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1 TANGENCY PROPERTY AND PRIOR-SATURATION POINTS IN 2 MINIMAL TIME PROBLEMS IN THE PLANE*

T. BAYEN^{\dagger} AND O. COTS^{\ddagger}

Abstract. In this paper, we consider minimal time problems governed by control-affine-systems 4 5 in the plane, and we focus on the synthesis problem in presence of a singular locus that involves 6 a saturation point for the singular control. After giving sufficient conditions on the data ensuring 7 occurrence of a prior-saturation point and a switching curve, we show that the bridge (i.e., the optimal8 bang arc issued from the singular locus at this point) is tangent to the switching curve at the priorsaturation point. This property is proved using the Pontryagin Maximum Principle that also provides 9 a set of non-linear equations that can be used to compute the prior-saturation point. These issues 10 11 are illustrated on a fed-batch model in bioprocesses and on a Magnetic Resonance Imaging (MRI) 12model for which minimal time syntheses for the point-to-point problem are discussed.

13 Key words. Geometric optimal control, Minimum time problems, Singular arcs,

14 AMS subject classifications. 49J15, 49N35, 70G45

3

1. Introduction. In this paper, we consider minimal time problems governed by single-input control-affine-systems in the plane

$$\dot{x}(t) = f(x(t)) + u(t) g(x(t)), \quad |u(t)| \le 1,$$

where $f, g: \mathbb{R}^2 \to \mathbb{R}^2$ are smooth vector fields. Syntheses for such problems have been investigated a lot in the literature (see, *e.g.*, [6, 12, 18, 23, 25, 24]). In particular, an exhaustive description of the various encountered singularities can be found in [12], as well as an algorithm leading to the determination of optimal paths. It is worth mentioning that even though many techniques exist in this setting, the computation of an optimal feedback synthesis (global) remains in general difficult because of the occurrence of geometric loci such as singular arcs, switching curves, cut-loci...

Our aim in this work is to focus on the notion of *singular arc* which appears in 22 the synthesis when the switching function (the scalar product between the adjoint 23vector and the controlled vector field q) vanishes over a time interval. In that case, 24the corresponding singular control u_s (which allows the associated trajectory to stay 25on the singular locus) can be expressed in feedback form $x \mapsto u_s[x]$. However, it 26may happen that u_s becomes non admissible, *i.e.*, $x \mapsto u_s[x]$ takes values above the 27maximal value for the control (namely 1 here). Such a situation naturally appears 28 in several application models, see, e.g., [2, 3, 15, 20]. In that case, we say that 29a saturation phenomenon occurs. The occurence of such a phenomenon implies the 30 31 following (non-intuitive) property that, if a singular arc is optimal, then it should leave the singular locus at a so-called *prior-saturation point* before reaching the saturation 32 point. This property has been studied in the literature in various situations such as 33 for control-affine systems in dimension 2 and 4 (see, e.q., [21, 22, 10, 3] and references 34 herein).

Our main goal in this paper is to provide new qualitative properties on the minimum time synthesis in presence of a saturation point. More precisely, our objective is twofold:

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- We first give a set of conditions on the system that ensure occurence of prior-saturation showing that, under certain assumptions, the system leaves the singular arc at this point (before reaching the saturation point) with the maximal value for the control, see Proposition 3.3. This last arc is usually called *bridge* following the terminology as in [9, 7] (see also [11, 6]).
- Second, we introduce a shooting function that allows an effective computation
 of the prior-saturation point. This mapping is used to show our main result
 (Theorem 4.6) which can be stated as follows: when the system exhibits a
 switching curve emanating from the prior-saturation point, then this curve is
 tangent to the bridge (in the cotangent bundle) at this point.

The tangency property (in the state space) has been pointed out in several application models (see, *e.g.*, [3, 9]). To the best of our knowledge, this property has not been addressed previously in this general setting in the literature. It allows to better understand the construction of optimal paths locally at the prior-saturation point.

The paper is structured as follows: in Section 2, we recall classical expressions 53 and properties of singular controls for control-affine-systems in the plane introducing 54the saturation phenomenon. In Section 3, we provide a set of conditions involving the target set and the system ensuring the occurrence of the prior-saturation phenomenon. 56 In Section 4, we show the tangency property between the switching curve emanating from a prior-saturation point and the bridge, and we describe how to compute the 58 prior-saturation point thanks to a shooting function constructed via the Hamiltonian lifts of f and q. Finally, we depict this geometrical property in Section 5 for a fed-61 batch model [17, 3] and MRI model [9, 7]. This allows us to illustrate the notion of bridge in various contexts: first, when it connects a component of the singular locus to another one (see the MRI-model in Section 5 and [9, 7]), and then when it connects 63 a component of a singular locus to an extended target set (see the fed-batch model 64 in Section 5 and [17, 3]). 65

2. Saturation phenomenon. The purpose of this section is to recall some facts about minimum time control problems in the plane that will allow us to introduce the saturation phenomenon. Throughout the paper, the standard inner product in \mathbb{R}^2 is written $a \cdot b$ for $a, b \in \mathbb{R}^2$, and a^{\perp} denotes the vector $a^{\perp} := (-a_2, a_1)$ orthogonal to a. The interior of a subset $S \subset \mathbb{R}^n$, $n \geq 2$, is denoted by Int(S).

2.1. Pontryagin's Principle. We start by applying the classical optimality conditions provided by the Pontryagin Maximum Principle (PMP), see [19]. Let $f, g: \mathbb{R}^2 \to \mathbb{R}^2$ be two vector fields of class C^{∞} , and consider the controlled dynamics:

74 (2.1)
$$\dot{x}(t) = f(x(t)) + u(t) g(x(t)),$$

with admissible controls in the set

 $\mathcal{U} \coloneqq \{ u : [0, +\infty) \to [-1, 1] ; u \text{ meas.} \}.$

Given an initial point $x_0 \in \mathbb{R}^2$ and a non-empty closed subset $\mathcal{T} \subset \mathbb{R}^2$, we focus on the problem of driving (2.1) in minimal time from x_0 to the target set \mathcal{T} :

77 (2.2)
$$\inf_{u \in \mathcal{U}} T_u \quad \text{s.t.} \ x_u(T_u) \in \mathcal{T},$$

where $x_u(\cdot)$ denotes the unique solution of (2.1) associated with the control u such that $x_u(0) = x_0$, and $T_u \in [0, +\infty]$ is the first entry time of $x_u(\cdot)$ into the target set \mathcal{T} . We

suppose hereafter that optimal trajectories exist¹ and we wish to apply the PMP on (2.2). The Hamiltonian associated with (2.2) is the function $H : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as

$$H(x, p, p^0, u) \coloneqq p \cdot f(x) + u p \cdot g(x) + p^0.$$

78 If u is an optimal control and x_u is the associated trajectory steering x_0 to the target 79 set \mathcal{T} in time $T_u \ge 0$, the following conditions are fulfilled:

• There exist $p^0 \leq 0$ and an absolutely continuous function $p: [0, T_u] \to \mathbb{R}^2$ satisfying the adjoint equation

82 (2.3)
$$\dot{p}(t) = -\nabla_x H(x_u(t), p(t), p^0, u(t))$$
 a.e. $t \in [0, T_u]$.

