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Tangency property and prior-saturation points in minimal time problems in the plane

T. Bayen*, O. Cots†

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Abstract

In this paper, we consider minimal time problems governed by control-affine-systems in the plane, and we focus on the synthesis problem in presence of a singular locus that involves a saturation point for the singular control. After giving sufficient conditions on the data ensuring occurence of a prior-saturation point and a switching curve, we show that the bridge (*i.e.*, the optimal bang arc issued from the singular locus at this point) is tangent to the switching curve at the prior-saturation point. This property is proved using the Pontryagin Maximum Principle that also provides a set of non-linear equations that can be used to compute the prior-saturation point. These issues are illustrated on a fed-batch model in bioprocesses and on a Magnetic Resonance Imaging (MRI) model for which minimal time syntheses for the point-to-point problem are discussed.

Keywords: Geometric optimal control, Minimum time problems, Singular arcs.

1 Introduction

In this paper, we consider minimal time problems governed by single-input control-affine-systems in the plane

$$\dot{x}(t) = f(x(t)) + u(t)\, g(x(t)), \quad |u(t)| \leq 1,$$

where $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ are smooth vector fields. Syntheses for such problems have been investigated a lot in the literature (see, *e.g.*, [6, 14, 20, 25, 27, 26]). In particular, an exhaustive description of the various encountered singularities can be found in [14], as well as an algorithm leading to the determination of optimal paths. It is worth mentioning that even though many techniques exist in this setting, the computation of an optimal feedback synthesis (global) remains in general difficult because of the occurence of geometric loci such as singular arcs, switching curves, cut-loci...

Our aim in this work is to focus on the notion of *singular arc* which appears in the synthesis when the switching function (the scalar product between the adjoint vector and the controlled vector field g) vanishes over a time interval. In the context of control-affine-systems, singular arcs can generically be explicitly retrieved by solving non-degenerate linear equations (see, *e.g.*, [8]). Besides, in the two-dimensional case, the corresponding singular control u_s (which allows the associated trajectory to stay on the singular locus) can be expressed in feedback form $x \mapsto u_s[x]$. However, it may happen that u_s becomes non admissible, *i.e.*, $x \mapsto u_s[x]$ takes values above the maximal value for the control (namely 1 here). Such a situation naturally appears in several application models, see, *e.g.*, [2, 3, 17, 22]. In that case, we say that a *saturation phenomenon* occurs. The occurence of such a phenomenon implies the following (non-intuitive) property that, if a singular arc is optimal, then it should leave the singular locus at a so-called *prior-saturation point* before reaching the saturation point. This property has been studied in the literature in various situations such as for control-affine systems in dimension 2 and 4 (see, *e.g.*, [23, 24, 12, 3] and references herein).

Our main goal in this paper is to provide new qualitative properties on the minimum time synthesis in presence of a saturation point. More precisely, our objective is twofold:

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- We first give a set of conditions on the system that ensure occurence of prior-saturation showing that, under certain assumptions, the system leaves the singular arc at this point (before reaching the saturation point) with the maximal value for the control, see Proposition 3.1. This last arc is usually called *bridge* following the terminology as in [11, 9] (see also [13, 6]).
- Second, we introduce a shooting function that allows an effective computation of the prior-saturation point. This mapping is used to show our main result (Theorem 4.1) which can be stated as follows: when the system exhibits a switching curve emanating from the prior-saturation point, then this curve is tangent to the bridge (in the cotangent bundle) at this point.

The tangency property (in the state space) has been pointed out in several application models (see, *e.g.*, [3, 11]). To the best of our knowledge, this property has not been addressed previously in this general setting in the literature. It allows to better understand the construction of optimal paths locally at the prior-saturation point.

The paper is structured as follows: in Section 2, we recall classical expressions and properties of singular controls for control-affine-systems in the plane introducing the saturation phenomenon. In Section 3, we provide a set of conditions involving the target set and the system ensuring the occurence of the prior-saturation phenomenon. In Section 4, we show the tangency property between the switching curve emanating from a prior-saturation point and the bridge, and we describe how to compute the prior-saturation point thanks to a shooting function constructed via the Hamiltonian lifts of f and g. Finally, we depict this geometrical property in Section 5 for a fed-batch model [19, 3] and MRI model [11, 9]. This allows us to illustrate the notion of bridge in various contexts: first, when it connects a component of the singular locus to another one (see the MRI-model in Section 5 and [11, 9]), and then when it connects a component of a singular locus to an extended target set (see the fed-batch model in Section 5 and [19, 3]).

2 Saturation phenomenon

The purpose of this section is to recall some facts about minimum time control problems in the plane that will allow us to introduce the saturation phenomenon. Throughout the paper, the standard inner product in $\mathbb{R}^2$ is written $a \cdot b$ for $a, b \in \mathbb{R}^2$, and $a^\perp$ denotes the vector $a^\perp := (-a_2, a_1)$ orthogonal to a. The interior of a subset $S \subset \mathbb{R}^n$, $n \geq 2$, is denoted by $\mathrm{Int}(S)$.

2.1 Pontryagin's Principle

We start by applying the classical optimality conditions provided by the Pontryagin Maximum Principle (PMP), see [21]. Let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ be two vector fields of class C^∞, and consider the controlled dynamics:

$$\dot{x}(t) = f(x(t)) + u(t)\, g(x(t)), \tag{2.1}$$

with admissible controls in the set

$$\mathcal{U} := \{u : [0, +\infty) \to [-1, 1]\ ;\ u \text{ meas.}\}.$$

Given an initial point $x_0 \in \mathbb{R}^2$ and a non-empty closed subset $\mathcal{T} \subset \mathbb{R}^2$, we focus on the problem of driving (2.1) in minimal time from x_0 to the target set $\mathcal{T}$:

$$\inf_{u \in \mathcal{U}} T_u \quad \text{s.t. } x_u(T_u) \in \mathcal{T}, \tag{2.2}$$

where $x_u(\cdot)$ denotes the unique solution of (2.1) associated with the control u such that $x_u(0) = x_0$, and $T_u \in [0, +\infty]$ is the first entry time of $x_u(\cdot)$ into the target set $\mathcal{T}$. We suppose hereafter that optimal trajectories exist[1] and we wish to apply the PMP on (2.2). The Hamiltonian associated with (2.2) is the function $H : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as

$$H(x, p, p^0, u) := p \cdot f(x) + u\, p \cdot g(x) + p^0.$$

If u is an optimal control and x_u is the associated trajectory steering x_0 to the target set $\mathcal{T}$ in time $T_u \geq 0$, the following conditions are fulfilled:

[1] If the target can be reached from x_0 and if f, g have linear growth, then (2.2) admits an optimal solution, thanks to Filippov's Existence Theorem, see, *e.g.*, [28].

- There exist $p^0 \leq 0$ and an absolutely continuous function $p : [0, T_u] \to \mathbb{R}^2$ satisfying the adjoint equation

$$\dot{p}(t) = -\nabla_x H(x_u(t), p(t), p^0, u(t)) \quad \text{a.e. } t \in [0, T_u]. \tag{2.3}$$

- The pair $(p^0, p(\cdot))$ is non-zero.
- The optimal control u satisfies the *Hamiltonian condition*

$$u(t) \in \operatorname{argmax}_{\omega \in [-1,1]} H(x_u(t), p(t), p^0, \omega) \quad \text{a.e. } t \in [0, T_u]. \tag{2.4}$$

- At the terminal time, the *transversality condition*[2] $p(T_u) \in -N_{\mathcal{T}}(x_u(T_u))$ is fulfilled.

Recall that an extremal $(x_u(\cdot), p(\cdot), p^0, u(\cdot))$ satisfying (2.1) and (2.3)-(2.4) is *abnormal* whenever $p^0 = 0$ and *normal* whenever $p^0 \neq 0$. In the latter case, we take $p^0 = -1$ and the corresponding extremal is denoted by $(x_u(\cdot), p(\cdot), u(\cdot))$ and we shall then write $H(x, p, u)$ in place of $H(x, p, p^0, u)$. Since T_u is free and (2.1) is autonomous, the Hamiltonian H is zero along any extremal: for a.e. $t \in [0, T_u]$,

$$H = p(t) \cdot f(x_u(t)) + u(t)p(t) \cdot g(x_u(t)) + p^0 = 0. \tag{2.5}$$

The *switching function* ϕ is defined as

$$\phi(t) := p(t) \cdot g(x_u(t)), \quad t \in [0, T_u], \tag{2.6}$$

and it gives us (thanks to (2.4)) the following control law:

$$\begin{cases} \phi(t) > 0 & \Rightarrow \quad u(t) = +1, \\ \phi(t) < 0 & \Rightarrow \quad u(t) = -1. \end{cases} \tag{2.7}$$

A *switching time* is an instant $t_c \in (0, T_u)$ such that the control u is discontinuous at time t_c. We say that the corresponding extremal trajectory has a *switching point* at time t_c. Of particular interest is the case when there is a time interval $[t_1, t_2]$ such that the switching function vanishes over this interval, *i.e.*,

$$\phi(t) = p(t) \cdot g(x_u(t)) = 0, \quad t \in I.$$

We then say that the extremal trajectory has a *singular arc* over $[t_1, t_2]$. Note that we shall suppose such an extremal to be normal, *i.e.*, $p^0 \neq 0$. Indeed, recall from [14, Prop. 2 p.49] that under generic conditions, abnormal extremals are bang-bang. By differentiating ϕ twice w.r.t. t, one gets

$$\dot{\phi}(t) = p(t) \cdot [f, g](x_u(t)), \quad t \in [0, T_u],$$

where $[f, g](x)$ is the Lie bracket of f and g at point x, and

$$\ddot{\phi}(t) = p(t) \cdot [f, [f, g]](x_u(t)) + u(t)\, p(t) \cdot [g, [f, g]](x_u(t)) \quad \text{a.e. } t \in [0, T_u].$$

