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**To cite this version:**

Loïc Bourdin, Anas Bouali, Térence Bayen. Minimum time problem for the double integrator with a loss control region. 2023. hal-03928967v2

**HAL Id: hal-03928967**

**https://univ-avignon.hal.science/hal-03928967v2**

Preprint submitted on 10 Jan 2023

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# Minimum time problem for the double integrator with a loss control region

Térence Bayen* Anas Bouali† Loïc Bourdin‡

January 10, 2023

**Abstract**

In this paper we address the minimum time problem to reach the origin for the double integrator but, in contrast with the classical version of this problem, the control is constrained to be frozen as long as the corresponding state belongs to a given region of the state space called *loss control region*. This situation prevents switches to occur in the loss control region and, therefore, a new analysis has to be performed. To this aim we prove a Pontryagin maximum principle adapted to this setting. The necessary conditions involve a costate that admits discontinuity jumps at the interface between the loss control region and its complement, and an *averaged Hamiltonian gradient condition* to determine the optimal constant value of the control in the loss control region. The purpose of this paper is to highlight the use of this principle and also the new behaviors that are observed in this new setting, such as the lacks of dynamical programming principle, feedback expression and saturation of the control constraint set.

## 1 Introduction

Geometric control theory is developed since the sixties and it now plays a central role in optimal control theory. Based on the Pontryagin maximum principle [26] and differential geometry, it gathers mathematical tools and methods to determine optimal controls and to synthesize feedback expressions [1, 10, 11, 27]. Several well known examples of optimal control problem illustrate various phenomena observed in that field. We can cite for instance the minimum time problem for the double integrator for which every optimal control is bang-bang with zero or one switching time (depending on the initial condition), or for the harmonic oscillator for which every optimal control is bang-bang with a finite (but possibly large) number of switching times. We should also mention the classical Fuller's problem [29] for which every optimal control is bang-bang with an infinite number of switching times on a finite time interval.

**Objective.** The objective of this paper is to study a variant of the minimum time problem for the double integrator in which the control is constrained to be constant as long as the corresponding state belongs to a given region of the state space. We speak of a *loss control region* (that is, a subset of the state space in which the control is *frozen* to the last assigned value before entering this region, and as long as the state belongs to this region). The consideration of optimal control problems involving loss control regions is motivated by various applications. For instance, in the context of aerospace, this question arises in order to take into account the shadow effect in the low-thrust transfer problem [21, 23]. Other examples arise when considering optimal control problems in the setting of viability theory [2]. Indeed, in order to reduce operating costs, constant controls can be applied whenever the system belongs to a safety zone, typically the viability kernel (see, *e.g.*, time crisis problems [9]). Our choice to focus in this paper on the double integrator is twofold. First, as far as we know, optimal control problems including loss control regions have not been treated in the literature yet. Therefore, the adaptation of an academic problem to this new setting could serve the community to highlight the construction of optimal paths in that context (see, *e.g.*, a related study [18] in

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which the double integrator is investigated under a linear pathwise constraint). Second, we shall see that the analysis of optimal trajectories for the double integrator in this new setting is more involved than in the usual case and it requires the use of an adapted methodology.

**Methodology.** Our methodology is to follow the approach of our previous works [6, 7]. Precisely, first-order necessary optimality conditions (in a Pontryagin form) for hybrid[1] optimal control problems involving regionally switching parameters are obtained in [6]. As a particular case, the paper [7] provides a Pontryagin maximum principle for optimal control problems involving loss control regions. This principle provides a so-called *averaged Hamiltonian gradient condition*[2] to determine the optimal constant value of the control whenever the state belongs to a loss control region, as well as the usual Hamiltonian maximization condition whenever the state belongs to the other regions. Since our framework is related to hybrid optimal control problems, we recall that the costate obtained in the principle admits discontinuity jumps.

It is worth mentioning that the framework in [6, 7] does not allow terminal state constraints. Since the minimal time problem for the double integrator involves an endpoint constraint, we cannot resort to the results of [6, 7]. Therefore, in the present paper, we first prove an adapted version of the Pontryagin maximum principle for a general minimum time problem involving an arbitrary loss control region and endpoint constraints. We refer to Proposition 2.2 whose proof is based on an augmentation procedure in the spirit of [17]. Note that our framework involves a partition of the state space and thus the use of such an augmentation technique requires a careful study to relate a solution to the original problem to a (local) solution to the augmented problem. This is made possible thanks to an hypothesis made on the velocity set at the boundary of the loss control region (in line with the usual *transverse assumptions* found in hybrid settings [6, 8, 17, 23]). We emphasize that Proposition 2.2 is established under quite strong hypotheses (see Remark 2.3 for details). However these hypotheses are all satisfied by the double integrator with a loss control region which constitutes the major focus of the present work. Therefore Proposition 2.2 is sufficient for our purposes in this paper. The extension of Proposition 2.2 to a more general setting should be the subject of further research papers.

**Main result (Theorem 3.1) and new observations.** Applying Proposition 2.2 to the minimum time problem for the double integrator with a loss control region, we prove that every optimal trajectory visits at most once the loss control region and then, thanks to the averaged Hamiltonian gradient condition, we are able to determine the corresponding optimal constant value of the control. The synthesis for each initial condition is given in Theorem 3.1 and the corresponding optimal trajectories are depicted in Figure 5. At this occasion, we observe new behaviors with respect to the classical setting (that is, without loss control region). For example, some optimal trajectories (for different initial conditions) cross each other, which implies that the classical *dynamical programming principle* does not hold true and that the optimal control cannot be expressed as a *feedback*. Furthermore, in contrary again to the classical setting, the optimal control takes *moderated values*, that is, values in the interior of the control constraint set (which is thus not *saturated*). We refer to Remarks 2.2 and 4.1 for details. These new phenomena raise many questions and open new challenges to address (theoretically and/or numerically) optimal control problems with loss control regions in view of applications.

**Organization of the paper.** This paper is organized as follows. In Section 2, we recall the well known solution to the classical minimum time problem for the double integrator. Next we state a version of the Pontryagin maximum principle adapted to a minimum time problem with a loss control region (see Proposition 2.2 whose proof is postponed in Appendix A). In Section 3, our main result (Theorem 3.1) is stated, providing an exact analytical solution to the minimum time problem for the double integrator with a loss control region. Its proof (based on Proposition 2.2) is given immediately after, being divided into several cases arising in the application of Proposition 2.2. Section 4 gives a list of additional comments on Theorem 3.1 and its proof. We conclude with open questions and perspectives about optimal control problems with

[1]Here the terminology *hybrid* means that the control system is described by a regionally heterogeneous dynamics in the spirit of [23].

[2]This necessary condition is well known in the context of *sampled-data controls* (that is, piecewise constant controls). We refer to [12, 13, 14] and references therein. However note that, in the present work, as in [6, 7], the situation is more involved since the constancy intervals of the control depends on the state position, and not on the independent time variable.

loss control regions (such as controllability/reachability issues, existence results, Hamilton-Jacobi-Bellman equation, etc.). Finally, Appendix A contains the proof of Proposition 2.2.

# 2 Preliminaries

Let us start with some basic notations and functional framework. In this paper, for any positive integer $d \in \mathbb{N}^\star$, we denote by $\langle \cdot, \cdot \rangle_{\mathbb{R}^d}$ (resp. $\| \cdot \|_{\mathbb{R}^d}$) the standard inner product (resp. Euclidean norm) of $\mathbb{R}^d$. For any subset $\mathrm{X} \subset \mathbb{R}^d$, we denote by $\overline{\mathrm{X}}$ its closure. Furthermore, for any extended-real number $r \in [1, \infty]$ and any real interval $I \subset \mathbb{R}$, we denote by:

- $\mathrm{L}^r(I, \mathbb{R}^d)$ the usual Lebesgue space of $r$-integrable functions defined on $I$ with values in $\mathbb{R}^d$, endowed with its usual norm $\| \cdot \|_{\mathrm{L}^r}$;
- $\mathrm{C}(I, \mathbb{R}^d)$ the standard space of continuous functions defined on $I$ with values in $\mathbb{R}^d$, endowed with its standard uniform norm $\| \cdot \|_{\mathrm{C}}$;
- $\mathrm{AC}(I, \mathbb{R}^d)$ the subspace of $\mathrm{C}(I, \mathbb{R}^d)$ of absolutely continuous functions.

Now take $I = [0, T]$ for some $T > 0$. Recall that a partition of the interval $[0, T]$ is a set $\mathbb{T} = \{\tau_k\}_{k=0,\dots,N}$ such that $0 = \tau_0 < \tau_1 < \ldots < \tau_{N-1} < \tau_N = T$ for some $N \in \mathbb{N}^\star$. In this paper a function $p : [0, T] \to \mathbb{R}^d$ is said to be *piecewise absolutely continuous*, with respect to a partition $\mathbb{T} = \{\tau_k\}_{k=0,\dots,N}$ of the interval $[0, T]$, if $p$ is continuous at 0 and $T$ and the restriction of $p$ over each open interval $(\tau_{k-1}, \tau_k)$ admits an extension over $[\tau_{k-1}, \tau_k]$ that is absolutely continuous. If so, $p$ admits left and right limits at each $\tau_k$ with $k \in \{1, \dots, N-1\}$, denoted respectively by $p^-(\tau_k)$ and $p^+(\tau_k)$. Finally, in this paper, we denote by:

- $\mathrm{PAC}_{\mathbb{T}}([0, T], \mathbb{R}^d)$ the space of piecewise absolutely continuous functions, with respect to a partition $\mathbb{T}$ of the interval $[0, T]$, with values in $\mathbb{R}^d$.

## 2.1 Reminders on the classical minimum time problem for the double integrator

Recall that the classical minimum time problem for the double integrator [27] is given by

$$\begin{array}{ll} \text{minimize} & T, \\ \text{subject to} & (x, u, T) \in \mathrm{AC}([0, T], \mathbb{R}^2) \times \mathrm{L}^\infty([0, T], \mathbb{R}) \times (0, +\infty), \\ & \dot{x}_1(t) = x_2(t), \quad \text{a.e. } t \in [0, T], \\ & \dot{x}_2(t) = u(t), \quad \text{a.e. } t \in [0, T], \\ & x(0) = x^0, \quad x(T) = 0_{\mathbb{R}^2}, \\ & u(t) \in [-1, 1], \quad \text{a.e. } t \in [0, T], \end{array} \tag{CP}$$

where $x^0 \in \mathbb{R}^2 \backslash \{0_{\mathbb{R}^2}\}$. As usual in the literature, $x = (x_1, x_2) \in \mathrm{AC}([0, T], \mathbb{R}^2)$ is called the *state* (or the *trajectory*), $u \in \mathrm{L}^\infty([0, T], \mathbb{R})$ is called the *control* and $T > 0$ is called the *final time*. Using the classical Filippov approach [20], it can be proved that Problem (CP) admits (at least) one solution. Then, from the classical Pontryagin maximum principle [26], it can be proved that Problem (CP) admits exactly one solution and its description can be separated into four cases according to the position of the initial condition $x^0$ in the partition $\mathbb{R}^2 \backslash \{0_{\mathbb{R}^2}\} = \Gamma_0 \cup \Omega_1 \cup \Gamma_1 \cup \Omega_0$ (see Figure 1) where

$$\Gamma_0 := \left\{ \left( \frac{1}{2} x_2^2, x_2 \right) \mid x_2 < 0 \right\} \quad \text{and} \quad \Gamma_1 := \left\{ \left( -\frac{1}{2} x_2^2, x_2 \right) \mid x_2 > 0 \right\},$$

and where $\Omega_1$ (resp. $\Omega_0$) stands for the strict epigraph (resp. strict hypograph) of $\Gamma_0 \cup \Gamma_1 \cup \{0_{\mathbb{R}^2}\}$.[3]

[3]The notation of the sets $\Gamma_0$, $\Omega_1$, $\Gamma_1$ and $\Omega_0$ may be non-intuitive with regards to other notations found in the literature. However they will be convenient and consistent with the setting developed in the next Section 3 (see Figure 4).

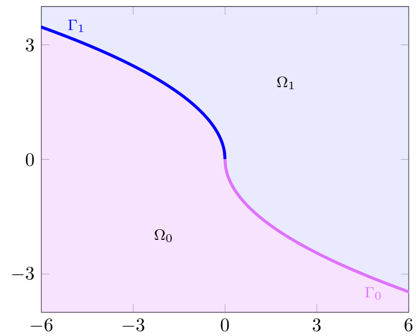


**Figure 1:** Partition of $\mathbb{R}^2\backslash\{0_{\mathbb{R}^2}\}$ arising from the analysis of Problem (CP) (see Proposition 2.1).

Precisely the following well known proposition is established [27].