• The pair $(p^0, p(\cdot))$ is non-zero.

• The optimal control *u* satisfies the *Hamiltonian condition*

85 (2.4)
$$u(t) \in \operatorname{argmax}_{\omega \in [-1,1]} H(x_u(t), p(t), p^0, \omega)$$
 a.e. $t \in [0, T_u]$.

• At the terminal time, the transversality condition² $p(T_u) \in -N_{\mathcal{T}}(x_u(T_u))$ is fulfilled.

Recall that an extremal $(x_u(\cdot), p(\cdot), p^0, u(\cdot))$ satisfying (2.1) and (2.3)-(2.4) is *abnormal* whenever $p^0 = 0$ and *normal* whenever $p^0 \neq 0$. In the latter case, we take $p^0 = -1$ and the corresponding extremal is denoted by $(x_u(\cdot), p(\cdot), u(\cdot))$ and we shall then write H(x, p, u) in place of $H(x, p, p^0, u)$. Since T_u is free and (2.1) is autonomous, the Hamiltonian H is zero along any extremal: for a.e. $t \in [0, T_u]$,

93 (2.5)
$$H = p(t) \cdot f(x_u(t)) + u(t)p(t) \cdot g(x_u(t)) + p^0 = 0.$$

94 The switching function ϕ is defined as

95 (2.6)
$$\phi(t) \coloneqq p(t) \cdot g(x_u(t)), \quad t \in [0, T_u],$$

and it gives us (thanks to (2.4)) the following control law:

97 (2.7)
$$\begin{cases} \phi(t) > 0 \Rightarrow u(t) = +1, \\ \phi(t) < 0 \Rightarrow u(t) = -1. \end{cases}$$

A switching time is an instant $t_c \in (0, T_u)$ such that the control u is discontinuous at time t_c . We say that the corresponding extremal trajectory has a switching point at time t_c . Of particular interest is the case when there is a time interval $[t_1, t_2]$ such that the switching function vanishes over this interval, *i.e.*,

$$\phi(t) = p(t) \cdot g(x_u(t)) = 0, \quad t \in I.$$

We then say that the extremal trajectory has a singular arc over $[t_1, t_2]$. Note that we shall suppose such an extremal to be normal, *i.e.*, $p^0 \neq 0$. Indeed, recall from [12, Prop. 2 p.49] that under generic conditions, abnormal extremals are bang-bang. By differentiating ϕ twice w.r.t. t, one gets

$$\phi(t) = p(t) \cdot [f, g](x_u(t)), \quad t \in [0, T_u],$$

¹If the target can be reached from x_0 and if f, g have linear growth, then (2.2) admits an optimal solution, thanks to Filippov's Existence Theorem, see, *e.g.*, [26].

²Here, $N_{\mathcal{T}}(x)$ stands for the (Mordukovitch) limiting normal cone to \mathcal{T} at point $x \in \mathcal{T}$, see [26]. It coincides with the normal cone in the sense of convex analysis when \mathcal{T} is convex.

where [f, g](x) is the Lie bracket of f and g at point x, and

$$\phi(t) = p(t) \cdot [f, [f, g]](x_u(t)) + u(t) p(t) \cdot [g, [f, g]](x_u(t)) \quad \text{a.e. } t \in [0, T_u].$$

⁹⁸ The singular locus Δ_{SA} (in the state space) is defined as the (possibly empty) subset ⁹⁹ of \mathbb{R}^2

100 (2.8)
$$\Delta_{SA} \coloneqq \{ x \in \mathbb{R}^2 ; \det(g(x), [f, g](x)) = 0 \}.$$

For future reference, we set $\delta_{SA}(x) := \det(g(x), [f,g](x))$ for $x \in \mathbb{R}^2$. Note that if an extremal is singular over a time interval $[t_1, t_2]$, then one has $x_u(t) \in \Delta_{SA}$ for any $t \in [t_1, t_2]$ because $p(\cdot)$ must be non-zero and orthogonal to the vector space span $\{g(x_u(t)), [f,g](x_u(t))\}$ over $[t_1, t_2]$. The singular control u_s is then the value of the control for which the trajectory stays on the singular locus Δ_{SA} . Supposing then that $\phi(t) = \dot{\phi}(t) = 0$ over $[t_1, t_2]$ gives:

107 (2.9)
$$u_s(t) \coloneqq -\frac{p(t) \cdot [f, [f, g]](x_u(t))}{p(t) \cdot [g, [f, g]](x_u(t))}, \quad t \in [0, T_u],$$

provided that $p(t) \cdot [g, [f, g]](x_u(t))$ is non zero for $t \in [t_1, t_2]$. This expression of the singular control does not guarantee that u_s is admissible, that is, $u_s(t) \in [-1, 1]$:

- When we have $u_s(t) \in [-1, 1]$, the point $x_u(t)$ is said hyperbolic if $p(t) \cdot [g, [f, g]](x_u(t)) > 0$, and elliptic if $p(t) \cdot [g, [f, g]](x_u(t)) < 0$ (see [11, 6]).
- When we have $|u_s(t)| > 1$ for some instant t, we say that a saturation phenomenon occurs and that the corresponding points of the singular locus are parabolic (see [11, 6]).

Our purpose in what follows is precisely to investigate properties of the synthesis of optimal paths when saturation occurs. To this end, we suppose in the rest of the paper that extremals are normal, *i.e.*, $p^0 \neq 0$ (we take hereafter $p^0 = -1$).