The *singular locus* Δ_{SA} (in the state space) is defined as the (possibly empty) subset of $\mathbb{R}^2$

$$\Delta_{SA} := \{x \in \mathbb{R}^2 \ ; \ \det(g(x), [f, g](x)) = 0\}. \tag{2.8}$$

For future reference, we set $\delta_{SA}(x) := \det(g(x), [f, g](x))$ for $x \in \mathbb{R}^2$. Note that if an extremal is singular over a time interval $[t_1, t_2]$, then one has $x_u(t) \in \Delta_{SA}$ for any $t \in [t_1, t_2]$ because $p(\cdot)$ must be non-zero and orthogonal to the vector space $\operatorname{span}\{g(x_u(t)), [f, g](x_u(t))\}$ over $[t_1, t_2]$. The *singular control* u_s is then the value of the control for which the trajectory stays on the singular locus Δ_{SA}. Supposing then that $\phi(t) = \dot{\phi}(t) = 0$ over $[t_1, t_2]$ gives:

$$u_s(t) := -\frac{p(t) \cdot [f, [f, g]](x_u(t))}{p(t) \cdot [g, [f, g]](x_u(t))}, \quad t \in [0, T_u], \tag{2.9}$$

provided that $p(t) \cdot [g, [f, g]](x_u(t))$ is non zero for $t \in [t_1, t_2]$. This expression of the singular control does not guarantee that u_s is admissible, that is, $u_s(t) \in [-1, 1]$:

[2] Here, $N_{\mathcal{T}}(x)$ stands for the (Mordukovitch) limiting normal cone to $\mathcal{T}$ at point $x \in \mathcal{T}$, see [28]. It coincides with the normal cone in the sense of convex analysis when $\mathcal{T}$ is convex.

- When we have $u_s(t) \in [-1, 1]$, the point $x_u(t)$ is said *hyperbolic* if $p(t) \cdot [g, [f, g]](x_u(t)) > 0$, and *elliptic* if $p(t) \cdot [g, [f, g]](x_u(t)) < 0$ (see [13, 6]).
- When we have $|u_s(t)| > 1$ for some instant t, we say that a *saturation phenomenon* occurs and that the corresponding points of the singular locus are *parabolic* (see [13, 6]).

Our purpose in what follows is precisely to investigate properties of the synthesis of optimal paths when saturation occurs. To this end, we suppose in the rest of the paper that extremals are normal, *i.e.*, $p^0 \neq 0$ (we take hereafter $p^0 = -1$).

2.2 Singular control and saturation phenomenon

In this part, we derive classical expressions of the singular control in terms of feedback that will allow us to introduce saturation points (in terms of the data defining the system). The *collinearity set* associated with (2.1) is the (possibly empty) subset of $\mathbb{R}^2$ defined as

$$\Delta_0 := \{x \in \mathbb{R}^2 \; ; \; \det(f(x), g(x)) = 0\}. \tag{2.10}$$

Define two functions $\delta_0, \psi : \mathbb{R}^2 \to \mathbb{R}$ as $\delta_0(x) := \det(f(x), g(x))$, $x \in \mathbb{R}^2$, and

$$\psi(x) := -\frac{\det(g(x), [f, [f, g]](x))}{\det(g(x), [g, [f, g]](x))}, \quad x \in \mathbb{R}^2. \tag{2.11}$$

The singular control can be then expressed as follows.

Lemma 2.1. *Suppose that $\Delta_{SA} \neq \emptyset$, that $x \mapsto \det(g(x), [g, [f, g]](x))$ is non-zero over Δ_{SA}, and consider a singular arc defined over an interval $[t_1, t_2]$. Then, one has:*

$$u_s(t) = \psi(x(t)), \quad t \in [t_1, t_2], \tag{2.12}$$

where $x(\cdot)$ is the corresponding singular trajectory such that $x(t) \in \Delta_{SA}$ for $t \in [t_1, t_2]$.

Proof. The proof is classical and combines (2.9) together with the equalities

$$\begin{aligned} -\delta_0(x(t))\, p(t) \cdot [f, [f, g]](x(t)) &= \det(g(x(t)), [f, [f, g]](x(t))), \\ -\delta_0(x(t))\, p(t) \cdot [g, [f, g]](x(t)) &= \det(g(x(t)), [g, [f, g]](x(t))), \end{aligned}$$

see, *e.g.*, [14, Lemma 10]. □

Remark 2.1. *Steady-state singular points are defined as the points $x^\star \in \Delta_{SA} \cap \Delta_0$ such that $g(x^\star) \neq 0$, see* [14, 4] *(if $\Delta_{SA} \cap \Delta_0 \neq \emptyset$). Such points are equilibria of* (2.1) *with $u = \psi(x)$. A singular arc defined over a time interval $[t_1, t_2]$ does not contain such a point because $f(x(t))$ and $g(x(t))$ must be linearly independent over $[t_1, t_2]$. But, it can contain points $x^\star \in \Delta_{SA} \cap \Delta_0$ such that $g(x^\star) = 0$.*

In the sequel, we consider a parametrization of the set Δ_{SA} as follows. This will be useful to introduce the notion of saturation point.

Lemma 2.2. *Suppose that Δ_{SA} is non-empty and that $x \mapsto \det(g(x), [g, [f, g]](x))$ is non-zero over Δ_{SA}. Then, near each point $x_0 \in \Delta_{SA} \backslash \Delta_0$, the set $\Delta_{SA} \backslash \Delta_0$ can be locally parametrized by a one-to-one parametrization $\zeta : J \to \Delta_{SA}$, $\tau \mapsto \zeta(\tau)$ of class C^1, where J is an interval of $\mathbb{R}$.*

Proof. For $x \notin \Delta_0$, one has $\mathrm{span}\{f(x), g(x)\} = \mathbb{R}^2$, hence, there exist $\alpha(x), \beta(x) \in \mathbb{R}$ such that

$$[f, g](x) = \alpha(x) f(x) + \beta(x) g(x). \tag{2.13}$$

By taking the determinant between $g(x)$ and $[f, g](x)$, and then between $f(x)$ and $[f, g](x)$, we find that for $x \notin \Delta_0$,

$$\alpha(x) = -\frac{\det(g(x), [f, g](x))}{\delta_0(x)} \quad \text{and} \quad \beta(x) = \frac{\det(f(x), [f, g](x))}{\delta_0(x)}.$$

By computing $[g,[f,g]](x)$ thanks to (2.13), we get

$$\det(g(x),[g,[f,g]](x)) = -\delta_0(x)\nabla\alpha(x)\cdot g(x), \quad x \notin \Delta_0.$$

Since $x \mapsto \det(g(x),[g,[f,g]](x))$ is non-zero over Δ_{SA}, the preceding equality implies that the scalar product $\nabla\alpha(x)\cdot g(x)$ is non-zero. On the other hand, Δ_{SA} is defined by the implicit equation $\delta_{SA}(x) = 0$. Next, observe that

$$x \notin \Delta_0 \;\Rightarrow \delta_{SA}(x) = -\alpha(x)\delta_0(x). \tag{2.14}$$

By taking the derivative, we find that for $x \notin \Delta_0$, one has

$$\nabla\delta_{SA}(x) = -\delta_0(x)\nabla\alpha(x) - \alpha(x)\nabla\delta_0(x).$$

Therefore, for $x \in \Delta_{SA}\backslash\Delta_0$, we obtain $\nabla\delta_{SA}(x) = -\delta_0(x)\nabla\alpha(x)$. We can then conclude that for any point $x_0 \in \Delta_{SA}\backslash\Delta_0$, the partial derivative $\partial_1\alpha(x_0)$ (w.r.t. x_1) or $\partial_2\alpha(x_0)$ (w.r.t. x_2) is non-zero. We are then in a position to apply the implicit function theorem to δ_{SA} locally at each point $x_0 \in \Delta_{SA}\backslash\Delta_0$, which implies the desired property. □

When $\Delta_{SA} \cap \Delta_0$ is non-empty, the set $\Delta_{SA}\backslash\Delta_0$ can be then partitioned into (maximal) connected one dimensional submanifolds of Δ_{SA}, according to the previous lemma. Hence, we write this set as

$$\Delta_{SA}\backslash\Delta_0 = \bigcup_{k\in K} \gamma_k,$$

where K is an index set. Hereafter, to shorten, the terminology "component" refers to a (maximal) connected one-dimensional manifold γ_k with $k \in K$. Under the assumptions of Lemma 2.2, given a component γ_k of Δ_{SA}, there is a parametrization ζ such that

$$\gamma_k := \{\zeta(\tau)\ ;\ \tau \in J\},$$

where $\zeta : J \to \mathbb{R}^2$ is C^1-mapping (injective) and J is an interval.

Definition 2.1. *A point $x^* := \zeta(\tau^*)$ with $\tau^* \in \mathrm{Int}(J)$ is called saturation point if $|\psi(x^*)| = 1$ and:*

- *either, $|\psi(\zeta(\tau))| < 1$ for any $\tau \in J$ such that $\tau < \tau^*$, and $|\psi(\zeta(\tau))| > 1$ for any $\tau \in J$ such that $\tau > \tau^*$,*
- *or $|\psi(\zeta(\tau))| > 1$ for any $\tau \in J$ such that $\tau < \tau^*$, and $|\psi(\zeta(\tau))| < 1$ for any $\tau \in J$ such that $\tau > \tau^*$.*

Note that, in this paper, we shall mainly consider those saturation points such that $\psi(x^*) = 1$ where ψ is increasing over γ_k.

Our next aim is to study the optimality of singular arcs in presence of a saturation point.

