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Minimum time synthesis for the double integrator including a loss control region

Térence Bayen* Anas Bouali† Loïc Bourdin‡

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Abstract

In this paper, we address the minimum time problem to reach the origin for the double integrator, and, in addition to the classical version of this problem, we assume the existence of a subset Ω of the state space such that the control should remain constant as long as the state remains in Ω . Such a set corresponds to a *loss-control region*, and, in particular, it prevents switching points to occur in Ω . This framework entails the use of a spatially hybrid maximum principle including parameters in order to characterize optimal trajectories. The purpose of this paper is to highlight the use of this principle in this setting. We provide an exact analytical description of optimal trajectories using the necessary optimality conditions that involve a jump on the covector at the interface between Ω and its complement, and the averaged gradient condition for determining an optimal constant control value in Ω .

1 Introduction

Geometric control theory is developed since the sixties and it now plays a central role in optimal control theory. Based on the Pontryagin Maximum Principle [21] and differential geometry, it provides mathematical tools in order to determine optimal controls and to synthesize optimal feedback controls related to a given cost functional [1, 10, 11, 22]. For minimum time problems to reach the origin, several well-known examples in the literature illustrate various phenomena arising in this framework. We can cite for instance the double integrator problem for which every optimal control has zero or one switching point, the harmonic oscillator for which every optimal control has a finite number of switching points (depending on the initial condition), and Fuller's problem which provides the most standard example of an optimal synthesis that exhibits an infinite number of switching points over a finite time interval.

The objective of this paper is to study a variant of the double integrator problem in which the control should be constant as long as the corresponding state belongs to a given subset $\Omega \subset \mathbb{R}^2$ of the state space called a *loss-control region* (or *non-control region*). This problem can be written as follows:

$$\begin{aligned} & \text{minimize} && T, \\ & \text{subject to} && \ddot{x}(t) = u(t), \text{ a.e. } t \in [0, T], \\ & && (x(0), \dot{x}(0)) = x_0 \in \mathbb{R}^2 \quad \text{and} \quad (x(T), \dot{x}(T)) = (0, 0), \\ & && u(t) \in [-1, 1] \quad \text{a.e. } t \in [0, T], \quad u \text{ is constant in } \Omega, \text{ and } T \in \mathbb{R}_+, \end{aligned} \tag{1.1}$$

where the initial condition is fixed. The consideration of optimal control problems involving non-control regions (*i.e.*, subsets of the state space in which the control is “frozen” to a value to be determined as long as the state belongs to such a set) is motivated by various applications. For instance, in the context of aerospace, this question arises in order to take into account the shadow effect in the the low-thrust transfer problem [18, 20]. Other examples arise when considering optimal control problems in the setting of viability theory [2]. In this context, in order to reduce operating costs, constant controls can be applied whenever the system belongs to

*Avignon Université, Laboratoire de Mathématiques d'Avignon (EA 2151) F-84018 terence.bayen@univ-avignon.fr

†Avignon Université, Laboratoire de Mathématiques d'Avignon (EA 2151) F-84018 anas.bouali@univ-avignon.fr

‡Institut de recherche XLIM. UMR CNRS 7252. Université de Limoges, France. loic.bourdin@unilim.fr

a safety zone (typically, the viability kernel), see, *e.g.*, time crisis problems [8]. Our choice to focus in this paper on the double integrator is twofold. First, we shall see that, depending on the choice of Ω , the analysis of optimal trajectories can be much more involved than in the usual case. Second, as far as we know, problems like (1.1) have not been treated in the literature. So, the study that we propose in this paper could serve in the community of optimal control in order to highlight the construction of optimal paths in this framework (see, *e.g.*, [17] in which the double integrator problem is also investigated but with a pathwise constraint).

At this step, we can observe that (1.1) depends on the choice of the set Ω . For instance, if the optimal feedback for the classical double integrator problem¹ is constant in Ω , then, both problems with Ω (*i.e.*, Problem (1.1)) and without Ω are obviously equivalent. In this paper, we shall be interested in studying (1.1) in the case where Ω is a half space. Remind that the switching locus for the classical double integrator problem is the union of two half parabola, hence, such a non-control set necessary intersects this locus leading necessarily to a different analysis than for the classical double integrator problem. Throughout this paper, we shall study (1.1) for this subset Ω , but, other similar choices for the non-control set could be possible leading to a similar analysis. Our objective is also to solve (1.1) for every initial condition for a given choice of Ω leading that way to an optimal synthesis in this setting.

Our methodology to solve (1.1) is to follow the approach of [6] which provides the necessary optimality conditions (in Pontryagin's form) for optimal control problems involving loss-control regions. The main result of [6] is an averaged gradient condition² to characterize optimal constant values of the control (whenever the state belongs to a loss control region) as well as the (usual) Hamiltonian maximization condition in the complement of those regions. It is worth mentioning that the framework of [6] is a particular instance of [5] in which a spatially³ hybrid framework with regionally constant parameters is considered, but without terminal constraints. Another important feature mentioned in [6] is that the system may visit more than once a given stratum, but, the constant value of the control may change. Since (1.1) involves terminal constraints on the state and [5, 6] do not consider this issue, we shall rather use the necessary conditions developed in [7] which considers regionally hybrid optimal control problems including initial-terminal constraints and a constant parameter. The use of [7] in place of [5] will be made possible by proving an important property related to optimal solutions of (1.1), namely that every optimal trajectory of (1.1) visits at most once the non-control set Ω . The paper is structured as follows.

- In Section 2, we consider in a general framework a Mayer optimal control problem including a terminal constraint and a loss-control region. We provide the necessary optimality conditions for an optimal solution that visits at most once time the set Ω . This result will be used in the next section.
- In Section 3, we provide an exact analytical solution to (1.1) when Ω is a half-plane (Theorem 3.2). Doing so, we first prove that an optimal solution visits at most once the set Ω . Next, the optimal synthesis is obtained analytically by application of the hybrid maximum principle (HMP) with parameter and the application of the averaged gradient condition as in [7]. We end-up by giving some remarks about the Hamilton-Jacobi-Bellman Equation (HJB) in this setting.

Finally, we would like to point out that, for more general controlled systems, if an optimal path visits more than once the non-control set, then, we do not know which principle (in Pontryagin's form) to apply in order to write the necessary optimality conditions. Indeed, this would require the use of a spatially hybrid maximum principle with regionally constant parameter (as in [7, 6]) together with terminal constraints, which is not available in the literature to our best knowledge.

2 A general framework

In this section, we consider a general Mayer optimal control problem including a loss control region Ω and a terminal constraint set. Based on [7], we provide first order necessary optimality conditions for an optimal solution visiting at most once the non-control region Ω .

Given $d \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{+\infty\}$, $L^r(I, \mathbb{R}^d)$ stands for the Lebesgue space of r -integrable functions defined on I with values in \mathbb{R}^d , endowed with its usual norm $\|\cdot\|_{L^r}$ and $AC(I, \mathbb{R}^d)$ denotes the space of absolutely

¹*i.e.*, without considering u constant whenever the state is in Ω .

²This necessary condition occurs in optimal control problems involving a constant parameter [9, 12, 14].

³In contrast with *switched systems*, see, *e.g.*, [16], a change of dynamics only occurs regionally, *i.e.*, whenever the state of the system goes from a region to another one.

continuous functions. Also, $\text{Int}(A)$ denotes the interior of a subset $A \subset \mathbb{R}^k$, ∂A its boundary, \bar{A} its closure, $A^c := \mathbb{R}^n \setminus A$ its complement, and $\langle \cdot, \cdot \rangle_{\mathbb{R}^k}$ stands for the inner product in \mathbb{R}^k where $k \in \mathbb{N}^*$. The (convex) normal cone to a convex subset $A \subset \mathbb{R}^k$ is defined as $N_A(x) := \{\xi \in \mathbb{R}^k ; \langle \xi, y - x \rangle_{\mathbb{R}^k} \leq 0 ; \forall y \in A\}$ for $x \in A$. Let $m, n, d \in \mathbb{N}^*$, $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ of class C^1 , $g : \mathbb{R}^n \rightarrow \mathbb{R}^d$ of class C^1 , and $\phi : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ of class C^1 . We consider the optimal control problem:

$$\begin{aligned}
& \text{minimize} && \phi(T, x(T)), \\
& \text{subject to} && (x, u, T) \in \text{AC}([0, T], \mathbb{R}^n) \times L^\infty([0, T], \mathbb{R}^n) \times \mathbb{R}_+, \\
& && \dot{x}(t) = f(x(t), u(t)) \quad \text{a.e. } t \in [0, T], \\
& && x(0) = x_0 \quad \text{and} \quad g(x(T)) \in S, \\
& && u(t) \in U \quad \text{a.e. } t \in [0, T] \text{ and } u \text{ is constant in } \Omega,
\end{aligned} \tag{2.1}$$