118 **2.2. Singular control and saturation phenomenon.** In this part, we derive 119 classical expressions of the singular control in terms of feedback that will allow us to 120 introduce saturation points (in terms of the data defining the system). The *collinearity* 121 *set* associated with (2.1) is the (possibly empty) subset of \mathbb{R}^2 defined as

122 (2.10)
$$\Delta_0 \coloneqq \{x \in \mathbb{R}^2 ; \det(f(x), g(x)) = 0\}.$$

123 Define two functions $\delta_0, \psi : \mathbb{R}^2 \to \mathbb{R}$ as $\delta_0(x) \coloneqq \det(f(x), g(x)), x \in \mathbb{R}^2$, and

124 (2.11)
$$\psi(x) \coloneqq -\frac{\det(g(x), [f, [f, g]](x))}{\det(g(x), [g, [f, g]](x))}, \quad x \in \mathbb{R}^2.$$

125

126 LEMMA 2.1. Suppose that $\Delta_{SA} \neq \emptyset$, that $x \mapsto \det(g(x), [g, [f, g]](x))$ is non-zero 127 over Δ_{SA} , and consider a singular arc defined over an interval $[t_1, t_2]$. Then, one has:

128 (2.12)
$$u_s(t) = \psi(x(t)), \quad t \in [t_1, t_2],$$

129 where $x(\cdot)$ is the corresponding singular trajectory such that $x(t) \in \Delta_{SA}$ for $t \in [t_1, t_2]$. Proof. Since we supposed extremals to be normal, for every $t \in [t_1, t_2]$, the family $\{f(x(t)), g(x(t))\}$ is a basis of \mathbb{R}^2 (using that H = 0). Using that H = 0 and $p^0 = -1$, we obtain that $p(t) \cdot f(x(t)) = 1$ for $t \in [t_1, t_2]$ which gives $p(t) = -\frac{g^{\perp}(x(t))}{\delta_0(x(t))}, t \in [t_1, t_2]$.

We also deduce (decomposing [f, [f, g]](x(t)) and [g, [f, g]](x(t)) on $\{f(x(t)), g(x(t))\}$) that

$$\begin{split} p(t) \cdot [f, [f, g]](x(t)) &= \det(g(x(t)), [f, [f, g]](x(t)))\Lambda(x(t)), \\ p(t) \cdot [g, [f, g]](x(t)) &= \det(g(x(t)), [g, [f, g]](x(t)))\Lambda(x(t)), \end{split}$$

130 where $\Lambda(x) \coloneqq \frac{g^{\perp}(x) \cdot f(x)}{\delta_0(x)^2}, x \notin \Delta_0$. Moreover, $g^{\perp}(x(t)) \cdot f(x(t)) = -\delta_0(x(t))$ and thus, 131 this scalar product is non-zero because extremals are normal $(p^0 \neq 0)$. This ends the 132 proof using (2.9).

133 Remark 2.2. Steady-state singular points are defined as the points $x^* \in \Delta_{SA} \cap \Delta_0$ 134 such that $g(x^*) \neq 0$, see [12, 4] (if $\Delta_{SA} \cap \Delta_0 \neq \emptyset$). Such points are equilibria of (2.1) 135 with $u = \psi(x)$. A singular arc defined over a time interval $[t_1, t_2]$ does not contain 136 such a point because f(x(t)) and g(x(t)) must be linearly independent over $[t_1, t_2]$. 137 But, it can contain points $x^* \in \Delta_{SA} \cap \Delta_0$ such that $g(x^*) = 0$.

To introduce the notion of saturation point, it is convenient to consider a parametrization of Δ_{SA} as follows. When $\Delta_{SA} \cap \Delta_0$ is non-empty, $\Delta_{SA} \setminus \Delta_0$ can be divided into several subsets (called components hereafter). Hence, we write this set as

$$\Delta_{SA} \backslash \Delta_0 = \bigcup_{k \in K} \gamma_k,$$

138 where K is an index set.

139 LEMMA 2.3. Suppose that Δ_{SA} is non-empty and that $x \mapsto \det(g(x), [g, [f, g]](x))$ 140 is non-zero over Δ_{SA} . Then, each component γ_k of $\Delta_{SA} \setminus \Delta_0$ can be parametrized by 141 a one-to-one parametrization $\zeta : J \to \gamma_k, \tau \mapsto \zeta(\tau)$ of class C^1 , where J is an interval 142 of \mathbb{R} .

143 Proof. For $x \notin \Delta_0$, one has span $\{f(x), g(x)\} = \mathbb{R}^2$, hence, there exist $\alpha(x), \beta(x) \in \mathbb{R}^4$ 144 \mathbb{R} such that

145 (2.13)
$$[f,g](x) = \alpha(x)f(x) + \beta(x)g(x).$$

By taking the determinant, we find that for $x \notin \Delta_0$,

$$\alpha(x) = -\frac{\det(g(x), [f, g](x))}{\delta_0(x)} \text{ and } \beta(x) = \frac{\det(f(x), [f, g](x))}{\delta_0(x)}.$$

Consider now a component γ_k of $\Delta_{SA} \setminus \Delta_0$ and $x \in \gamma_k$. By computing [f, [f, g]](x) thanks to (2.13), we get

$$\det(g(x), [g, [f, g]](x)) = -\delta_0(x)\nabla\alpha(x) \cdot g(x), \quad x \in \gamma_k$$

Since $x \mapsto \det(g(x), [g, [f, g]](x))$ is non-zero over Δ_{SA} , the preceding equality implies 146 147 that the scalar product $\nabla \alpha(x) \cdot g(x)$ is non-zero. On the other hand, γ_k is defined by the implicit equation $\delta_{SA}(x) = 0$. Observe that for $x \notin \Delta_0$, $\delta_{SA}(x) = -\alpha(x)\delta_0(x)$. 148By taking the derivative, we find that for $x \notin \Delta_0$, one has $\nabla \delta_{SA}(x) = -\delta_0(x) \nabla \alpha(x) - \delta_0(x) \nabla \alpha(x)$ 149 $\alpha(x)\nabla\delta_0(x)$. Therefore, for $x \in \gamma_k$, we obtain $\nabla\delta_{SA}(x) = -\delta_0(x)\nabla\alpha(x)$. We can 150conclude that for any point $x \in \gamma_k$, the partial derivative $\partial_1 \alpha(x)$ (w.r.t. x_1) or $\partial_2 \alpha(x)$ 151 $(w.r.t. x_2)$ is non-zero. We are then in a position to apply the implicit function theorem 152153to δ_{SA} locally at each point $x \in \gamma_k$, which then implies the desired property.