**Proposition 2.1.** If $(x^\dagger, u^\dagger, T^\dagger)$ is the unique solution to Problem (CP), then an overview description of $(x^\dagger, u^\dagger)$ over the interval $[0, T^\dagger]$, according to the position of the initial condition $x^0$ in the partition $\mathbb{R}^2\backslash\{0_{\mathbb{R}^2}\} = \Gamma_0 \cup \Omega_1 \cup \Gamma_1 \cup \Omega_0$, can be summarized as follows:

| Position of $x^0$ | Overview description of $\begin{matrix} x^\dagger(t) \\ u^\dagger(t) \end{matrix}$ | Figure |
|---|---|---|
| $\Gamma_0$ | $\begin{matrix} \Gamma_0 \\ 1 \end{matrix}$ | 2a |
| $\Omega_1$ | $\begin{matrix} \Omega_1 & \rightsquigarrow & \Gamma_0 \\ -1 & \rightsquigarrow & 1 \end{matrix}$ | 2b |
| $\Gamma_1$ | $\begin{matrix} \Gamma_1 \\ -1 \end{matrix}$ | 2c |
| $\Omega_0$ | $\begin{matrix} \Omega_0 & \rightsquigarrow & \Gamma_1 \\ 1 & \rightsquigarrow & -1 \end{matrix}$ | 2d |

For example, the second case of the above table can be read as follows: if $x^0 \in \Omega_1$, then there exists a *switching time* $\sigma^\dagger \in (0, T^\dagger)$ such that $x^\dagger(t) \in \Omega_1$ and $u^\dagger(t) = -1$ over $(0, \sigma^\dagger)$, and $x^\dagger(t) \in \Gamma_0$ and $u^\dagger(t) = 1$ over $(\sigma^\dagger, T^\dagger)$.

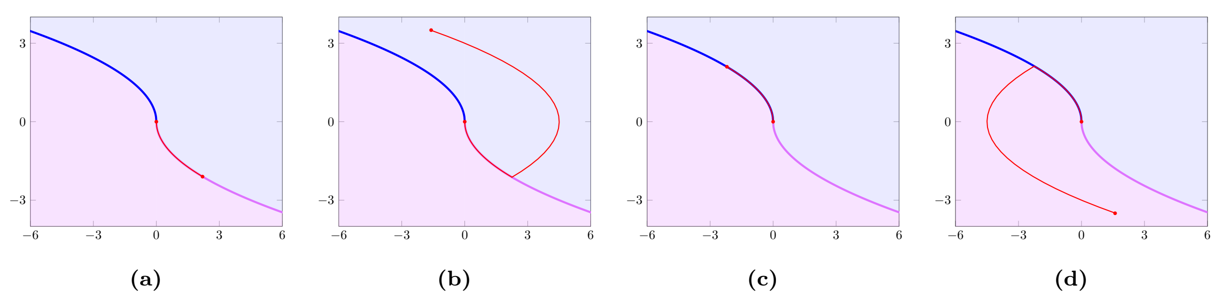


**Figure 2:** Optimal trajectories (in red) in the four cases of Proposition 2.1.

Our objective in the present work is to state and prove a similar result to Proposition 2.1, but when adding a so-called *loss control region* in the control system. We refer to the next Section 2.2 for a general presentation of this new concept and to Section 3 for a specification to the double integrator.

**Remark 2.1.** Note that Proposition 2.1 is not as complete as it could be. Indeed the expressions of the final time $T^\dagger$ and of the (possible) switching time $\sigma^\dagger$ and switching state $x^\dagger(\sigma^\dagger)$, in function of the initial condition $x^0$, are not explicitly provided. Nevertheless these expressions can be easily obtained. To this aim define $\chi(\cdot, x^0, \mu) : \mathbb{R} \to \mathbb{R}^2$ as the unique solution to the control system, associated with the initial condition $x^0 \in \mathbb{R}^2$ and with the control constantly equal to $\mu \in \mathbb{R}$, whose explicit expression is given by

$$\chi(t, x^0, \mu) = \left( x_1^0 + x_2^0 t + \frac{\mu}{2} t^2, x_2^0 + \mu t \right), \tag{E}$$

for all $t \in \mathbb{R}$. For example, if $x^0 \in \Omega_1$, it holds from Proposition 2.1 that $x^\dagger(t) = \chi(t, x^0, -1)$ over $[0, \sigma^\dagger]$ and $x^\dagger(t) = \chi(t - \sigma^\dagger, x^\dagger(\sigma^\dagger), 1)$ over $[\sigma^\dagger, T^\dagger]$. Hence, in the case $x^0 \in \Omega_1$, one can easily deduce from (E) and simple computations that

$$\sigma^\dagger = x_2^0 + \sqrt{\frac{1}{2}(x_2^0)^2 + x_1^0}, \quad x^\dagger(\sigma^\dagger) = \left( \frac{1}{2} \left( \frac{1}{2}(x_2^0)^2 + x_1^0 \right), -\sqrt{\frac{1}{2}(x_2^0)^2 + x_1^0} \right), \quad T^\dagger = x_2^0 + 2\sqrt{\frac{1}{2}(x_2^0)^2 + x_1^0}.$$

Therefore a complete and detailed description of the unique solution $(x^\dagger, u^\dagger, T^\dagger)$ to Problem (CP), in function of the initial condition $x^0$, can be easily derived from Proposition 2.1, (E) and simple computations.

**Remark 2.2.** Consider the framework of Proposition 2.1. In the present classical setting (that is, without loss control region), it is well known that:

(i) As usual with a classical minimal time problem, the *dynamical programming principle* holds true, in the sense that $(x^\dagger, u^\dagger)$ is not only the fastest way to reach the origin $0_{\mathbb{R}^2}$ from $x^0$, but also the fastest way to reach the origin $0_{\mathbb{R}^2}$ from any intermediate point $x^\dagger(s)$ with $s \in (0, T^\dagger)$, and also the fastest way to reach $x^\dagger(s)$ from $x^0$.

(ii) The optimal control $u^\dagger$ can be expressed as a *feedback control* (that is, as a function of the instantaneous position) given by

$$u^\dagger(t) = \begin{cases} -1 & \text{if } x^\dagger(t) \in \Gamma_1 \cup \Omega_1, \\ +1 & \text{if } x^\dagger(t) \in \Gamma_0 \cup \Omega_0, \end{cases}$$

over $(0, T^\dagger)$.

(iii) Furthermore, as is often the case with a classical optimal control problem for which the Hamiltonian is affine with respect to the control and without singular arc, the optimal control $u^\dagger$ *saturates* the control constraint set $[-1, 1]$, in the sense that it does not take any *moderated value* in the interior $(-1, 1)$.

As we will see in the next Section 3, these three well known properties are broken when considering a loss control region in the control system (see Remark 4.1).

## 2.2 Pontryagin maximum principle for a general minimum time problem with a loss control region

Let $n \in \mathbb{N}^\star$ be a positive integer and consider a state space partition $\mathbb{R}^n = \overline{\mathrm{X}}_1 \cup \overline{\mathrm{X}}_2$ where $\mathrm{X}_1$, $\mathrm{X}_2$ are two disjoint nonempty open subsets of $\mathbb{R}^n$ called *regions*. In the sequel we denote by $\partial \mathrm{X} := \overline{\mathrm{X}}_1 \cap \overline{\mathrm{X}}_2$ and we assume that there exists a $\mathrm{C}^1$ description map $F : \mathbb{R}^n \to \mathbb{R}$ such that

$$\mathrm{X}_1 = \{x \in \mathbb{R}^n \mid F(x) > 0\}, \quad \partial \mathrm{X} = \{x \in \mathbb{R}^n \mid F(x) = 0\}, \quad \mathrm{X}_2 = \{x \in \mathbb{R}^n \mid F(x) < 0\}.$$

Consider the control system given by

$$\begin{cases} (x, u, T) \in \mathrm{AC}([0, T], \mathbb{R}^n) \times \mathrm{L}^\infty([0, T], \mathbb{R}^m) \times (0, +\infty), \\ \dot{x}(t) = f(x(t), u(t)), \quad \text{a.e. } t \in [0, T], \\ \mathrm{X}_2 \text{ is a loss control region,} \end{cases} \tag{CS}$$

where the dynamics $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is of class $\mathrm{C}^1$. The novelty in the control system (CS) is that $\mathrm{X}_2$ is a *loss control region*, in the sense that the control value $u(t)$ is *frozen* (that is, cannot be modified) in the region $\mathrm{X}_2$. In other words, the control value $u(t)$ remains constant on the intervals for which the state position $x(t)$ belongs to $\mathrm{X}_2$. The precise definition of a solution to (CS) is given as follows.

**Definition 2.1** (Solution to (CS))**.** A triplet $(x,u,T) \in \mathrm{AC}([0,T],\mathbb{R}^n) \times \mathrm{L}^\infty([0,T],\mathbb{R}^m) \times (0,+\infty)$ is said to be a *solution* to (CS) if the following conditions are satisfied:

(i) It holds that $\dot{x}(t) = f(x(t),u(t))$ for almost every $t \in [0,T]$.

(ii) There exists a partition $\mathbb{T} = \{\tau_k\}_{k=0,\dots,N}$ of the interval $[0,T]$ such that $x$ is alternatively, over the open intervals $(\tau_{k-1},\tau_k)$, with values in $\mathrm{X}_1$ and then with values in $\mathrm{X}_2$. We denote by $\mathcal{I}_1$ (resp. $\mathcal{I}_2$) the set of indexes $k \in \{1,\dots,N\}$ such that $x$ is with values in $\mathrm{X}_1$ (resp. in $\mathrm{X}_2$) over $(\tau_{k-1},\tau_k)$.

(iii) For all $k \in \mathcal{I}_2$, there exists $\mu_k \in \mathbb{R}^m$ such that $u(t) = \mu_k$ for almost every $t \in (\tau_{k-1},\tau_k)$.

Our aim in this section is to derive first-order necessary optimality conditions in a Pontryagin form for the general minimum time problem with a loss control region given by

$$\begin{array}{ll} \text{minimize} & T, \\ \text{subject to} & (x,u,T) \in \mathrm{AC}([0,T],\mathbb{R}^n) \times \mathrm{L}^\infty([0,T],\mathbb{R}^m) \times (0,+\infty), \\ & \dot{x}(t) = f(x(t),u(t)), \quad \text{a.e. } t \in [0,T], \\ & x(0) = x^0, \quad x(T) = x^{\mathrm{targ}}, \\ & u(t) \in \mathrm{U}, \quad \text{a.e. } t \in [0,T], \\ & \mathrm{X}_2 \text{ is a loss control region,} \end{array} \tag{GP}$$

where the initial condition $x^0 \in \mathbb{R}^n$ and the target $x^{\mathrm{targ}} \in \mathbb{R}^n$ are distinct and U is a nonempty compact convex subset of $\mathbb{R}^m$. A triplet $(x,u,T) \in \mathrm{AC}([0,T],\mathbb{R}^n) \times \mathrm{L}^\infty([0,T],\mathbb{R}^m) \times (0,+\infty)$ is said to be *admissible* for Problem (GP) if it satisfies all the constraints of Problem (GP) (in particular it has to be a solution to (CS) in the sense of Definition 2.1). Finally such an admissible triplet is said to be a *solution* to Problem (GP) if it minimizes the final time among all admissible triplets.

Recall that the *normal cone* to U at some point $u \in \mathrm{U}$ is defined by

$$\mathrm{N}_{\mathrm{U}}[u] := \{u'' \in \mathbb{R}^m \mid \forall u' \in \mathrm{U},\ \langle u'', u'-u\rangle_{\mathbb{R}^m} \leq 0\},$$

and that the *Hamiltonian* $H : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ associated with Problem (GP) is defined by

$$H(x,u,p) := \langle p, f(x,u)\rangle_{\mathbb{R}^n},$$

for all $(x,u,p) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$. We are now in a position to provide a Pontryagin maximum principle for Problem (GP) under the *transverse assumption* given by

$$\forall (x,u) \in (\partial\mathrm{X} \backslash \{x^{\mathrm{targ}}\}) \times \mathrm{U}, \quad \langle \nabla F(x), f(x,u)\rangle_{\mathbb{R}^n} \neq 0. \tag{$\mathcal{A}$}$$

**Proposition 2.2** (Pontryagin maximum principle for Problem (GP))**.** Under the transverse assumption ($\mathcal{A}$), if $(x^\star,u^\star,T^\star)$ is a solution to Problem (GP), associated with a partition $\mathbb{T}^\star = \{\tau_k^\star\}_{k=0,\dots,N}$ of the interval $[0,T^\star]$, then there exists a nontrivial pair $(p,p^0) \in \mathrm{PAC}_{\mathbb{T}^\star}([0,T^\star],\mathbb{R}^n) \times \mathbb{R}_+$ satisfying:

(i) The *Hamiltonian system* $\dot{x}^\star(t) = \nabla_p H(x^\star(t),u^\star(t),p(t))$ and $-\dot{p}(t) = \nabla_x H(x^\star(t),u^\star(t),p(t))$ for almost every $t \in [0,T^\star]$.

(ii) The *Hamiltonian maximization condition* $u^\star(t) \in \arg\max_{\omega \in \mathrm{U}} H(x^\star(t),\omega,p(t))$ for almost every $t \in (\tau_{k-1}^\star,\tau_k^\star)$, for all $k \in \mathcal{I}_1^\star$.