where the initial condition $x_0 \in \mathbb{R}^n$ is fixed, U is a non-empty closed convex subset of \mathbb{R}^m , S is a non-empty closed convex subset of \mathbb{R}^d with a boundary of class C^1 , and the so-called *non-control* set $\Omega \subset \mathbb{R}^n$ is open with a boundary of class C^1 . Let us insist on the fact that the constraint on the control u means that $t \mapsto u(t)$ should remain constant (to some unknown value to be determined) as long as $x(t) \in \Omega$. If for instance, $x(t) \in \Omega$ over $(\tau_1, \tau_2) \cup (\tau_3, \tau_4)$ (with $\tau_1 < \tau_2 < \tau_3 < \tau_4$) and $x(t) \notin \Omega$ over $[0, T] \setminus ((\tau_1, \tau_2) \cup (\tau_3, \tau_4))$, this means that for a.e. $t \in (\tau_1, \tau_2)$, resp., $t \in (\tau_3, \tau_4)$, one has $u(t) = \bar{u}_1$, resp., $u(t) = \bar{u}_2$ for some $\bar{u}_1, \bar{u}_2 \in U$ (not necessarily equal). In what follows, we also do not discuss the existence of an optimal control of (2.1) which is not our purpose here (we shall simply assume the existence of an optimal control). By admissible solution, we mean a triplet $(x, u, T) \in \text{AC}([0, T], \mathbb{R}^n) \times L^\infty([0, T], \mathbb{R}^n) \times \mathbb{R}_+$ satisfying the constraints of Problem 2.1. Before stating the HMP in our setting, let us introduce the notion of transverse crossing time.

Definition 2.1. *Given an admissible solution (x, u, T) of (2.1), we say that $t_i \in (0, T)$ is a transverse crossing time if there is $\alpha > 0$ such that for every time $t \in [t_i - \alpha, t_i]$, resp. $(t_i, t_i + \alpha]$, one has $x(t) \in \Omega$, resp. $x(t) \in \Omega^c$, if u is left and right continuous at $t = t_i$ (with values $u(t_i^\pm)$), and if the transverse condition holds true:*

$$\langle \nabla F_i(x(t_i)), f(x(t_i), u(t_i^-)) \rangle_{\mathbb{R}^n} > 0 \quad \text{and} \quad \langle \nabla F_i(x(t_i)), f(x(t_i), u(t_i^+)) \rangle_{\mathbb{R}^n} > 0,$$

where $F_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is a C^1 mapping describing locally the boundary of Ω .

Define the Hamiltonian associated with (2.1) as $H(x, p, u) := \langle p, f(x, u) \rangle_{\mathbb{R}^n}$ where $p \in \mathbb{R}^n$.

Theorem 2.1. *Let $x_0 \in \mathbb{R}^n \setminus \partial\Omega$ and let (x^*, u^*, T^*) be an optimal solution of (2.1) such that $x^*(T^*) \notin \partial\Omega$ and such that g is submersive at $x^*(T^*)$. Suppose that x^* has at most two consecutive transverse crossing times $t_1, t_2 \in (0, T^*)$ respectively from Ω^c into Ω and from Ω into Ω^c . Then, there is $p^0 \in \{0, 1\}$ and a piecewise absolutely continuous function $p : [0, T^*] \rightarrow \mathbb{R}^n$ with $(p^0, p(\cdot)) \neq 0$ and such that:*

- the Hamiltonian system is satisfied:

$$\dot{x}^*(t) = \nabla_p H(x^*(t), p(t), u^*(t)) \quad \text{and} \quad -\dot{p}(t) = \nabla_x H(x^*(t), p(t), u^*(t)) \quad \text{a.e. } t \in [0, T^*],$$

- there is $\xi \in N_S(g(x^*(T^*)))$ such that the transversality condition holds true:

$$-p(T^*) = p^0 \nabla_x \phi(T^*, x^*(T^*)) + \nabla g(x^*(T^*))^\top \xi,$$

- the Hamiltonian condition holds true:

$$u^*(t) \in \arg \max_{\omega \in U} H(x^*(t), p(t), \omega) \quad \text{a.e. } t \in (0, T^*) \setminus [t_1, t_2],$$

- for $i = 1, 2$, there is $\nu_i \in \mathbb{R}$ such that p has at most two discontinuity points at $t = t_i$:

$$p(t_i^+) - p(t_i^-) = \nu_i \nabla F_i(x^*(t_i)), \quad i = 1, 2,$$

- there is $\bar{u} \in U$ satisfying $u^*(t) = \bar{u}$ for a.e. $t \in [t_1, t_2]$ together with the inclusion:

$$\int_{t_1}^{t_2} \nabla_u H(x^*(t), p(t), \bar{u}) dt \in N_U(\bar{u}).$$

Proof. Since we only consider optimal solutions of (2.1) visiting at most once the set Ω , we can transform the controlled dynamics in (2.1) in $n + 1$ space variables (x, \bar{u}) adding the equation $\dot{\bar{u}} = 0$ over the time period. This transformation yields the controlled system in \mathbb{R}^{n+1}

$$\begin{cases} \dot{x}(t) = \tilde{f}(x(t), u(t), \bar{u}(t)), \\ \dot{\bar{u}}(t) = 0, \end{cases} \quad \text{a.e. } t \in [0, T], \quad (2.2)$$

where the hybrid dynamics $\tilde{f} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is defined as

$$\tilde{f}(x, u, \bar{u}) := \begin{cases} f(x, u) & \text{if } x \in \Omega^c, \\ f(x, \bar{u}) & \text{if } x \in \Omega. \end{cases}$$

Because t_i , $i = 1, 2$, are transverse crossing times, g is submersive at $x^*(T^*)$, and $x_0, x^*(T^*) \notin \partial\Omega$, we are in a position to apply the regionally HMP ([7, Theorem 3.1]) to Problem (2.1) with the dynamics (2.2) in place of f . This provides the existence of a piecewise absolutely continuous adjoint vector p as in Theorem 2.1. \square

Remark 2.1. (i). *The scalar ν_i in the jump of p at $t = t_i$ can be explicitated using that the Hamiltonian function $t \mapsto H(x^*(t), p(t), u^*(t))$ is constant a.e. over $[0, T^*]$ (we refer to [7] for this property).*

(ii). *Theorem 2.1 also covers the case where x^* has zero or a single (transverse) crossing time.*

(iii). *It may be restrictive to suppose that an optimal path has at most two crossing times, however, depending on the example under consideration, this property can be verified as we do for the double integrator (see Section 3). Of course, this property depends on the dynamics and on the set Ω .*

(iv). *If an optimal path visits more than once the set Ω , the necessary optimality conditions would require the use of a HMP including a regionally constant parameter as in [5]. Indeed, in that case, an optimal path may enter the set Ω over different sub-intervals of $[0, T^*]$ with different values for the constant control. Observe that in [5], the necessary optimality conditions do not involve terminal conditions in contrast with the HMP in [7] (which does not cover the case of regionally constant parameters). Hence, to the best of our knowledge, it is not clear how to write necessary optimality conditions for optimal control problems such as (2.1) with a nominal trajectory visiting more than two times the set Ω . However, it is to expect that the HMP as in [7] can be extended to the setting of hybrid optimal control problems including regionally constant parameters using augmentation techniques as in [5] (see also, e.g., [3]).*

3 Construction of the optimal synthesis

3.1 Statement of the problem and preliminary properties

Given some initial condition $x^0 := (x_1^0, x_2^0) \in \mathbb{R}^2$, the minimal time control problem governed by the double integrator can be stated as

$$\begin{aligned} & \text{minimize} && T, \\ & \text{subject to} && \begin{aligned} \dot{x}_1(t) &= x_2(t) & \text{a.e. } t \in [0, T], \\ \dot{x}_2(t) &= u(t) & \text{a.e. } t \in [0, T], \\ (x_1(0), x_2(0)) &= (x_1^0, x_2^0) \in \mathbb{R}^2 & \text{and } (x_1(T), x_2(T)) = (0, 0), \\ u(t) &\in [-1, 1] & \text{a.e. } t \in [0, T] \text{ and } T \in \mathbb{R}_+. \end{aligned} \end{aligned} \quad (3.1)$$

As usual, T (or T_u) is the first entry time of an admissible trajectory into the target point and $u \in L^\infty([0, T], [-1, 1])$. We now recall the minimal time synthesis for (3.1), see Fig. 1. Let $\Gamma := \Gamma_- \cup \Gamma_+$ where

$$\Gamma_- := \left\{ \left(-\frac{x_2^2}{2}, x_2 \right); x_2 \geq 0 \right\}; \quad \Gamma_+ := \left\{ \left(\frac{x_2^2}{2}, x_2 \right); x_2 \leq 0 \right\},$$

and let $\Gamma^>$, resp. $\Gamma^<$ be the (strict) epigraph, resp. hypograph of Γ .