Under the assumptions of Lemma 2.3, given a component γ_k of Δ_{SA} , there is a parametrization ζ such that

$$\gamma_k \coloneqq \{\zeta(\tau) \; ; \; \tau \in J\},\$$

where $\zeta: J \to \mathbb{R}^2$ is C^1 -mapping (injective) and J is an interval.

155 DEFINITION 2.4. A point $x^* \coloneqq \zeta(\tau^*)$ with $\tau^* \in \text{Int}(J)$ is called saturation point 156 if $\psi(x^*) = 1$, $\psi(\zeta(\tau)) \in (-1, 1)$ for any $\tau \in J$ such that $\tau < \tau^*$, and $\psi(\zeta(\tau)) > 1$ for 157 any $\tau \in J$ such that $\tau > \tau^*$.

As well, we can define saturation points x^* such that $\psi(x^*) = -1$, that is, when the lower bound of the admissible control set is saturated. Our next aim is to study the optimality of singular arcs in presence of a saturation point.

3. Existence of a prior-saturation point. In this section, we show that a prior-saturation phenomenon can occur whenever the system exhibits a saturation point. We start by introducing our main assumptions.

- 164 ASSUMPTION 3.1. The system (2.1) satisfies the following hypotheses:
- 165 (i) One has $\Delta_0 = \emptyset$ and $\delta_0(x) < 0$ for all $x \in \mathbb{R}^2$.
- (ii) The set Δ_{SA} is non-empty, simply connected, and has exactly one saturation point x^* with $\psi(x^*) = 1$.
- 168 (iii) Along the singular locus, the strict Legendre-Clebsch optimality condition is 169 satisfied, that is, any singular extremal $(x_u(\cdot), p(\cdot), u(\cdot))$ defined over $[t_1, t_2]$ 170 satisfies:

171 (3.1)
$$\frac{\partial}{\partial u} \frac{d^2}{dt^2} \frac{\partial H}{\partial u}(x_u(t), p(t), u(t)) > 0, \quad \forall t \in [t_1, t_2].$$

(iv) If Γ_{-} is the forward semi-orbit of (2.1) with u = -1 with the initial condition x^* at time 0, then

174 (3.2)
$$\mathcal{T} \cap \Gamma_{-} = \emptyset.$$

175 (v) The target \mathcal{T} is reachable from every point $x_0 \in \mathbb{R}^2$.

Remark 3.1. (i) The hypothesis $\Delta_0 = \emptyset$ is not restrictive since we could restrict our analysis to a component γ of Δ_{SA} in place of Δ_{SA} .

(ii) By the previous computations, we can observe that (3.1) is equivalent to

$$\det(g(x), [g, [f, g]](x)) > 0, \quad \forall x \in \Delta_{SA}.$$

- 176 Recall that, under the strict Legendre-Clebsch condition, the singular arc is a *turnpike*,
- *i.e.*, it is time-minimizing in every neighborhood of a hyperbolic point of Δ_{SA} , [11]. This property can be retrieved by the clock form argument [13].

Under Assumption 3.1, the singular locus Δ_{SA} is written $\Delta_{SA} \coloneqq \zeta(J)$ where $J \subset \mathbb{R}$ is an interval and $\zeta : J \to \Delta_{SA}$ is a C^1 -mapping. In addition, Δ_{SA} partitions the state space into two simply connected (open) subsets Δ_{SA}^{\pm} :

$$\Delta_{SA}^{+} \coloneqq \{ x \in \mathbb{R}^2 ; \det(g(x), [f, g](x)) > 0 \}$$

$$\Delta_{SA}^{-} \coloneqq \{ x \in \mathbb{R}^2 ; \det(g(x), [f, g](x)) < 0 \}$$

Given a normal extremal $(x_u(\cdot), p(\cdot), u(\cdot))$, the function

$$t \mapsto \gamma_u(t) \coloneqq \beta(x_u(t)) - \alpha(x_u(t))u(t), \quad t \in [0, T_u],$$

179 is well-defined since $\Delta_0 = \emptyset$.

180 LEMMA 3.2. Along a normal extremal $(x_u(\cdot), p(\cdot), u(\cdot))$, the switching function ϕ 181 satisfies the ODE

182 (3.3)
$$\phi(t) = \gamma_u(t)\phi(t) + \alpha(x_u(t))$$
 a.e. $t \in [0, T_u]$.

183 *Proof.* The proof follows using the expression of ϕ and the fact that the Hamil-184 tonian H is constant equal to zero.

The next proposition shows that an extremal trajectory containing a singular arc until the point x^* is not optimal.

187 PROPOSITION 3.3. Suppose that Assumption 3.1 holds true, and consider an op-188 timal trajectory steering x_0 to the target \mathcal{T} in time T_u . Then, the corresponding 189 extremal $(x_u(\cdot), p(\cdot), u(\cdot))$ does not contain a singular arc defined over a time interval 190 $[t_1, t_2]$ such that $x_u(t_2) = x^*$.

191 Proof. Suppose by contradiction that there is a time interval $[t_1, t_2]$ such that the 192 trajectory is singular over $[t_1, t_2]$ with $x_u(t_2) = x^*$. We claim that, at time t_2 , the 193 vector $f(x_u(t_2)) + g(x_u(t_2))$ is tangent to Δ_{SA} . Indeed, it is enough to check that the 194 vector $f(x^*) + g(x^*)$ is orthogonal to $\nabla \delta_{SA}(x^*) = -\delta_0(x^*) \nabla \alpha(x^*)$. As we have seen 195 in the proof of Lemma 2.3, one has for $x \in \Delta_{SA}$:

196 (3.4)
$$\begin{aligned} \det(g(x), [g, [f, g]](x)) &= -\delta_0(x) \nabla \alpha(x) \cdot g(x), \\ \det(g(x), [f, [f, g]](x)) &= -\delta_0(x) \nabla \alpha(x) \cdot f(x). \end{aligned}$$

These equalities imply that

$$-\delta_0(x^*)\nabla\alpha(x^*)\cdot(f(x^*)+g(x^*)) = \det(g(x^*), [g, [f, g]](x^*)) + \det(g(x^*), [f, [f, g]](x^*)).$$

197 Since $\psi(x^*) = 1$, the right member of the above equality is zero which shows the 198 claim. In addition, for $x \in \Delta_{SA}$, (3.4) implies the equalities

199
$$\nabla \delta_{SA}(x) \cdot (f(x) + g(x)) = \det(g(x), [g, [f, g]](x))(1 - \psi(x)),$$

$$\nabla \delta_{SA}(x) \cdot (f(x) - g(x)) = \det(g(x), [g, [f, g]](x))(-1 - \psi(x)).$$

Consider now the unique solution x_{-} of (2.1) with u = -1 starting from x^{*} at time t_{2} . This trajectory enters into the set Δ_{SA}^{-} for $t > t_{2}$, t close to t_{2} , because one has $\nabla \delta_{SA}(x^{*}) \cdot (f(x^{*}) - g(x^{*})) < 0$. Going back to the optimal trajectory, there are now two possibilities for $x_{u}(\cdot)$. In a right neighborhood of $t = t_{2}$, either $x_{u}(\cdot)$ enters into Δ_{SA}^{+} or into Δ_{SA}^{-} (because the singular control becomes non admissible).

