Problem (CP\*) and thus the version of the PMP that can be  
 438 applied to it, and finally the version of the HMP obtained on the original pair  $(x^*, u^*)$ .

(ii) For simplicity, Definition 2.1 allows trajectories  $x$  such that  $x(0)$  and  $x(T)$   
 belong to regions only (and not to their boundaries). This restriction may limit  
 the scope of our results. To overcome this restriction, some adjustments have to be  
 performed. For instance, consider the framework of Theorem 3.1 with  $x^*(0) \in E_1$   
 and  $x^*(T) \in \partial E_N$  (other cases can be handled similarly). To deal with this situation,  
 one has to add in Definition 3.2 the existence of a local  $C^1$  description  $F_N$  of  $\partial E_N$  in  
 a neighborhood of  $x^*(T)$  and an adapted transverse condition of the form

$$\langle \nabla F_N^*(x^*(T)), f_N^*(x^*(T), u^*(t)) \rangle_{\mathbb{R}^n} \geq \beta_N, \quad \text{a.e. } t \in [T - \alpha_N, T],$$

439 with  $\alpha_N > 0$  and  $\beta_N > 0$ . Then the augmented problem (CP\*) must be adjusted care-  
 440 fully by adding the inequality constraint  $F_N^*(y_N(1)) \leq 0$  to keep the validity of Propo-  
 441 sition 3.1. Finally, adapting the submersiveness hypothesis (involving both  $g$  and  $F_N^*$ ),  
 442 applying Theorem 2.1 and inverting the augmentation procedure, the conclusion of  
 443 Theorem 3.1 remains valid, but with an additional term of the form  $\zeta \nabla F_N^*(x^*(T))$   
 444 with  $\zeta \geq 0$  in the expression of  $-p(T)$ .

445 (iii) In addition to the comments made in the previous Item (ii), we would like  
 446 to emphasize that certain cases where  $x^*(0)$  and  $x^*(T)$  belong to boundaries of the  
 447 regions can be treated without the adjusted procedure discussed above. For instance,  
 448 if the initial condition is fixed on a boundary, then no information is expected for  $p(0)$   
 449 and, furthermore, with the approach developed in this paper, only perturbations of the  
 450 control over  $[\varepsilon, T]$  for small  $\varepsilon > 0$  are considered. Hence the corresponding perturbed  
 451 trajectories coincide with the nominal trajectory over  $[0, \varepsilon]$  and thus satisfy the initial

452 condition. Another example is provided with minimum time problems where the target  
 453 belongs to a boundary of a region. In that context, a simple dynamical programming  
 454 argument can eliminate the need of a transverse condition at  $T$  (see [4]).

455 (iv) Here we focus on possible extensions and perspectives of our work.

- 456 • First, one may consider a setting where the subdynamics  $h_j : \mathbb{R}^n \times \mathbb{R}^{m_j} \rightarrow \mathbb{R}^n$   
 457 have possibly different control dimensions  $m_j \in \mathbb{N}^*$  and with possibly different  
 458 control constraint sets  $U_j \subset \mathbb{R}^{m_j}$ . This generalized context is interesting  
 459 to impose specific values for the control in certain regions (for example, by  
 460 taking  $U_j = \{0_{\mathbb{R}^{m_j}}\}$  for some  $j \in \mathcal{J}$ ). We believe that our methodology can  
 461 be adapted to this framework without any major difficulty.
- 462 • Second, one may consider an extended setting that includes a *regionally*  
 463 *switching parameter* (see [2]), meaning that the control system depends on  
 464 a parameter that remains constant in each region but can change its value  
 465 when the state crosses a boundary. This framework enables us to handle, as  
 466 a specific case, control systems with *loss control regions* (see [3, 4]). This  
 467 extension is the subject of a work in progress by the authors.

468 **3.4. A counterexample.** Consider the framework of Proposition 3.1. This  
 469 section is dedicated to an explicit counterexample showing that the triplet  $(y^*, v^*, \mathbb{T}^*)$   
 470 is not a  $L^1$ -local solution to Problem (CP\*) in general. To this aim consider the  
 471 two-dimensional case  $n = 2$ , the state space partition  $\mathbb{R}^2 = \overline{X_1} \cup \overline{X_2}$  where  $X_1 :=$   
 472  $(-\infty, 1) \times \mathbb{R}$  and  $X_2 := (1, +\infty) \times \mathbb{R}$ , and the hybrid optimal control problem given by

$$\begin{aligned}
 & \text{minimize} && -(x_1(2) - 2)^3 - \rho x_2(2), \\
 & \text{subject to} && (x, u) \in \text{AC}([0, 2], \mathbb{R}^2) \times L^\infty([0, 2], \mathbb{R}), \\
 473 \text{ (HP}_{\text{ex}}) &&& \dot{x}(t) = h(x(t), u(t)), \quad \text{a.e. } t \in [0, 2], \\
 &&& x(0) = 0_{\mathbb{R}^2}, \\
 &&& u(t) \in [-1, 1], \quad \text{a.e. } t \in [0, 2],
 \end{aligned}$$

474 where the spatially heterogeneous dynamics  $h : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2$  is defined by

$$475 \quad h(x, u) := \begin{cases} \left( 1, ((1 - x_1)^+)^2 \right), & \text{if } x \in X_1, \\ \left( u, ((1 - x_1)^+)^2 \right), & \text{if } x \in X_2, \end{cases}$$

476 for all  $x = (x_1, x_2) \in X_1 \cup X_2$  and all  $u \in \mathbb{R}$ , and where  $\rho > 96$ .