Theorem 3.1. *The minimal time synthesis can be described in feedback form over \mathbb{R}^2 as follows:*

$$u^\dagger[x_1, x_2] := \begin{cases} -1 & \text{if } x \in \Gamma^> \cup \Gamma_-, \\ +1 & \text{if } x \in \Gamma^< \cup \Gamma_+. \end{cases} \quad (3.2)$$

The control $u^\dagger[\cdot]$ is an optimal feedback for (3.1) and for every initial condition $x^0 \in \mathbb{R}^2$, (3.1) has a unique open loop solution $u_{x^0}^\dagger(\cdot)$ up to a set of zero measure such that $u^\dagger(t) = u^\dagger[x(t)]$ for a.e. $t \in [0, T_{u^\dagger}]$.

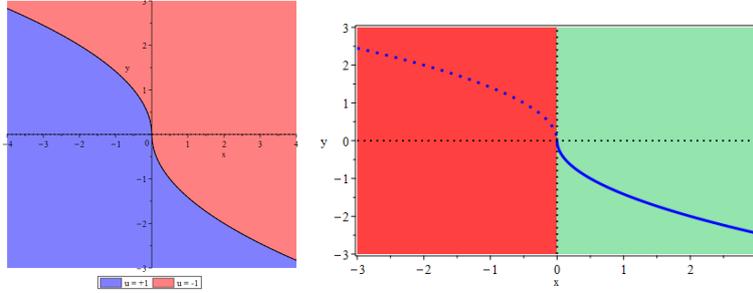


Figure 1: *Fig. left:* optimal synthesis for the double integrator (Theorem 3.1). In the blue region (including Γ_+), the feedback equals +1 and in the red region the feedback equals -1 (including Γ_-). *Fig. right:* plot of the non-control zone Ω_1 in red.

As a consequence, every optimal control $u_{x^0}^\dagger(\cdot)$ either has zero switching point (if $x^0 \in \Gamma$) or it has exactly one switching point on the switching curve Γ . We now turn to the minimal time control problem involving a non-control region. Throughout this section, we consider as non-control set the set Ω_1 defined as

$$\Omega_1 := \{(x_1, x_2) \in \mathbb{R}^2 ; x_1 < 0\},$$

and we denote by $\Omega_2 := \mathbb{R}^2 \setminus \Omega_1$ its complement, see Fig. 1. The variant of (3.1) we investigate reads as follows:

$$\begin{aligned} & \text{minimize} && T, \\ & \text{subject to} && \dot{x}_1(t) = x_2(t) \quad \text{a.e. } t \in [0, T], \\ & && \dot{x}_2(t) = u(t) \quad \text{a.e. } t \in [0, T], \\ & && (x_1(0), x_2(0)) = (x_1^0, x_2^0) \in \mathbb{R}^2 \text{ and } (x_1(T), x_2(T)) = (0, 0), \\ & && u(t) \in [-1, 1] \quad \text{a.e. } t \in [0, T], \text{ } u \text{ is constant in } \Omega_1, \text{ and } T \in \mathbb{R}_+. \end{aligned} \quad (3.3)$$

Let us remind that on each (maximal) interval of time where the trajectory belongs to Ω_1 , the control u should remain constant equal to some unknown value to be determined. We do not know in advance these time intervals and their numbers. Also, there is no need to specify the dynamics on the boundary since trajectories enter into Ω_1 or into Ω_2 transversally namely because $\langle (1, 0), (x_2, u) \rangle_{\mathbb{R}^2} = x_2$ is always non-zero (except if the trajectory has reached the origin). To prove the existence of an optimal control to (3.3), one cannot straightforwardly apply Fillipov's existence Theorem. However, we can adapt this theorem to (3.3) once we know that every admissible trajectory visiting at least twice Ω_1 is not optimal (see Lemma 3.1). Next, we denote by $v(\cdot)$, $w(\cdot)$ the value functions (defined over \mathbb{R}^2) associated respectively with (3.1) and (3.3). Since every measurable control $u(\cdot)$ with values in $[-1, 1]$ is admissible for (3.1), we have for every $x^0 \in \mathbb{R}^2$,

$$v(x^0) \leq w(x^0).$$

Given $x^0 \in \mathbb{R}^2$, we denote by $\gamma_{x^0}^\dagger$ the optimal path associated with $u_{x^0}^\dagger$, by $\gamma_{x^0}^*$ the graph of an optimal solution of (3.3) associated with an optimal control denoted by $u_{x^0}^*(\cdot)$. In what follows, we denote by \mathcal{B}_\pm a bang arc \pm (such that over a certain time interval, one has $u(t) = +1$ or $u(t) = -1$), and we also denote by \mathcal{P} an arc of trajectory for which there is a maximal time interval $[t_0, t_1]$ such that $x(t) \in \Omega_1$ for every $t \in (t_0, t_1)$. The corresponding constant control value is then denoted by $\lambda \in (-1, 1)$. An important remark about the dynamics is that the system can enter into Ω_1 , resp. Ω_2 only through the semi-axis $\{x_2 < 0\}$, resp. $\{x_2 > 0\}$. We give now an important property related to the number of times Ω_1 is visited by $\gamma_{x^0}^*$.

Lemma 3.1. *For every $x^0 \in \mathbb{R}^2$, $\gamma_{x^0}^*$ has at most two crossing times respectively from Ω_2 into Ω_1 and from Ω_1 into Ω_2 .*

Proof. From Theorem 3.1, the result is clear if $x^0 \in \Gamma^> \cup \Gamma$. Next, let $x^0 \in \Gamma^<$. If the origin is reached from Ω_1 with a constant value of the parameter, then, $\gamma_{x^0}^*$ has no crossing time and the result is also clear. Now, suppose that the trajectory $x(\cdot)$ does not reach the origin from the set Ω_1 . Then, it necessarily leaves Ω_1 through the positive x_2 -axis, say at time $t_1 \geq 0$. Suppose now that the trajectory bypasses the origin and leaves Ω_2 through the semi-axis $\{x_2 < 0\}$. By optimality of $u_{x(t_1)}^\dagger$ for (3.1), we have,

$$w(x^0) - t_1 > v(x(t_1)).$$

Consider now the trajectory defined as the concatenation of $\gamma_{x^0}^*|_{[0, t_1]}$ and $\gamma_{x(t_1)}^\dagger$. The preceding inequality shows that its cost is strictly less than the cost of $\gamma_{x^0}^*$. This contradiction ends the proof. \square

Note that once an optimal trajectory $x(\cdot)$ enters into Ω_2 through the semi-axis $\{x_1 = 0\}$ at some instant $t = t_1$, then, an optimal control is $u_{x(t_1)}^\dagger$ over the remaining time interval. Observe also that the target point belongs to the boundary of Ω_1 and that Γ_+ is tangent to the axis $\{x_1 = 0\}$ at the origin. So, in order to apply Theorem 2.1, it is desirable to reformulate (3.3) with a terminal pay-off and an augmented target set such that the terminal state does not belong to $\partial\Omega_1$. Doing so, we consider the optimal control problem:

$$\begin{aligned} & \text{minimize} && \tau - x_2(\tau), \\ & \text{subject to} && \dot{x}_1(t) = x_2(t) \quad \text{a.e. } t \in [0, \tau], \\ & && \dot{x}_2(t) = u(t) \quad \text{a.e. } t \in [0, \tau], \\ & && (x_1(0), x_2(0)) = (x_1^0, x_2^0) \in \mathbb{R}^2 \quad \text{and } g(x(\tau)) \in \{0\} \times \mathbb{R}_-, \\ & && u(t) \in [-1, 1] \quad \text{a.e. } t \in [0, \tau], \quad u \text{ is constant in } \Omega_1, \quad \text{and } \tau \in \mathbb{R}_+, \end{aligned} \tag{3.4}$$

where $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ $(x_1, x_2) := (x_1 - \frac{x_2^2}{2}, x_2)$. Observe that $x \in \Gamma_+$ if and only if $g(x) \in S$ and that g is submersive everywhere.

Lemma 3.2. *Let $x^0 := (0, x_2^0)$ be an initial condition such that $x_2^0 > 0$. Then, (3.3) and (3.4) are equivalent.*

Proof. First, if a point $x = (x_1, x_2) \in \Gamma_+$, then, the time to reach the origin with $u = +1$ equals $-x_2$. Now, an optimal triplet $(x^\dagger, u^\dagger, T^\dagger)$ of (3.3) defines an admissible path for (3.4). On the other hand, if the restriction of u^\dagger to the interval $[0, t_2]$ (where t_2 is the first entry time of x^\dagger into Γ_+), is not optimal for (3.4), then, there is a solution $(\tilde{x}, \tilde{u}, \tilde{\tau})$ of (3.4) such that $\tilde{x}_2(\tilde{\tau}) \in \Gamma_+$ and such that $\tilde{\tau} - \tilde{x}_2(\tilde{\tau}) < t_2 - x_2^\dagger(t_2) = T^\dagger$. This inequality contradicts the optimality of u^\dagger for (3.3), hence, u^\dagger is optimal for (3.4) which ends the proof. \square

Now, problem (3.4) is such that the terminal state does not belong to the interface between Ω_1 and Ω_2 . In addition, g is submersive, so, Theorem 2.1 can be applied. The rest of the analysis is organized as follows. First, we consider initial conditions in the set $\Gamma^> \cup \Gamma \cup \Omega_2$ for which classical tools can be applied. Next, we consider initial conditions in the set $\Omega_1 \cap \Gamma^<$ and we apply the HMP provided by Theorem 2.1 on (3.3) thanks to Lemma 3.1 and Lemma 3.2.