3 Existence of a prior-saturation point

In this section, we show that a prior-saturation phenomenon can occur whenever the system exhibits a saturation point. We start by introducing our main assumptions.

Assumption 3.1. *The system (2.1) satisfies the following hypotheses:*

(i) *One has $\Delta_0 = \emptyset$ and $\delta_0(x) < 0$ for all $x \in \mathbb{R}^2$ so that the singular locus Δ_{SA} is written $\Delta_{SA} := \zeta(J)$ where $J \subset \mathbb{R}$ is an interval and $\zeta : J \to \Delta_{SA}$ is a C^1-mapping.*

(ii) *The set Δ_{SA} is non-empty, simply connected, and has exactly one saturation point $x^* = \zeta(\tau^*)$ with $\psi(x^*) = 1$ and $\tau^* \in J$. Moreover, $\tau \mapsto \psi(\zeta(\tau))$ is increasing over J.*

(iii) *Along the singular locus, the strict (generalized) Legendre-Clebsch optimality condition is satisfied, that is, any singular extremal $(x_u(\cdot), p(\cdot), u(\cdot))$ defined over $[t_1, t_2]$ satisfies:*

$$\frac{\partial}{\partial u}\frac{d^2}{dt^2}\frac{\partial H}{\partial u}(x_u(t), p(t), u(t)) > 0, \quad \forall t \in [t_1, t_2]. \tag{3.1}$$

(iv) *If* Γ_- *is the forward semi-orbit of* (2.1) *with* $u = -1$ *with the initial condition* x^* *at time* 0*, then*

$$\mathcal{T} \cap \Gamma_- = \emptyset. \tag{3.2}$$

(v) *The target* $\mathcal{T}$ *is reachable from every point* $x_0 \in \mathbb{R}^2$*.*

Remark 3.1. (i) *The hypothesis* $\Delta_0 = \emptyset$ *could be weakened considering in place of* $\mathbb{R}^2$ *an optimally*[3] *invariant subset* Ω *(by the dynamics) containing only one component* γ_k *and the target set. In addition, we would require the condition* $\delta_0 < 0$ *in* Ω*. For simplicity, we supposed here that* $\Omega = \mathbb{R}^2$*.*
(ii) *By the previous computations, we can observe that* (3.1) *is equivalent to*

$$\forall x \in \Delta_{SA},\ |\psi(x)| \leq 1 \ \Rightarrow\ \det(g(x), [g, [f, g]](x)) > 0.$$

Recall that, under the strict Legendre-Clebsch condition, the singular arc is a turnpike, i.e., it is time-minimizing in every neighborhood of a hyperbolic point of Δ_{SA}*,* [13]*. This property can be retrieved by the clock form argument* [15]*.*
(iii) In general, the saturation point x^* *is not unique (this depends on the case study). Whereas in the two examples of Section 5 it is unique, let us mention* [7] *in which several saturation points as well as bridges occur in the context of chemical reactors.*

Under Assumption 3.1, Δ_{SA} partitions the state space into two simply connected (open) subsets $\Delta_{SA}^{\pm}$:

$$\begin{aligned} \Delta_{SA}^{+} &:= \{x \in \mathbb{R}^2\ ;\ \det(g(x), [f, g](x)) > 0\}, \\ \Delta_{SA}^{-} &:= \{x \in \mathbb{R}^2\ ;\ \det(g(x), [f, g](x)) < 0\}. \end{aligned}$$

Given a normal extremal $(x_u(\cdot), p(\cdot), u(\cdot))$, the function

$$t \mapsto \gamma_u(t) := \beta(x_u(t)) - \alpha(x_u(t))u(t), \quad t \in [0, T_u],$$

is also well-defined since $\Delta_0 = \emptyset$.

Lemma 3.1. *Along a normal extremal* $(x_u(\cdot), p(\cdot), u(\cdot))$*, the switching function* ϕ *satisfies the ODE*

$$\dot{\phi}(t) = \gamma_u(t)\phi(t) + \alpha(x_u(t)) \quad \text{a.e. } t \in [0, T_u]. \tag{3.3}$$

Proof. The proof follows using the expression of $\dot{\phi}$ and the fact that the Hamiltonian H is constant equal to zero. □

The next proposition shows that an extremal trajectory containing a singular arc until the point $x^\star$ is not optimal.

Proposition 3.1. *Suppose that Assumption 3.1 holds true, and consider an optimal trajectory steering* x_0 *to the target* $\mathcal{T}$ *in time* T_u*. Then, the corresponding extremal* $(x_u(\cdot), p(\cdot), u(\cdot))$ *does not contain a singular arc defined over a time interval* $[t_1, t_2]$ *such that* $x_u(t_2) = x^*$*.*

Proof. Suppose by contradiction that there is a time interval $[t_1, t_2]$ such that the trajectory is singular over $[t_1, t_2]$ with $x_u(t_2) = x^*$. First, note that at time t_2, the vector $f(x_u(t_2)) + g(x_u(t_2))$ is tangent to Δ_{SA}. Indeed, since $x_u(t) \in \Delta_{SA}$ for $t \in [t_1, t_2]$, the vector $f(x_u(t)) + \psi(x_u(t))g(x_u(t))$ is tangent to Δ_{SA} at every time $t \in [t_1, t_2]$. The result then follows because at time t_2, one has $\psi(x_u(t_2)) = 1$.

From lemma 2.2, observe that for $x \in \Delta_{SA}$, one has:

$$\begin{aligned} \det(g(x), [g, [f, g]](x)) &= -\delta_0(x)\, \nabla\alpha(x) \cdot g(x), \\ \det(g(x), [f, [f, g]](x)) &= -\delta_0(x)\, \nabla\alpha(x) \cdot f(x). \end{aligned} \tag{3.4}$$

We deduce the following equalities:

$$\begin{aligned} \nabla\delta_{SA}(x) \cdot (f(x) + g(x)) &= \det(g(x), [g, [f, g]](x))\,(1 - \psi(x)), \\ \nabla\delta_{SA}(x) \cdot (f(x) - g(x)) &= \det(g(x), [g, [f, g]](x))\,(-1 - \psi(x)). \end{aligned} \tag{3.5}$$

[3] By optimally invariant subset, we mean a subset $\Omega \subset \mathbb{R}^2$ such that for any initial condition $x_0 \in \Omega$, an optimal trajectory stays in Ω.

Going back to the optimal trajectory, under the strict Legendre-Clebsch condition[4], there is no chattering phenomenon for the optimal control at time t_2. This excludes the trajectory to have an infinite number of switching times in a right neighborhood of $t = t_2$. Because the singular arc becomes non admissible, the trajectory is also not singular in a right neighborhood of t_2.

There remain now two possibilities for the trajectory $x_u(\cdot)$:
- either one has $u = +1$ over some time interval $[t_2, t_2 + \varepsilon)$ with $\varepsilon > 0$,
- or one has $u = -1$ over some time interval $[t_2, t_2 + \varepsilon)$ with $\varepsilon > 0$.

Suppose first that $u = +1$ over $[t_2, t_2 + \varepsilon)$. Thanks to Assumption 3.1 (ii), one has $\psi(\zeta(\tau)) > 1$ for every $\tau > \tau^*$ and Remark 3.1 (ii) implies that

$$\det(g(\zeta(\tau)), [g, [f, g]](\zeta(\tau))) > 0.$$

From the expression of ψ given by (2.11), we deduce the inequalities

$$\det(g(\zeta(\tau)), [g, [f, g]](\zeta(\tau))) > 0 \ \text{ and } \ -\det(f(\zeta(\tau)), [g, [f, g]](\zeta(\tau))) > 0,$$

for every $\tau > \tau^*$. Taking into account (3.5), we obtain that

$$\tau > \tau^* \ \Rightarrow \ \nabla\delta_{SA}(\zeta(\tau)) \cdot (f(\zeta(\tau)) + g(\zeta(\tau))) < 0.$$

One can then infer that for $\tau > \tau^*$, each vector $f(\zeta(\tau)) + g(\zeta(\tau))$ at $\zeta(\tau) \in \Delta_{SA}$ points into Δ_{SA}^-. It follows that the corresponding trajectory necessarily enters into Δ_{SA}^- implying (recall (2.14))

$$\det(g(x_u(t)), [f, g](x_u(t))) = -\delta_0(x_u(t))\alpha(x_u(t)) < 0$$

for $t \in (t_2, t_2 + \varepsilon]$. From (3.3), since $\phi(t_2) = \dot{\phi}(t_2) = 0$, we obtain that $\phi(t) < 0$ for $t \in (t_2, t_2 + \varepsilon]$. From Pontryagin's Principle, $\phi < 0$ implies that $u(t) = -1$. We thus have a contradiction.

It follows that the optimal trajectory necessarily satisfies $u = -1$ in some time interval $(t_2, t_2 + \varepsilon]$, and thus it enters into the set Δ_{SA}^- because one has $\nabla\delta_{SA}(x^*) \cdot (f(x^*) - g(x^*)) < 0$.

From Assumption 3.1, the forward semi-orbit with $u = -1$ starting from x^* does not reach the target set. Hence, $x_u(\cdot)$ must have a switching point to $u = +1$ in Δ_{SA}^- or it must reach Δ_{SA} with the control $u = -1$. We see from (3.3) that the first case is not possible because at a switching time t_c such that $x_u(t_c) \in \Delta_{SA}^-$, we would have $\dot{\phi}(t_c) \geq 0$ in contradiction with $\alpha(x_u(t_c)) < 0$.