**3.4.1. A global solution  $(x^*, u^*)$  to Problem (HP<sub>ex</sub>).** In view of the definition  
 of  $h$  in the region  $X_1$  and following Definition 2.1, any admissible pair  $(x, u)$  for  
 Problem (HP<sub>ex</sub>) has exactly one crossing time  $\tau_1 = 1$ , and satisfies  $x_1(t) = t$  for  
 all  $t \in [0, 1]$  and  $x_1(t) > 1$  for all  $t \in (1, 2]$ . Moreover an easy computation shows that

$$x_2(t) = \begin{cases} \frac{1}{3}((t-1)^3 + 1) & \text{if } t \in [0, 1], \\ \frac{1}{3} & \text{if } t \in [1, 2], \end{cases}$$

for all  $t \in [0, 2]$ . Since the value  $x_2(2)$  is fixed to  $\frac{1}{3}$  for any admissible pair, Prob-  
 lem (HP<sub>ex</sub>) simply amounts to maximize the value of  $x_1(2)$ . In view of the definition  
 of  $h$  in the region  $X_2$ , one can easily deduce that a global solution  $(x^*, u^*)$  to Prob-  
 lem (HP<sub>ex</sub>) is given by

$$x_1^*(t) := t, \quad x_2^*(t) := \begin{cases} \frac{1}{3}((t-1)^3 + 1) & \text{if } t \in [0, 1], \\ \frac{1}{3} & \text{if } t \in [1, 2], \end{cases} \quad u^*(t) := 1,$$

477 for all  $t \in [0, 2]$ , and the corresponding optimal cost is given by  $\mathcal{C}^* := -\frac{\rho}{3}$ . Furthermore  
 478 one can observe that the pair  $(x^*, u^*)$  is a regular solution to the corresponding hybrid  
 479 control system (Definition 3.2) with exactly one crossing time  $\tau_1^* = 1$ .

480 **3.4.2. The corresponding triplet  $(y^*, v^*, \mathbb{T}^*)$ .** Now consider the framework of  
 481 Proposition 3.1. The corresponding triplet  $(y^*, v^*, \mathbb{T}^*)$  is given by

$$482 \quad (y_1^1)^*(s) = s, \quad (y_2^1)^*(s) = s + 1, \quad (y_1^2)^*(s) = \frac{1}{3}((s-1)^3 + 1), \quad (y_2^2)^*(s) = \frac{1}{3},$$

483 and  $v_1^*(s) = v_2^*(s) = 1$  for all  $s \in [0, 1]$ , and  $\mathbb{T}^* = \{0, 1, 2\}$ . As expected the  
 484 triplet  $(y^*, v^*, \mathbb{T}^*)$  is admissible for the classical optimal control problem with parameter

$$\begin{aligned} & \text{minimize} && -(y_2^1(1) - 2)^3 - \rho y_2^2(1), \\ & \text{subject to} && (y, v, \mathbb{T}) \in \text{AC}([0, 1], \mathbb{R}^4) \times \text{L}^\infty([0, 1], \mathbb{R}^2) \times \mathbb{R}^3, \\ & && \dot{y}_1^1(s) = \tau_1, \quad \text{a.e. } s \in [0, 1], \\ & && \dot{y}_1^2(s) = \tau_1((1 - y_1^1(s))^+)^2, \quad \text{a.e. } s \in [0, 1], \\ 485 \text{ (CP}_{\text{ex}}^*) & && \dot{y}_2^1(s) = (2 - \tau_1)v_2(s), \quad \text{a.e. } s \in [0, 1], \\ & && \dot{y}_2^2(s) = (2 - \tau_1)((1 - y_2^1(s))^+)^2, \quad \text{a.e. } s \in [0, 1], \\ & && y_1^1(0) = 0, \quad y_1^2(0) = 0, \quad y_1^1(1) - 1 = 0, \\ & && y_2^1(0) - y_1^1(1) = 0, \quad y_2^2(0) - y_1^2(1) = 0, \\ & && \tau_0 = 0, \quad \tau_1 \in [0, 2], \quad \tau_2 = 2, \\ & && v_1(s), v_2(s) \in [-1, 1], \quad \text{a.e. } s \in [0, 1], \end{aligned}$$

486 with the cost  $\mathcal{C}^* = -\frac{\rho}{3}$ .

**3.4.3. The triplet  $(y^*, v^*, \mathbb{T}^*)$  is not a  $L^1$ -local solution to Problem  $(\text{CP}_{\text{ex}}^*)$ .**  
 For any  $\varepsilon > 0$  small enough, we introduce the triplet  $(y^\varepsilon, v^\varepsilon, \mathbb{T}^\varepsilon)$  defined by  $(y_1^1)^\varepsilon :=$   
 $(y_1^1)^*$ ,  $(y_1^2)^\varepsilon := (y_1^2)^*$ ,  $v_1^\varepsilon = v_1^*$ ,  $\mathbb{T}^\varepsilon = \mathbb{T}^*$ , and by

$$(y_2^1)^\varepsilon(s) := \begin{cases} s + 1, & \text{if } s \in [0, \varepsilon], \\ 2\varepsilon - s + 1, & \text{if } s \in [\varepsilon, 3\varepsilon], \\ s - 4\varepsilon + 1, & \text{if } s \in [3\varepsilon, 1], \end{cases} \quad v_2^\varepsilon(s) := \begin{cases} 1, & \text{if } s \in [0, \varepsilon], \\ -1, & \text{if } s \in [\varepsilon, 3\varepsilon], \\ 1, & \text{if } s \in [3\varepsilon, 1], \end{cases}$$

and

$$(y_2^2)^\varepsilon(s) := \begin{cases} \frac{1}{3}, & \text{if } s \in [0, 2\varepsilon], \\ \frac{1}{3}((s - 2\varepsilon)^3 + 1), & \text{if } s \in [2\varepsilon, 3\varepsilon], \\ \frac{1}{3}((s - 4\varepsilon)^3 + 2\varepsilon^3 + 1), & \text{if } s \in [3\varepsilon, 4\varepsilon], \\ \frac{1}{3}(2\varepsilon^3 + 1), & \text{if } s \in [4\varepsilon, 1], \end{cases}$$

487 for all  $s \in [0, 1]$ . One can easily conclude that the triplet  $(y^*, v^*, \tau^*)$  is not a  $L^1$ -local  
 488 solution to Problem  $(\text{CP}_{\text{ex}}^*)$  since:

- 489 – The triplet  $(y^\varepsilon, v^\varepsilon, \tau^\varepsilon)$  is admissible for Problem  $(\text{CP}_{\text{ex}}^*)$  for any  $\varepsilon > 0$ .
- 490 – It holds that  $\lim_{\varepsilon \rightarrow 0} (\|y^\varepsilon - y^*\|_{\text{C}} + \|v^\varepsilon - v^*\|_{\text{L}^1} + \|\mathbb{T}^\varepsilon - \mathbb{T}^*\|_{\mathbb{R}^3}) = 0$ .
- For any  $\varepsilon > 0$ , the cost  $\mathcal{C}^\varepsilon$  associated with the triplet  $(y^\varepsilon, v^\varepsilon, \mathbb{T}^\varepsilon)$  is given by

$$\mathcal{C}^\varepsilon = -\frac{\rho}{3} - \left( \frac{2\rho}{3} - 64 \right) \varepsilon^3 < -\frac{\rho}{3} = \mathcal{C}^*.$$