(iii) The *averaged Hamiltonian gradient condition* $\int_{\tau_{k-1}^\star}^{\tau_k^\star} \nabla_u H(x^\star(t),\mu_k^\star,p(t))\,dt \in \mathrm{N}_{\mathrm{U}}[\mu_k^\star]$ for all $k \in \mathcal{I}_2^\star$.

(iv) The *discontinuity jump condition* $p^+(\tau_k^\star) - p^-(\tau_k^\star) = \nu_k \nabla F(x^\star(\tau_k^\star))$ for some $\nu_k \in \mathbb{R}$, for all $k \in \{1,\dots,N-1\}$.

(v) The *constancy Hamiltonian condition* $H(x^\star(t),u^\star(t),p(t)) = p^0$ for almost every $t \in [0,T^\star]$.

The proof of Proposition 2.2 is postponed in Appendix A. It is based on an augmentation technique and the application of the classical Pontryagin maximum principle for local solutions to a classical (that is, without loss control region) augmented optimal control problem involving parameters and endpoint constraints.

**Remark 2.3.** Hereafter we provide a list of comments on Proposition 2.2 and its proof.

(i) First of all, we emphasize that Proposition 2.2 is established under strong hypotheses such as the transverse assumption $(\mathcal{A})$, the topological assumptions made on the control constraint set U or the global descriptions of the regions $X_1$ and $X_2$. However these hypotheses are all satisfied in the context of the double integrator with a loss control region considered in the next Section 3 which constitutes the central part of the present work. Therefore Proposition 2.2 is sufficient for our purposes in this paper. We also emphasize that, in this paper, we do not consider in Definition 2.1 the possibility of an infinite number of instants $\tau_k$ (in the spirit of a chattering phenomenon [29]). The extension of Proposition 2.2 to more general contexts (including also general Bolza costs, not only minimum time problems) should be the subject of future research works.

(ii) The transverse assumption $(\mathcal{A})$ has a geometrical interpretation. It implies that, for any admissible triplet $(x, u, T)$ for Problem (GP), if the trajectory $x$ crosses the boundary $\partial X$, then it does not cross it *tangentially*. This assumption plays a central role in the proof of Proposition 2.2 in order to guarantee that the reverse procedure of the augmentation technique produces (at least locally) admissible triplets for the original Problem (GP). We refer to Appendix A for details. We also emphasize that, in the next Section 3, the non-equality in the transverse assumption $(\mathcal{A})$ is not satisfied at $x^{\mathrm{targ}}$. Fortunately, since we consider here a minimum time problem (and not a general Bolza cost), the non-equality in the transverse assumption $(\mathcal{A})$ is not mandatory at $x^{\mathrm{targ}}$ thanks to a basic dynamical programming argument. We refer to Appendix A for details. To conclude on the transverse assumption $(\mathcal{A})$, we mention that weaker assumptions could be considered. For example, one could consider a transverse assumption on the solution $(x^\star, u^\star, T^\star)$ only (and not everywhere). However, as explained in the previous item, it is not our objective here to provide a Pontryagin maximum principle for very general optimal control problems with loss control regions. Proposition 2.2 is sufficient for our purposes in this paper.

(iii) From linearity, the nontrivial pair $(p, p^0)$ in Proposition 2.2 is defined up to a positive multiplicative constant. When the pair is *normal* (that is, when $p^0 \neq 0$), we renormalize it so that $p^0 = 1$.

(iv) The averaged Hamiltonian gradient condition is well known in the context of *sampled-data controls* (that is, piecewise constant controls). We refer to [12, 13, 14] and references therein. In the present context, the control is imposed to be constant on intervals for which the state position lies in the loss control region. Therefore it is not surprising that the averaged Hamiltonian gradient condition appears in Proposition 2.2 as first-order necessary optimality condition on these *constancy intervals*. However note that our setting here is more involved than the framework of sampled-data controls since the constancy intervals for the control are determined by the state position $x(t)$, and not by the (independent) time variable $t$.

(v) The discontinuity jump condition on the *costate* $p$ is well known in the literature on *hybrid maximum principles* (in which, for example, authors consider control systems with spatially heterogeneous dynamics). We refer to [5, 23] and references therein. As also well known, when the control $u^\star$ admits left and right limits at $\tau_k^\star$ for all $k \in \{1, \dots, N-1\}$, denoted respectively by $(u^\star)^-(\tau_k^\star)$ and $(u^\star)^+(\tau_k^\star)$, the constancy Hamiltonian condition allows to obtain (forward and backward) expressions for $\nu_k$ given by

$$\nu_k = -\frac{\langle p^\pm(\tau_k^\star), f(x^\star(\tau_k^\star), (u^\star)^+(\tau_k^\star)) - f(x^\star(\tau_k^\star), (u^\star)^-(\tau_k^\star))\rangle_{\mathbb{R}^n}}{\langle \nabla F(x^\star(\tau_k^\star)), f(x^\star(\tau_k^\star), (u^\star)^\pm(\tau_k^\star))\rangle_{\mathbb{R}^n}},$$

for all $k \in \{1, \dots, N-1\}$.

One can conclude from Items (iv) and (v) that the present framework of loss control region can be seen, in some sense, as a mix of two well known topics in the literature, namely the sampled-data controls and the hybrid control systems.

# 3 Main result and its proof

In this section we focus on the minimum time problem for the double integrator with a loss control region given by

$$\begin{array}{ll} \text{minimize} & T, \\ \text{subject to} & (x,u,T) \in \mathrm{AC}([0,T],\mathbb{R}^2) \times \mathrm{L}^\infty([0,T],\mathbb{R}) \times (0,+\infty), \\ & \dot{x}_1(t) = x_2(t), \quad \text{a.e. } t \in [0,T], \\ & \dot{x}_2(t) = u(t), \quad \text{a.e. } t \in [0,T], \\ & x(0) = x^0, \quad x(T) = 0_{\mathbb{R}^2}, \\ & u(t) \in [-1,1], \quad \text{a.e. } t \in [0,T], \\ & \mathrm{X}_2 \text{ is a loss control region,} \end{array} \tag{P}$$

where $x^0 \in \mathbb{R}^2 \backslash \{0_{\mathbb{R}^2}\}$ and where the state space $\mathbb{R}^2 = \overline{\mathrm{X}}_1 \cup \overline{\mathrm{X}}_2$ has been partitioned (see Figure 3) with

$$\mathrm{X}_1 := \{x \in \mathbb{R}^2 \mid x_1 > 0\}, \quad \partial\mathrm{X} = \{x \in \mathbb{R}^2 \mid x_1 = 0\}, \quad \mathrm{X}_2 := \{x \in \mathbb{R}^2 \mid x_1 < 0\}.$$

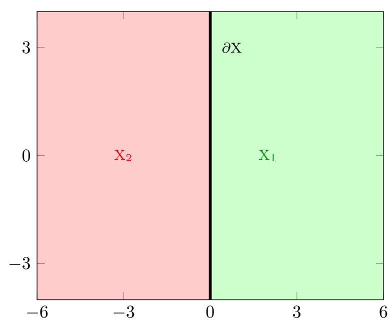


**Figure 3:** Partition of $\mathbb{R}^2$ into a "non-control region" (in red) and a "control region" (in green).

In the case where $x^0 \in \Gamma_0 \cup \Omega_1 \cup \Gamma_1$, the unique solution $(x^\dagger, u^\dagger, T^\dagger)$ to Problem (CP) is admissible for Problem (P) (since the control $u^\dagger$ remains frozen in the region $\mathrm{X}_2$, see Proposition 2.1 and Figure 2) and therefore it is clear that $(x^\dagger, u^\dagger, T^\dagger)$ is the unique solution to Problem (P). On the contrary, when $x^0 \in \Omega_0$, the unique solution $(x^\dagger, u^\dagger, T^\dagger)$ to Problem (CP) is not admissible for Problem (P) (since the control $u^\dagger$ requires a switch from $-1$ to $+1$ on the curve $\Gamma_1 \subset \mathrm{X}_2$, see Proposition 2.1 and Figure 2). Hence a rigorous analysis has to be performed in order to determine the candidate solution to Problem (P) in the case $x^0 \in \Omega_0$. This is the objective of the present section. To state and prove our main result (see Theorem 3.1 below), we need to introduce several elements:

- The positive real number $\theta := \frac{1}{1+\sqrt{2}} > 0$ introduced to simplify notations.
- The partition $\mathbb{R}^2 \backslash \{0_{\mathbb{R}^2}\} = \cup_{i=1}^6 (\Gamma_{i-1} \cup \Omega_i)$ (see Figure 4) where $\Gamma_0$, $\Omega_1$ and $\Gamma_1$ have already been defined in Section 2.1, where $\Gamma_3 := \{(x_1, 0) \mid x_1 < 0\}$ and $\Gamma_5 := \{(0, x_2) \mid x_2 < 0\}$, where

$$\Gamma_2 := \left\{ \left( -\frac{1}{2\theta} x_2^2, x_2 \right) \mid x_2 > 0 \right\}, \quad \Gamma_4 := \left\{ \left( -\frac{1}{\theta} x_2^2, x_2 \right) \mid x_2 < 0 \right\},$$

  and where $\Omega_i$ stands for the open region delimited by $\Gamma_{i-1}$ and $\Gamma_i$ for all $i \in \{1, \ldots, 6\}$ (with $\Gamma_6 := \Gamma_0$ by convention).

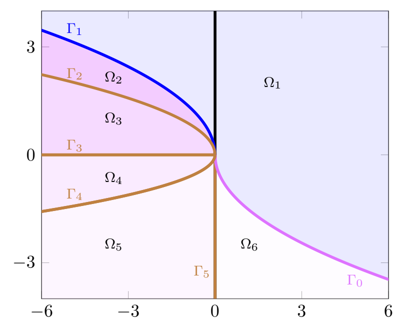


**Figure 4:** Partition of $\mathbb{R}^2\backslash\{0_{\mathbb{R}^2}\}$ arising from the analysis of Problem (P) (see Theorem 3.1).

- The three real numbers

$$\lambda(x^0) := \frac{(x_2^0)^2}{2x_1^0} \quad \text{and} \quad \lambda^{\pm}(x^0) := \sqrt{\theta}\left(\sqrt{\theta} \pm 2\sqrt{-\lambda(x^0)}\right),$$

introduced for any initial condition $x^0 \in \Omega_0 \cap \mathrm{X}_2$, for which $\lambda(x^0) \leq 0$. We refer to Remark 3.1 for additional comments on these numbers.

**Theorem 3.1.** If $(x^\star, u^\star, T^\star)$ is a solution to Problem (P), then an overview description of $(x^\star, u^\star)$ over the interval $[0, T^\star]$, according to the position of the initial condition $x^0$ in the partition $\mathbb{R}^2\backslash\{0_{\mathbb{R}^2}\} = \cup_{i=1}^6(\Gamma_{i-1} \cup \Omega_i)$, can be summarized as follows:

| Position of $x^0$ | Overview description of $x^\star(t)$ / $u^\star(t)$ | $\mu^\star \in (-1,1)$ | $N$ | Figure |
|---|---|---|---|---|
| $\Gamma_0$ | $\Gamma_0$<br>$1$ | | 1 | 5a |
| $\Omega_1$ | $\Omega_1 \rightsquigarrow \Gamma_0$<br>$-1 \rightsquigarrow 1$ | | 1 or 2 | 5b |
| $\Gamma_1$ | $\Gamma_1$<br>$-1$ | | 1 | 5c |
| $\Omega_2$ | $\Omega_2$<br>$\mu^\star$ | $\lambda(x^0)$ | 1 | 5d |
| $\Gamma_2$ | $\Gamma_2$<br>$\mu^\star$ | $\lambda(x^0) = \lambda^-(x^0)$ | 1 | 5e |
| $\Omega_3$ | $\mathrm{X}_2 \rightsquigarrow \Omega_1 \cap \mathrm{X}_1 \rightsquigarrow \Gamma_0$<br>$\mu^\star \rightsquigarrow -1 \rightsquigarrow 1$ | $\lambda^-(x^0)$ | 2 | 5f |
| $\Gamma_3$ | $\mathrm{X}_2 \rightsquigarrow \Omega_1 \cap \mathrm{X}_1 \rightsquigarrow \Gamma_0$<br>$\mu^\star \rightsquigarrow -1 \rightsquigarrow 1$ | $\lambda^-(x^0) = \lambda^+(x^0)$ | 2 | 5g |
| $\Omega_4$ | $\mathrm{X}_2 \rightsquigarrow \Omega_1 \cap \mathrm{X}_1 \rightsquigarrow \Gamma_0$<br>$\mu^\star \rightsquigarrow -1 \rightsquigarrow 1$ | $\lambda^+(x^0)$ | 2 | 5h |
| $\Gamma_4$ | $\mathrm{X}_2 \rightsquigarrow \Omega_1 \cap \mathrm{X}_1 \rightsquigarrow \Gamma_0$<br>$1 \rightsquigarrow -1 \rightsquigarrow 1$ | | 2 | 5i |
| $\Omega_5$ | $\mathrm{X}_2 \rightsquigarrow \Omega_1 \cap \mathrm{X}_1 \rightsquigarrow \Gamma_0$<br>$1 \rightsquigarrow -1 \rightsquigarrow 1$ | | 2 | 5j |
| $\Gamma_5$ | $\mathrm{X}_2 \rightsquigarrow \Omega_1 \cap \mathrm{X}_1 \rightsquigarrow \Gamma_0$<br>$1 \rightsquigarrow -1 \rightsquigarrow 1$ | | 2 | 5k |
| $\Omega_6$ | $\Omega_6 \rightsquigarrow \mathrm{X}_2 \rightsquigarrow \Omega_1 \cap \mathrm{X}_1 \rightsquigarrow \Gamma_0$<br>$1 \rightsquigarrow 1 \rightsquigarrow -1 \rightsquigarrow 1$ | | 3 | 5l |

The column $N$ allows to know how many *crossing times* (from $\mathrm{X}_1$ to $\mathrm{X}_2$, or from $\mathrm{X}_2$ to $\mathrm{X}_1$) are observed for the trajectory $x^\star$ (that is, $N-1$). For example, if $x^0 \in \Omega_3$, then the trajectory $x^\star$ has only one crossing time $\tau_1^\star$ from $\mathrm{X}_2$ to $\mathrm{X}_1$. Precisely, in the case $x^0 \in \Omega_3$, there exist $0 < \tau_1^\star < \sigma^\star < T^\star$ such that $x^\star(t) \in \mathrm{X}_2$ and $u^\star(t) = \mu^\star$ over $(0, \tau_1^\star)$, and $x^\star(t) \in \Omega_1 \cap \mathrm{X}_1$ and $u^\star(t) = -1$ over $(\tau_1^\star, \sigma^\star)$, and $x^\star(t) \in \Gamma_0$ and $u^\star(t) = 1$ over $(\sigma^\star, T^\star)$.