Suppose now that $x_u(\cdot)$ reaches Δ_{SA} at some point $x := \zeta(\tau)$ with $\tau < \tau^*$. Then, we obtain $\nabla\delta_{SA}(x) \cdot (f(x) - g(x)) < 0$ since $\psi(x) > -1$. But, as $x_u(\cdot)$ reaches Δ_{SA} with $u = -1$ at point x, the trajectory enters into the set $\Delta_{SA} \cup \Delta_{SA}^+$ and we must have $\nabla\delta_{SA}(x) \cdot (f(x) - g(x)) \geq 0$ ($\nabla\delta_{SA}(x)$ is collinear to the outward normal vector to Δ_{SA} at point x). This gives a contradiction. In the same way, the trajectory cannot reach a point $x \in \Delta_{SA}$ such that $x = \zeta(\tau)$ with $\tau > \tau^*$.

We can conclude that for any time $t \geq t_2$, one has $u(t) = -1$, but then, the optimal trajectory cannot reach the target set which is a contradiction (Assumption 3.1 (iv)). This concludes the proof. □

As an example, if $x_0 := \zeta(\tau_0)$ belongs to the singular locus with $\tau_0 < \tau^*$, and if an optimal trajectory starting from x_0 contains a singular arc, then the trajectory should leave the singular locus before reaching x^*. Let us insist on the fact that this property of leaving the singular locus before reaching x^* relies on the fact that the optimal trajectory should contain a singular arc. In the fed-batch model presented in Section 5.1, this property can be easily verified (see [3]).

We now introduce the following definition (in line with [17, 23, 24]). Hereafter, the notation $\mathcal{S}_{[\tau_0', \tau_0]}$ denotes the singular arc comprised between the points $\zeta(\tau_0')$ and $\zeta(\tau_0)$ with $\tau_0' \leq \tau_0 < \tau^*$.

Definition 3.1. *Let $\tau_0 < \tau^*$. A point $x_e := \zeta(\tau_e) \in \Delta_{SA}$ with $\tau_0 < \tau_e < \tau^*$ is called a prior-saturation point if the singular arc $\mathcal{S}_{[\tau_0, \tau]}$ ceases to be optimal for $\tau \geq \tau_e$, that is,*

$$\tau_e = \sup\{\tau \in J \ ; \ \mathcal{S}_{[\tau_0, \tau]} \text{ is optimal}\}.$$

This definition makes sense only for initial conditions $\zeta(\tau_0)$ with $\tau_0 < \tau^*$ because for $\tau_0 \geq \tau^*$, optimal controls are not singular (since the singular control is non-admissible). Note that x_e a priori depends on the initial condition $\zeta(\tau_0)$ on the singular locus (hence, we shall write $x_e = x_e(\tau_0)$ hereafter if necessary).

[4]Following for instance [24, Section 2.8.4], chattering occurs for singular arcs of higher order (at least 2) for which the Legendre-Clebsch condition is not fulfilled.

Proposition 3.2. *Suppose that Assumption 3.1 holds true and that there are $\tau_1, \tau_2 \in J$ with $\tau_1 < \tau_2 < \tau^*$ such that any optimal trajectory starting from $\zeta(\tau_0)$ with $\tau_0 \in [\tau_1, \tau_2)$ contains a singular arc $\mathcal{S}_{[\tau_0,\tau_2]}$. Then, one has $x_e(\tau_0) = x_e(\tau_2)$ for every $\tau_0 \in [\tau_1, \tau_2]$.*

Proof. Consider an optimal trajectory γ starting from $\zeta(\tau_0) \in \Delta_{SA}$ with $\tau_0 \in [\tau_1, \tau_2]$. The hypothesis of the proposition implies that this trajectory passes though the point $\zeta(\tau_2)$. In addition, from the initial condition $\zeta(\tau_2)$, an optimal trajectory necessarily leaves the singular locus at a prior saturation point denoted by $\bar{x}_e$. It follows that the optimal trajectory starting from $\zeta(\tau_0)$ also leaves Δ_{SA} at the same point $\bar{x}_e$. Otherwise, γ would leave Δ_{SA} at a point $x_e \neq \bar{x}_e$. Since γ is optimal, the optimal trajectory from $\zeta(\tau_2)$ would also leave Δ_{SA} at x_e implying a contradiction. □

This property implies that for every initial conditions $x_0 := \zeta(\tau_0) \in \Delta_{SA}$ such that $\tau_0 \in [\tau_1, \tau_2]$, then the corresponding optimal path has a singular arc until the point $x_e(\tau_2)$ and a switching point at this point.

Remark 3.2. *Such a situation (in particular the existence of an interval $[\tau_1, \tau_2]$ as in Proposition 3.2) is encountered in the fed-batch model presented in Section 5.1 (see also [3]). Note also that for initial conditions $\zeta(\tau_0) \in \Delta_{SA}$ with $\tau_0 < \tau^*$, τ_0 close to τ^*, we can expect that $\tau_e(\tau_0) = \tau_0$, i.e., the singular arc (altough being admissible) is no longer optimal.*

Remark 3.3. *In addition to Assumption 3.1 (in particular (3.2)), if we suppose that $\mathcal{T}$ is not reachable with the constant control $u = -1$ from those points of Δ_{SA} located between x_e and x^* (i.e. corresponding to $\tau \in [\tau_e, \tau^*]$), then the maximal value for the control $u = +1$ is locally optimal from the prior-saturation point x_e. In other words, the bridge (the last arc leaving Δ_{SA}) corresponds to $u = +1$. This can be proved by using similar arguments as for proving Proposition 3.2. Since the singular arc is a turnpike, this additional hypothesis also implies the existence of a switching curve emanating from x_e. Our next aim is precisely to investigate more into details geometric properties of optimal paths at the point x_e.*

4 Tangency property and prior-saturation phenomenon

The aim of this section is to prove the tangency property as stated in Theorem 4.1.

4.1 Introduction to prior-saturation lift and tangency property

In this section, we first introduce the concept of *prior-saturation lift* and discuss its local uniqueness. We also provide a set of nonlinear equations allowing to compute prior-saturation lifts given by the PMP. We end this section with an introduction to the tangency property on an example.

Definition 4.1. *Let x_e be a prior-saturation point. Any point z_e in the cotangent space at x_e is called a* prior-saturation lift *of x_e.*

To introduce the computation of prior-saturation lifts given by the PMP, let us start with an example. Consider a target set $\mathcal{T} := \{x_f\}$, $x_f \in \mathbb{R}^2$, with an optimal trajectory of the form $\sigma_-\sigma_s\sigma_+$, where σ_-, σ_+ and σ_s are arcs, respectively, with control $u = -1$, $u = +1$ and $u = u_s$, where u_s is the singular control. Assume that the switching point between the singular arc σ_s (supposed to be non-empty) and the positive bang arc σ_+ is a prior-saturation point. The PMP[5] gives necessary optimality conditions satisfied by this extremal trajectory that we can write as a system of nonlinear equations, the so-called *shooting equations*. Actually, the shooting equations, that we present hereinafter, will give us the initial adjoint vector and the switching times together with the switching points. From this, we can retrieve the extremal trajectory. Before defining this set of equations, we need to introduce some notation. We define the Hamiltonian lifts associated with f and g as

$$H_f(z) := p \cdot f(x) \; ; \; H_g(z) := p \cdot g(x),$$

where $z := (x, p)$ belongs to the cotangent bundle. All the other Hamiltonian lifts in the rest of the paper are defined like this. Define also the Hamiltonians $H_\pm := H_f \pm H_g$ and $H_s := H_f + u_s H_g$, where u_s is viewed here as a function of z:

$$u_s(z) := -\frac{p \cdot [f, [f, g]](x)}{p \cdot [g, [f, g]](x)} = -\frac{H_{[f,[f,g]]}(z)}{H_{[g,[f,g]]}(z)}. \tag{4.1}$$

[5]Since $\mathcal{T} := \{x_f\}$ is a point, there is no transversality condition at the terminal time.

For any Hamiltonian H we define the *Hamiltonian system* $\overrightarrow{H} := (\partial_p H, -\partial_x H)$, and finally, we introduce the *exponential mapping* $\exp(t\varphi)(z_0)$ as the solution at time t of the differential equation $\dot{z}(s) = \varphi(z(s))$ with initial condition $z(0) = z_0$, where φ is supposed to be smooth. The shooting equations are then given by

$$S(y) = 0, \quad y := (p_0, t_1, t_2, t_f, z_1, z_2) \in \mathbb{R}^{n+3+(2n)\times 2}, \quad n := 2,$$

where the *shooting function* is defined by

$$S(y) := \begin{pmatrix} H_g(z_1) \\ H_{[f,g]}(z_1) \\ H_+(\exp((t_f - t_2)\overrightarrow{H_+})(z_2)) + p^0 \\ \pi(\exp((t_f - t_2)\overrightarrow{H_+})(z_2)) - x_f \\ z_1 - \exp(t_1 \overrightarrow{H_-})(x_0, p_0) \\ z_2 - \exp((t_2 - t_1)\overrightarrow{H_s})(z_1) \end{pmatrix}, \tag{4.2}$$

where $\pi(x, p) := x$, and where $x_0 \in \mathbb{R}^2$ is given and $p^0 = -1$ in the normal case. The two first equations mean that the trajectory is entering the singular locus at z_1. Hence, the second arc is a singular arc. The third equation takes into account the free terminal time. It could be replaced by $H_-(x_0, p_0) + p^0 = 0$ since the maximized Hamiltonian is constant along the extremal. The fourth equation implies that the last bang arc reaches the target $\mathcal{T} = \{x_f\}$ at the final time t_f, and the last two equations are the so-called *matching conditions* which give the switching points. Let consider a solution $y^* := (p_0, t_1, t_2, t_f, z_1, z_2)$ to $S(y) = 0$ associated with the optimal trajectory described before. Then, the point $\pi(z_2)$ is a prior-saturation point, and so, z_2 is a prior-saturation lift.