491 **Appendix A. Proof of Proposition 3.1.** Consider the framework of Propo-  
 492 sition 3.1 and let us prove that the triplet  $(y^*, v^*, \mathbb{T}^*)$  is a  $L^1_{[\varepsilon, 1-\varepsilon]}$ -local solution to

493 Problem (CP\*) for any  $0 < \varepsilon < \frac{1}{2}$ . Therefore let  $0 < \varepsilon < \frac{1}{2}$  and  $R \geq \|v^*\|_{L^\infty}$ . Our aim  
 494 is to prove that there exists  $\eta > 0$  such that  $\phi^*(y^*(0), y^*(1), \mathbb{T}^*) \leq \phi^*(y(0), y(1), \mathbb{T})$  for  
 495 any triplet  $(y, v, \mathbb{T})$  that is admissible for Problem (CP\*) and satisfying

$$496 \quad (\text{A.1}) \quad \begin{cases} \|y - y^*\|_{\mathbb{C}} + \|v - v^*\|_{L^1} + \|\mathbb{T} - \mathbb{T}^*\|_{\mathbb{R}^{N+1}} \leq \eta, \\ \|v\|_{L^\infty} \leq R, \\ v(s) = v^*(s) \text{ a.e. } s \in [0, \varepsilon] \cup [1 - \varepsilon, 1]. \end{cases}$$

497 To this aim we need to introduce several technical positive parameters:

498  $(\mathfrak{P}_1)$  Let  $\underline{\theta} := \min_{k \in \{1, \dots, N\}} |\tau_k^* - \tau_{k-1}^*| > 0$  and  $\bar{\theta} := \max_{k \in \{1, \dots, N\}} |\tau_k^* - \tau_{k-1}^*| > 0$ .

$(\mathfrak{P}_2)$  From the transverse conditions (see Definition 3.2) and the (uniform) con-  
 tinuities of the functions  $\nabla F_k^*$  and  $f_k^*$  on compact sets, there exist  $0 < \nu \leq$   
 $\min_{k \in \{1, \dots, N-1\}} \nu_k^*$  and  $0 < \alpha \leq \min\{\frac{\theta}{3}, \min_{k \in \{1, \dots, N-1\}} \alpha_k^*\}$  such that

$$\begin{cases} \langle \nabla F_k^*(z), f_k^*(z, u^*(t)) \rangle_{\mathbb{R}^n} > 0, & \text{a.e. } t \in [\tau_k^* - \alpha, \tau_k^*), \\ \langle \nabla F_k^*(z), f_{k+1}^*(z, u^*(t)) \rangle_{\mathbb{R}^n} > 0, & \text{a.e. } t \in (\tau_k^*, \tau_k^* + \alpha], \end{cases}$$

499 for all  $z \in \bar{\mathbb{B}}_{\mathbb{R}^n}(x^*(\tau_k^*), \nu)$  and all  $k \in \{1, \dots, N-1\}$ .

500  $(\mathfrak{P}_3)$  From continuity of  $y^*$  over  $[0, 1]$ , there exists  $0 < \chi < \frac{1}{2}$  such that  $\|y_k^*(s) -$   
 501  $y_k^*(0)\|_{\mathbb{R}^n} \leq \frac{\nu}{2}$  for all  $s \in [0, \chi]$  and  $\|y_k^*(s) - y_k^*(1)\|_{\mathbb{R}^n} \leq \frac{\nu}{2}$  for all  $s \in [1 - \chi, 1]$ ,  
 502 for all  $k \in \{1, \dots, N\}$ .

503  $(\mathfrak{P}_4)$  Define  $\gamma := \frac{\theta}{3} \min\{\varepsilon, \chi, \frac{\alpha}{\bar{\theta}}\} > 0$  and  $r := \frac{\gamma}{\bar{\theta} + \theta} > 0$ . Note that  $0 < \gamma \leq \alpha \leq \frac{\theta}{3}$   
 504 and  $0 < r < \frac{1}{2}$ .

505  $(\mathfrak{P}_5)$  From continuity of  $y^*$ , from (3.3) and the openness of the regions  $E_k^*$ , there  
 506 exists  $\delta > 0$  such that

$$507 \quad \begin{cases} \bar{\mathbb{B}}_{\mathbb{R}^n}(y_1^*(s), \delta) \subset E_1^*, & \forall s \in [0, 1 - r], \\ \bar{\mathbb{B}}_{\mathbb{R}^n}(y_k^*(s), \delta) \subset E_k^*, & \forall s \in [r, 1 - r], \quad \forall k \in \{2, \dots, N-1\}, \\ \bar{\mathbb{B}}_{\mathbb{R}^n}(y_N^*(s), \delta) \subset E_N^*, & \forall s \in [r, 1]. \end{cases}$$

508 We are now in a position to continue the proof. To this aim let  $\eta := \min\{\frac{\theta}{3}, \frac{\nu}{2}, \delta\} > 0$   
 509 and  $(y, v, \mathbb{T})$  be an admissible triplet for Problem (CP\*) satisfying (A.1). Our aim is  
 510 to prove that  $\phi^*(y^*(0), y^*(1), \mathbb{T}^*) \leq \phi^*(y(0), y(1), \mathbb{T})$ .