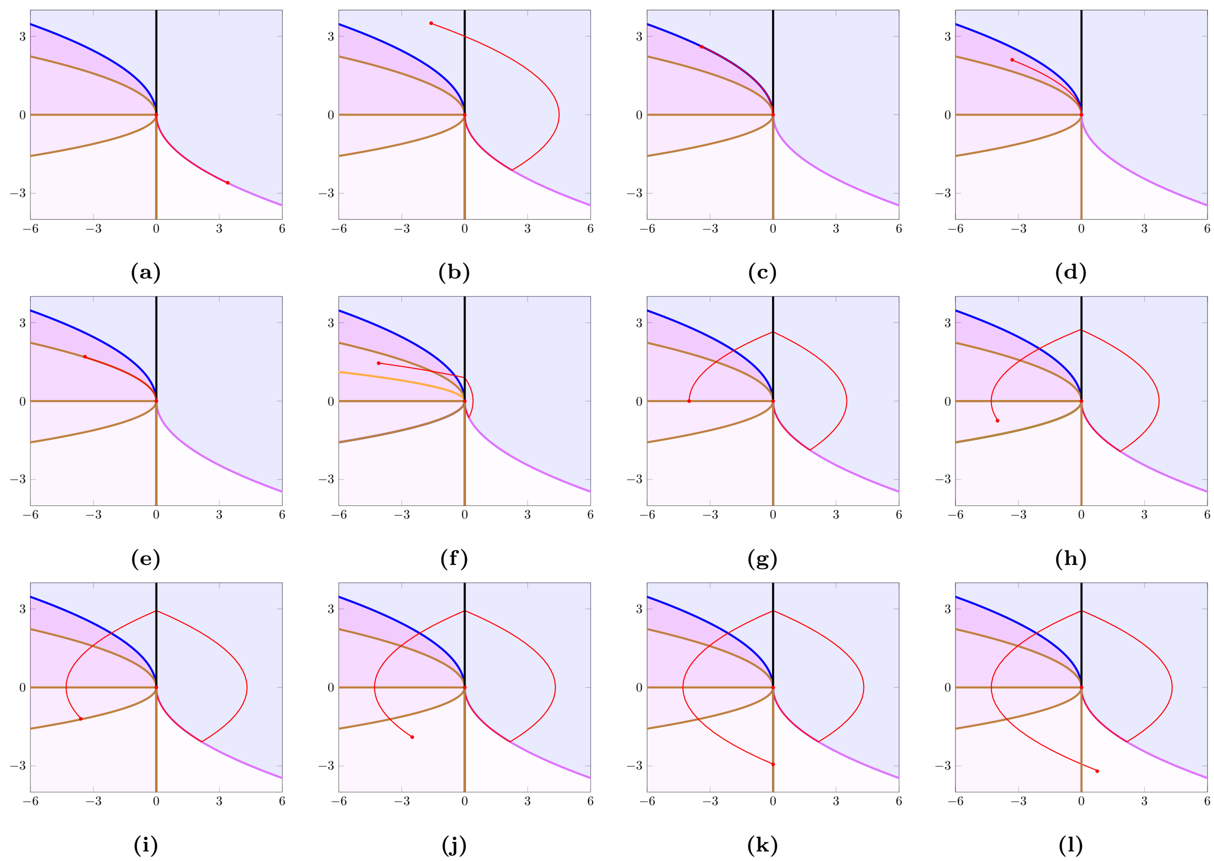


**Figure 5:** Optimal trajectories (in red) in the twelve cases of Theorem 3.1.

The results of Theorem 3.1 will be commented in Section 4.1. The rest of this section is dedicated to its proof which is based on the Pontryagin maximum principle stated in Proposition 2.2. To this aim let us fix a solution $(x^\star, u^\star, T^\star)$ to Problem (P), associated with a partition $\mathbb{T}^\star = \{\tau_k^\star\}_{k=0,\dots,N}$ of the interval $[0, T^\star]$, and let us denote by $(p, p^0) \in \mathrm{PAC}_{\mathbb{T}^\star}([0, T^\star], \mathbb{R}^n) \times \mathbb{R}_+$ the nontrivial pair provided by Proposition 2.2 (whose hypotheses are all satisfied).

**Remark 3.1.** Before going any further in the proof of Theorem 3.1, we need to emphasize several facts.

(i) Note that $\Gamma_0 \subset \mathrm{X}_1$ and $\Gamma_1 \subset \mathrm{X}_2$, and that $\Omega_1$ intersects both $\mathrm{X}_1$ and $\mathrm{X}_2$. Also note that $\Omega_0 \cap \mathrm{X}_2 = \Omega_2 \cup \Gamma_2 \cup \Omega_3 \cup \Gamma_3 \cup \Omega_4 \cup \Gamma_4 \cup \Omega_5$, that $\Omega_0 \cap \partial\mathrm{X} = \Gamma_5$ and that $\Omega_0 \cap \mathrm{X}_1 = \Omega_6$.

(ii) For any initial condition $x^0 \in \Omega_0 \cap \mathrm{X}_2$, it holds that $\lambda(x^0) \leq 0$ (with equality if and only if $x^0 \in \Gamma_3$) and $\lambda^+(x^0) > 0$. Note that, if $x^0 \in \Gamma_1$ (resp. $x^0 \in \Gamma_2$, $x^0 \in \Gamma_3$, $x^0 \in \Gamma_4$), then $\lambda(x^0) = -1$ (resp. $\lambda^-(x^0) = \lambda(x^0)$, $\lambda^+(x^0) = \lambda^-(x^0)$, $\lambda^+(x^0) = 1$). Also note that, if $x^0 \in \Omega_2 \cup \Gamma_2$ (resp. $x^0 \in \Gamma_2 \cup \Omega_3 \cup \Gamma_3$, $x^0 \in \Gamma_3 \cup \Omega_4$), then $\lambda(x^0) \in (-1, 1)$ (resp. $\lambda^-(x^0) \in (-1, 1)$, $\lambda^+(x^0) \in (-1, 1)$).

(iii) Consider the framework of Proposition 2.1 in the case where $x^0 \in \Omega_1 \cap \partial\mathrm{X}$. In that context, from Remark 2.1, it holds that

$$\sigma^\dagger = \left(1 + \frac{\sqrt{2}}{2}\right) x_2^0, \quad x_2^\dagger(\sigma^\dagger) = -\frac{\sqrt{2}}{2} x_2^0, \quad T^\dagger = \frac{x_2^0}{\theta}.$$

In the sequel we denote by $(x^\dagger(\cdot,x^0),u^\dagger(\cdot,x^0),T^\dagger(x^0))$ the unique solution $(x^\dagger,u^\dagger,T^\dagger)$ to Problem (CP) corresponding to such an initial condition $x^0\in\Omega_1\cap\partial\mathrm{X}$.

We are now in a position to pursue the proof of Theorem 3.1 by separating the cases according to the position of the initial condition $x^0$ in the partition of $\mathbb{R}^2\backslash\{0_{\mathbb{R}^2}\}$ depicted in Figure 4. First, recall that the first three cases of Theorem 3.1 (that is, when $x^0\in\Gamma_0\cup\Omega_1\cup\Gamma_1$) are trivial since, in these cases, the unique solution $(x^\dagger,u^\dagger,T^\dagger)$ to Problem (CP) is admissible for Problem (P) and thus $(x^\star,u^\star,T^\star)=(x^\dagger,u^\dagger,T^\dagger)$ and we refer to Proposition 2.1 for the corresponding overview description. Note that, in the case $x^0\in\Omega_1$ (since $\Omega_1$ intersects both $\mathrm{X}_1$ and $\mathrm{X}_2$), we have $N=2$ (resp. $N=1$) if $x_1^0<0$ (resp. $x_1^0\geq 0$).

In the sequel we will focus only on the case $x^0\in\Omega_0$ and we separate it into three subcases given by $x^0\in\Omega_0\cap\mathrm{X}_2$ (see Section 3.1), $x^0\in\Omega_0\cap\partial\mathrm{X}$ (see Section 3.2) and $x^0\in\Omega_0\cap\mathrm{X}_1$ (see Section 3.3).

## 3.1 The case $x^0\in\Omega_0\cap\mathrm{X}_2$

Here we focus on the case $x^0\in\Omega_0\cap\mathrm{X}_2$. Since $x^\star(0)=x^0\in\mathrm{X}_2$, we get that $x^\star(t)\in\mathrm{X}_2$ over $[0,\tau_1^\star)$. Moreover, since $x^\star(T^\star)=0_{\mathbb{R}^2}$, we get that $x_1^\star(\tau_1^\star)=0$ (independently of $N=1$ or $N\geq 2$). Since $\mathrm{X}_2$ is a loss control region, let us denote by $\mu^\star\in[-1,1]$ the constant value of $u^\star$ over $(0,\tau_1^\star)$. Therefore it holds that $x^\star(t)=\chi(t,x^0,\mu^\star)$ over $[0,\tau_1^\star]$ (see Remark 2.1). From (E) and simple computations, one can easily derive the following lemma.

**Lemma 3.1** (Case $x^0\in\Omega_0\cap\mathrm{X}_2$)**.** The following five properties are satisfied:

$$\text{(i)}\quad (x_2^0)^2-2\mu^\star x_1^0\geq 0,\qquad \text{(ii)}\quad (x_2^0,\mu^\star)\notin\mathbb{R}_-^2,\qquad \text{(iii)}\quad \mu^\star\neq -1,$$

$$\text{(iv)}\quad \tau_1^\star=\begin{cases}\dfrac{\sqrt{(x_2^0)^2-2\mu^\star x_1^0}-x_2^0}{\mu^\star} & \text{if } \mu^\star\neq 0,\\ -\dfrac{x_1^0}{x_2^0} & \text{if } \mu^\star=0.\end{cases},\qquad \text{(v)}\quad x_2^\star(\tau_1^\star)=\sqrt{(x_2^0)^2-2\mu^\star x_1^0}\geq 0.$$

*Proof.* (i) Since $x_1^\star(\tau_1^\star)=0$, the discriminant of $x_1^\star(t)$ is nonnegative. (ii) By contradiction, if $(x_2^0,\mu^\star)\notin\mathbb{R}_-^2$, then one would obtain that $x_1^\star(\tau_1^\star)\leq x_1^0<0$ which is absurd. (iii) By contradiction, if $\mu^\star=-1$, then, from the previous two items, one would obtain $(x_2^0)^2+2x_1^0\geq 0$ and $x_2^0>0$, which contradicts $x^0\in\Omega_0\cap\mathrm{X}_2$. (iv)(v) Separating the cases $\mu^\star>0$, $\mu^\star=0$ and $\mu^\star<0$ (note that $x_2^0>0$ in the last two cases), one can easily derive from (E) and simple computations the above expressions of $\tau_1^\star$ and $x_2^\star(\tau_1^\star)$. ☐

From Lemma 3.3 we deduce that $x^\star(\tau_1^\star)\in(\Omega_1\cap\partial\mathrm{X})\cup\{0_{\mathbb{R}^2}\}$. Precisely we obtain two cases:

1. Either $x_2^\star(\tau_1^\star)=0$. In that case $x^\star(\tau_1^\star)=0_{\mathbb{R}^2}$ and thus $T^\star=\tau_1^\star$ and $N=1$. Furthermore, from Lemma 3.3, it holds that $\mu^\star=\lambda(x^0)\leq 0$ and thus this situation is possible only for $x_2^0>0$.