3.2 Optimal synthesis in $\Gamma^> \cup \Gamma \cup \Omega_2$

For initial conditions in $\Gamma \cup \Gamma^>$, problems (3.1) and (3.3) are equivalent as shows the next result.

Proposition 3.1. *For every $x^0 \in \Gamma^> \cup \Gamma$, an optimal solution to (3.3) is $u_{x^0}^\dagger$.*

Proof. Take $x^0 \in \Gamma^> \cap \Omega_1$. Then, the path generated by $u_{x^0}^\dagger$ is admissible for (3.3) since no switching occurs in Ω_1 . It follows that $w(x^0) \leq v(x^0)$ and thus $w(x^0) = v(x^0)$. Hence, u^\dagger is optimal. Similarly, for every initial condition $x^0 \in \Gamma_-$, the path generated by $u_{x^0}^\dagger$ has no switching, thus, we conclude in the same way. Finally, if $x^0 \in \Omega_2$ is such that $x^0 \in \Gamma^> \cup \Gamma$, then the path generated by $u_{x^0}^\dagger$ is also admissible for (3.3) which implies $w(x^0) \leq v(x^0)$, whence the result in this case. \square

For initial conditions in $\Gamma^<$ such that $x_1^0 \geq 0$, the synthesis is as follows.

Proposition 3.2. *If $x^0 \in \Gamma^<$ is such that $x_1^0 \geq 0$, then, the trajectory $\gamma_{x^0}^*$ is of type $\mathcal{B}_+\mathcal{B}_-\mathcal{B}_+$. The two switching points occur on the semi-axis $\{x_2 > 0\}$ and on Γ_+ .*

Proof. Step 1: consider an initial $x^0 = (0, x_2^0)$ such that $x_2^0 < 0$. From Lemma 3.1, the trajectory $\gamma_{x^0}^*$ is of type $\mathcal{P}\mathcal{B}_-\mathcal{B}_+$ and it reaches the semi-axis $\{x_2 > 0\}$ at some time $t = t_1$ from which the optimal control is $u = u_{x(t_1)}^\dagger$. Thus, the trajectory $\gamma_{x^0}^*$ is as follows : the control remains constant over $[0, t_1]$ (equal to some positive value $\lambda > 0$), then the control is $u(t) = -1$ until the trajectory reaches Γ_+ at some time $t = t_2$, and finally the origin is reached at some terminal time $\tau(x_2^0, \lambda)$. Simple computations give $x_2(t_1) = -x_2^0$, $t_1 = -\frac{2x_2^0}{\lambda}$, $x_2(t_2) = -\frac{x_2(t_1)}{\sqrt{2}}$, $t_2 - t_1 = -x_2(t_2)$, and $\tau(x_2^0, \lambda) - t_2 = -x_2(t_2)$. This gives

$$\tau(x_2^0, \lambda) = -x_2^0 \left[1 + \sqrt{2} + \frac{2}{\lambda} \right]. \quad (3.5)$$

Clearly $\lambda \mapsto \tau(x_2^0, \lambda)$ is minimal over $(0, 1]$ for $\lambda = 1$ which proves the desired result.

Step 2: consider an initial $x^0 = (x_1^0, x_2^0) \in \Gamma^<$ such that $x_1^0 > 0$ and let $(x(\cdot), u(\cdot))$ be an admissible pair for (3.3) starting from x^0 . Since the origin cannot be reached from x^0 while remaining in the set Ω_2 , the trajectory necessarily enters into Ω_1 (at some point located on the semi-axis $\{x_2 < 0\}$). Next, we denote by θ_u the first entry time into Ω_1 :

$$\theta_u := \min\{t > 0 ; x_1(t) = 0\},$$

so that $x(\theta_u)$ belongs to the semi-axis $\{x_2 < 0\}$. When entering into Ω_1 , the control remains constant to some unknown value λ until the trajectory exits Ω_1 at some time $t = t_2$ on the semi-axis $\{x_2 > 0\}$. We know from step 1 that for $t > t_2$, the trajectory is necessarily of type $\mathcal{B}_-\mathcal{B}_+$ (the switching occurring on Γ_+). Using (3.5), the terminal time associated with this trajectory is $\theta_u - \tau(x_u(\theta_u), 1)$ (note that we necessarily have $\lambda = 1$ from the monotonicity of τ w.r.t. λ , see (3.5)). It thus remains to determine the value of an optimal control u over $[0, \theta_u]$. Taking into account the preceding property, (3.3) is equivalent to the optimal control problem:

$$\inf_{u(\cdot) \in [-1, 1]} \psi(\theta_u, x(\theta_u)) := \theta_u - x_2(\theta_u)(3 + \sqrt{2}) \quad \text{s.t. } x_1(\theta_u) = 0.$$

We are in a position to apply the classical PMP⁴ to this problem [23]. If u is an optimal control, there is $\theta_u \geq 0$, a covector $p : [0, \theta_u] \rightarrow \mathbb{R}^2$, and $p_0 \leq 0$ such that $(p(\cdot), p_0) \neq 0$ and

$$\begin{cases} \dot{p}_1 &= 0, \\ \dot{p}_2 &= -p_1, \end{cases}$$

In addition, the Hamiltonian condition implies that $u(t) = \text{sign}(p_2(t))$ for a.e. $t \in [0, \theta_u]$ (thanks to the non-triviality condition, we can easily verify that singular arcs are not possible). Since the terminal time is free, we have the terminal condition (on the maximized Hamiltonian):

$$p_1 x_2(\theta_u) + |p_2(\theta_u)| = -p_0 \nabla_t \psi(\theta_u, x(\theta_u)) = -p_0,$$

Finally, the following transversality condition is fulfilled : there exists $\gamma \in \mathbb{R}$ such that

$$p(\theta_u) - p_0 \nabla_x \psi(\theta_u, x(\theta_u)) = \gamma(1, 0)^\top.$$

We deduce that $p_1 = \gamma$ and $p_2(\theta_u) = -p_0(3 + \sqrt{2})$, thus:

$$p_1 = \frac{-p_0 - |p_2(\theta_u)|}{x_2(\theta_u)} = \frac{-p_0 + p_0(3 + \sqrt{2})}{x_2(\theta_u)} = \frac{\sqrt{2}(1 + \sqrt{2})p_0}{x_2(\theta_u)} \leq 0.$$

We can conclude that p_2 is given by:

$$p_2(t) = -p_1(t - \theta_u) + p_2(\theta_u) = -\frac{\sqrt{2}(1 + \sqrt{2})p_0}{x_2(\theta_u)}(t - \theta_u) - p_0(3 + \sqrt{2}).$$

Because $x_2(\theta_u) < 0$, we deduce that $p_2 > 0$ over $[0, \theta_u]$ which allows to conclude that $u(t) = 1$ a.e. over $[0, \theta_u]$ as wanted. This ends the proof. \square

⁴Pontryagin Maximum Principle.

Remark 3.1. For x^0 as in Proposition 3.1, our approach shows that it is not necessary to apply the HMP at this step. Observe also that the minimal time synthesis possesses similarities with the one for (3.3), but, since no switching of the control may occur in Ω_1 , the optimal trajectory $x(\cdot)$ must re-enter into Ω_2 at some time $t = t_1$ from which the optimal control is $u_{x(t_1)}^\dagger$. Another interesting point is that the minimal time problem to reach the semi-axis $\{x_2 < 0\}$ consists in taking $u(t) = -1$ (this easily follows from the PMP). So, for these initial conditions, the solution to (3.3) does not consist in reaching Ω_1 in minimal time.

3.3 Optimal synthesis in $\Omega_1 \cap \Gamma^<$

We now turn to initial conditions in the set $\Omega_1 \cap \{x_1 < 0\}$. This case requires a careful analysis. In particular, we will use the averaged gradient condition and the jump of the covector to determine for every initial condition an optimal constant value of the control (while the trajectory remains in Ω_1) for each initial condition. Let us introduce a number

$$\lambda_{x^0} := \frac{(x_2^0)^2}{2x_1^0},$$

whenever $x_1^0 \neq 0$. We start by giving the possible structure of optimal paths.