Let us now discuss the uniqueness of the prior-saturation lift, considering for instance, a smooth and local one-parameter family of initial conditions $x_0(\alpha)$, $\alpha \in (-\varepsilon, \varepsilon)$, $\varepsilon > 0$, in relation with the construction of optimal syntheses (see section 5) and in relation with Proposition 3.2. Let us denote by $y^*(\alpha) := (p_0(\alpha), t_1(\alpha), t_2(\alpha), t_f(\alpha), z_1(\alpha), z_2(\alpha))$ a family of solutions to the equation $S(y, \alpha) = 0$ for $\alpha \in (-\varepsilon, \varepsilon)$, where $S(\cdot, \alpha)$ is defined as (4.2) but with the initial condition $x_0(\alpha)$ in place of x_0. Let us assume that for any $\alpha \in (-\varepsilon, \varepsilon)$, the corresponding trajectory is an optimal trajectory of the form $\sigma_-\sigma_s\sigma_+$. In addition, suppose that the lengths t_1, $t_2 - t_1$ and $t_f - t_2$ are positive, that is, each arc is defined on a time interval of positive length. In this setting, for any α, we have $x_e := \pi(z_2(0)) = \pi(z_2(\alpha))$, that is, the prior-saturation point x_e is locally unique. This is related to Proposition 3.2 and illustrated on Figure 1. Besides, under some other assumptions, see Proposition 4.1 and remark 4.1, the prior-saturation lift $z_e := z_2(0)$ is also locally unique and so in this case we have also $z_2(0) = z_2(\alpha)$ for any $\alpha \in (-\varepsilon, \varepsilon)$.

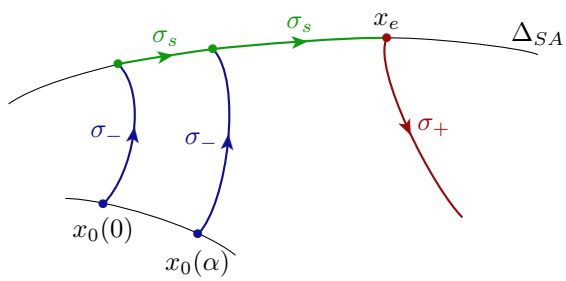

Figure 1: Local uniqueness of the prior-saturation point x_e.

Assuming that the prior-saturation lift z_e is locally unique, we can compute it with a set of equations excerpt from the shooting equations but with some minor modifications. Roughly speaking, the main idea is to consider the particular case where the initial condition is the prior-saturation point, that is such that $x_0 = x_e$. In this case, we have $t_1 = t_2 = 0$ and $z_1 = z_2 = (x_e, p_0) = z_e$. With these considerations in mind we introduce

$$F_{\text{ex}}(t_b, z_b) := \begin{pmatrix} H_g(\exp(-t_b\overrightarrow{H_+})(z_b)) \\ H_{[f,g]}(\exp(-t_b\overrightarrow{H_+})(z_b)) \\ H_+(z_b) + p^0 \\ \pi(z_b) - x_f \end{pmatrix},$$

with $F_{\text{ex}} : \mathbb{R}^5 \to \mathbb{R}^5$. Note that the exponential mapping is here computed by backward integration. Hence, with the preceding notation, we have $z_b = \exp((t_f - t_2)\overrightarrow{H_+})(z_2) = \exp(t_f \overrightarrow{H_+})(z_e)$ and $t_b = t_f - t_2 = t_f$. At the end, the prior-saturation lift is simply given by

$$z_e = \exp(-t_b \overrightarrow{H_+})(z_b),$$

for a couple (t_b, z_b) solution of $F_{\text{ex}} = 0$.

Tangency property. We end this section with an introduction to the tangency property. Let us start with solutions of the form $\sigma_- \sigma_s \sigma_+$, considering a smooth and local one-parameter family of initial conditions $x_0(\alpha)$, $\alpha \in (-\varepsilon, \varepsilon)$, $\varepsilon > 0$, but assuming that for $\alpha = 0$, the optimal solution is of the form $\sigma_- \sigma_s \sigma_+$ with σ_s reduced to a single point, that is, $t_2(0) - t_1(0) = 0$, with

$$y^*(0) := (p_0(0), t_1(0), t_2(0), t_f(0), z_1(0), z_2(0)),$$

the solution to the associated shooting equations, still denoted $S(y, \alpha) = 0$. Assume also that for $\alpha > 0$, we are in the previous case, that is one has $t_2(\alpha) - t_1(\alpha) > 0$ with $y^*(\alpha) := (p_0(\alpha), t_1(\alpha), t_2(\alpha), t_f(\alpha), z_1(\alpha), z_2(\alpha))$ the corresponding solution of $S(\cdot, \alpha) = 0$. The prior-saturation lift is thus given by $z_e = z_2(\alpha)$ for $\alpha \in [0, \varepsilon)$.

The idea is now to consider the case where there is a bifurcation in the structure of the optimal trajectories when $\alpha = 0$. We thus assume that for $\alpha \in (-\varepsilon, 0)$, the solutions are of the form $\sigma_- \sigma_+$ and we denote by $z_1(\alpha)$ the switching point (in the cotangent bundle) between the two arcs. In this setting, there exists a switching locus in the optimal synthesis denoted $\Sigma_- \cup \Sigma_0$, where

$$\Sigma_- := \{z_1(\alpha);\ \alpha \in (-\varepsilon, 0]\} \quad \text{and} \quad \Sigma_0 := \{z_1(0) = z_e\}.$$

The aim of the next section is to prove that the semi-orbit Γ_+ of $\dot{z} = \overrightarrow{H_+}(z)$ starting from z_e is tangent to the switching curve $\Sigma_- \cup \Sigma_0$ at the prior-saturation lift z_e in a general frame. This is precisely the *tangency property* (see Fig. 2).

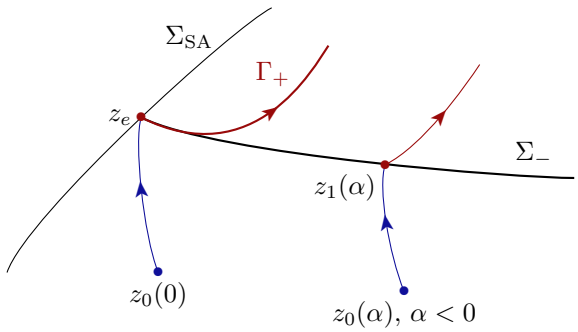

Figure 2: Illustration of the tangency property between Γ_+ and $\Sigma_- \cup \Sigma_0$ at the prior-saturation lift z_e. The *singular locus* in the cotangent bundle is $\Sigma_{\text{SA}} := \{z \in \mathbb{R}^{2n}\ ;\ H_g(z) = H_{[f,g]}(z) = 0\}$.

4.2 Proof of the tangency property

From a general point of view, we shall assume that the prior-saturation lift is given by solving a set of nonlinear equations

$$F(t_b, z_b, \lambda) = 0$$

with

$$F(t_b, z_b, \lambda) := \begin{pmatrix} H_{[f,g]}(\exp(-t_b \overrightarrow{H_+})(z_b)) \\ G(t_b, z_b, \lambda) \end{pmatrix}, \tag{4.3}$$

where $\lambda \in \mathbb{R}^k$ is a vector of $k \in \mathbb{N}$ parameters, where F is a function from $\mathbb{R}^{5+k}$ to $\mathbb{R}^{5+k}$ and where $G : \mathbb{R}^{5+k} \to \mathbb{R}^{4+k}$ is defined by

$$G(t_b, z_b, \lambda) := \begin{pmatrix} H_g(\exp(-t_b \overrightarrow{H_+})(z_b)) \\ H_+(z_b) + p^0 \\ \Psi(z_b, \lambda) \end{pmatrix}, \tag{4.4}$$

with $\Psi : \mathbb{R}^{4+k} \to \mathbb{R}^{2+k}$ a given function and $p^0 = -1$ considering the normal case. We assume that all the functions F, G and Ψ are smooth. It is important to notice that the mapping Ψ does not depend on t_b and that we can replace H_+ by H_- without any loss of generality. In the previous example from section 4.1, we have (with a slight abuse of notation) $\Psi(z_b) = \pi(z_b) - x_f$ which corresponds to the simplest case where there are no transversality conditions and no additional parameters, that is $k = 0$. For a more complex structure of the form $\sigma_-\sigma_s\sigma_+\sigma_-$, the parameter λ would be the last switching time between the σ_+ and σ_- arcs. In this case, Ψ would contain the additional switching condition $H_g = 0$ at this time.

Let $(t_b^*, z_b^*, \lambda^*) \in \mathbb{R}^{5+k}$ be a solution to the equation $F = 0$ and define

$$z_e := \exp(-t_b^* \overrightarrow{H_+})(z_b^*) \in \Sigma_{\mathrm{SA}} := \{z \in \mathbb{R}^{2n} \ ; \ H_g(z) = H_{[f,g]}(z) = 0\}. \tag{4.5}$$

We introduce the following assumptions at the point z_e.

Assumption 4.1. *We have $H_{[g,[f,g]]}(z_e) \neq 0$ and $u_s(z_e) < 1$ with u_s the singular control given by* (4.1)*.*

Assumption 4.2. *The matrix*

$$\begin{bmatrix} \dfrac{\partial G}{\partial z_b}(t_b^*, z_b^*, \lambda^*) & \dfrac{\partial G}{\partial \lambda}(t_b^*, z_b^*, \lambda^*) \end{bmatrix} \in \mathrm{GL}_{4+k}(\mathbb{R}),$$

i.e., it is invertible in $\mathbb{R}^{(4+k)\times(4+k)}$.