2. Either $x_2^\star(\tau_1^\star)>0$. In that case $x^\star(\tau_1^\star)\in\Omega_1\cap\partial\mathrm{X}$ and, from a basic dynamical programming argument, it holds that $x^\star=x^\dagger(\cdot-\tau_1^\star,x^\star(\tau_1^\star))$ and $u^\star=u^\dagger(\cdot-\tau_1^\star,x^\star(\tau_1^\star))$ over $(\tau_1^\star,T^\star)$ (see Remark 3.1 for the notation). From Proposition 2.1 and Remark 3.1, we deduce that $x^\star(t)\in\Omega_1\cap\mathrm{X}_1$ and $u^\star(t)=-1$ over $(\tau_1^\star,\sigma^\star)$, and $x^\star(t)\in\Gamma_0$ and $u^\star(t)=1$ over $(\sigma^\star,T^\star)$, where $\sigma^\star=\tau_1^\star+(1+\frac{\sqrt{2}}{2})x_2^\star(\tau_1^\star)$, $x_2^\star(\sigma^\star)=-\frac{\sqrt{2}}{2}x_2^\star(\tau_1^\star)<0$ and $T^\star=\tau_1^\star+\frac{x_2^\star(\tau_1^\star)}{\theta}$. In particular, in that case, we have $T^\star=\tau_2^\star$ and $N=2$.

In the second case above, we already have a quite complete description of $(x^\star,u^\star)$ over $(\tau_1^\star,T^\star)$. Therefore we only need to determine the constant value $\mu^\star\in[-1,1]$ of the optimal control $u^\star$ over $(0,\tau_1^\star)$. Our aim in the next lemma is to reduce the possibilities of values for $\mu^\star$ in that case. This lemma, whose proof is based on the application of the Pontryagin maximum principle stated in Proposition 2.2, allows to discriminate four values.

**Lemma 3.2** (Case $x^0\in\Omega_0\cap\mathrm{X}_2$)**.** If $x_2^\star(\tau_1^\star)>0$, then $\mu^\star\in\{0,\lambda^-(x^0),\lambda^+(x^0),1\}$.

*Proof.* We only deal with the case $x_2^0 > 0$ (the other cases $x_2^0 = 0$ and $x_2^0 < 0$ are similar). Since $N = 2$ and from the Pontryagin maximum principle stated in Proposition 2.2 (precisely from the Hamiltonian system and the discontinuity jump condition), we get that

$$p_1(t) = \begin{cases} p_{11} & \text{if} \quad t \in [0, \tau_1^\star), \\ p_{12} & \text{if} \quad t \in (\tau_1^\star, T^\star], \end{cases} \quad \text{and} \quad p_2(t) = \begin{cases} -p_{11}t + p_2(0) & \text{if} \quad t \in [0, \tau_1^\star], \\ -p_{12}(t - \tau_1^\star) - p_{11}\tau_1^\star + p_2(0) & \text{if} \quad t \in [\tau_1^\star, T^\star], \end{cases}$$

with $p_{11}$, $p_{12} \in \mathbb{R}$. From the Hamiltonian maximization condition, since $x^\star(t) \in \mathrm{X}_1$ over $(\tau_1^\star, T^\star)$ and $u^\star$ changes its value at $\sigma^\star$, we deduce that $p_2(\sigma^\star) = 0$. From the Hamiltonian constancy (considered at 0, $\tau_1^\star$ and $\sigma^\star$), we obtain that

$$p_{11}x_2^0 + p_2(0)\mu^\star = p^0, \quad p_{11}x_2^\star(\tau_1^\star) + p_2(\tau_1^\star)\mu^\star = p^0, \quad p_{12}x_2^\star(\tau_1^\star) - p_2(\tau_1^\star) = p^0, \quad p_{12}x_2^\star(\sigma^\star) = p^0.$$

From these four equalities, one can easily prove in the one hand that

$$p_{12} = p_{11} + \frac{p_2(\tau_1^\star)}{x_2^\star(\tau_1^\star)}(1 + \mu^\star) = p_{11} + \frac{p_2(0) - p_{11}\tau_1^\star}{x_2^\star(\tau_1^\star)}(1 + \mu^\star), \tag{3.1}$$

and, in the other hand, using the nontriviality of the pair $(p, p^0)$, that $p^0 \neq 0$ (by contradiction). In the sequel we take $p^0 = 1$ (see Remark 2.3) and we assume that $\mu^\star \notin \{0, 1\}$. Therefore it only remains to prove that $\mu^\star \in \{\lambda^-(x^0), \lambda^+(x^0)\}$. Since $\mu^\star \notin \{-1, 1\}$ (see Lemma 3.3), the averaged Hamiltonian gradient condition gives $\int_0^{\tau_1^\star} p_2(t)dt = 0$ and thus $p_2(0) = \frac{p_{11}}{2}$ (and thus $p_{11} \neq 0$ by contradiction). Using it in the equality $p_2(\sigma^\star) = 0$, we obtain that $-p_{12}(\sigma^\star - \tau_1^\star) - \frac{p_{11}}{2}\tau_1^\star = 0$. Replacing the value $p_{12}$ from (3.1) and the value $\sigma^\star = \tau_1^\star + (1 + \frac{\sqrt{2}}{2})x_2^\star(\tau_1^\star)$ (and dividing by $p_{11} \neq 0$ and by $x_2^\star(\tau_1^\star) \neq 0$), one can obtain that

$$\frac{2}{\theta}x_2^\star(\tau_1^\star) - \tau_1^\star \left(1 + \frac{\mu^\star}{\theta}\right) = 0.$$

Replacing the values $x_2^\star(\tau_1^\star)$ and $\tau_1^\star$ obtained in Lemma 3.3 (and dividing by $x_2^0 > 0$), we obtain that

$$\left(\frac{\mu^\star}{\theta} - 1\right)\sqrt{1 - \frac{\mu^\star}{\lambda(x^0)}} = -\left(1 + \frac{\mu^\star}{\theta}\right).$$

Squaring this last equality (and dividing by $\mu^\star \neq 0$), we obtain that

$$\left(\frac{\mu^\star}{\theta}\right)^2 - 2\left(\frac{\mu^\star}{\theta}\right) + \left(1 + \frac{4\lambda(x^0)}{\theta}\right) = 0.$$

which admits two solutions given by $\lambda^-(x^0)$ and $\lambda^+(x^0)$. The proof is complete. □

Finally, according to the previous analysis and using the equality $T^\star = \tau_1^\star + \frac{x_2^\star(\tau_1^\star)}{\theta}$, we can summarize the situation as follows:

(i) If $x_2^0 < 0$, then $\mu^\star \geq 0$ and $\mu^\star \in \{\lambda^-(x^0), \lambda^+(x^0), 1\}$ and

$$T^\star = -x_2^0\left(\left(\frac{1}{\mu^\star} + \frac{1}{\theta}\right)\sqrt{1 - \frac{\mu^\star}{\lambda(x^0)}} + \frac{1}{\mu^\star}\right). \tag{3.2}$$

(ii) If $x_2^0 = 0$, then $\mu^\star \in \{\theta, 1\}$ and

$$T^\star = \left(\frac{1}{\mu^\star} + \frac{1}{\theta}\right)\sqrt{-2\mu^\star x_1^0}. \tag{3.3}$$

(iii) If $x_2^0 > 0$, then $\mu^\star \in \{0, \lambda(x^0), \lambda^-(x^0), \lambda^+(x^0), 1\}$ and

$$T^\star = \begin{cases} x_2^0\left(\left(\frac{1}{\mu^\star} + \frac{1}{\theta}\right)\sqrt{1 - \frac{\mu^\star}{\lambda(x^0)}} - \frac{1}{\mu^\star}\right) & \text{if} \quad \mu^\star \neq 0, \\ x_2^0\left(\frac{1}{\theta} - \frac{1}{2\lambda(x^0)}\right) & \text{if} \quad \mu^\star = 0. \end{cases} \tag{3.4}$$

By comparing the value of $T^\star$ in function of the possibilities of value of $\mu^\star$, we get the following proposition which concludes the proof in the case $x^0 \in \Omega_0 \cap \mathrm{X}_2 = \Omega_2 \cup \Gamma_2 \cup \Omega_3 \cup \Gamma_3 \cup \Omega_4 \cup \Gamma_4 \cup \Omega_5$.

**Proposition 3.1.** It holds that:

| If $x^0 \in$ | $\Omega_2$ | $\Gamma_2$ | $\Omega_3$ | $\Gamma_3$ | $\Omega_4$ | $\Gamma_4$ | $\Omega_5$ |
|---|---|---|---|---|---|---|---|
| Then $\mu^\star =$ | $\lambda(x^0)$ | $\lambda(x^0) = \lambda^-(x^0)$ | $\lambda^-(x^0)$ | $\lambda^-(x^0) = \lambda^+(x^0)$ | $\lambda^+(x^0)$ | $\lambda^+(x^0) = 1$ | 1 |

*Proof.* In this proof we denote by $\mathcal{T}(\alpha)$ the value of $T^\star$ given in (3.2), (3.3) and (3.4) if $\mu^\star = \alpha$.

- Take $x^0 \in \Omega_2$. It holds that $-\frac{1}{2\theta}(x_2^0)^2 < x_1^0 < -\frac{1}{2}(x_2^0)^2$ and thus $-1 < \lambda(x^0) < -\theta$. In the one hand we deduce that $\mathcal{T}(0) > x_2^0(\frac{3}{2} + \sqrt{2})$, $\mathcal{T}(1) > x_2^0(1 + 2\sqrt{2})$ and $\mathcal{T}(\lambda(x^0)) < x_2^0(1 + \sqrt{2})$ and thus $\mu^\star \neq 0$ and $\mu^\star \neq 1$. In the other hand we deduce $\lambda^+(x^0) > 3\theta > 1$, and thus $\mu^\star \neq \lambda^+(x^0)$. By studying the quotient
$$\frac{\lambda^-(x^0)}{\lambda(x^0)} = \sqrt{\frac{\theta}{-\lambda(x^0)}}\left(2 - \sqrt{\frac{\theta}{-\lambda(x^0)}}\right),$$
one can also obtain that $-1 < \lambda(x^0) < \lambda^-(x^0) < -\theta$ and thus $\mathcal{T}(\lambda^-(x^0)) > \mathcal{T}(\lambda(x^0))$. We conclude that $\mu^\star = \lambda(x^0)$.
- Take $x^0 \in \Gamma_2$. Similar to the first item.
- Take $x^0 \in \Gamma_3$. In that case it holds that $x_2^0 = 0$ and thus $\mu^\star \in \{\theta, 1\}$. Since $(\sqrt{\theta} - 1)^2 > 0$, one can easily obtain that $\mathcal{T}(\theta) < \mathcal{T}(1)$ and thus $\mu^\star = \theta = \lambda^-(x^0) = \lambda^+(x^0)$.
- Take $x^0 \in \Gamma_4 \cup \Omega_5$. In that case it holds that $-\frac{1}{\theta}(x_2^0)^2 \leq x_1^0 < 0$ and thus $\lambda(x^0) \leq -\frac{\theta}{2}$. One can deduce that $\lambda^+(x^0) \geq 1$ and $\lambda^-(x^0) < 0$. Therefore $\mu^\star = 1$ (and recall that $\lambda^+(x^0) = 1$ in the case $x^0 \in \Gamma_4$).

The cases $x^0 \in \Omega_3$ and $x^0 \in \Omega_4$ can be treated similarly but with more involved computations. For the sake of conciseness, these cases are omitted. □

## 3.2 The case $x^0 \in \Omega_0 \cap \partial\mathrm{X}$

Here we focus on the case $x^0 \in \Omega_0 \cap \partial\mathrm{X}$. This section is very similar (and even simpler) to the previous one, except that some minor adjustments have to be performed since $x_1^0 = 0$ and thus $\lambda(x^0)$ is not defined. Therefore, in this section, the proof is sketched.

From continuity of $\dot{x}_1^\star = x_2^\star$ and since $x_1^\star(0) = 0$ and $x_2^\star(0) = x_0^2 < 0$, we deduce that $x^\star(t) \in \mathrm{X}_2$ over $(0, \tau_1^\star)$. Since $x^\star(T^\star) = 0_{\mathbb{R}^2}$, we get that $x_1^\star(\tau_1^\star) = 0$ (independently of $N = 1$ or $N \geq 2$). Since $\mathrm{X}_2$ is a loss control region, let us denote by $\mu^\star \in [-1, 1]$ the constant value of $u^\star$ over $(0, \tau_1^\star)$. Therefore it holds that $x^\star(t) = \chi(t, x^0, \mu^\star)$ over $[0, \tau_1^\star]$ (see Remark 2.1). From (E) and simple computations, one can easily derive the following lemma.