511 *Step 1.* Since  $0 = \tau_0^* < \tau_1^* < \dots < \tau_{N-1}^* < \tau_N^* = T$  and  $\mathbb{T} \in \Delta$  with  $\|\mathbb{T} -$   
 512  $\mathbb{T}^*\|_{\mathbb{R}^{N+1}} \leq \eta \leq \frac{\theta}{3}$ , one can easily deduce that  $0 = \tau_0 < \tau_1 < \dots < \tau_{N-1} < \tau_N = T$ .  
 513 Therefore we are in a position to define  $(x, u) \in \text{AC}([0, T], \mathbb{R}^n) \times L^\infty([0, T], \mathbb{R}^m)$  by

$$514 \quad (\text{A.2}) \quad x(t) := y_k \left( \frac{t - \tau_{k-1}}{\tau_k - \tau_{k-1}} \right) \quad \text{and} \quad u(t) := v_k \left( \frac{t - \tau_{k-1}}{\tau_k - \tau_{k-1}} \right),$$

515 for all  $t \in [\tau_{k-1}, \tau_k]$  and all  $k \in \{1, \dots, N\}$ . Note that  $x$  is well defined since  $y_{k+1}(0) =$   
 516  $y_k(1)$  for all  $k \in \{2, \dots, N\}$  (from admissibility of the triplet  $(y, v, \mathbb{T})$ ). Observe  
 517 that  $(y_1(0), y_N(1)) = (x(0), x(T))$  and recall that  $(y_1^*(0), y_N^*(1)) = (x^*(0), x^*(T))$ .  
 518 Therefore, from the definition of  $\phi^*$  (see Section 3.2) and since  $(x^*, u^*)$  is a global  
 519 solution to Problem (HP), to obtain that  $\phi^*(y^*(0), y^*(1), \mathbb{T}^*) \leq \phi^*(y(0), y(1), \mathbb{T})$ , we  
 520 only need to prove that the pair  $(x, u)$  is admissible for Problem (HP).

521 From admissibility of the triplet  $(y, v, \mathbb{T})$ , it is clear that  $g(x(0), x(T)) \in S$   
 522 and  $u(t) \in U$  for almost every  $t \in [0, T]$ . Therefore it only remains to prove that  $(x, u)$   
 523 is a solution to the hybrid control system (HS) (see Definition 3.1). From (A.2) and  
 524 the admissibility of the triplet  $(y, v, \mathbb{T})$ , one can easily obtain that

$$525 \quad (\text{A.3}) \quad \dot{x}(t) = f_k^*(x(t), u(t)), \quad \text{a.e. } t \in (\tau_{k-1}, \tau_k),$$



for all  $k \in \{1, \dots, N\}$ . Therefore, to conclude the proof, we only need to prove that

$$\begin{cases} x(t) \in E_1^*, & \forall t \in [\tau_0, \tau_1), \\ x(t) \in E_k^*, & \forall t \in (\tau_{k-1}, \tau_k), \quad \forall k \in \{2, \dots, N-1\}, \\ x(t) \in E_N^*, & \forall t \in (\tau_{N-1}, \tau_N]. \end{cases}$$

526 This is exactly our goal in the next two steps.

*Step 2.* Since  $\|\mathbb{T} - \mathbb{T}^*\|_{\mathbb{R}^{N+1}} \leq \eta \leq \frac{\theta}{3}$ , note that  $\tau_k - \tau_{k-1} \leq \bar{\theta} + 2\eta \leq \bar{\theta} + \underline{\theta}$  for all  $k \in \{1, \dots, N\}$ . Hence, since moreover  $r := \frac{\gamma}{\bar{\theta} + \underline{\theta}}$ , observe that

$$\begin{cases} \frac{t - \tau_0}{\tau_1 - \tau_0} \in [0, 1 - r], & \forall t \in [\tau_0, \tau_1 - \gamma], \\ \frac{t - \tau_{k-1}}{\tau_k - \tau_{k-1}} \in [r, 1 - r], & \forall t \in [\tau_{k-1} + \gamma, \tau_k - \gamma], \quad \forall k \in \{2, \dots, N-1\}, \\ \frac{t - \tau_{N-1}}{\tau_N - \tau_{N-1}} \in [r, 1], & \forall t \in [\tau_{N-1} + \gamma, \tau_N]. \end{cases}$$

As a consequence, from (A.2) and  $(\mathfrak{P}_5)$ , and since  $\|y_k - y_k^*\|_C \leq \|y - y^*\|_C \leq \eta \leq \delta$ , one can easily obtain that

$$\begin{cases} x(t) \in E_1^*, & \forall t \in [\tau_0, \tau_1 - \gamma], \\ x(t) \in E_k^*, & \forall t \in [\tau_{k-1} + \gamma, \tau_k - \gamma], \quad \forall k \in \{2, \dots, N-1\}, \\ x(t) \in E_N^*, & \forall t \in [\tau_{N-1} + \gamma, \tau_N]. \end{cases}$$

527 Therefore, to conclude the proof, it only remains to prove that  $x(t) \in E_k^*$  for all  $t \in$   
528  $[\tau_k - \gamma, \tau_k)$  and  $x(t) \in E_{k+1}^*$  for all  $t \in (\tau_k, \tau_k + \gamma]$ , for all  $k \in \{1, \dots, N-1\}$ . This is  
529 the objective of the following last step.

530 *Step 3.* Let us start with two observations. First, since  $\|\mathbb{T} - \mathbb{T}^*\|_{\mathbb{R}^{N+1}} \leq \eta \leq \frac{\theta}{3}$ , it  
531 holds that  $|\tau_k - \tau_{k-1}| \geq \frac{\theta}{3}$  for all  $k \in \{1, \dots, N\}$ . Second, since  $\gamma := \frac{\theta}{3} \min\{\varepsilon, \chi, \frac{\alpha}{\bar{\theta}}\}$ ,  
532 one can get that

$$533 \frac{t - \tau_{k-1}}{\tau_k - \tau_{k-1}} \in [1 - \varepsilon, 1], \quad \frac{t - \tau_{k-1}}{\tau_k - \tau_{k-1}} \in [1 - \chi, 1], \quad \tau_{k-1}^* + (\tau_k^* - \tau_{k-1}^*) \frac{t - \tau_{k-1}}{\tau_k - \tau_{k-1}} \in [\tau_k^* - \alpha, \tau_k^*],$$

534 for all  $t \in [\tau_k - \gamma, \tau_k]$  and all  $k \in \{1, \dots, N-1\}$ . We deduce the following results:

- 535 (i) Since  $v_k(s) = v_k^*(s)$  for almost every  $s \in [1 - \varepsilon, 1]$ , one can easily obtain  
536 from (A.2) and (3.1) that  $u(t) = u^*(\tau_{k-1}^* + (\tau_k^* - \tau_{k-1}^*) \frac{t - \tau_{k-1}}{\tau_k - \tau_{k-1}})$ , with  $\tau_{k-1}^* +$   
537  $(\tau_k^* - \tau_{k-1}^*) \frac{t - \tau_{k-1}}{\tau_k - \tau_{k-1}} \in [\tau_k^* - \alpha, \tau_k^*]$ , for almost every  $t \in [\tau_k - \gamma, \tau_k]$  and  
538 all  $k \in \{1, \dots, N-1\}$ .
- 539 (ii) Since  $\|y_k - y_k^*\|_C \leq \|y - y^*\|_C \leq \eta \leq \frac{\nu}{2}$ , one can easily obtain from (A.2),  
540 from the equality  $x^*(\tau_k^*) = y_k^*(1)$  and from  $(\mathfrak{P}_3)$  that  $x(t) \in \bar{B}_{\mathbb{R}^n}(x^*(\tau_k^*), \nu)$   
541 for all  $t \in [\tau_k - \gamma, \tau_k]$  and all  $k \in \{1, \dots, N-1\}$ .
- (iii) We obtain from (A.3), from the previous two items and from  $(\mathfrak{P}_2)$  that the  
derivative of  $F_k^* \circ x$  satisfies

$$\left\langle \nabla F_k^*(x(t)), f_k^* \left( x(t), u^* \left( \tau_{k-1}^* + (\tau_k^* - \tau_{k-1}^*) \frac{t - \tau_{k-1}}{\tau_k - \tau_{k-1}} \right) \right) \right\rangle_{\mathbb{R}^n} > 0,$$

542 for almost every  $t \in [\tau_k - \gamma, \tau_k)$  and all  $k \in \{1, \dots, N-1\}$ . From admissibility  
543 of the triplet  $(y, v, \mathbb{T})$  and (A.2), we know that  $F_k^*(x(\tau_k)) = F_k^*(y_k(1)) = 0$   
544 for all  $k \in \{1, \dots, N-1\}$ . As a consequence we obtain that  $F_k^*(x(t)) < 0$  for  
545 all  $t \in [\tau_k - \gamma, \tau_k)$  which implies from Definition 3.2, since  $x(t) \in \bar{B}_{\mathbb{R}^n}(x^*(\tau_k^*), \nu)$   
546 and  $\nu \leq \nu_k^*$ , that  $x(t) \in E_k^*$  for all  $t \in [\tau_k - \gamma, \tau_k)$  and all  $k \in \{1, \dots, N-1\}$ .

547 Following the same strategy one can obtain that  $x(t) \in E_{k+1}^*$  for all  $t \in (\tau_k, \tau_k + \gamma]$   
 548 and all  $k \in \{1, \dots, N-1\}$ . The proof of Proposition 3.1 is complete.

549 **Appendix B. Proof of Theorem 3.1.** Consider the framework of Theorem 3.1.  
 550 From Proposition 3.1, the corresponding triplet  $(y^*, v^*, \mathbb{T}^*)$  constructed in Section 3.2  
 551 is a  $L^1_{\square}$ -local solution to Problem (CP\*). Before applying Theorem 2.1, we need to  
 552 verify that  $g^*$  is submersive at  $(y^*(0), y^*(1), \mathbb{T}^*)$ . From the definition of the function  $g^*$   
 553 (see Section 3.2), note that the matrix  $\nabla g^*(y^*(0), y^*(1), \mathbb{T}^*) \in \mathbb{R}^{(nN+nN+(N+1)) \times \ell^*}$  is

$$554 \begin{array}{c} \left[ \begin{array}{c|c|c|c} \nabla_1 g(y_1^*(0), y_N^*(1)) & 0_{\mathbb{R}^{n \times n(N-1)}} & 0_{\mathbb{R}^{n \times (N-1)}} & 0_{\mathbb{R}^{n \times (N+1)}} \\ \hline 0_{\mathbb{R}^{n(N-1) \times \ell}} & \text{Id}_{\mathbb{R}^{n(N-1) \times n(N-1)}} & 0_{\mathbb{R}^{n(N-1) \times (N-1)}} & 0_{\mathbb{R}^{n(N-1) \times (N+1)}} \\ \hline 0_{\mathbb{R}^{n(N-1) \times \ell}} & -\text{Id}_{\mathbb{R}^{n(N-1) \times n(N-1)}} & \begin{array}{c} \nabla F_1^*(y_1^*(1)) \\ \vdots \\ \nabla F_{N-1}^*(y_{N-1}^*(1)) \end{array} & 0_{\mathbb{R}^{n(N-1) \times (N+1)}} \\ \hline \nabla_2 g(y_1^*(0), y_N^*(1)) & 0_{\mathbb{R}^{n \times n(N-1)}} & 0_{\mathbb{R}^{n \times (N-1)}} & 0_{\mathbb{R}^{n \times (N+1)}} \\ \hline 0_{\mathbb{R}^{(N+1) \times \ell}} & 0_{\mathbb{R}^{(N+1) \times n(N-1)}} & 0_{\mathbb{R}^{(N+1) \times (N-1)}} & \text{Id}_{\mathbb{R}^{(N+1) \times (N+1)}} \end{array} \right] \end{array}$$

555 From Definition 3.2, it holds that  $\nabla F_k^*(y_k^*(1)) = \nabla F_k^*(x^*(\tau_k^*)) \neq 0_{\mathbb{R}^n}$  for all  $k \in$   
 556  $\{1, \dots, N-1\}$ . Since  $g$  is submersive at  $(x^*(0), x^*(T)) = (y_1^*(0), y_N^*(1))$ , one can easily  
 557 conclude that  $g^*$  is submersive at  $(y^*(0), y^*(1), \mathbb{T}^*)$ .