Lemma 3.3. (i) If $x^0 \in \Gamma^<$ is such that $x_1^0 < 0$ and $x_2^0 > 0$, then, $\gamma_{x^0}^*$ is of type \mathcal{P} (and the control equals λ_{x^0}) or of type $\mathcal{PB}_-\mathcal{B}_+$ (and $u_{x^0}^*(t) = u_{x(t_1)}^\dagger(t)$ for $t \geq t_1$ where t_1 is the entry time of $\gamma_{x^0}^*$ into Ω_2).

(ii) If $x^0 \in \Gamma^<$ is such that $x_1^0 < 0$ and $x_2^0 \leq 0$, (i) holds true except that only the structure $\mathcal{PB}_-\mathcal{B}_+$ is possible.

Proof. Let $x^0 \in \Gamma^<$ be such that $x_1^0 < 0$. If $\gamma_{x^0}^*$ reaches the origin with a constant control value λ , then, we necessarily have $\lambda = \lambda_{x^0}$ and $x_2^0 > 0$. Otherwise, the control should remain constant until the trajectory leaves Ω_1 through the semi-axis $\{x_2 > 0\}$ at some time $t = t_1$. Since the trajectory reaches the origin without leaving Ω_2 (Lemma 3.1), the optimal control from $x(t_1)$ is $u = u_{x(t_1)}^\dagger$ until reaching the origin. \square

In the case where the trajectory enters into Ω_2 before reaching the origin, we must determine the constant control value for every initial condition in Ω_1 . This is the purpose of the next proposition. Remind that for $x_1^0 < 0$, then $\lambda_{x^0} \in (-1, 0]$ and that $\lambda_{x^0} = -1$ if and only if $x^0 \in \Gamma_-$. Also, we set $c_0 := \sqrt{2} - 1$

Lemma 3.4. Let $x^0 \in \Omega_1 \cap \Gamma^<$. If $\gamma_{x^0}^*$ is of type $\mathcal{PB}_-\mathcal{B}_+$, then the constant control value λ satisfies $\lambda \in \{0, \lambda_{x^0}^+, \lambda_{x^0}^-, 1\}$ where

$$\lambda_{x^0}^\pm := \sqrt{c_0} \left(\sqrt{c_0} \pm 2\sqrt{-\lambda_{x^0}} \right). \quad (3.6)$$

The terminal time $\mathcal{T}_{x^0}(\lambda)$ of an extremal of type \mathcal{P} or $\mathcal{PB}_-\mathcal{B}_+$ is

$$\begin{aligned} x_2^0 > 0 &\Rightarrow \begin{cases} \mathcal{T}_{x^0}(\lambda) = x_2^0 \left[\left(1 + \sqrt{2} + \frac{1}{\lambda}\right) \sqrt{1 - \frac{\lambda}{\lambda_{x^0}}} - \frac{1}{\lambda} \right] & \text{for } \lambda \in \{\lambda_{x^0}, \lambda_{x^0}^\pm, 1\}, \\ \mathcal{T}_{x^0}(0) = x_2^0 \left[1 + \sqrt{2} - \frac{1}{2\lambda_{x^0}} \right] & \text{for } \lambda = 0, \end{cases} \\ x_2^0 = 0 &\Rightarrow \mathcal{T}_{x^0}(\lambda) = \left(1 + \sqrt{2} + \frac{1}{\lambda}\right) \left(\sqrt{-2\lambda x_1^0} - \frac{1}{\lambda}\right) & \text{for } \lambda \in \{\sqrt{2} - 1, 1\} \\ x_2^0 < 0 &\Rightarrow \mathcal{T}_{x^0}(\lambda) = -x_2^0 \left[\left(1 + \sqrt{2} + \frac{1}{\lambda}\right) \sqrt{1 - \frac{\lambda}{\lambda_{x^0}}} - \frac{1}{\lambda} \right] & \text{for } \lambda \in \{\lambda_{x^0}^\pm, 1\} \end{aligned} \quad (3.7)$$

Proof. Observe first that for every $x^0 \in \Omega_1 \cap \Gamma^<$, the control -1 is never admissible (the corresponding path never reaches Ω_2 or the origin).

Suppose first that $x_2^0 > 0$ and that $\lambda \neq 1$. Consider an extremal $\mathcal{PB}_-\mathcal{B}_+$ that switches at some time $t = t_1$ on the semi-axis $\{x_2 > 0\}$ and on the parabola Γ_+ at some time $t_2 > t_1$. We denote by λ the constant control value over $[0, t_1)$ and by $\mathcal{T}_{x^0}(\cdot)$ the terminal time (as a function of $\lambda \in [-1, 1]$). If p is an adjoint vector defined over $[0, \mathcal{T}_{x^0}(\lambda)]$, then, it satisfies the adjoint equation

$$\begin{cases} \dot{p}_1 &= 0, \\ \dot{p}_2 &= -p_1. \end{cases}$$

We know that p_2 is continuous at $t = t_1$ whereas p_1 is discontinuous, hence, we set $p_1 := p_1^-$ over $[0, t_1)$ and $p_1 := p_1^+$ over $(t_1, \mathcal{T}_{x^0}(\lambda)]$.

Step 1: Analysis over $[0, t_1)$. We set $p_2^0 := p_2(0)$. The extremal fulfills the conditions

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \lambda \end{cases} ; \begin{cases} \dot{p}_1 &= 0 \\ \dot{p}_2 &= -p_1^- \end{cases} ; H = p_1^- x_2 + p_2 \lambda - 1 = 0 ; \int_0^{t_1} p_2(t) dt = 0,$$

where the last condition follows from the averaged gradient condition. Using, this condition, the conservation of H , $x_1(t_1) = 0$, and $p_2(t) = -p_1^- t + p_2^0$, we get

$$p_2^0 = \frac{p_1^- t_1}{2}; \quad p_1^- = \frac{1}{x_2^0 + \frac{\lambda t_1}{2}}; \quad x_2(t_1) = x_2^0 \sqrt{1 - \frac{\lambda}{\lambda_{x^0}}}; \quad p_2(t_1) = -\frac{p_1^- t_1}{2}; \quad t_1 = \frac{x_2^0}{\lambda} \left(\sqrt{1 - \frac{\lambda}{\lambda_0}} - 1 \right).$$

(Note that the denominator in p_1^- is non-zero because $H = 0$).

Step 2: Analysis over (t_1, t_2) . Over this interval, the extremal fulfills $u(t) = -1$, a.e. $t \in [t_1, t_2]$ and

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -1 \end{cases}; \quad \begin{cases} \dot{p}_1 &= 0 \\ \dot{p}_2 &= -p_1^+ \end{cases}; \quad H = p_1^+ x_2 - p_2 - 1 = 0,$$

until the trajectory reaches Γ_- . Using the constancy of H , we obtain:

$$p_1^+ = p_1^- + \frac{p_2(t_1)(1 + \lambda)}{x_2(t_1)} = p_1^- \left[1 - \frac{(1 + \lambda)t_1}{2x_2(t_1)} \right]. \quad (3.8)$$

Integrating the state equation also gives $x_1 = -\frac{x_2^2}{2} + \frac{x_2^2(t_1)}{2}$. Since at time t_2 , the trajectory crosses Γ_+ , we obtain $\frac{x_2^2(t_2)}{2} = -\frac{x_2(t_2)^2}{2} + \frac{x_2(t_1)^2}{2}$ implying that

$$x_2(t_2) = -\frac{x_2(t_1)}{\sqrt{2}}; \quad t_2 - t_1 = \left(1 + \frac{1}{\sqrt{2}} \right) x_2(t_1).$$

Now, t_2 is a switching time from \mathcal{B}^- to \mathcal{B}^+ . Using that $p_2(t) = -p_1^+(t - t_1) + p_2(t_1)$ for $t \in [t_1, t_2]$, we get:

$$\begin{aligned} p_2(t_2) = 0 &\iff -p_1^+(t_2 - t_1) + p_2(t_1) = 0 \\ &\iff -p_1^- \left[1 - \frac{(1 + \lambda)t_1}{2x_2(t_1)} \right] \frac{(1 + \sqrt{2})}{\sqrt{2}} x_2(t_1) - \frac{p_1^- t_1}{2} = 0 \\ &\iff (2x_2(t_1) - (\lambda + 1)t_1)(1 + 1/\sqrt{2}) + t_1 = 0 \\ &\iff 2 \left(1 + \frac{1}{\sqrt{2}} \right) x_2(t_1) + t_1 \left(1 - (1 + \lambda)(1 + 1/\sqrt{2}) \right) = 0 \\ &\iff 2 \left(1 + \frac{1}{\sqrt{2}} \right) x_2(t_1) - t_1 \left(\frac{1}{\sqrt{2}} + \lambda \left(1 + \frac{1}{\sqrt{2}} \right) \right) = 0. \end{aligned} \quad (3.9)$$

Finally, replacing $x_2(t_1)$ and t_1 by their expression in terms of λ and x^0 yields the equation

$$\left(\lambda \left(1 + \frac{1}{\sqrt{2}} \right) - \frac{1}{\sqrt{2}} \right) \sqrt{1 - \frac{\lambda}{\lambda_{x^0}}} + \frac{1}{\sqrt{2}} + \lambda \left(1 + \frac{1}{\sqrt{2}} \right) = 0. \quad (3.10)$$

Squaring this equation yields a cubic equation for which $\lambda = 0$ is always a solution. We obtain $\lambda = \lambda_{x^0}^\pm$ after solving the remaining quadratic polynomial.