**Lemma 3.3** (Case $x^0 \in \Omega_0 \cap \partial\mathrm{X}$)**.** The following three properties are satisfied:
$$\text{(i)} \quad \mu^\star > 0, \qquad \text{(ii)} \quad \tau_1^\star = \frac{-2x_2^0}{\mu^\star}, \qquad \text{(iii)} \quad x_2^\star(\tau_1^\star) = -x_2^0 > 0.$$
In particular it holds that $x^\star(\tau_1^\star) = -x^0 \in \Omega_1 \cap \partial\mathrm{X}$.

Since $x^\star(\tau_1^\star) = -x^0 \in \Omega_1 \cap \partial\mathrm{X}$, it holds from a basic dynamical programming argument that $x^\star = x^\dagger(\cdot - \tau_1^\star, x^\star(\tau_1^\star))$ and $u^\star = u^\dagger(\cdot - \tau_1^\star, x^\star(\tau_1^\star))$ over $(\tau_1^\star, T^\star)$ (see Remark 3.1 for the notation). From Proposition 2.1 and Remark 3.1, we deduce that $x^\star(t) \in \Omega_1 \cap \mathrm{X}_1$ and $u^\star(t) = -1$ over $(\tau_1^\star, \sigma^\star)$, and $x^\star(t) \in \Gamma_0$ and $u^\star(t) = 1$ over $(\sigma^\star, T^\star)$, where $\sigma^\star = \tau_1^\star + (1 + \frac{\sqrt{2}}{2})x_2^\star(\tau_1^\star)$, $x_2^\star(\sigma^\star) = -\frac{\sqrt{2}}{2}x_2^\star(\tau_1^\star) < 0$ and $T^\star = \tau_1^\star + \frac{x_2^\star(\tau_1^\star)}{\theta}$. In particular it holds that $T^\star = \tau_2^\star$ and $N = 2$. Hence we already have a quite complete description of $(x^\star, u^\star)$ over $(\tau_1^\star, T^\star)$. Therefore we only need to determine the constant value $\mu^\star \in [-1, 1]$ of the optimal control $u^\star$ over $(0, \tau_1^\star)$. To this aim one can follow the same steps than the proof of Lemma 3.2, except that one should assume by contradiction that $\mu^\star \neq 1$ (recall that $\mu^\star > 0$ from Lemma 3.3). At the step of replacing the values $x_2^\star(\tau_1^\star)$ and $\tau_1^\star$ from Lemma 3.3, one obtains $\frac{1}{\mu^\star} = 0$ which is absurd. We get the following proposition which concludes the proof in the case $x^0 \in \Omega_0 \cap \partial\mathrm{X} = \Gamma_5$.

**Proposition 3.2** (Case $x^0 \in \Omega_0 \cap \partial\mathrm{X}$)**.** It holds that $\mu^\star = 1$ and $T^\star = -x_2^0(2 + \frac{1}{\theta})$.

### 3.3 The case $x^0 \in \Omega_0 \cap \mathrm{X}_1$

Here we focus on the case $x^0 \in \Omega_0 \cap \mathrm{X}_1$. This section is different from the previous two sections since our proof here is based, not only on a basic dynamical programming argument and the results of the previous section, but also on the application of the classical Pontryagin maximum principle on a classical (that is, without loss control region) optimal control problem. To this aim we first establish the next lemma.

**Lemma 3.4** (Case $x^0 \in \Omega_0 \cap \mathrm{X}_1$)**.** It holds that $x^\star(t) \in \Omega_0 \cap \mathrm{X}_1$ over $[0,\tau_1^\star)$ and $x^\star(\tau_1^\star) \in \Omega_0 \cap \partial\mathrm{X}$.

*Proof.* In the one hand, since $x^\star(0) = x^0 \in \mathrm{X}_1$, we get that $x^\star(t) \in \mathrm{X}_1$ over $[0,\tau_1^\star)$. Moreover, since $x^\star(T^\star) = 0_{\mathbb{R}^2}$, we get that $x_1^\star(\tau_1^\star) = 0$ (independently of $N = 1$ or $N \geq 2$). On the other hand, from the control system and since the control $u^\star$ is with values in $[-1,1]$, one has $x_1^\star(t) \leq \chi_1(t)$ and $x_2^\star(t) \leq \chi_2(t)$ over $[0,T^\star]$, where $\chi := \chi(\cdot,x^0,1)$ (see Remark 2.1). Defining $r := -x_2^0 - \sqrt{(x_2^0)^2 - 2x_1^0} > 0$, from (E) and simple computations, one can easily obtain that $\chi_2(t) < 0 < \chi_1(t) < \frac{1}{2}\chi_2(t)^2$ (that is $\chi(t) \in \Omega_0 \cap \mathrm{X}_1$) over $[0,r)$ and that $\chi_2(r) < 0 = \chi_1(r)$ (that is $\chi(r) \in \Omega_0 \cap \partial\mathrm{X}$). Firstly, one can deduce that $\tau_1^\star \leq r$ (indeed, if $r < \tau_1^\star$, then $0 < x_1^\star(r) \leq \chi_1(r) = 0$ which is absurd). Secondly, one obtains that $x_2^\star(t) \leq \chi_2(t) < 0 < x_1^\star(t) \leq \chi_1(t) < \frac{1}{2}\chi_2(t)^2 \leq \frac{1}{2}x_2^\star(t)^2$ (and thus $x^\star(t) \in \Omega_0 \cap \mathrm{X}_1$) over $[0,\tau_1^\star)$ and $x_2^\star(\tau_1^\star) \leq \chi_2(\tau_1^\star) < 0 = x_1^\star(\tau_1^\star)$ (and thus $x^\star(\tau_1^\star) \in \Omega_0 \cap \partial\mathrm{X}$). □

From Lemma 3.4, it holds that $x^\star(\tau_1^\star) \in \Omega_0 \cap \partial\mathrm{X}$. From a basic dynamical programming argument, it holds from the previous section that $x^\star = \chi(\cdot - \tau_1^\star, x^\star(\tau_1^\star), 1)$ and $u^\star = 1$ over $(\tau_1^\star,\tau_2^\star)$, that $x^\star(\tau_2^\star) \in \Omega_1 \cap \partial\mathrm{X}$ and that $x^\star = x^\dagger(\cdot - \tau_2^\star, x^\star(\tau_2^\star))$ and $u^\star = u^\dagger(\cdot - \tau_2^\star, x^\star(\tau_2^\star))$ over $(\tau_2^\star, T^\star)$ (see Remark 3.1 for the notation). From Proposition 2.1 and Remark 3.1, we deduce that $x^\star(t) \in \Omega_1 \cap \mathrm{X}_1$ and $u^\star(t) = -1$ over $(\tau_2^\star,\sigma^\star)$, and $x^\star(t) \in \Gamma_0$ and $u^\star(t) = 1$ over $(\sigma^\star, T^\star)$, where $\sigma^\star = \tau_2^\star + (1+\frac{\sqrt{2}}{2})x_2^\star(\tau_2^\star)$, $x_2^\star(\sigma^\star) = -\frac{\sqrt{2}}{2}x_2^\star(\tau_2^\star) < 0$ and $T^\star = \tau_2^\star + \frac{x_2^\star(\tau_2^\star)}{\theta}$. In particular, in that case, we have $T^\star = \tau_3^\star$ and $N = 3$.

From the previous section, it also holds that $\tau_2^\star = \tau_1^\star - 2x_2^\star(\tau_1^\star)$ and $x_2^\star(\tau_2^\star) = -x_2^\star(\tau_1^\star)$. As a consequence we obtain that $T^\star = \tau_1^\star - (2+\frac{1}{\theta})x_2^\star(\tau_1^\star)$. We deduce that the triplet $(x^\star,u^\star,\tau_1^\star)$ is a solution to the classical (that is, without loss control region) optimal control problem given by

$$\begin{aligned} &\text{minimize} && \tau_1 - (3+\sqrt{2})x_2(\tau_1), \\ &\text{subject to} && (x,u,\tau_1) \in \mathrm{AC}([0,\tau_1],\mathbb{R}^2) \times \mathrm{L}^\infty([0,\tau_1],\mathbb{R}) \times (0,+\infty), \\ &&& \dot{x}_1(t) = x_2(t), \quad \text{a.e. } t \in [0,\tau_1], \\ &&& \dot{x}_2(t) = u(t), \quad \text{a.e. } t \in [0,\tau_1], \\ &&& x(0) = x^0, \quad x_1(\tau_1) = 0, \\ &&& u(t) \in [-1,1], \quad \text{a.e. } t \in [0,\tau_1]. \end{aligned}$$

Applying the classical Pontryagin maximum principle, there exists a nontrivial pair $(q,q^0) \in \mathrm{AC}([0,\tau_1^\star],\mathbb{R}^2) \times \mathbb{R}_+$ such that $-\dot{q}_2 = q_1$ is constant (and thus $q_2(t) = q_2(\tau_1^\star) + q_1(\tau_1^\star - t)$ is affine) over $[0,\tau_1^\star]$ and $q_1 x_2^\star(t) + q_2(t)u^\star(t) = q^0$ over $[0,\tau_1^\star]$ (and thus $q_2$ vanishes at most one time over $[0,\tau_1^\star]$ by contradiction), but also $u^\star(t) = \mathrm{sign}(q_2(t))$ over $[0,\tau_1^\star]$ and $q_2(\tau_1^\star) = q^0(3+\sqrt{2})$. Since $x_2^\star(\tau_1^\star) < 0$, we deduce that

$$q_1 = \frac{q^0}{x_2^\star(\tau_1^\star)}(1 - (3+\sqrt{2})u_-^\star(t_1^\star)),$$

and thus $q^0 \neq 0$ (by contradiction) that we renormalize so that $q^0 = 1$. We obtain that $q_2(\tau_1^\star) > 0$, $u_-^\star(\tau_1^\star) = 1$ and thus $q_1 > 0$. Finally we get that $q_2(t) > 0$ and thus $u^\star(t) = 1$ over $[0,\tau_1^\star]$, which concludes the proof in the case $x^0 \in \Omega_0 \cap \mathrm{X}_1 = \Omega_6$.

# 4 Comments and perspectives

This section is dedicated to comments on Theorem 3.1 and its proof (Section 4.1) and to several perspectives about the concept of loss control region for further research works (Section 4.2).

## 4.1 Comments on Theorem 3.1 and its proof

**Remark 4.1.** In connection with Remark 2.2, we emphasize that several well known properties observed in the classical (that is, without loss control region) minimal time problem for the double integrator are broken when considering a loss control region in the control system. First of all, we observe that some optimal trajectories obtained in Theorem 3.1 (from different initial conditions) intersect each other (see Figure 6). We deduce that, in the presence of a loss control region in the control system, the dynamical programming principle does not hold true and that the optimal control $u^\star$ cannot be expressed as a feedback.

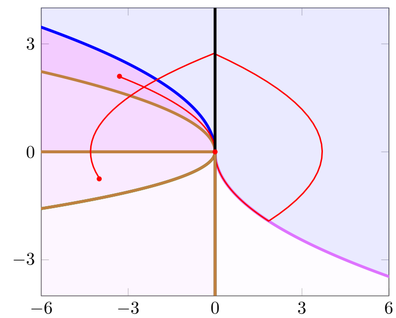


**Figure 6:** Illustration of intersecting optimal trajectories in Theorem 3.1.

Furthermore we observe that, for initial conditions in $\Omega_2 \cup \Gamma_2 \cup \Omega_3 \cup \Gamma_3 \cup \Omega_4$, the optimal control $u^\star$ takes a moderated value $\mu^\star$ in the interior $(-1, 1)$ of the control constraint set, and therefore does not saturate it.

**Remark 4.2.** In this remark we comment on the different behaviors observed in Theorem 3.1.

(i) For initial conditions in $\Omega_2 \cup \Gamma_2$, the optimal control $u^\star$ consists in taking a moderated value $\mu^\star \in (-1, 1)$ until reaching the origin $0_{\mathbb{R}^2}$. This behavior differs from the optimal strategies observed in classical (that is, without loss control region) minimum time problems (such as double integrator or harmonic oscillator). Indeed, for classical minimum time problems governed by affine systems with respect to the control, the target is usually reached by a so-called *bang-bang control* (apart singular arc and Fuller's phenomenon).

(ii) For initial conditions in $\Omega_3 \cup \Gamma_3 \cup \Omega_4$, the optimal control $u^\star$ takes a moderated value $\mu^\star \in (-1, 1)$ until reaching $\Omega_1 \cap \partial \mathrm{X}$ and then is bang-bang until reaching the origin $0_{\mathbb{R}^2}$. This analysis reveals that a moderated value can be associated with a bang-bang policy. Again, this feature differs from what is observed in classical settings.

(iii) For initial conditions in $\Omega_3$, let us introduce the set $\Sigma$ defined by

$$\Sigma := \left\{ \left( -\frac{2}{\theta} x_2^2, x_2 \right) \mid x_2 > 0 \right\},$$

which corresponds to the set of points $x^0 \in \Omega_3$ such that $\lambda^-(x^0) = 0$. Therefore, for initial conditions in $\Omega_3$, we observe the three situations illustrated in Figure 7 in which the curve $\Sigma$ is depicted in orange.