**B.1. Application of Theorem 2.1.** Let us introduce the Hamiltonian  $\mathcal{H} : \mathbb{R}^{nN} \times \mathbb{R}^{mN} \times \mathbb{R}^{N+1} \times \mathbb{R}^{nN} \rightarrow \mathbb{R}$  associated with Problem (CP\*) given by

$$\mathcal{H}(y, v, \mathbb{T}, q) := \langle q, f^*(y, v, \mathbb{T}) \rangle_{\mathbb{R}^{nN}} = \sum_{k=1}^N \langle q_k, (\tau_k - \tau_{k-1}) f_k^*(y_k, v_k) \rangle_{\mathbb{R}^n},$$

558 for all  $y = (y_1, \dots, y_N) \in \mathbb{R}^{nN}$ ,  $v = (v_1, \dots, v_N) \in \mathbb{R}^{mN}$ ,  $\mathbb{T} = \{\tau_0, \dots, \tau_N\} \in \mathbb{R}^{N+1}$   
 559 and  $q = (q_1, \dots, q_N) \in \mathbb{R}^{nN}$ . From Theorem 2.1, there exists a nontrivial pair  $(q, q^0) \in$   
 560  $\text{AC}([0, 1], \mathbb{R}^{nN}) \times \mathbb{R}_+$  satisfying:

- 561 (i) the Hamiltonian system  $\dot{y}^*(s) = \nabla_q \mathcal{H}(y^*(s), v^*(s), \mathbb{T}^*, q(s))$  and  $-\dot{q}(s) =$   
 562  $\nabla_y \mathcal{H}(y^*(s), v^*(s), \mathbb{T}^*, q(s))$  for almost every  $s \in [0, 1]$ ;  
 563 (ii) the endpoint transversality condition

$$564 \left( \begin{array}{c} q(0) \\ -q(1) \\ \int_0^1 \nabla_{\mathbb{T}} \mathcal{H}(y^*(s), v^*(s), \mathbb{T}^*, q(s)) ds \end{array} \right) = q^0 \nabla \phi^*(y^*(0), y^*(1), \mathbb{T}^*) + \nabla g^*(y^*(0), y^*(1), \mathbb{T}^*) \tilde{\xi},$$

- 565 for some  $\tilde{\xi} \in N_{S^*}[g^*(y^*(0), y^*(1), \mathbb{T}^*)]$ ;  
 566 (iii) the Hamiltonian maximization condition  $v^*(s) \in \arg \max_{\tilde{\omega} \in U^N} \mathcal{H}(y^*(s), \tilde{\omega}, \mathbb{T}^*, q(s))$   
 567 for almost every  $s \in [0, 1]$ .

**B.2. Introduction of the nontrivial pair  $(p, p^0)$ .** Since the pair  $(q, q^0)$  is not  
 trivial, it is clear that the pair  $(p, p^0) \in \text{PAC}_{\mathbb{T}^*}([0, T], \mathbb{R}^n) \times \mathbb{R}_+$  defined by  $p^0 := q^0$

and

$$p(t) := \begin{cases} q_1 \left( \frac{t-\tau_0^*}{\tau_1^*-\tau_0^*} \right), & \forall t \in [\tau_0^*, \tau_1^*), \\ q_k \left( \frac{t-\tau_{k-1}^*}{\tau_k^*-\tau_{k-1}^*} \right), & \forall t \in (\tau_{k-1}^*, \tau_k^*), \quad \forall k \in \{2, \dots, N-1\}, \\ q_N \left( \frac{t-\tau_{N-1}^*}{\tau_N^*-\tau_{N-1}^*} \right), & \forall t \in (\tau_{N-1}^*, \tau_N^*], \end{cases}$$

is not trivial.

**B.3. Hamiltonian system and Hamiltonian maximization condition of Theorem 3.1.** From the above Items (i) and (iii) and from (3.2), the Hamiltonian system and the Hamiltonian maximization condition of Theorem 3.1 are satisfied.

**B.4. Endpoint transversality condition of Theorem 3.1.** From the definitions of  $g^*$  and  $S^*$  (see Section 3.2) and since  $\tilde{\xi} \in N_{S^*}[g^*(y^*(0), y^*(1), \mathbb{T}^*)]$ , we can write  $\tilde{\xi} := (\xi, \xi^2, \xi^3, \xi^4) \in \mathbb{R}^\ell \times \mathbb{R}^{n(N-1)} \times \mathbb{R}^{N-1} \times \mathbb{R}^{N+1}$  with

$$\xi \in N_S[g(y_1^*(0), y_N^*(1))] \quad \text{and} \quad \xi^4 \in N_\Delta[\mathbb{T}^*].$$

Since  $(y_1^*(0), y_N^*(1)) = (x^*(0), x^*(T))$ , note that  $\xi \in N_S[g(x^*(0), x^*(T))]$ . Furthermore, from the first two components of the above Item (ii), from the expression of  $\nabla g^*(y^*(0), y^*(1), \mathbb{T}^*)$  given at the beginning of Appendix B and from the expression of  $\nabla \phi^*(y^*(0), y^*(1), \mathbb{T}^*)$  (see Section 3.2 for the definition of  $\phi^*$ ), we obtain that

$$p(0) = q_1(0) = q^0 \nabla_1 \phi(y_1^*(0), y_N^*(1)) + \nabla_1 g(y_1^*(0), y_N^*(1)) \xi \\ = p^0 \nabla_1 \phi(x^*(0), x^*(T)) + \nabla_1 g(x^*(0), x^*(T)) \xi,$$

and

$$-p(T) = -q_N(1) = q^0 \nabla_2 \phi(y_1^*(0), y_N^*(1)) + \nabla_2 g(y_1^*(0), y_N^*(1)) \xi \\ = p^0 \nabla_2 \phi(x^*(0), x^*(T)) + \nabla_2 g(x^*(0), x^*(T)) \xi.$$

Therefore the endpoint transversality condition of Theorem 3.1 is proved.

**B.5. Discontinuity condition of Theorem 3.1.** From the first two components of the above Item (ii), from the expression of  $\nabla g^*(y^*(0), y^*(1), \mathbb{T}^*)$  given at the beginning of Appendix B and from the expression of  $\nabla \phi^*(y^*(0), y^*(1), \mathbb{T}^*)$  (see Section 3.2 for the definition of  $\phi^*$ ), we obtain that

$$\forall k \in \{2, \dots, N\}, \quad q_k(0) = \xi_{k-1}^2 \\ \text{and} \quad \forall k \in \{1, \dots, N-1\}, \quad -q_k(1) = -\xi_k^2 + \xi_k^3 \nabla F_k^*(y_k^*(1)).$$

We deduce that

$$p^+(\tau_k^*) - p^-(\tau_k^*) = q_{k+1}(0) - q_k(1) = \xi_k^3 \nabla F_k^*(y_k^*(1)) = \xi_k^3 \nabla F_k^*(x^*(\tau_k^*)),$$

for all  $k \in \{1, \dots, N-1\}$ . Therefore the discontinuity condition of Theorem 3.1 is satisfied with  $\sigma_k := \xi_k^3$  for all  $k \in \{1, \dots, N-1\}$ .