Suppose first that $x_u(\cdot)$ enters into Δ_{SA}^+ . Then, there is $\varepsilon > 0$ such that one has $\alpha(x_u(t)) > 0$ for $t \in (t_2, t_2 + \varepsilon]$. It follows from (3.3) that one has u = +1 on this interval. But the velocity set being convex, we obtain a contradiction with the nonadmissibility of the singular control at x^* (because x_- enters into Δ_{SA}^-). It follows that the optimal trajectory necessarily enters into the set Δ_{SA}^- . But then, since $\alpha < 0$ in Δ_{SA}^- , (3.3) implies that u = -1 in some time interval $(t_2, t_2 + \varepsilon]$.

From Assumption 3.1, the forward semi-orbit with u = -1 starting from x^* does not reach the target set. Hence, $x_u(\cdot)$ must have a switching point to u = +1 in $\Delta_{SA}^$ or it must reach Δ_{SA} with the control u = -1. We see from (3.3) that the first case is not possible because at a switching time t_c such that $x_u(t_c) \in \Delta_{SA}^-$, we would have $\dot{\phi}(t_c) \ge 0$ in contradiction with $\alpha(x_u(t_c)) < 0$.

Suppose now that $x_u(\cdot)$ reaches Δ_{SA} at some point $x \coloneqq \zeta(\tau)$ with $\tau < \tau^*$. Then, we obtain $\nabla \delta_{SA}(x) \cdot (f(x) - g(x)) < 0$ since $\psi(x) > -1$. But, as $x_u(\cdot)$ reaches Δ_{SA} with u = -1 at point x, the trajectory enters into the set $\Delta_{SA} \cup \Delta_{SA}^+$ and we must have $\nabla \delta_{SA}(x) \cdot (f(x) - g(x)) \ge 0$ ($\nabla \delta_{SA}(x)$ is collinear to the outward normal vector to Δ_{SA} at point x). This gives a contradiction. In the same way, the trajectory cannot reach a point $x \in \Delta_{SA}$ such that $x = \zeta(\tau)$ with $\tau > \tau^*$.

We can conclude that for any time $t \ge t_2$, one has u(t) = -1, but then, the optimal trajectory cannot reach the target set which is a contradiction (Assumption 3.1 (iv)). This concludes the proof.

As an example, if $x_0 := \zeta(\tau_0)$ belongs to the singular locus with $\tau_0 < \tau^*$, and if an optimal trajectory starting from x_0 contains a singular arc, then the trajectory should leave the singular locus before reaching x^* . Let us insist on the fact that this property of leaving the singular locus before reaching x^* relies on the fact that the optimal trajectory should contain a singular arc. In the fed-batch model presented in Section 5.1, this property can be easily verified (see [3]).

We now introduce the following definition (in line with [15, 21, 22]). Hereafter, the notation $S_{[\tau'_0,\tau_0]}$ denotes a singular arc passing through the points $\zeta(\tau'_0)$ and $\zeta(\tau_0)$ with $\tau'_0 \leq \tau_0 < \tau^*$.

236 DEFINITION 3.4. Let $\tau_0 < \tau^*$. A point $x_e \coloneqq \zeta(\tau_e) \in \Delta_{SA}$ with $\tau_0 < \tau_e < \tau^*$ 237 is called a prior-saturation point if the singular arc $S_{[\tau_0,\tau]}$ ceases to be optimal for 238 $\tau \ge \tau_e$.

This definition makes sense only for initial conditions $\zeta(\tau_0)$ with $\tau_0 < \tau^*$ because for $\tau_0 \ge \tau^*$, optimal controls are not singular (since the singular control is nonadmissible). We highlight the dependency of x_e w.r.t. initial conditions $\zeta(\tau_0) \in \Delta_{SA}$ as follows.

PROPOSITION 3.5. Suppose that Assumption 3.1 holds true and that there are $\tau_1, \tau_2 \in J$ with $\tau_1 < \tau_2 < \tau^*$ such that any optimal trajectory starting from $\zeta(\tau_0)$ with $\tau_0 \in [\tau_1, \tau_2)$ contains a singular arc $S_{[\tau_0, \tau_2]}$. Then, for any initial condition $\tau_0 \in [\tau_1, \tau_2)$, one has $x_e = \zeta(\tau_e)$ with

247 (3.5)
$$\tau_e \coloneqq \sup\{\tau \in J \; ; \; \mathcal{S}_{[\tau_1,\tau]} \text{ is optimal}\} \in [\tau_2,\tau^*).$$

Moreover, for any $\tau_0 \in [\tau_e, \tau^*]$ an optimal trajectory starting at $\zeta(\tau_0)$ leaves the singular locus at $\zeta(\tau_0)$.

250 Proof. Let $E := \{\tau \in J ; S_{[\tau_1,\tau]} \text{ is optimal}\}\ \text{and } F := \{\tau \in J ; S_{[\tau_0,\tau]} \text{ is optimal}\}\$ 251 where $\tau_0 \in [\tau_1, \tau_2)$ is fixed. Take a point $\tau \in F$. Then, from our assumption, $S_{[\tau_1,\tau]}$ 252 is also optimal (by concatenation) which shows that $\tau \in E$. On the other hand, if 253 $\tau \in E$, then $S_{[\tau_0,\tau]}$ remains optimal (as a sub-arc). It follows that E = F and, in 254 addition, since x_e is defined as the point such that $S_{[\tau_1,\tau]}$ ceases to be optimal, we 255 obtain (3.5). From Proposition 3.3, we have $\tau_e < \tau^*$.