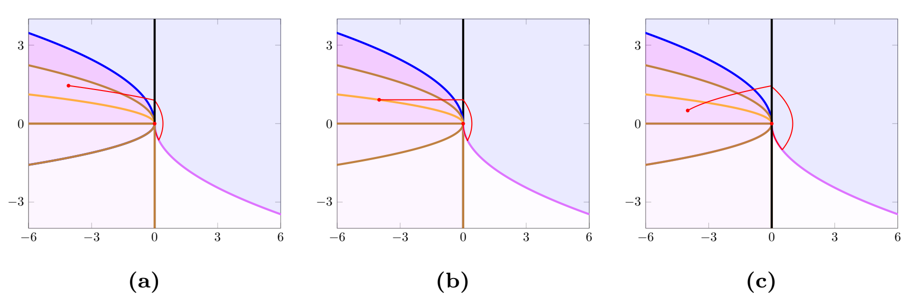


**Figure 7:** Three situations for initial conditions in $\Omega_3$.

In Figure 7(b), we observe that the part of the trajectory $x^\star$ in the region $\mathrm{X}_2$ is an horizontal segment. This is due to the fact that, when $x^0 \in \Sigma$, it holds that $u^\star(t) = \mu^\star = \lambda^-(x^0) = 0$ in the region $\mathrm{X}_2$. Finally, contrary to what Figures 7(a) and 7(c) above might suggest, the part of the trajectory $x^\star$ in the region $\mathrm{X}_2$ is a not a segment, but a parabolic curve.

(iv) For initial conditions in $\Omega_3 \cup \Gamma_3 \cup \Omega_4 \cup \Gamma_4 \cup \Omega_5 \cup \Gamma_5 \cup \Omega_6$, we observe that the optimal control $u^\star$ admits two switching times. The structure of the optimal control in the presence of a loss control region is thus more complex than in the classical setting (for which every optimal control has at most one switching time).

(v) For initial conditions in $\Omega_6$, we point out a non-intuitive property. Indeed it can be proved from the classical Pontryagin maximum principle that the fastest way to reach $\Gamma_5$ from $\Omega_6$ consists in taking $u(t) = -1$. However, from Theorem 3.1, the optimal control $u^\star$ from an initial condition in $\Omega_6$ consists in taking $u^\star(t) = +1$ until reaching $\Gamma_5$. We deduce that the optimal strategy in Theorem 3.1 from an initial condition in $\Omega_6$ does not consist in reaching $\Gamma_5$ in minimal time.

**Remark 4.3.** As it is shown in the proof of Theorem 3.1, every optimal trajectory visits the loss control region $\mathrm{X}_2$ at most one time. In view of this behavior, a direct analysis (that is, without using the Pontryagin maximum principle stated in Proposition 2.2) may lead to the same Theorem 3.1. Nevertheless our approach should also apply to more complicate situations in which the optimal trajectory would visit a loss control region more than one time. In particular it could be used to tackle a loss control region in a minimal time problem associated with the harmonic oscillator, or in an optimal control problem associated with an oscillatory controlled system (such as the Lotka-Volterra system [25]).

## 4.2 Several perspectives

In this paper we have investigated the minimal time problem for the double integrator with a loss control region given by $\mathrm{X}_2 := \{x \in \mathbb{R}^2 \mid x_1 < 0\}$. Of course this study could be extended to many different loss control regions, such as the ones depicted in Figure 8.

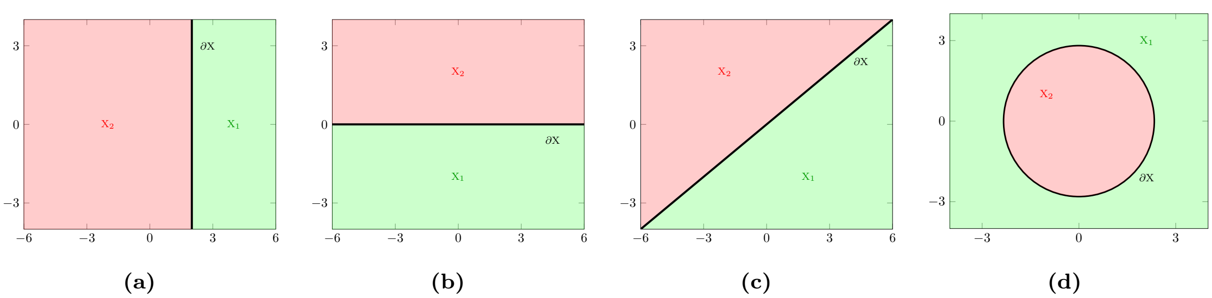


**Figure 8:** Illustration of other possible loss control regions.

As mentioned in Remark 4.3, this study could be extended also to other control systems than the double integrator, such as the harmonic oscillator, Zermelo-type models, controlled Lotka-Volterra systems, etc. In this section our aim is to discuss several perspectives in view of a general (theoretical and/or numerical) treatment of optimal control problems including loss control regions. These objectives are of course out of the scope of the present paper and could be the subject of further research works.

**Controllability/reachability.** When adding a loss control region in the control system, it is clear that the set of admissible controls is reduced. As a consequence, controllability issues may appear. Typically, for a minimum time problem, depending on the choice of the loss control region, the target may not be reachable. Therefore a natural question concerns the robustness of the reachability of a target under the presence of a loss control region. We refer to [15] for a similar study in the context of control sampling. From a more general point of view, one could be interested in finding sufficient conditions on the control system, the target and the loss control region to ensure the reachability of the target. As a natural first step, one may look for including loss control regions in the classical Kalman theory about controllability of linear control systems.

**Existence of an optimal control.** In this paper note that the existence of a solution to Problem (P) has not been investigated. From a general point of view, one may be interested in extending the classical Filippov's existence theorem [20] to the context of loss control regions (for minimal time problems or more general Bolza optimal control problems). We believe that, if one is able to give an upper bound on the number of times the state visits the loss control region, then existence of an optimal control could be ensured under standard hypotheses (such as compactness of the set of admissible triplets trajectory/control/final time and convexity of the so-called *augmented velocities set*).

**Pontryagin maximum principle.** Proposition 2.2 provides first-order necessary optimality conditions in a Pontryagin form for a general minimum time problem including a loss control region, but under strong hypotheses (see Remark 2.3). This result was sufficient to investigate Problem (P) in Section 3. In future works, we shall extend Proposition 2.2 to a more general setting. First we want to cover the case of a general Bolza optimal control problem including mixed initial-final state constraints. Second, the transverse assumption ($\mathcal{A}$) does not hold in general. Therefore we want to extend Proposition 2.2 under a weaker transverse assumption (involving only the optimal pair $(x^\star, u^\star)$ for example). This could be done by using an augmentation technique as in the proof of Proposition 2.2. It would serve to solve more involved application models involving loss control regions, from a theoretical approach as well as by using numerical tools as explained below.

**Numerical methods.** There are two predominant kinds of numerical methods in classical optimal control theory. In one hand, *direct numerical methods* consist in a full discretization of the optimal control problem which results into a constrained finite-dimensional optimization problem that can be solved using standard numerical optimization algorithms. On the other hand, *indirect numerical methods* consist in the numerical solving by a shooting method of the boundary value problem satisfied by the pair state/costate given by the Pontryagin maximum principle. We emphasize that neither method is inherently better to the other. For a detailed discussion on the advantages and drawbacks of each method, we refer to [28, pp. 178-179]. A challenge to solve application problems involving loss control regions would be to extend direct/indirect numerical methods to that context. The main focus would be the possibility to constrain the control to be constant without knowing in advance when and how many times the corresponding state visits the loss control region. Furthermore note that the extension of indirect numerical methods is anyway conditioned in a first place by the extension of the Pontryagin maximum principle mentioned above.

**Some insights into the HJB equation.** In the literature, it is well known [3, 16, 27] how to define the Hamilton-Jacobi-Bellman (HJB) equation associated with the classical Problem (CP). As well, the characterization of its value function $V$ as the unique solution (in a certain sense) to the HJB equation is also well known. In contrast, when considering a loss control region, it is not clear how to define a HJB equation associated with Problem (P) and also if the corresponding value function $W$ is a solution (in a certain sense) to this extended HJB equation. The aim of this paragraph is to give an insight into this question. In the sequel, in order to ease the notations, we write $x$ in place of $x^0$ for the initial condition. Recall that the value function $V$ associated with Problem (CP) is given by

$$V(x) = \begin{cases} 2\sqrt{\frac{x_2^2}{2} + x_1} + x_2 & \text{if } x \in \Omega_1, \\ 2\sqrt{\frac{x_2^2}{2} - x_1} - x_2 & \text{if } x \in \Omega_0, \end{cases}$$

and that it is continuous and $C^1$-piecewise. Moreover, setting $H : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ the corresponding Hamiltonian defined by $H(x,u,p) := p_1 x_2 + p_2 u$ for all $(x,u,p) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$, the value function $V$ can be characterized as the unique continuous and $C^1$-piecewise solution to the HJB equation

$$1 + \min_{u \in [-1,1]} H(x, u, \nabla V(x)) = 0, \quad x \in \Omega_1 \cup \Omega_0,$$

that can be rewritten as

$$1 + \partial_1 V(x_1,x_2)x_2 - |\partial_2 V(x_1,x_2)| = 0, \quad x \in \Omega_1 \cup \Omega_0. \tag{HJB$_{\text{CP}}$}$$

Going back to our setting, one can show (from simple computations and from the results obtained in the proof of Theorem 3.1) that the value function $W$ associated with Problem (P) is continuous, $\mathrm{C}^1$-piecewise and that it fulfills the equalities

$$\begin{aligned} 1+\partial_1 W(x_1,x_2)x_2 - |\partial_2 W(x_1,x_2)| = 0, &\quad \text{if } x\in\Omega_1,\\ 1+\partial_1 W(x_1,x_2)x_2 + \partial_2 W(x_1,x_2)\mu^\star(x) = 0, &\quad \text{if } x\in\Omega_2\cup\Omega_3\cup\Omega_4\cup\Omega_5\cup\Omega_6, \end{aligned} \tag{HJB$_\mathrm{P}$}$$

where $\mu^\star(x)$ is given in Theorem 3.1 for $x\in\Omega_2\cup\Omega_3\cup\Omega_4$ and $\mu^\star(x)=1$ for $x\in\Omega_5\cup\Omega_6$. Note that both HJB equations ($\mathrm{HJB_{CP}}$) and ($\mathrm{HJB_P}$) are the same in $\Omega_1$ (since Problems (CP) and (P) coincide for initial conditions in $\Omega_1$). On the contrary, when $x\notin\Omega_1$, note that the term $\min_{u\in[-1,1]} H(x,u,\nabla V(x))$ in ($\mathrm{HJB_{CP}}$) is replaced by $H(x,\mu^\star(x),\nabla V(x))$ in ($\mathrm{HJB_P}$).

Future works should investigate how to properly define a HJB equation when considering an optimal control problem involving a loss control region, as well as a characterization of the value function as the unique solution (in a certain sense) to this extended HJB equation. To this aim, a possible way could be to consider an augmented technique (as in the proof of Proposition 2.2), to apply the classical methodology [4, 22, 24] to the augmented system and try to the reverse the augmentation procedure.

# A Proof of Proposition 2.2

In this appendix we prove Proposition 2.2 by separating the two cases $x^{\mathrm{targ}}\notin\partial\mathrm{X}$ and $x^{\mathrm{targ}}\in\partial\mathrm{X}$. In the sequel, when $(\mathcal{Z},\mathrm{d}_\mathcal{Z})$ is a metric set, we denote by $\overline{\mathrm{B}}_\mathcal{Z}(z,r)$ the standard closed ball of $\mathcal{Z}$ centered at $z\in\mathcal{Z}$ and of radius $r>0$.

## A.1 The case $x^{\mathrm{targ}}\notin\partial\mathrm{X}$.

Assume that $x^{\mathrm{targ}}\notin\partial\mathrm{X}$ and let $(x^\star,u^\star,T^\star)$ be a solution to Problem (GP), associated with a partition $\mathbb{T}^\star=\{\tau_k^\star\}_{k=0,\dots,N}$ of the interval $[0,T^\star]$.