**B.6. Hamiltonian constancy condition of Theorem 3.1.** From the Hamiltonian system and the maximization condition and applying [21, Theorem 2.6.1] on each interval  $[\tau_{k-1}^*, \tau_k^*]$ , we obtain that, for all  $k \in \{1, \dots, N\}$ , there exists a constant  $c_k \in \mathbb{R}$  such that  $\langle p(t), f_k^*(x^*(t), u^*(t)) \rangle_{\mathbb{R}^n} = c_k$  for almost every  $t \in [\tau_{k-1}^*, \tau_k^*]$ .

602 Furthermore, from the definition of  $\Delta$  (see Section 3.2) and since  $0 = \tau_0^* < \tau_1^* < \dots <$   
 603  $\tau_{N-1}^* < \tau_N^* = T$ , we deduce from  $\xi^4 \in N_\Delta[\mathbb{T}^*]$  that all components of  $\xi^4$  are zero,  
 604 except possibly the first and last components. Thus, from the third component of the  
 605 above Item (ii), from the expression of  $\nabla g^*(y^*(0), y^*(1), \mathbb{T}^*)$  given at the beginning of  
 606 Appendix B and from the expression of  $\nabla \phi^*(y^*(0), y^*(1), \mathbb{T}^*)$  (see Section 3.2 for the  
 607 definition of  $\phi^*$ ), we obtain that

$$608 \int_0^1 \langle q_{k+1}(s), f_{k+1}^*(y_{k+1}^*(s), v_{k+1}^*(s)) \rangle_{\mathbb{R}^n} ds = \int_0^1 \langle q_k(s), f_k^*(y_k^*(s), v_k^*(s)) \rangle_{\mathbb{R}^n} ds,$$

609 for all  $k \in \{1, \dots, N-1\}$ . From affine changes of time variable, we obtain that

$$610 \frac{1}{\tau_{k+1}^* - \tau_k^*} \int_{\tau_k^*}^{\tau_{k+1}^*} \langle p(t), f_{k+1}^*(x^*(t), u^*(t)) \rangle_{\mathbb{R}^n} dt = \frac{1}{\tau_k^* - \tau_{k-1}^*} \int_{\tau_{k-1}^*}^{\tau_k^*} \langle p(t), f_k^*(x^*(t), u^*(t)) \rangle_{\mathbb{R}^n} dt,$$

611 for all  $k \in \{1, \dots, N-1\}$ . From constancy of the above two integrands, we deduce  
 612 that  $c_{k+1} = c_k$  for all  $k \in \{1, \dots, N-1\}$ . Therefore the Hamiltonian constancy  
 613 condition is satisfied and the proof of Theorem 3.1 is complete.

614 **Appendix C. Nonadmissibility of needle-like perturbations.** Here we  
 615 prove that needle-like perturbations of the control are not admissible (in a sense to  
 616 precise) in our setting of spatially heterogeneous dynamics. This is a major difference  
 617 with respect to the classical optimal control theory. Consider the one-dimensional  
 618 case  $n = 1$ , the state space partition  $\mathbb{R} = \overline{X_1} \cup \overline{X_2}$ , where  $X_1 := (-\infty, 1)$  and  
 619  $X_2 := (1, +\infty)$ , and the hybrid control system given by

$$620 (\text{HS}_{\text{ex}}) \quad \dot{x}(t) = h(x(t), u(t)), \quad \text{for a.e. } t \in [0, 2],$$

621 where the spatially heterogeneous dynamics  $h : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is defined by  $h(x, u) := u$   
 622 if  $x \in X_1$  and by  $h(x, u) := -u$  if  $x \in X_2$ . Now consider the trajectory  $x$  given  
 623 by  $x(t) = t$  for all  $t \in [0, 2]$  and the corresponding control  $u$  given by  $u(t) = 1$   
 624 over  $[0, 1]$  and  $u(t) = -1$  over  $(1, 2]$ . Note that all conditions from Definitions 3.1  
 625 and 3.2 are satisfied, with  $\tau_1 = 1$  as unique crossing time. For any small  $\alpha > 0$ ,  
 626 denote by  $x_\alpha$  the solution to the hybrid control system  $(\text{HS}_{\text{ex}})$  associated with the  
 627 initial condition  $x_\alpha(0) = x(0) = 0$  and the needle-like perturbation  $u_\alpha : [0, 2] \rightarrow \mathbb{R}$   
 628 of  $u$  defined by  $u_\alpha(t) = -1$  over  $(\frac{1}{2} - \alpha, \frac{1}{2}]$  and by  $u_\alpha(t) = u(t)$  elsewhere. Then the  
 629 perturbed trajectory  $x_\alpha$  satisfies  $x_\alpha(t) \in \overline{X_1}$  over the whole interval  $[0, 2]$  and thus  $x_\alpha$   
 630 does not uniformly converge to  $x$  over  $[0, 2]$  when  $\alpha \rightarrow 0$ .

631

## REFERENCES

- 632 [1] G. BARLES, A. BRIANI, AND E. TRÉLAT, *Value function for regional control problems via*  
 633 *dynamic programming and Pontryagin maximum principle*, Math. Control Relat. Fields, 8  
 634 (2018), pp. 509–533.  
 635 [2] T. BAYEN, A. BOUALI, AND L. BOURDIN, *Hybrid maximum principle with regionally switching*  
 636 *parameter*, submitted, hal-03638701, (2022).  
 637 [3] T. BAYEN, A. BOUALI, AND L. BOURDIN, *Optimal control problems with non-control regions:*  
 638 *necessary optimality conditions*, IFAC-PapersOnLine, 55 (2022), pp. 68–73.  
 639 [4] T. BAYEN, A. BOUALI, AND L. BOURDIN, *Minimum time problem for the double integrator with*  
 640 *a loss control region*, submitted, hal-03928967v2, (2023).  
 641 [5] T. BAYEN AND L. PFEIFFER, *Second-order analysis for the time crisis problem*, J. Convex Anal.,  
 642 27 (2020), pp. 139–163.  
 643 [6] M. BERGOUNIOUX AND L. BOURDIN, *Pontryagin maximum principle for general Caputo fractional*  
 644 *optimal control problems with Bolza cost and terminal constraints*, ESAIM Control Optim.  
 645 Calc. Var., 26 (2020).