Finally, for every $\tau_0 \in [\tau_e, \tau^*)$, a singular arc $S_{[\tau_0, \tau'_0]}$ with $\tau_0 < \tau'_0 < \tau^*$ cannot be optimal, since otherwise, this would contradict the definition of τ_e . It follows that for every $\tau_0 \in [\tau_e, \tau^*]$, an optimal path cannot contain a singular arc, implying the desired property.

This property implies in particular that for every initial conditions $x_0 \coloneqq \zeta(\tau_0) \in \Delta_{SA}$ such that $\tau_0 \in [\tau_1, \tau_2]$, then the corresponding optimal path has a singular arc until the point x_e and a switching point at this point.

263 Remark 3.6. In addition to Assumption 3.1 (in particular (3.2)), if we suppose 264 that \mathcal{T} is not reachable with the constant control u = -1 from those points of Δ_{SA} 265 located between x_e and x^* (*i.e.* corresponding to $\tau \in [\tau_e, \tau^*]$), then the maximal value 266 for the control u = +1 is locally optimal from the prior-saturation point x_e . In other 267 words, the *bridge* (the last arc leaving Δ_{SA}) corresponds to u = +1. This can be 268 proved by using similar arguments as for proving Proposition 3.5. Since the singular 269 arc is a turnpike, this additional hypothesis also implies the existence of a switching 270 curve emanating from x_e . Our next aim is precisely to investigate more into details 271 geometric properties of optimal paths at the point x_e .

4. Tangency property and prior-saturation phenomenon. The aim of this section is to prove the tangency property as stated in Theorem 4.6.

4.1. Introduction to prior-saturation lift and tangency property. In this section, we first introduce the concept of *prior-saturation lift* and discuss its local uniqueness. We also provide a set of nonlinear equations allowing to compute priorsaturation lifts given by the PMP. We end this section with an introduction to the tangency property on an example.

DEFINITION 4.1. Let x_e be a prior-saturation point. Any point z_e in the cotangent space at x_e is called a prior-saturation lift of x_e .

To introduce the computation of prior-saturation lifts given by the PMP, let us start with an example. Consider a target set $\mathcal{T} := \{x_f\}, x_f \in \mathbb{R}^2$, with an optimal trajectory of the form $\sigma_-\sigma_s\sigma_+$, where σ_- , σ_+ and σ_s are arcs, respectively, with control u = -1, u = +1 and $u = u_s$, where u_s is the singular control. Assume that the optimal trajectory is unique and that the switching point between the singular arc σ_s (supposed to be non-empty) and the positive bang arc σ_+ is a prior-saturation point. The PMP³ gives necessary optimality conditions satisfied by this extremal trajectory that we can write as a system of nonlinear equations, the so-called *shooting* equations. We introduce some notation to define this set of shooting equations. We define the Hamiltonian lifts associated with f and g as

$$H_f(z) \coloneqq p \cdot f(x) ; H_g(z) \coloneqq p \cdot g(x),$$

where z := (x, p) belongs to the cotangent bundle. All the others Hamiltonian lifts in the rest of the paper are defined like this. Define also the Hamiltonians $H_{\pm} := H_f \pm H_g$ and $H_s := H_f + u_s H_g$, where u_s is viewed here as a function of z:

284 (4.1)
$$u_s(z) \coloneqq -\frac{p \cdot [f, [f, g]](x)}{p \cdot [g, [f, g]](x)} = -\frac{H_{[f, [f, g]]}(z)}{H_{[g, [f, g]]}(z)}$$

For any Hamiltonian H we define the Hamiltonian system $\vec{H} := (\partial_p H, -\partial_x H)$, and finally, we introduce the exponential mapping $\exp(t\varphi)(z_0)$ as the solution at time t of the differential equation $\dot{z}(s) = \varphi(z(s))$ with initial condition $z(0) = z_0$, where φ is supposed to be smooth. The shooting equations are then given by

289
$$S(y) = 0, \quad y \coloneqq (p_0, t_1, t_2, t_f, z_1, z_2) \in \mathbb{R}^{n+3+(2n)\times 2}, \quad n \coloneqq 2,$$

290 where the *shooting function* is defined by

291 (4.2)
$$S(y) \coloneqq \begin{pmatrix} H_g(z_1) \\ H_{[f,g]}(z_1) \\ H_+(\exp((t_f - t_2)\overrightarrow{H_+})(z_2)) + p^0 \\ \pi(\exp((t_f - t_2)\overrightarrow{H_+})(z_2)) - x_f \\ z_1 - \exp(t_1\overrightarrow{H_-})(x_0, p_0) \\ z_2 - \exp((t_2 - t_1)\overrightarrow{H_s})(z_1) \end{pmatrix}$$

where $\pi(x, p) \coloneqq x$, and where $x_0 \in \mathbb{R}^2$ is given and $p^0 = -1$ in the normal case. The two first equations mean that the trajectory is entering the singular locus at z_1 .

³Since $\mathcal{T} \coloneqq \{x_f\}$ is a point, there is no transversality condition at the terminal time.

Hence, the second arc is a singular arc. The third equation takes into account the 294 free terminal time. It could be replaced by $H_{-}(x_0, p_0) + p^0 = 0$ since the maximized 295Hamiltonian is constant along the extremal. The fourth equation implies that the last 296bang arc reaches the target $\mathcal{T} = \{x_f\}$ at the final time t_f , and the last two equations 297 are the so-called matching conditions (which are not required but improve numerical 298 stability). Given a solution $y^* \coloneqq (p_0, t_1, t_2, t_f, z_1, z_2)$ to S(y) = 0 associated with the 299unique optimal trajectory, the point $\pi(z_2)$ is then a prior-saturation point thanks to 300 our hypotheses, and so, z_2 is a prior-saturation lift. 301

Let us now discuss the uniqueness of the prior-saturation lift, considering for instance, a smooth and local one-parameter family of initial conditions $x_0(\alpha)$, $\alpha \in$ $(-\varepsilon, \varepsilon)$, $\varepsilon > 0$, in relation with the construction of optimal syntheses (see section 5) and in relation with Proposition 3.5. Let us assume that for any $\alpha \in (-\varepsilon, \varepsilon)$, the unique optimal trajectory is of the form $\sigma_{-}\sigma_{s}\sigma_{+}$ and denote by

$$y^*(\alpha) \coloneqq (p_0(\alpha), t_1(\alpha), t_2(\alpha), t_f(\alpha), z_1(\alpha), z_2(\alpha))$$

the corresponding solution to the equation

$$S(y,\alpha) = 0,$$

where $S(\cdot, \alpha)$ is defined as (4.2) but with the initial condition $x_0(\alpha)$ in place of x_0 . In addition, suppose that the lengths $t_1, t_2 - t_1$ and $t_f - t_2$ are positive, that is, each arc is defined on a time interval of positive length. In this setting, for any α , we have $x_e := \pi(z_2(0)) = \pi(z_2(\alpha))$, that is, the prior-saturation point x_e is locally unique. This is related to Proposition 3.5 and illustrated on Figure 1. Besides, whenever the prior-saturation lift $z_e := z_2(0)$ is also locally unique, see Proposition 4.3 and remark 4.2, we have $z_2(0) = z_2(\alpha)$ for any $\alpha \in (-\varepsilon, \varepsilon)$.