**Step 1: augmentation proceduce.** Define $(y^\star,v^\star,\lambda^\star)\in\mathrm{AC}([0,1],\mathbb{R}^{nN})\times\mathrm{L}^\infty([0,1],\mathbb{R}^{mN_1})\times\mathbb{R}^{mN_2}$ by

$$\left\{\begin{array}{ll} y_k^\star(s):=x^\star(\tau_{k-1}^\star+(\tau_k^\star-\tau_{k-1}^\star)s) & \text{for all } s\in[0,1] \text{ and all } k\in\{1,\dots,N\},\\ v_k^\star(s):=u^\star(\tau_{k-1}^\star+(\tau_k^\star-\tau_{k-1}^\star)s) & \text{for all } s\in[0,1] \text{ and all } k\in\mathcal{I}_1^\star,\\ \lambda_k^\star:=\overline{u}_k^\star & \text{for all } k\in\mathcal{I}_2^\star, \end{array}\right.$$

where $N_1:=\mathrm{card}(\mathcal{I}_1^\star)$ and $N_2:=\mathrm{card}(\mathcal{I}_2^\star)$. It is clear that the quadruplet $(y^\star,v^\star,\lambda^\star,\mathbb{T}^\star)$ is admissible for the classical (that is, without loss control region) augmented optimal control problem involving parameters and endpoint constraints given by

$$\begin{array}{ll} \text{minimize} & \tau_N,\\ \text{subject to} & (y,v,\lambda,\mathbb{T})\in\mathrm{AC}([0,1],\mathbb{R}^{nN})\times\mathrm{L}^\infty([0,1],\mathbb{R}^{mN_1})\times\mathbb{R}^{mN_2}\times\mathbb{R}^{N+1},\\ & \dot{y}(s)=g(y(s),v(s),\lambda,\mathbb{T}),\quad \text{a.e. } s\in[0,1],\\ & v(s)\in\mathrm{U}^{N_1},\quad \text{a.e. } s\in[0,1],\\ & (\lambda,\mathbb{T})\in\mathrm{U}^{N_2}\times\Delta,\\ & y_1(0)=x^0,\quad y_N(1)=x^{\mathrm{targ}},\\ & y_k(0)=y_{k-1}(1),\quad \text{for all } k\in\{2,\dots,N\},\\ & F(y_k(1))=0,\quad \text{for all } k\in\{1,\dots,N-1\}, \end{array} \tag{AP}$$

where $g=(g_k)_{k=1,\dots,N}:\mathbb{R}^{nN}\times\mathbb{R}^{mN_1}\times\mathbb{R}^{mN_2}\times\mathbb{R}^{N+1}\to\mathbb{R}^{nN}$ is defined by

$$g_k(y,v,\lambda,\mathbb{T}):=\left\{\begin{array}{ll} (\tau_k-\tau_{k-1})f(y_k,v_k) & \text{if } k\in\mathcal{I}_1^\star,\\ (\tau_k-\tau_{k-1})f(y_k,\lambda_k) & \text{if } k\in\mathcal{I}_2^\star, \end{array}\right.$$

for all $(y,v,\lambda,\mathbb{T})\in\mathbb{R}^{nN}\times\mathbb{R}^{mN_1}\times\mathbb{R}^{mN_2}\times\mathbb{R}^{N+1}$ and all $k\in\{1,\dots,N\}$, and where $\Delta:=\{\mathbb{T}=\{\tau_k\}_{k=0,\dots,N}\in\mathbb{R}^{N+1}\mid 0=\tau_0\leq\tau_1\leq\dots\leq\tau_{N-1}\leq\tau_N\}$ is a nonempty closed convex subset of $\mathbb{R}^{N+1}$.

**Step 2: the quadruplet $(y^\star, v^\star, \lambda^\star, \mathbb{T}^\star)$ is a local solution to Problem** (AP)**.** Let us prove that there exists $\eta > 0$ such that $\tau_N^\star \leq \tau_N$ for any quadruplet $(y, v, \lambda, \mathbb{T})$ admissible for Problem (AP) satisfying

$$\|y - y^\star\|_{\mathrm{C}} + \|v - v^\star\|_{\mathrm{L}^1} + \|\lambda - \lambda^\star\|_{\mathbb{R}^{nN_1}} + \|\mathbb{T} - \mathbb{T}^\star\|_{\mathbb{R}^{N+1}} \leq \eta.$$

To this aim let $\eta > 0$ and $(y, v, \lambda, \mathbb{T})$ be an admissible triplet admissible for Problem (AP) satisfying the above inequality. In the sequel we explain how to reduce $\eta > 0$ (step by step, and independently of the quadruplet $(y, v, \lambda, \mathbb{T})$) to obtain that $\tau_N^\star \leq \tau_N$.

(i) First one has to reduce $\eta > 0$ so that $0 = \tau_0 < \tau_1 < \ldots < \tau_{N-1} < \tau_N$ and then one can correctly define $(x, u, T) \in \mathrm{AC}([0,T], \mathbb{R}^n) \times \mathrm{L}^\infty([0,T], \mathbb{R}^m) \times (0, +\infty)$ by

$$x(t) := y_k\left(\frac{t - \tau_{k-1}}{\tau_k - \tau_{k-1}}\right) \text{ for all } t \in [\tau_{k-1}, \tau_k] \text{ and all } k \in \{1, \ldots, N\},$$

and

$$u(t) := \begin{cases} v_k\left(\dfrac{t - \tau_{k-1}}{\tau_k - \tau_{k-1}}\right) & \text{for a.e. } t \in (\tau_{k-1}, \tau_k) \text{ if } k \in \mathcal{I}_1^\star, \\ \lambda_k & \text{for a.e. } t \in (\tau_{k-1}, \tau_k) \text{ if } k \in \mathcal{I}_2^\star, \end{cases} \qquad \text{for all } k \in \{1, \ldots, N\},$$

and $T := \tau_N$. To obtain that $\tau_N^\star \leq \tau_N$, which is equivalent to $T^\star \leq T$, it is sufficient to prove that the triplet $(x, u, T)$ is admissible for Problem (GP). This is our aim in the next step.

(ii) It is clear that the triplet $(x, u, T)$ satisfies $\dot{x}(t) = f(x(t), u(t))$ and $u(t) \in \mathrm{U}$ for almost every $t \in [0, T]$, and $x(0) = x^0$ and $x(T) = x^{\mathrm{targ}}$. Therefore, since $u$ is constant over the intervals $(\tau_{k-1}, \tau_k)$ when $k \in \mathcal{I}_2^\star$, it only remains to prove that $x$ is with values in $\mathrm{X}_1$ (resp. in $\mathrm{X}_2$) over the intervals $(\tau_{k-1}, \tau_k)$ when $k \in \mathcal{I}_1^\star$ (resp. when $k \in \mathcal{I}_2^\star$). This is possible by reducing $\eta > 0$ and by using the transverse assumption $(\mathcal{A})$, the compactness of $\mathrm{U}$, the fact that $x^{\mathrm{targ}} \notin \partial \mathrm{X}$ and the openness of the regions $\mathrm{X}_1$ and $\mathrm{X}_2$.

**Step 3: application of the classical Pontryagin maximum principle.** Consider the Hamiltonian $\widetilde{H} : \mathbb{R}^{nN} \times \mathbb{R}^{mN_1} \times \mathbb{R}^{mN_2} \times \mathbb{R}^{N+1} \times \mathbb{R}^{nN} \to \mathbb{R}$ associated with Problem (AP) defined by

$$\begin{aligned}\widetilde{H}(y, v, \lambda, \mathbb{T}, q) &:= \langle q, g(y, v, \lambda, \mathbb{T}) \rangle_{\mathbb{R}^{nN}} \\ &= \sum_{k \in \mathcal{I}_1^\star} (\tau_k - \tau_{k-1}) \langle q_k, f(y_k, v_k) \rangle_{\mathbb{R}^n} + \sum_{k \in \mathcal{I}_2^\star} (\tau_k - \tau_{k-1}) \langle q_k, f(y_k, \lambda_k) \rangle_{\mathbb{R}^n},\end{aligned}$$

for all $(y, v, \lambda, \mathbb{T}, q) \in \mathbb{R}^{nN} \times \mathbb{R}^{mN_1} \times \mathbb{R}^{mN_2} \times \mathbb{R}^{N+1} \times \mathbb{R}^{nN}$. From the classical Pontryagin maximum principle applied to the quadruplet $(y^\star, v^\star, \lambda^\star, \mathbb{T}^\star)$, there exists a nontrivial pair $(q, q^0) \in \mathrm{AC}([0,1], \mathbb{R}^{nN}) \times \mathbb{R}_+$ such that:

(i) It holds that $-\dot{q}(s) = \nabla_y \widetilde{H}(y^\star(s), v^\star(s), \lambda^\star, \mathbb{T}^\star, q(s))$ for almost every $s \in [0,1]$.

(ii) It holds that $v^\star(s) \in \arg\max_{\omega \in \mathrm{U}^{N_1}} \widetilde{H}(y^\star(s), \omega, \lambda^\star, \mathbb{T}^\star, q(s))$ for almost every $s \in [0,1]$.

(iii) It holds that

$$\int_0^1 \nabla_\lambda \widetilde{H}(y^\star(s), v^\star(s), \lambda^\star, \mathbb{T}^\star, q(s))\, ds \in \mathrm{N}_{\mathrm{U}^{N_2}}[\lambda^\star].$$

(iv) It holds that $q_{k+1}(0) - q_k(1) = \nu_k \nabla F(y_k^\star(1))$ for some $\nu_k \in \mathbb{R}$ for all $k \in \{1, \ldots, N-1\}$.

(v) It holds that

$$\int_0^1 \nabla_{\mathbb{T}} \widetilde{H}(y^\star(s), v^\star(s), \lambda^\star, \mathbb{T}^\star, q(s))\, ds \in q^0 e + \mathrm{N}_\Delta[\mathbb{T}^\star],$$

where $e = (0, \ldots, 0, 1)^\top \in \mathbb{R}^{N+1}$.

**Step 4: construction of the nontrivial pair** $(p, p^0)$**.** Define $p^0 := q^0 \in \mathbb{R}_+$ and $p \in \mathrm{PAC}_{\mathbb{T}^\star}([0,T^\star],\mathbb{R}^n)$ by $p(0) := q_1(0)$, $p(T^\star) := q_N(1)$ and by

$$p(t) := q_k\left(\frac{t-\tau^\star_{k-1}}{\tau^\star_k-\tau^\star_{k-1}}\right) \text{ for all } t \in (\tau^\star_{k-1},\tau^\star_k) \text{ and all } k \in \{1,\dots,N\}.$$

From nontriviality of the pair $(q, q^0)$, it is clear that the pair $(p, p^0)$ is also nontrivial. Then the first four above items allows to obtain the first four items of Proposition 2.2. At this step, one can obtain that, for all $k \in \{1,\dots,N\}$, there exists $c_k \in \mathbb{R}$ such that $H(x^\star(t), u^\star(t), p(t)) = c_k$ for almost every $t \in (\tau^\star_{k-1},\tau^\star_k)$. Indeed, the case $k \in \mathcal{I}^\star_1$ is obtained from the Hamiltonian maximization condition over $(\tau^\star_{k-1},\tau^\star_k)$ and [19, Theorem 2.6.3], and the case $k \in \mathcal{I}^\star_2$ is easily obtained from the constancy of $u^\star$ and the Hamiltonian system over $(\tau^\star_{k-1},\tau^\star_k)$.

Finally the above fifth item allows to obtain, in one hand, that $c_k = c_{k-1}$ for all $k \in \{2,\dots,N\}$ and, in the other hand, that $c_N = q^0$, which concludes the proof of Proposition 2.2 in the case $x^{\mathrm{targ}} \notin \partial \mathrm{X}$.

## A.2 The case $x^{\mathrm{targ}} \in \partial\mathrm{X}$

Assume that $x^{\mathrm{targ}} \in \partial\mathrm{X}$ and let $(x^\star, u^\star, T^\star)$ be a solution to Problem (GP), associated with a partition $\mathbb{T}^\star = \{\tau^\star_k\}_{k=0,\dots,N}$ of the interval $[0,T^\star]$. For any $\varepsilon > 0$ small enough (precisely $0 < \varepsilon < T^\star - \tau^\star_{N-1}$), we denote by $T_\varepsilon := T^\star - \varepsilon$ and, from a standard dynamical programming argument, one can easily see that $(x^\star, u^\star, T_\varepsilon)$ is a solution to Problem (GP) when replacing $x^{\mathrm{targ}}$ by $x^{\mathrm{targ}}_\varepsilon := x^\star(T_\varepsilon) \notin \partial\mathrm{X}$. Therefore one can follow exactly the proof of Proposition 2.2 detailed in the previous subsection. For any $\varepsilon > 0$ small enough, it provides the existence of a nontrivial pair $(q_\varepsilon, q^0_\varepsilon) \in \mathrm{AC}([0,1],\mathbb{R}^{nN}) \times \mathbb{R}_+$ satisfying the above five items (just replacing $T^\star$ by $T_\varepsilon$ everywhere). Using the fact that the nontrivial pair $(q_\varepsilon, q^0_\varepsilon)$ can be renormalized (since it is defined up to a positive multiplicative constant), compactness arguments and the fact that $T_\varepsilon \to T^\star$ when $\varepsilon \to 0$, one can obtain the existence of a nontrivial pair $(q, q^0) \in \mathrm{AC}([0,1],\mathbb{R}^{nN}) \times \mathbb{R}_+$ satisfying the above five items (with $T^\star$, and not with $T_\varepsilon$). Finally the proof of Proposition 2.2 is concluded in a similar way than Step 4 of the previous subsection.

# Acknowledgements

The authors would like to thank Prof. Olivier Cots (INP-ENSEEIHT & IRIT, Toulouse, France) for fruitful exchanges.

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