Step 3: It remains to compute the terminal time. If $\lambda \neq 0$, the previous computations imply that

$$t_1 = \frac{x_2^0}{\lambda} \left(\sqrt{1 - \frac{\lambda}{\lambda_0}} - 1 \right) \quad \text{and} \quad \mathcal{T}_{x_0}(\lambda) - t_2 = -x_2(t_2) = \frac{x_2(t_1)}{\sqrt{2}}.$$

It follows that $\mathcal{T}_{x_0}(\lambda) = (t_2 - t_1) + t_1 + \frac{x_2(t_1)}{\sqrt{2}}$, therefore we get that

$$\mathcal{T}_{x_0}(\lambda) = (1 + \sqrt{2})x_2(t_1) + t_1 \implies \mathcal{T}_{x_0}(\lambda) = x_2^0 \left[\left(1 + \sqrt{2} + \frac{1}{\lambda} \right) \sqrt{1 - \frac{\lambda}{\lambda_{x^0}}} - \frac{1}{\lambda} \right].$$

which is the desired expression. Note that it is still valid in the case of an extremal of type \mathcal{P} for which $\lambda = \lambda_{x^0}$. If $\lambda = 0$, a direct computation shows that $\mathcal{T}_{x_0}(0) = x_2^0 \left(1 + \sqrt{2} - \frac{1}{2\lambda_{x^0}} \right)$. This concludes the proof of the lemma whenever $x_2^0 > 0$.

Next, we suppose that $x_2^0 < 0$ and that $\lambda \neq 1$. Note that in this case, we necessarily have $\lambda > 0$ (in such a way that the trajectory can enter into Ω_2). Then, we proceed as previously except that the expressions of $x_2(t_1)$ and t_1 must be replaced by

$$x_2(t_1) = -x_2^0 \sqrt{1 - \frac{\lambda}{\lambda_{x^0}}}; \quad t_1 = \frac{x_2(t_1) - x_2^0}{\lambda} = \frac{-x_2^0 \sqrt{1 - \frac{\lambda}{\lambda_{x^0}}} - x_2^0}{\lambda}.$$

We can check that λ is also a solution to (3.10) which is unchanged. Using the previous expressions of $x_2(t_1)$ and t_1 , the terminal time becomes $\mathcal{T}_{x_0}(\lambda) = -x_2^0 \left[\left(1 + \sqrt{2} + \frac{1}{\lambda}\right) \sqrt{1 - \frac{\lambda}{\lambda_{x^0}}} + \frac{1}{\lambda} \right]$

Finally, for $x_2^0 = 0$, only $\lambda > 0$ is possible. If $\lambda \neq 1$, we find that $\lambda_{x^0}^+ = \lambda_{x^0}^- = \sqrt{2} - 1$ replacing $x_2(t_1)$ and t_1 respectively by $\sqrt{-2\lambda x_1^0}$ and $\sqrt{-2\lambda x_1^0}/\lambda$ in (3.9). The computation of the terminal time is similar as in the two other cases. \square

We now examine the possibility for an optimal trajectory to be such that $u(t) = +1$ in Ω_1 .

Lemma 3.5. *Given $x_0 \in \Omega_1$, if $u = 1$ is optimal in Ω_1 , then, $x_0 \in \{(x_1, x_2) \in \mathbb{R}_- \times \mathbb{R}_-; x_2^2 < x_1(1 - \sqrt{2})\}$.*

Proof. It is enough to consider an extremal of type $\mathcal{B}_+ \mathcal{B}_- \mathcal{B}_+$ starting in the set $\Omega_1 \cap \Gamma^<$. Let us denote by (p_1^-, p_2^0) the initial condition of the covector $p(\cdot)$ and by p_1^+ the value of p_1 in Ω_2 . By similar computations as in the previous lemma, we get from the constancy of the Hamiltonian $p_1^- = \frac{1 - p_2(t_1)}{x_2(t_1)}$ and $p_2(t_1) = -\frac{(2 + \sqrt{2})}{\sqrt{2}}$ (note that this value of $p_2(t_1)$ does not depend on the initial condition). Now,

$$\int_0^{t_1} p_2(t) dt = t_1 \left(1 - p_1^- \left(x_2^0 + \frac{t_1}{2} \right) \right).$$

Case 1: $x_2^0 > 0$. In that case, $p_1^- = \frac{\sqrt{2}(\sqrt{2}+1)}{x_2(t_1)}$, $t_1 = x_2(t_1) - x_2^0$, and $x_2(t_1) = x_2^0 \sqrt{1 - \frac{1}{\lambda_{x^0}}}$ which gives

$$\int_0^{t_1} p_2(t) dt < 0.$$

This contradicts the fact that $u = 1$ over $[0, t_1]$. Hence, this case is not possible.

Case 2: $x_2^0 = 0$. In that case, $p_1^- = \frac{1 + \sqrt{2}}{\sqrt{-x_1^0}}$ and $x_2(t_1) = t_1 = \sqrt{-2x_1^0}$ which gives

$$\int_0^{t_1} p_2(t) dt = -\frac{t_1}{\sqrt{2}} < 0.$$

This contradicts the fact that $u = 1$ over $[0, t_1]$. Hence, this case is also not possible.

Case 3: $x_2^0 < 0$. Consider an optimal trajectory of type $\mathcal{B}_+ \mathcal{B}_- \mathcal{B}_+$. If $u = 1$ over $[0, t_1]$, then,

$$\int_0^{t_1} p_2(t) dt = t_1 \left(p_2^0 - p_1^- \frac{t_1}{2} \right) = t_1 \left(1 - p_1^- \left(x_2^0 + \frac{t_1}{2} \right) \right).$$

From the constancy of the Hamiltonian we obtain $p_1^- = \frac{1 - p_2(t_1)}{x_2(t_1)}$, and using (3.8) we get $p_2(t_1) = -\frac{(2 + \sqrt{2})}{\sqrt{2}}$.

This implies that $p_1^- = \frac{2 + \sqrt{2}}{|x_2^0| \sqrt{1 - \frac{1}{\lambda_{x^0}}}}$. Therefore, for $x_2^0 < 0$, we find that

$$\int_0^{t_1} p_2(t) dt = \frac{t_1}{2 \sqrt{1 - \frac{1}{\lambda_{x^0}}}} \left(-\sqrt{2} \sqrt{1 - \frac{1}{\lambda_{x^0}}} + \sqrt{2} + 2 \right).$$

Note that $\frac{t_1}{2 \sqrt{1 - \frac{1}{\lambda_{x^0}}}} > 0$ and that $\lambda_{x^0} \mapsto -\sqrt{2} \sqrt{1 - \frac{1}{\lambda_{x^0}}} + \sqrt{2} + 2$ is positive over $(-\infty, \frac{1 - \sqrt{2}}{2})$. Hence, we deduce that for $x_2^0 < 0$, if $u = 1$ over $[0, t_1]$, then,

$$\int_0^{t_1} p_2(t) dt > 0 \iff \lambda_{x^0} < \frac{1 - \sqrt{2}}{2}.$$

In conclusion, if $u = 1$ in Ω_1 , then, one has $x_0 \in \{(x_1, x_2) \in \mathbb{R}_- \times \mathbb{R}_- ; x_2^2 < x_1(1 - \sqrt{2})\}$ (because for $x_2^0 < 0$, the inequality $\lambda_{x^0} < \frac{1-\sqrt{2}}{2}$ is equivalent to $x_2^2 < x_1(1 - \sqrt{2})$). This ends the proof. \square

To conclude our study, we have to determine for every initial condition in $\Omega_1 \cap \Gamma^<$ which constant control value is optimal. Doing so, let us introduce the sets⁵

$$\begin{aligned}\Omega_1^d &:= \left\{ (x_1, x_2) \in \mathbb{R}_- \times \mathbb{R}_+ ; -\frac{\sqrt{2}+1}{2}x_2^2 \leq x_1 < -\frac{1}{2}x_2^2 \right\}, \\ \Omega_1^- &:= \left\{ (x_1, x_2) \in \mathbb{R}_- \times \mathbb{R}_+ ; x_1 < -\frac{(\sqrt{2}+1)x_2^2}{2} \right\}, \\ \Omega_1^+ &:= \left\{ (x_1, x_2) \in \mathbb{R}_- \times \mathbb{R}_- ; x_1 \leq -(\sqrt{2}+1)x_2^2 \right\},\end{aligned}$$

and we also set $\Omega_1^r := (\Omega_1 \cap \Gamma^<) \setminus (\Omega_1^d \cup \Omega_1^+ \cup \Omega_1^-)$.