FIG. 1. Local uniqueness of the prior-saturation point x_e .

Assuming that the prior-saturation lift z_e is locally unique, we can compute it with a set of equations excerpt from the shooting equations but with some minor modifications. Roughly speaking, the main idea is to consider the particular case where the initial condition is the prior-saturation point, that is such that $x_0 = x_e$. In this case, we have $t_1 = t_2 = 0$ and $z_1 = z_2 = (x_e, p_0) = z_e$. This emphasizes the fact that what happens before the prior-saturation lift is useless to compute it. With these considerations in mind we introduce

$$F_{\mathrm{ex}}(t_b, z_b) \coloneqq \begin{pmatrix} H_g(\mathrm{exp}(-t_b \overrightarrow{H_+})(z_b)) \\ H_{[f,g]}(\mathrm{exp}(-t_b \overrightarrow{H_+})(z_b)) \\ H_+(z_b) + p^0 \\ \pi(z_b) - x_f \end{pmatrix},$$

where $F_{\text{ex}} : \mathbb{R}^5 \to \mathbb{R}^5$, and where we use the notation t_b , z_b (b stands for bridge) in relation with the concept of *bridge* defined in [9] and detailed in the MRI example in Section 5.2. Note that the exponential mapping is here computed by backward integration. Hence, with the preceding notation, we have $z_b = \exp((t_f - t_2)\overline{H_+})(z_2) =$ $\exp(t_f \overline{H_+})(z_e)$ and $t_b = t_f - t_2 = t_f$. At the end, the prior-saturation lift is simply given by

$$z_e = \exp(-t_b H_+)(z_b),$$

324 for a couple (t_b, z_b) solution of $F_{\text{ex}} = 0$.

Tangency property. We end this section with an introduction to the tangency property. Let us start with solutions of the form $\sigma_{-}\sigma_{s}\sigma_{+}$, considering a smooth and local one-parameter family of initial conditions $x_{0}(\alpha)$, $\alpha \in (-\varepsilon, \varepsilon)$, $\varepsilon > 0$, but assuming that for $\alpha = 0$, the optimal solution is of the form $\sigma_{-}\sigma_{s}\sigma_{+}$ with σ_{s} reduced to a single point, that is, $t_{2}(0) - t_{1}(0) = 0$, with

$$y^*(0) \coloneqq (p_0(0), t_1(0), t_2(0), t_f(0), z_1(0), z_2(0)),$$

the solution to the associated shooting equations, still denoted $S(y, \alpha) = 0$. Assume also that for $\alpha > 0$, we are in the previous case, that is one has $t_2(\alpha) - t_1(\alpha) > 0$ with $y^*(\alpha) := (p_0(\alpha), t_1(\alpha), t_2(\alpha), t_f(\alpha), z_1(\alpha), z_2(\alpha))$ the corresponding solution of $S(\cdot, \alpha) = 0$. The prior-saturation lift is thus given by $z_e = z_2(\alpha)$ for $\alpha \in [0, \varepsilon)$.

The idea is now to consider the case where there is a bifurcation in the structure of the optimal trajectories when $\alpha = 0$. We thus assume that for $\alpha \in (-\varepsilon, 0)$, the solutions are of the form $\sigma_{-}\sigma_{+}$ and we denote by $z_{1}(\alpha)$ the switching point (in the cotangent bundle) between the two arcs. In this setting, there exists a switching locus in the optimal synthesis denoted $\Sigma_{-} \cup \Sigma_{0}$, where

334
$$\Sigma_{-} \coloneqq \{z_1(\alpha); \ \alpha \in (-\varepsilon, 0]\}$$
 and $\Sigma_0 \coloneqq \{z_1(0) = z_e\}.$

The aim of the next section is to prove that the semi-orbit Γ_+ of $\dot{z} = \overline{H_+}(z)$ starting from z_e is tangent to the switching curve $\Sigma_- \cup \Sigma_0$ at the prior-saturation lift z_e in a general frame. This is precisely the *tangency property* (see Fig. 2).



FIG. 2. Illustration of the tangency property between Γ_+ and $\Sigma_- \cup \Sigma_0$ at the prior-saturation lift z_e . The singular locus in the cotangent bundle is $\Sigma_{SA} := \{z \in \mathbb{R}^{2n} ; H_g(z) = H_{[f,g]}(z) = 0\}.$

337

4.2. Proof of the tangency property. From a general point of view, we shall assume that the prior-saturation lift is given by solving a set of nonlinear equations of the following form:

341 (4.3)
$$F(t_b, z_b, \lambda) \coloneqq \begin{pmatrix} H_{[f,g]}(\exp(-t_b \overline{H_+})(z_b)) \\ G(t_b, z_b, \lambda) \end{pmatrix},$$

where $\lambda \in \mathbb{R}^k$ is a vector of $k \in \mathbb{N}$ parameters, where F is a function from \mathbb{R}^{5+k} to \mathbb{R}^{5+k} and where $G : \mathbb{R}^{5+k} \to \mathbb{R}^{4+k}$ is defined by 342 343

344 (4.4)
$$G(t_b, z_b, \lambda) \coloneqq \begin{pmatrix} H_g(\exp(-t_b \overrightarrow{H_+})(z_b)) \\ H_+(z_b) + p^0 \\ \Psi(z_b, \lambda) \end{pmatrix},$$

with $\Psi : \mathbb{R}^{4+k} \to \mathbb{R}^{2+k}$ a given function and $p^0 = -1$ considering the normal case. We 345 assume that all the functions F, G and Ψ are smooth. It is important to notice that 346 the mapping Ψ does not depend on t_b and that we can replace H_+ by H_- without any 347 loss of generality. In the previous example from section 4.1, we have (with a slight 348 abuse of notation) $\Psi(z_b) = \pi(z_b) - x_f$ which corresponds to the simplest case where 349 there are no transversality conditions and no additional parameters, that is k = 0. 350For a more complex structure of the form $\sigma_{-}\sigma_{s}\sigma_{+}\sigma_{-}$, the parameter λ would be the last switching time between the σ_+ and σ_- arcs. In this case, Ψ would contain the 352 additional switching condition $H_g = 0$ at this time. Let $(t_b^*, z_b^*, \lambda^*) \in \mathbb{R}^{5+k}$ be a solution to the equation F = 0 and define 353