Remark 4.1. *Assumption 4.1 is related to the prior-saturation phenomenon while in combination with Assumption 4.2, it is related to the well-posedness of the shooting system $F = 0$. Besides, the point z_e is locally unique under these assumptions, according to the following result.*

Proposition 4.1. *Suppose that Assumptions 4.1 and 4.2 hold true. Then,*

$$F'(t_b^*, z_b^*, \lambda^*) \in \mathrm{GL}_{5+k}(\mathbb{R}).$$

Moreover z_e (defined by (4.5)*) is locally unique.*

Proof. The Jacobian of the mapping F at the point $(t_b^*, z_b^*, \lambda^*)$ is given by:

$$F'(t_b^*, z_b^*, \lambda^*) = \begin{bmatrix} -a & * & * \\ -b & \dfrac{\partial G}{\partial z_b}(t_b^*, z_b^*, \lambda^*) & \dfrac{\partial G}{\partial \lambda}(t_b^*, z_b^*, \lambda^*) \end{bmatrix},$$

where $a := H_{[f,[f,g]]}(z_e) + H_{[g,[f,g]]}(z_e)$ and $b := (H_{[f,g]}(z_e), 0, 0)$. Then, we have $F'(t_b^*, z_b^*, \lambda^*) \in \mathrm{GL}_{5+k}(\mathbb{R})$, noticing that $b = 0$ since $F(t_b^*, z_b^*, \lambda^*) = 0$, that $a \neq 0$ since $u_s(z_e) < 1$ and that $H_{[g,[f,g]]}(z_e) \neq 0$ by Assumption 4.1. By the inverse function theorem z_e is thus locally unique. □

Lemma 4.1. *Suppose that Assumption 4.2 holds true. Then, there exists $\varepsilon > 0$ and a C^1-map $t_b \mapsto \sigma(t_b) := (z_b(t_b), \lambda(t_b)) \in \mathbb{R}^{4+k}$ defined over $I_\varepsilon := (t_b^* - \varepsilon, t_b^* + \varepsilon)$, that satisfies*

$$\forall t_b \in I_\varepsilon, \quad G(t_b, \sigma(t_b)) = 0. \tag{4.6}$$

In addition, one has $\sigma(t_b^*) = (z_b^*, \lambda^*)$ and $\sigma'(t_b^*) = 0_{\mathbb{R}^{4+k}}$.

Proof. The existence of σ follows from the implicit function theorem applied to the mapping G at $(t_b^*, z_b^*, \lambda^*)$ which also gives (4.6). The derivative of σ is then obtained from (4.6):

$$\sigma'(t_b) = -\begin{bmatrix} \dfrac{\partial G}{\partial z_b}[t_b] & \dfrac{\partial G}{\partial \lambda}[t_b] \end{bmatrix}^{-1} \cdot \frac{\partial G}{\partial t_b}[t_b], \quad t_b \in I_\varepsilon,$$

where $[t_b]$ stands for $(t_b, \sigma(t_b))$. Since

$$\frac{\partial G}{\partial t_b}[t_b^*] = (H_{[f,g]}(z_e), 0_{\mathbb{R}^{3+k}}) = 0_{\mathbb{R}^{4+k}},$$

the result follows. □

Let us introduce the mapping $\varphi(t_b) := \exp(-t_b\overrightarrow{H_+})(z_b(t_b))$ for $t_b \in I_\varepsilon$ and define

$$\Sigma := \{\varphi(t_b)\ ;\ t_b \in I_\varepsilon\}. \tag{4.7}$$

Remark 4.2. *The curve Σ is a switching curve in the contangent bundle since one has $H_g(\varphi(t_b)) = 0$ by definition of G. However, this switching curve is not necessarily optimal, that is, the optimal synthesis, with respect to the initial condition, may not contain Σ. Let us stratify Σ according to $\Sigma = \Sigma_- \cup \Sigma_0 \cup \Sigma_+$, with*

$$\begin{aligned}
\Sigma_- &:= \{\varphi(t_b)\ ;\ t_b \in (t_b^* - \varepsilon, t_b^*)\},\\
\Sigma_0 &:= \{\varphi(t_b^*)\} = \{z_e\},\\
\Sigma_+ &:= \{\varphi(t_b)\ ;\ t_b \in (t_b^*, t_b^* + \varepsilon)\}.
\end{aligned}$$

A typical situation is when $\Sigma_- \cup \Sigma_0$ is contained in the optimal synthesis while Σ_+ is not optimal for local and/or global optimality reasons. See the end of Section 4.1 for an example of this typical situation.

Our first main result is given by Proposition 3.2 which states the existence of a prior-saturation point x_e in the state space under Assumption 3.1. Our second main result is the following.

Theorem 4.1. *Suppose the existence of a triple $(t_b^*, z_b^*, \lambda^*) \in \mathbb{R}^{5+k}$ such that $F(t_b^*, z_b^*, \lambda^*) = 0$, with F defined by (4.3) and set $z_e := \exp(-t_b^*\overrightarrow{H_+})(z_b^*)$. Suppose also that Assumption 4.2 holds true. Then, the switching curve Σ given by (4.7) is tangent at z_e to the forward semi-orbit Γ_+ of $\dot{z} = \overrightarrow{H_+}(z)$ starting from z_e.*

Proof. From Assumption 4.2 and by lemma 4.1, one can define the switching curve Σ by (4.7). To prove the tangency property, we have to show that $\varphi'(t_b^*)$ is collinear to $\overrightarrow{H_+}(z_e)$. For any $t_b \in I_\varepsilon$, we have

$$\varphi'(t_b) = -\overrightarrow{H_+}(\varphi(t_b)) + \exp(-t_b\overrightarrow{H_+})'(z_b(t_b)) \cdot z_b'(t_b).$$

By lemma 4.1, one has $\sigma'(t_b^*) = 0$ thus $z_b'(t_b^*) = 0$ and we get $\varphi'(t_b^*) = -\overrightarrow{H_+}(\varphi(t_b^*)) = -\overrightarrow{H_+}(z_e)$, which concludes the proof. □

Remark 4.3. *It is worth to mention that the tangency property is proved in the cotangent bundle, and thus it is also true in the state space at a prior saturation point (under the assumptions of Proposition 3.1).*

Setting $\xi(z) := (H_g(z), H_{[f,g]}(z))$ the singular locus Σ_{SA} can be written $\Sigma_{\mathrm{SA}} = \xi^{-1}(\{0_{\mathbb{R}^2}\})$, and we have the following relation between the singular locus and the switching curve.

Corollary 4.1. *Suppose that ξ is a submersion at z_e and that Assumptions 4.1 and 4.2 hold true. Then the switching curve Σ is transverse to the singular locus Σ_{SA} at z_e.*

Proof. Since ξ is a submersion at z_e, the singular locus Σ_{SA} is locally a regular submanifold of codimension two near z_e. Its tangent space at z_e is given by the kernel of the matrix $\xi'(z_e)$. But,

$$\begin{aligned}
\xi'(z_e)\,\varphi'(t_b^*) &= -\xi'(z_e)\,\overrightarrow{H_+}(z_e) \quad \text{(see the proof of Theorem 4.1)}\\
&= -\begin{pmatrix} \partial_x H_g(z_e) & \partial_p H_g(z_e) \\ \partial_x H_{[f,g]}(z_e) & \partial_p H_{[f,g]}(z_e) \end{pmatrix} \begin{pmatrix} \partial_p H_+(z_e) \\ -\partial_x H_+(z_e) \end{pmatrix}\\
&= -\begin{pmatrix} H_{[f,g]}(z_e) \\ H_{[f,[f,g]]}(z_e) + H_{[g,[f,g]]}(z_e) \end{pmatrix}\\
&\neq 0_{\mathbb{R}^2} \quad \text{(by Assumption 4.1)},
\end{aligned}$$

recalling that φ is given from lemma 4.1 by Assumption 4.2. □

Remark 4.4. *From Theorem 4.1, the tangency property holds even if the singular control at z_e is saturating. The main reason of the tangency property comes from the fact that z_e belongs to the singular locus Σ_{SA}. However, if the singular control at z_e is not saturating, for instance if z_e is a prior-saturation lift, then the switching curve Σ is transverse to the singular locus Σ_{SA} at z_e according to Corollary 4.1.*

5 Illustration of the prior-saturation phenomenon

The aim of this section is to develop two examples arising in the field of bioprocesses and magnetic resonance imaging respectively, that will highlight the various concepts introduced in Sections 3-4. For the related minimal time problems, we shall also briefly discuss the corresponding optimal syntheses that exhibit prior-saturation points and bridges.

5.1 The fed-batch model

A bioreactor operated in fed-batch is described by the controlled dynamics (see [19]):

$$\begin{cases} \dot{s} &= -\mu(s)\left(\frac{M}{v} + s_{in} - s\right) + \frac{Q_{max}(1+u)}{2v}(s_{in} - s), \\ \dot{v} &= \frac{Q_{max}}{2}(1+u), \end{cases} \tag{5.1}$$

where s_{in} and s denote respectively the input substrate and substrate concentrations, and v is the volume of the reactor[6]. The parameter $Q_{max} > 0$ is the maximal speed of the input pump (chosen large enough) so that $\frac{Q_{max}}{2}(1+u)$ represents the input flow rate, $u(\cdot)$ being the control variable with values in $[-1, 1]$. Finally, $M \in \mathbb{R}$ depends on the initial value of micro-organism concentration[7]. As in many engineering applications (see, *e.g.*, [5]), the kinetics μ of the reaction is of Haldane type, *i.e.*,

$$\mu(s) := \frac{\mu_h s}{K + s + \frac{s^2}{K_I}},$$

with a unique maximum $s^* = \sqrt{KK_I}$ (parameters μ_h, K, K_I are positive). We suppose hereafter that s^* belongs to $(0, s_{in})$. This hypothesis is not restrictive since from an applicative point of view, $s_{in} > 0$ represents the input substrate concentration (to be treated). Hence, it can be expected that this value is greater than s^*, see, *e.g.*, [5].