Proposition 3.3. *Let $x^0 \in \Gamma^< \cap \Omega_1$ and let λ be the optimal constant control value in Ω_1 . Then, one has:*

$$\begin{aligned}x^0 \in \Omega_1^d &\Rightarrow \lambda = \lambda_{x^0}, \\ x^0 \in \Omega_1^\pm &\Rightarrow \lambda = \lambda_{x^0}^\pm, \\ x^0 \in \Omega_1^r &\Rightarrow \lambda = 1.\end{aligned}$$

Proof. Given some initial condition $x^0 \in \Gamma^< \cap \Omega_1$, we denote by λ the constant control value until the trajectory exits Ω_1 or enters into the target point. Note that $\lambda = -1$ is always non-admissible.

Case 1: Suppose that $x_2^0 > 0$. Using Lemma 3.4, λ is necessarily such that $\lambda \in \{0, \lambda_{x^0}^\pm, \lambda_{x^0}, 1\}$. The value $\lambda = 1$ can be removed from Lemma 3.5.

(i) Suppose that $\lambda_{x^0} \in (-1, 1 - \sqrt{2})$. By studying $\lambda_{x^0} \mapsto \lambda_{x^0}^+$, we obtain that $\lambda_{x^0}^+$ is strictly greater than one over $[-1, 1 - \sqrt{2})$ so that λ necessary belongs to $\{0, \lambda_{x^0}^-, \lambda_{x^0}\}$. By comparison of the one real variable functions $\lambda_{x^0} \mapsto \mathcal{T}_{x^0}(\lambda_{x^0})$, $\lambda_{x^0} \mapsto \mathcal{T}_{x^0}(0)$, and $\lambda_{x^0} \mapsto \mathcal{T}_{x^0}(\lambda_{x^0}^-)$, we obtain the inequality

$$\mathcal{T}_{x^0}(\lambda_{x^0}) \leq \mathcal{T}_{x^0}(\lambda_{x^0}^-) < \mathcal{T}_{x^0}(0),$$

for every $\lambda_{x^0} \in (-1, 1 - \sqrt{2}]$. The left inequality above is strict only over $(-1, 1 - \sqrt{2})$ because the mapping $\lambda_{x^0} \mapsto \lambda_{x^0}^- - \lambda_{x^0}$ vanishes for $\lambda_{x^0} = 1 - \sqrt{2}$. Now, $x^0 \in \Omega_1^d$ if and only if $\lambda_{x^0} \in (-1, 1 - \sqrt{2})$. It follows that $\lambda = \lambda_{x^0}$ in Ω_1^d as wanted.

(ii) Suppose that $\lambda_{x^0} \in [1 - \sqrt{2}, \frac{1-\sqrt{2}}{2})$ which amounts to saying that $x^0 \in \Omega_1^-$ is such that $x_1^0 \geq -\frac{(x_2^0)^2}{\sqrt{2}-1}$. As in (i), one has $\lambda_{x^0}^+ > 1$ in $[1 - \sqrt{2}, \frac{1-\sqrt{2}}{2})$ so that we necessarily have $\lambda \in \{0, \lambda_{x^0}, \lambda_{x^0}^-\}$. We also verify that

$$\mathcal{T}_{x^0}(\lambda_{x^0}^-) \leq \mathcal{T}_{x^0}(\lambda_{x^0}) < \mathcal{T}_{x^0}(0),$$

for every $\lambda_{x^0} \in (1 - \sqrt{2}, \frac{1-\sqrt{2}}{2})$. As previously, the left inequality above is strict only over $(1 - \sqrt{2}, \frac{1-\sqrt{2}}{2})$. It follows that $\lambda = \lambda_{x^0}^-$ for those initial conditions of Ω_1^- that are such that $x_1^0 \geq -\frac{(x_2^0)^2}{\sqrt{2}-1}$.

(iii) Finally, let us take $\lambda_{x^0} \in [\frac{1-\sqrt{2}}{2}, 0)$ which amounts to saying that $x^0 \in \Omega_1^-$ is such that $x_1^0 \leq -\frac{(x_2^0)^2}{\sqrt{2}-1}$. We obtain (using similar computations as in (i) and (ii)) that $\lambda = \lambda_{x^0}^-$ remains optimal.

Case 2: Suppose that $x_2^0 = 0$. The optimal control value is necessarily such that $\lambda \in \{\lambda_{x^0}^\pm\}$. Since $\lambda_{x^0}^+ = \lambda_{x^0}^-$ in this case, one has $\lambda = \lambda_{x^0}^- = \lambda_{x^0}^+ = \sqrt{2} - 1$. We deduce from case 1 (ii)-(iii) and case 2 that $\lambda = \lambda_{x^0}^-$ in Ω_1^- as desired.

Case 3: Suppose that $x_2^0 < 0$. The optimal control value λ is necessarily such that $\lambda \in \{\lambda_{x^0}^\pm, 1\}$.

(i) Suppose that $\lambda_{x^0} \in [\frac{1-\sqrt{2}}{4}, 0)$ which amounts to consider those initial conditions in Ω_1^+ that are such that $x_1^0 \leq \frac{2(x_2^0)^2}{\sqrt{2}-1}$. By comparison of one real variable functions, we find that $\mathcal{T}_{x^0}(\lambda_{x^0}^+) < \mathcal{T}_{x^0}(\lambda_{x^0}^-)$ which implies that $\lambda = \lambda_{x^0}^+$ is optimal.

(ii) Suppose that $\lambda_{x^0} \in [\frac{1-\sqrt{2}}{2}, \frac{1-\sqrt{2}}{4})$ which amounts to consider those initial conditions in Ω_1^+ that are such that $-\frac{2(x_2^0)^2}{\sqrt{2}-1} < x_1^0 \leq -\frac{(x_2^0)^2}{\sqrt{2}-1}$. In this region, only $\lambda_{x^0}^+$ is possible. Combining (i) and (ii), we obtain that $\lambda = \lambda_{x^0}^+$ in Ω_1^+ as wanted (note that $\lambda_{x^0}^+$ equals one whenever $x_1 = -(\sqrt{2}+1)x_2^2$).

Finally, when $x^0 \in \Omega_1^r$, Lemma 3.5 implies that $u = 1$ in Ω_1 . \square

⁵The superscript d and r stand for “direct” and “rest” whereas the notation Ω_1^\pm is motivated by the result in Proposition 3.3.

As shows this proposition, the constant control value 0 is never optimal in Ω_1 although it is a candidate according to Lemma 3.6.

3.4 Optimal synthesis for (3.3)

The next theorem provides the optimal synthesis for (3.3) and summarizes the preceding results (Proposition 3.1 for initial conditions in $\Gamma^> \cup \Gamma$, Proposition 3.2 for initial conditions in $\Omega_2 \cap \Gamma^<$, and Proposition 3.3 for initial conditions in $\Omega_1^d \cup \Omega_1^\pm \cup \Omega_1^r$).

Theorem 3.2. (i) If $x^0 \in \Gamma^> \cup \Gamma$, then, $\gamma_{x^0}^*$ is of type $\mathcal{B}_-\mathcal{B}_+$ and $u_{x^0}^* = u_{x^0}^\dagger$.
(ii) If $x^0 \in (\Gamma^< \cap \Omega_2) \cup \Omega_1^+$, then $\gamma_{x^0}^*$ is of type $\mathcal{B}_+\mathcal{B}_-\mathcal{B}_+$ and

$$u_{x^0}^*(t) = \begin{cases} 1 & \text{if } t \in [0, t_1] \\ u_{x^*(t_1)}^*(t) & \text{if } t > t_1 \end{cases} \quad (3.11)$$

where t_1 is the exit time of Ω_1 .

(iii) If $x^0 \in \Omega_1^\pm$, then $\gamma_{x^0}^*$ is of type $\mathcal{PB}_-\mathcal{B}_+$ and

$$u_{x^0}^*(t) = \begin{cases} \lambda_{x^0}^\pm & \text{if } t \in [0, t_1] \\ u_{x^*(t_1)}^*(t) & \text{if } t > t_1, \end{cases} \quad (3.12)$$

where t_1 is the exit time of Ω_1 .

(iv) If $x^0 \in \Omega_1^d$, then $\gamma_{x^0}^*$ is of type \mathcal{P} and $u_{x^0}^*(t) = \lambda_{x^0}$.