- 646 [7] P. BETTIOL AND L. BOURDIN, *Pontryagin maximum principle for state constrained optimal*  
647 *sampled-data control problems on time scales*, ESAIM Control Optim. Calc. Var., 27 (2021).  
648 [8] V. BOLTYANSKI, *The maximum principle for variable structure systems*, Internat. J. Control, 77  
649 (2004), pp. 1445–1451.  
650 [9] J. F. BONNANS, *Course on optimal control*, OROC Ensta Paris-Tech and optimization master,  
651 U. Paris-Saclay, (2017).  
652 [10] J. F. BONNANS AND C. S. F. DE LA VEGA, *Optimal control of state constrained integral equations*,  
653 *Set-Valued Var. Anal.*, 18 (2010), pp. 307–326.  
654 [11] J. F. BONNANS, X. DUPUIS, AND L. PFEIFFER, *Second-order necessary conditions in Pontryagin*  
655 *form for optimal control problems*, SIAM J. Control Optim., 52 (2014), pp. 3887–3916.  
656 [12] L. BOURDIN, *Note on Pontryagin maximum principle with running state constraints and*  
657 *smooth dynamics—proof based on the Ekeland variational principle*, Research notes,  
658 arXiv:1604.04051, (2016).  
659 [13] L. BOURDIN AND E. TRÉLAT, *Pontryagin maximum principle for finite dimensional nonlinear*  
660 *optimal control problems on time scales*, SIAM Journal on Control and Optimization, 51  
661 (2013), pp. 3781–3813.  
662 [14] A. BRESSAN AND Y. HONG, *Optimal control problems on stratified domains*, Netw. Heterog.  
663 *Media*, 2 (2007), pp. 313–331.  
664 [15] A. BRESSAN AND B. PICCOLI, *Introduction to the mathematical theory of control*, vol. 2 of AIMS  
665 *Ser. Appl. Math.*, Springfield, MO, 2007.  
666 [16] L. CESARI, *Lagrange and Bolza Problems of optimal control and other problems*, Springer, New  
667 *York*, 1983, p. 196–205.  
668 [17] F. CLARKE, *Functional analysis, calculus of variations and optimal control*, vol. 264 of Grad.  
669 *Texts in Math.*, Springer, London, 2013.  
670 [18] F. H. CLARKE AND R. B. VINTER, *Optimal multiprocesses*, SIAM J. Control Optim., 27 (1989),  
671 pp. 1072–1091.  
672 [19] A. DMITRUK AND A. KAGANOVICH, *The hybrid maximum principle is a consequence of Pon-*  
673 *tryagin maximum principle*, Systems Control Lett., 57 (2008), pp. 964–970.  
674 [20] I. EKELAND, *Nonconvex minimization problems*, Bull. Amer. Math. Soc. (N.S.), 1 (1979),  
675 pp. 443–474.  
676 [21] H. O. FATTORINI, *Infinite dimensional optimization and control theory*, Cambridge University  
677 *Press*, 1999.  
678 [22] M. GARAVELLO AND B. PICCOLI, *Hybrid necessary principle*, SIAM J. Control Optim., 43 (2005),  
679 pp. 1867–1887.  
680 [23] T. HABERKORN AND E. TRÉLAT, *Convergence results for smooth regularizations of hybrid*  
681 *nonlinear optimal control problems*, SIAM J. Control Optim., 49 (2011), pp. 1498–1522.  
682 [24] X. J. LI AND J. M. YONG, *Optimal control theory for infinite-dimensional systems*, Systems  
683 *Control Found. Appl.*, Birkhäuser Boston, Inc., Boston, MA, 1995.  
684 [25] A. A. MILYUTIN AND N. P. OSMOLOVSKII, *Calculus of variations and optimal control*, vol. 180  
685 *of Transl. Math. Monogr.*, 1998.  
686 [26] A. PAKNIYAT AND P. E. CAINES, *On the hybrid minimum principle: The hamiltonian and*  
687 *adjoint boundary conditions*, IEEE Trans. Automat. Control, 66 (2020), pp. 1246–1253.  
688 [27] L. S. PONTRYAGIN, V. G. BOLTYANSKII, R. V. GAMKRELIDZE, AND E. F. MISHCHENKO, *The*  
689 *mathematical theory of optimal processes*, A Pergamon Press Book. The Macmillan Co.,  
690 *New York*, 1964.  
691 [28] P. RIEDINGER, C. LUNG, AND F. KRATZ, *An optimal control approach for hybrid systems*, Eur.  
692 *J. Control*, 9 (2003), pp. 449–458.  
693 [29] M. S. SHAIKH AND P. E. CAINES, *On the optimal control of hybrid systems: optimization of*  
694 *trajectories, switching times, and location schedules*, Springer Berlin Heidelberg, 2003.  
695 [30] M. S. SHAIKH AND P. E. CAINES, *On the hybrid optimal control problem: theory and algorithms*,  
696 *IEEE Trans. Automat. Control*, 52 (2007), pp. 1587–1603.  
697 [31] H. J. SUSSMANN, *A maximum principle for hybrid optimal control problems*, in Proceedings of  
698 *the 38th IEEE Conf. Decis. Control*, vol. 1, 1999, pp. 425–430.  
699 [32] A. J. VAN DER SCHAFT AND J. M. SCHUMACHER, *An introduction to hybrid dynamical systems*,  
700 *vol. 251*, Springer London, 2000.  
701 [33] R. VINTER, *Optimal control*, Systems & Control: Foundations & Applications, Birkhäuser  
702 *Boston, Inc.*, Boston, MA, 2000.  
703 [34] M. I. ZELIKIN AND V. F. BORISOV, *Theory of chattering control*, Systems & Control: Foundations  
704 *& Applications. With applications to astronautics, robotics, economics, and engineering*,  
705 *Birkhäuser Boston, Inc.*, Boston, MA, 1994.