This type of growth function models inhibition by substrate (microbial growth is limited when s is too large w.r.t. s^*). It is worth mentioning that the set $\mathcal{D} := (0, s_{in}] \times \mathbb{R}_+^*$ is invariant by (5.1). For waste water treatment purpose, the problem of interest is:

$$\inf_{u \in \mathcal{U}} T_u \quad \text{s.t.} \quad (s(T_u), v(T_u)) \in \mathcal{T}, \tag{5.2}$$

where $\mathcal{T} := (0, s_{ref}] \times \{v_{max}\}$ is the target set, $s_{ref} \ll s_{in}$ is a given threshold, and $v_{max} > 0$ denotes the maximal volume of the bioreactor. From a practical point of view, the goal is to treat a volume v_{max} of wasted water in minimal time. For more details about this system, we refer to [19, 3].

It appears that Problem (5.2) may exhibit a saturation phenomenon. Indeed, by using the PMP, we can check that there is a singular locus that is the line segment

$$\Delta_{SA} = \{s^*\} \times (0, v_{max}],$$

and that the singular control can be expressed in feedback form as

$$u_s[v] := \frac{\mu(s^*)\left[M + v(s_{in} - s^*)\right]}{(s_{in} - s^*)Q_{max}} - 1,$$

(writing $\dot{s} = 0$ along $s = s^*$). It follows that there exists a unique saturation point

$$x_{sat} = (s^*, v^*),$$

with $v^* := \frac{2Q_{max}}{\mu(s^*)} - \frac{M}{s_{in} - s^*}$ and $u_s[v^*] = 1$ if the following condition is fulfilled

$$0 < v^* < v_{max}. \tag{5.3}$$

[6] In contrast with the previous sections in which state variables are (x_1, x_2), we chose to adopt the notation (s, v) that is commonly used in the bioprocesses literature for fed-batch operations.

[7] Micro-organism concentration $X > 0$ can be expressed as a simple function of (s, v), namely $X = M/v + s_{in} - s$, thus (5.1) is enough to describe a bioreactor operated in fed-batch mode.

This typically happens when the volume of water to be trated, v_{max}, is large w.r.t. other parameters of the model (like in large scale waste water treatement industries), see [3]. Next, we suppose that (5.3) holds true.

At this step, we wish to know if prior-saturation occurs (according to Propositions 3.1 and 3.2). Doing so, let us check Assumption 3.1. One gets

$$\delta_0(s,v) = -\mu(s)(M/v + s_{in} - s)Q_{max}/2 = -\mu(s)XQ_{max}/2 < 0,$$

hence $\Delta_0 \cap \mathcal{D} = \emptyset$ and $\delta_0 < 0$ in $\mathcal{D}$. Now, the singular arc is of turnpike type and Legendre-Clebsch's optimality condition holds true because μ has a unique maximum for $s = s^*$, see [1], or a clock form argumentation in [19]. In addition, observe that, in the (s,v)-plane, trajectories of (5.1) with $u = -1$ are horizontal, hence, every arc with $u = -1$ and starting at a volume value $v_0 < v_{max}$ never reaches the target set $\mathcal{T}$. Finally, $\mathcal{T}$ is reachable from $\mathcal{D}$ taking the control $u = +1$ until reaching $v = v_{max}$ and then $u = -1$ until reaching s_{ref}.

Second, let us verify the hypotheses of Proposition 3.2. Doing so, let $v \mapsto \hat{s}(v)$ be the unique solution to the Cauchy problem

$$\frac{ds}{dv} = \left(-\frac{\mu(s)}{Q_{max}} \left[\frac{M}{v} + s_{in} - s \right] + \frac{s_{in} - s}{v} \right), \quad s(v_{max}) = s^*,$$

(the solution of (5.1) with $u = 1$ backward in time from (s^*, v_{max})). From [3], if there exists $v_* \in (0, v^*)$ such that $\hat{s}(v_*) = s^*$, then optimal paths starting at a volume value sufficiently small necessarily contain a singular arc (this actually follows using the PMP). Now, by using Cauchy-Lipschitz's Theorem, the existence of v_* is easy to verify when $M = 0$, and thus, it is also verified for small values of the parameter M (by a continuity argumentation). To pursue our analysis, we suppose next the existence of $v_* \in (0, v^*)$. We are then in a position to apply Propositions 3.1 and 3.2. It follows that there is a unique volume value $v_e \in (0, v^*)$ such that any singular arc starting at a volume value $v_0 < v_e$ will be optimal only until v_e. In addition, combining this result with a study of extremals using the PMP, we obtain that

- if the initial condition is (s^*, v_0) with $v_0 < v_e$, then the optimal path is of the form $\sigma_s \sigma_+^b \sigma_-$ (see below for the definition of σ_+^b);
- if the initial condition is (s^*, v_0) with $v_0 \geq v_e$, then the optimal path is of the form $\sigma_+ \sigma_-$;
- for any initial condition (s_{in}, v_0) with $v_e \leq v_0 < v_{max}$, the optimal path is of the form $\sigma_- \sigma_+ \sigma_-$ where the first switching time appears on a switching curve emanating from (s^*, v_e).

To determine the prior-saturation point $x_e = (s^*, v_e)$ numerically, we proceed as in Section 4. For this application model, it is convenient to introduce an *extended target set* as $\overline{\mathcal{T}} = (0, s_{in}] \times \{v_{max}\}$ (observe that for initial conditions on $\overline{\mathcal{T}}$, optimal paths are σ_- arcs). In this context, a *bridge* is defined as an arc σ_+ (denoted by σ_+^b) on $[0, t_b]$ such that

$$\phi(0) = \dot{\phi}(0) = \phi(t_b) = 0 \quad \text{and} \quad v(t_b) = v_{max},$$

where ϕ is the switching function defined by (2.6) and t_b is the time to steer x_e at time 0 to the extended target set $\overline{\mathcal{T}}$ with $u = +1$. To compute x_e, we need to compute the extremities of the bridge together with its length. Denoting by t_b^* the length of the bridge and by z_b^* its extremity in the cotangent bundle whose projection on the state space belongs to $\overline{\mathcal{T}}$, the point (t_b^*, z_b^*) is then a solution of the equation $F_{\text{bio}} = 0$ with

$$F_{\text{bio}}(t_b, z_b) := \begin{pmatrix} H_{[f,g]}(\exp(-t_b \overrightarrow{H_+})(z_b)) \\ H_g(\exp(-t_b \overrightarrow{H_+})(z_b)) \\ H_+(z_b) + p^0 \\ H_g(z_b) \\ v_b - v_{max} \end{pmatrix}, \tag{5.4}$$

where (s_b, v_b) is the projection of z_b on the state space and vector fields f, g are given by (5.1). From Theorem 4.1, the bridge is then tangent to the switching curve at x_e (the projection of Σ given by (4.7) onto the state space). To conclude this part, let us comment Fig. 3 on which the optimal synthesis is plotted in a neighborhood of the prior-saturation point:

- In black, the switching curve Σ^π emanates from the prior-saturation point. It is computed using the shooting functions F_{bio}.
- The synthesis is such that trajectories are horizontal ($u = -1$) until reaching Δ_{SA} or the switching curve. For initial conditions with a substrate concentration less than s^* and $v_0 \geq v_e$, then $u = 1$ is optimal until reaching $\overline{\mathcal{T}}$.

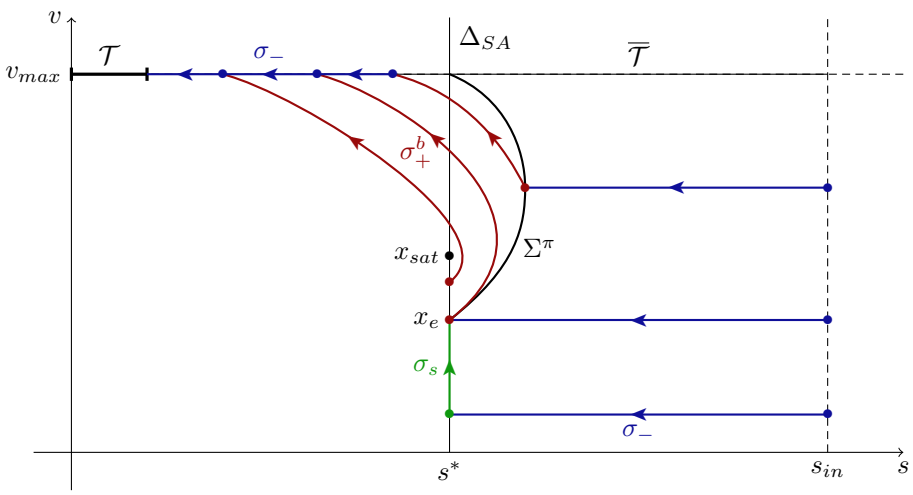

Figure 3: Minimal time synthesis for (5.2): the target set $\mathcal{T} = (0, s_{ref}] \times \{v_{max}\}$ is in black (left). The switching curve Σ^π (in black) is tangent to the bridge σ_+^b (in red) at x_e. Arcs with $u = +1$ (resp. $u = -1$) are depicted in red (resp. in blue).