In contrast with the classical double integrator, there are four different structures for $u_{x^0}^*$ depending on the initial condition:

$$\mathcal{P} ; \mathcal{B}_-\mathcal{B}_+ ; \mathcal{PB}_-\mathcal{B}_+ ; \mathcal{B}_+\mathcal{B}_-\mathcal{B}_+.$$

The partition of \mathbb{R}^2 arising from Theorem 3.2 separating the four structures for optimal paths is depicted on Fig.2. Optimal paths in cases (ii) and (iv) of Theorem 3.2 are illustrated on Fig. 3.

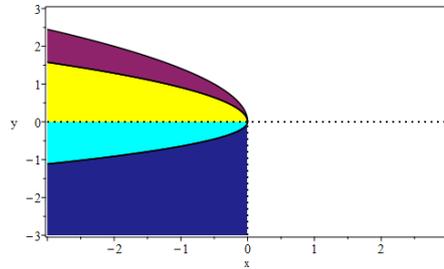


Figure 2: Plot of the various subsets arising in Theorem 3.2. The sets Ω_1^d , Ω_1^- , Ω_1^+ , and Ω_1^r are depicted respectively in magenta, yellow, cyan, and blue.

Remark 3.2. Our methodology can be adapted to other non-control regions such as other half planes. Of course, if the geometry of the non-control set is more involved (for instance, if Ω is not connected), the synthesis will be more delicate, nevertheless, for a fixed initial condition, we can apply the same tools based on the hybrid maximum principle with parameter to obtain informations on some optimal control.

3.5 Remarks about the HJB equation associated with (3.3)

We end-up by giving an expression of the value function w associated with (3.3). In order to ease the reading, we write x in place of x^0 for the initial condition. Recall that the value function v of (3.1) is given by

$$v(x) = \begin{cases} 2\sqrt{\frac{x_2^2}{2} - x_1 - x_2} & \text{if } x \in \Gamma_+ \cup \Gamma^>, \\ 2\sqrt{\frac{x_2^2}{2} + x_1 + x_2} & \text{if } x \in \Gamma_- \cup \Gamma^<, \end{cases} \quad (3.13)$$

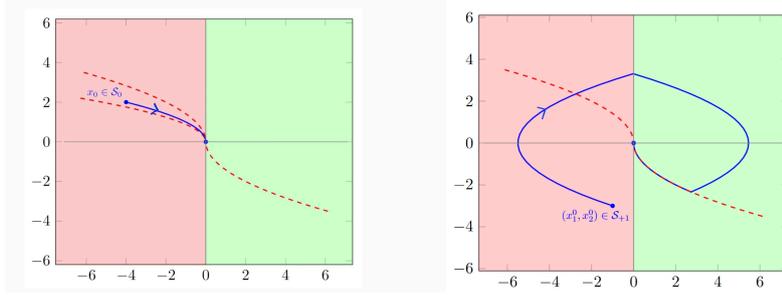


Figure 3: *Fig. left:* example of an optimal trajectory reaching $(0,0)$ from $x^0 \in \Omega_1^d$ without switching point neither crossing time (Theorem 3.2 (iv) ; *Fig. right:* example of an optimal trajectory of type $\mathcal{B}_+\mathcal{B}_-\mathcal{B}_+$ arising from $x^0 \in \Omega_1^r$ (Theorem 3.2 (ii)).

and that it is continuous and C^1 -piecewise. Moreover, it can be characterized as the unique solution to the HJB equation

$$1 + \partial_{x_1} v(x)x_2 - |\partial_{x_2} v(x)| = 0, \quad x \in \mathbb{R}^2 \setminus \Gamma, \quad (3.14)$$

which null at $x = 0$. We refer for instance to [15] for the C^1 -piecewise approach to the HJB equation. We set $s(x) := -\sqrt{x_2^2 - 2x_1}$ for $x \in \mathbb{R}^2$ such that $x_1 \in [0, x_2^2/2]$.

Proposition 3.4. *The value function w associated with (3.3) is continuous, C^1 -piecewise and it is given by:*

$$w(x) = \begin{cases} v(x) & \text{if } x \in \Gamma \cup \Gamma^>, \\ -2x_1/x_2 & \text{if } x \in \Omega_1^d, \\ \mathcal{T}_x(\lambda_x^\pm) & \text{if } x \in \Omega_1^\pm, \\ \mathcal{T}_x(1) & \text{if } x \in \Omega_1^r, \\ s(x) - x_2 + \mathcal{T}_{s(x)}(1) & \text{if } x \in \Omega_2 \cap \Gamma^<. \end{cases} \quad (3.15)$$

Proof. First, the expressions of w in $\Gamma \cup \Gamma^>$, in Ω_1^\pm , and in Ω_1^r follow from Proposition 3.3 and from the expression of $\mathcal{T}_x(\lambda)$ provided by Lemma 3.4. The expression of w in Ω_1^d is straightforward since the optimal control is constant equal to λ_x until reaching $(0,0)$. For $x \in \Omega_2 \cap \Gamma^<$, the time to reach the semi-axis $x_2 = 0$ at the point $(0, s(x))$ is $s(x) - x_2$ applying $u = 1$. Next, according to Proposition 3.2, the time to reach $(0,0)$ from the point $(0, s(x))$ can be written $s(x)(3 + \sqrt{2}) = \mathcal{T}_{s(x)}(1)$, whence the result for $x \in \Omega_2 \cap \Gamma^<$.

Next, to check the continuity of w , it is enough to verify its continuity at the various interfaces:

1. Continuity over Γ_- . It follows from the fact that $v(x) = x_2$ for $x \in \Gamma_-$.
2. Continuity over $\bar{\Omega}_1^- \cap \bar{\Omega}_1^d$. It follows from the expression of $\mathcal{T}_x(\lambda_x^-)$ using that $\lambda_x^- = \lambda_x$ in $\bar{\Omega}_1^- \cap \bar{\Omega}_1^d$.
3. Continuity over $\{x_2 < 0\}$. It follow from the expressions of $\mathcal{T}_x(\lambda_x^\pm)$ using the fact that $\lambda_{x^\pm} \rightarrow \sqrt{2} - 1$ as $x_2 \rightarrow 0$.
4. Continuity over $\bar{\Omega}_1^- \cap \bar{\Omega}_1^d$. It follows from the fact that $\lambda_x^+ \rightarrow 1$ when $x \in \Omega_1^+$ goes to some point of $\bar{\Omega}_1^- \cap \bar{\Omega}_1^d$.
5. Continuity over $\{x_2 < 0\}$. When $x \in \Omega_2 \cap \Gamma^<$ goes to some point of $\{x_2 < 0\}$, we have $-s(x) + x_2 \rightarrow 0$, thus $w(x) \rightarrow -x_2 + \mathcal{T}_x(1)$ which is equal to the left limit of w over the semi-axis $\{x_2 < 0\}$.
6. Continuity over Γ_- . One has $v(x) \rightarrow -x_2$ as x conerges to some point of Γ_+ . On the other hand, $s(x) \rightarrow 0$ whenever x goes to some point of Γ_+ . We deduce that $s(x) - x_2 + \mathcal{T}_{s(x)}(1) \rightarrow -x_2$ whenever x goes to some point of Γ_+ using that $\mathcal{T}_{s(x)}(1)$ goes to zero, whence the result.

Finally, the expressions defining w in each stratum of \mathbb{R}^2 given by (3.15) are of class C^1 , so, w is C^1 -piecewise. \square

Remark 3.3. *Concerning (3.3), we can check that w is also continuous and C^1 -piecewise. However, we do not know characterizations of the value function in the setting of optimal control problems including loss control regions such as (2.1). A possible way to characterize the value function in this setting could be to consider an augmented system (as for proving Theorem 2.1) and next to use characterizations of the value function over stratified domains as for instance in [4, 19].*

4 Conclusion

In this paper, we have studied an optimal control problem involving a “non-classical” constraint w.r.t. classical examples of minimum time control syntheses. The main difference is that we assumed the control function to be constant as long as the state belongs to some region of the state called non-control set. Because of the spatial constraint related to the constancy of the control, this framework also differs from sampled data optimal control problems such as in [13] in which the control is supposed to be piecewise constant (but along a given or free partition of the time period). Our analysis of the problem used the hybrid maximum principle including a regionally constant parameter as in [5, 7]. For each initial condition, we have computed analytically an optimal control leading that way to an optimal synthesis. We point out a difference with classical optimal control problems. Because the control function cannot be changed while the state of the system remains in the non-control region, the terminology feedback control is not adapted to our setting (remind that each initial condition in the non-control set defines a constant control value). Also, our synthesis shows that some optimal control values belong to the interior of the admissible control set in contrast with bang-bang solutions to classical optimal control problems (that do not involve singular arcs). We believe that the methodology employed in this paper could serve to study theoretically and numerically other optimal control problems including loss control regions (arising for instance in aerospace or in viability) and optimal solutions with more than one arc in the non-control set (such as for the harmonic oscillator). These questions will be the matter of future works.

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