5.2 The MRI model

In Nuclear Magnetic Resonance (NMR) *saturating* one chemical species consists in driving the magnetization vector representing the state to zero. In Magnetic Resonance Imaging (MRI) a challenging problem is to maximize the *constrast* between two observed species (for instance, healthy tissues and tumors) saturating one species. For the model, we consider an ensemble of spin-1/2 particles, excited by a radio-frequency (RF) field which is ideally assumed homogeneous, each spin of this ensemble being described by its magnetization vector whose dynamics is governed in a specific rotating frame, after some normalizations and considering the 2-dimensional case, by the *Bloch equation* [18]:

$$\begin{cases} \dot{x}_1 = -\Gamma x_1 - u\, x_2, \\ \dot{x}_2 = \gamma(1 - x_2) + u\, x_1, \end{cases} \tag{5.5}$$

where $x := (x_1, x_2)$ is the normalized magnetization vector, where (γ, Γ) is a couple of parameters satisfying the physical constraint $0 < \gamma \leq 2\Gamma$ and depending on the longitudinal and transversal relaxation constants specific to the observed species, and where u is the RF-field which plays the role of the control. The time-minimal problem of interest here is the following:

$$\inf_{u \in \mathcal{U}} T_u \quad \text{s.t.} \quad x_u(T_u, x_0) = O := (0, 0), \tag{5.6}$$

where the initial condition x_0 belongs to the set $B := \{(x_1, x_2) \in \mathbb{R}^2 \ ; \ x_1^2 + x_2^2 \leq 1\}$ called the *Bloch ball* and where $x_u(\cdot, x_0)$ is the unique solution of (5.5) such that $x_u(0, x_0) = x_0$.

Remark 5.1. *The problem of saturation in MRI is the problem* (5.6) *with* $x_0 = N$*, where* $N := (0, 1)$ *is the North pole of the Bloch ball. We refer to* [10, 9] *for more details about the saturation and contrast problems in MRI. In* [10]*, the following optimal synthesis is constructed: the authors give the optimal paths to go from* N *to any reachable point of the Bloch ball. Hence, the initial point is fixed to the North pole while the final point may be seen as a parameter. Here, we are interested in the converse problem, that is, the parameter is*

the initial condition and we want to steer the system to a given target, which is the origin O. The common problem in these two cases is the problem of saturation where the initial condition is N and where the target is O.

In this MRI application [10], the singular locus has a singularity at the intersection of two lines. Setting $\delta := \gamma - \Gamma$, the singular locus is described by

$$\Delta_{SA} = \Delta_{SA}^h \cup \Delta_{SA}^v,$$

where $\Delta_{SA}^h := \{x_2 = \gamma/(2\delta)\}$ is a horizontal line and where $\Delta_{SA}^v := \{x_1 = 0\}$ is the vertical axis. On the vertical axis, the singular control is zero while on the horizontal line, the singular control is given in feedback form by

$$u_s[x_1, x_2] := \gamma(2\Gamma - \gamma)/(2\delta x_1).$$

Considering only the half space $x_1 \leq 0$ of the Bloch ball (this is possible due to a discrete symmetry) and restricting (γ, Γ) to the interesting case $0 < 3\gamma \leq 2\Gamma$ (in this case, the horizontal line cuts the Bloch ball), there exists only one saturation point denoted by $x_{\text{sat}} \in \Delta_{SA}$. The point x_{sat} belongs to the set $\Delta_{SA}^h \cap \{x_1 < 0\}$, it satisfies $u_s(x_{\text{sat}}) = 1$, and it is given by

$$x_{\text{sat}} = \left(\frac{\gamma(2\Gamma - \gamma)}{2\delta}, \frac{\gamma}{2\delta} \right).$$

Following [10], we introduce the concept of bridge. An arc σ_+ or σ_- with control $u = +1$ or $u = -1$, is called a *bridge* on $[0, t_b]$ if its extremities correspond to non ordinary switching points, that is, if

$$\phi(0) = \dot{\phi}(0) = \phi(t_b) = \dot{\phi}(t_b) = 0,$$

where ϕ is the switching function defined by (2.6). According to [10], there exists a bridge with $u = +1$ denoted σ_+^b connecting Δ_{SA}^h and Δ_{SA}^v. We denote by $x_e := (x_{e,1}, x_{e,2})$ the extremity of the bridge on the horizontal line Δ_{SA}^h and we can now restrict the analysis to the following situation. We assume that the following conditions are satisfied by the couple of parameters (γ, Γ) (see Fig. 3 of [10] and the description that comes after for details):

- x_e belongs to the Bloch ball B (this implies in particular that $3\gamma \leq 2\Gamma$),
- $0 < \gamma$ (this comes from the physical constraint),
- $0 \leq (2\Gamma^2 - \gamma\Gamma + 1)\exp((\alpha - \gamma)t_0) - 2\delta$ (hence the origin O is reachable by a Bang-Singular sequence from x_{sat} and so also from x_e),

where $\alpha := \delta/2$ and $t_0 := \arctan(-\beta/\alpha)/\beta$ with $\beta := \sqrt{1 - \alpha^2}$. In this setting, for any initial condition $x_0 := (x_{0,1}, x_{0,2}) \in \Delta_{SA}^h \cap B$ such that $x_{0,1} \leq x_{e,1}$, the optimal trajectory (see [10]) is of the form $\sigma_s \sigma_+^b \sigma_0$, that is composed of a singular arc on Δ_{SA}^h followed by the bridge with $u = +1$ and ending with a singular arc σ_0 along Δ_{SA}^v with $u = 0$. The first singular arc reduces to a point if $x_0 = x_e$. At x_e, the singular control is not saturating, so, in conclusion, the point x_e is a prior-saturation point.

Remark 5.2. *In the MRI application, Assumption 3.1 is not exactly satisfied since the collinearity set Δ_0 is non-empty and plays a role in the optimal synthesis, such as the singularity of the singular locus at the intersection of the two lines. However, the singular arcs are turnpikes and Legendre-Clebsch optimality condition holds. Besides, there exists a prior-saturation point and so this case is more general than the fed-batch application. We will see hereinafter that the tangency property holds at the prior-saturation point and that the switching curve is transverse to the singular locus.*

We end this part by showing how to compute the prior-saturation point x_e and by giving the optimal synthesis near x_e for an initial condition on the horizontal singular line. To compute x_e, we need to compute the extremities of the bridge together with its length. Denoting by t_b^* the length of the bridge and by z_b^* the extremity of the bridge in the cotangent bundle whose projection on the state space belongs to Δ_{SA}^v, the point (t_b^*, z_b^*) is then a solution of the equation $F_{\text{mri}} = 0$ with

$$F_{\text{mri}}(t_b, z_b) := \begin{pmatrix} H_{[f,g]}(\exp(-t_b \overrightarrow{H_+})(z_b)) \\ H_g(\exp(-t_b \overrightarrow{H_+})(z_b)) \\ H_+(z_b) + p^0 \\ H_{[f,g]}(z_b) \\ H_g(z_b) \end{pmatrix}, \tag{5.7}$$

where the vector fields f and g are given by (5.5) and where the Hamiltonians, the Hamiltonian lifts and the Hamiltonian vector field are defined in Section 4.1. We recognize here a function of the form (4.3) without any additional parameter λ and so, $z_e := \exp(-t_b^* \overrightarrow{H_+})(z_b^*)$ is the prior-saturation lift such that $\pi(z_e) = x_e \in \Delta_{SA}^h$. Finally, the optimal synthesis near x_e is given on Fig. 4. The optimal solution from the initial condition $x_0 \in \Delta_{SA}^h$ is of the form $\sigma_s \sigma_+^b \sigma_0$. The red arc σ_+^b is the bridge starting from x_e, it is a part of the forward semi-orbit Γ_+ of $\dot{z} = \overrightarrow{H_+}(z)$ starting from z_e projected into the state space. The black curve Σ^π is the existing part in the optimal synthesis of the projection of the switching curve Σ defined by (4.7). According to the tangency property from Theorem 4.1, the arc σ_+^b is tangent to Σ^π at the prior-saturation point x_e. Note also that the switching curve Σ^π is transverse to the singular locus Δ_{SA}^h in accordance with Corollary 4.1.

Remark 5.3. *One can see from* (5.4) *and* (5.7) *that neither* F_{bio} *nor* F_{mri} *include any conditions about reaching the target. Indeed, to compute the prior-saturation lift we do not necessarily need the final conditions from the shooting functions. We only need a part of the shooting function which satisfies Assumptions 4.1 and 4.2. In these two applications, we have by Proposition 4.1 that the bridge is locally unique. Note that this gives us a new procedure to solve the shooting equations: we can first solve the reduced problem of computing the prior-saturation lift and then we can use this information to connect the initial condition to the prior-saturation lift and to connect the extremity of the bridge to the target.*

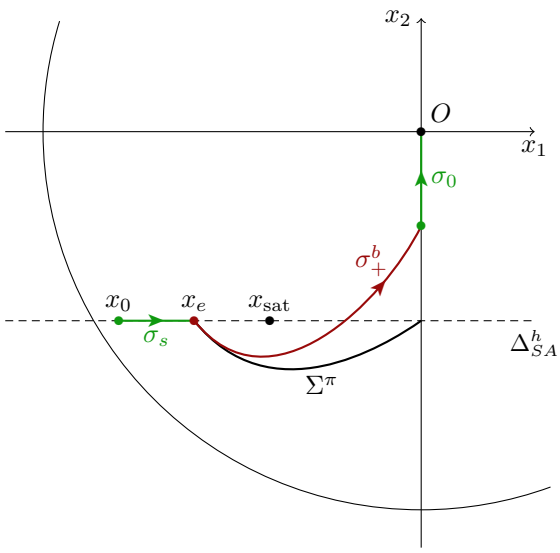

Figure 4: Optimal synthesis near the prior-saturation point x_e in the left part of the Bloch ball.

6 Conclusion

Even though the tangency property between the bridge and the switching curve provides useful informations on the minimum time synthesis when prior saturation occurs (typically, under assumptions of Proposition 3.1), it remains valid in a larger context (under the hypotheses of Theorem 4.1) and not only in the framework of saturation and prior-saturation of the singular control for affine-control systems in the plane. This property also appears in other settings such as in Lagrange control problems governed by one-dimensional systems, see, *e.g.*, [16]. Future works could then investigate prior-saturation phenomenon and the tangency property in other frameworks or in dimension $n \geq 3$.

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