354

355 (4.5)
$$z_e := \exp(-t_b^* \overrightarrow{H_+})(z_b^*) \in \Sigma_{\text{SA}} := \{ z \in \mathbb{R}^{2n} ; H_g(z) = H_{[f,g]}(z) = 0 \}.$$

We introduce the following assumptions at the point z_e . 356

ASSUMPTION 4.1. We have $u_s(z_e) < 1$ with u_s the singular control given by (4.1). 357

Assumption 4.2. The matrix 358

359
$$\left[\frac{\partial G}{\partial z_b}(t_b^*, z_b^*, \lambda^*) \quad \frac{\partial G}{\partial \lambda}(t_b^*, z_b^*, \lambda^*)\right] \in \mathrm{GL}_{4+k}(\mathbb{R})$$

i.e., it is invertible in $\mathbb{R}^{(4+k)\times(4+k)}$. 360

Remark 4.2. Assumption 4.1 is related to the prior-saturation phenomenon while 361 in combination with Assumption 4.2, it is related to the well-posedness of the shooting 362 system F = 0. Besides, the point z_e is locally unique under these assumptions, 363 according to the following result. 364

PROPOSITION 4.3. Suppose that Assumptions 4.1 and 4.2 hold true. Then, 365

366
$$F'(t_h^*, z_h^*, \lambda^*) \in \mathrm{GL}_{5+k}(\mathbb{R}).$$

Proof. The Jacobian of the mapping F at the point $(t_b^*, z_b^*, \lambda^*)$ is given by: 367

$$F'(t_b^*, z_b^*, \lambda^*) = \begin{bmatrix} -a & * & * \\ -b & \frac{\partial G}{\partial z_b}(t_b^*, z_b^*, \lambda^*) & \frac{\partial G}{\partial \lambda}(t_b^*, z_b^*, \lambda^*) \end{bmatrix},$$

where $a \coloneqq H_{[f,[f,g]]}(z_e) + H_{[g,[f,g]]}(z_e)$ and $b \coloneqq (H_{[f,g]}(z_e), 0, 0)$. Observe now that b = 0 since $F(t_b^*, z_b^*, \lambda^*) = 0$ and that $a \neq 0$ since $u_s(z_e) < 1$, which ends the proof. 369 370

LEMMA 4.4. Suppose that Assumption 4.2 holds true. Then, there exists $\varepsilon > 0$ 371 and a C^1 -map $t_b \mapsto \sigma(t_b) \coloneqq (z_b(t_b), \lambda(t_b)) \in \mathbb{R}^{4+k}$ defined over $I_{\varepsilon} \coloneqq (t_b^* - \varepsilon, t_b^* + \varepsilon)$, 372 that satisfies 373

374 (4.6)
$$\forall t_b \in I_{\varepsilon}, \quad G(t_b, \sigma(t_b)) = 0.$$

In addition, one has $\sigma(t_b^*) = (z_b^*, \lambda^*)$ and $\sigma'(t_b^*) = 0_{\mathbb{R}^{4+k}}$. 375

276 Proof. The existence of σ follows from the implicit function theorem applied to 277 the mapping G at $(t_b^*, z_b^*, \lambda^*)$ which also gives (4.6). The derivative of σ is then 278 obtained from (4.6):

379
$$\sigma'(t_b) = -\left[\frac{\partial G}{\partial z_b}[t_b] \quad \frac{\partial G}{\partial \lambda}[t_b]\right]^{-1} \cdot \frac{\partial G}{\partial t_b}[t_b], \quad t_b \in I_{\varepsilon},$$

380 where $[t_b]$ stands for $(t_b, \sigma(t_b))$. Since

381
$$\frac{\partial G}{\partial t_b}[t_b^*] = (H_{[f,g]}(z_e), 0_{\mathbb{R}^{3+k}}) = 0_{\mathbb{R}^{4+k}},$$

382 the result follows.

Let us introduce the mapping $\varphi(t_b) \coloneqq \exp(-t_b \overrightarrow{H_+})(z_b(t_b))$ for $t_b \in I_{\varepsilon}$ and define

384 (4.7)
$$\Sigma \coloneqq \{\varphi(t_b) \; ; \; t_b \in I_{\varepsilon}\}.$$

Remark 4.5. The curve Σ is a switching curve in the contangent bundle since one has $H_g(\varphi(t_b)) = 0$ by definition of G. However, this switching curve is not necessarily optimal, that is, the optimal synthesis, with respect to the initial condition, may not contain Σ . Let us stratify Σ according to $\Sigma = \Sigma_- \cup \Sigma_0 \cup \Sigma_+$, with

389

403

$$\begin{split} \Sigma_{-} &\coloneqq \{\varphi(t_b) \; ; \; t_b \in (t_b^* - \varepsilon, t_b^*)\},\\ \Sigma_0 &\coloneqq \{\varphi(t_b^*)\} = \{z_e\},\\ \Sigma_+ &\coloneqq \{\varphi(t_b) \; ; \; t_b \in (t_b^*, t_b^* + \varepsilon)\}. \end{split}$$

A typical situation is when $\Sigma_{-} \cup \Sigma_{0}$ is contained in the optimal synthesis while Σ_{+} is not optimal for local and/or global optimality reasons. See the end of Section 4.1 for an example of this typical situation.

Our first main result is given by Proposition 3.5 which states the existence of a prior-saturation point x_e in the state space under Assumption 3.1. Our second main result is the following.

THEOREM 4.6. Suppose the existence of a triple $(t_b^*, z_b^*, \lambda^*) \in \mathbb{R}^{5+k}$ such that $F(t_b^*, z_b^*, \lambda^*) = 0$, with F defined by (4.3) and set $z_e \coloneqq \exp(-t_b^* \overrightarrow{H_+})(z_b^*)$. Suppose also that Assumption 4.2 holds true. Then, the switching curve Σ given by (4.7) is tangent at z_e to the forward semi-orbit Γ_+ of $\dot{z} = \overrightarrow{H_+}(z)$ starting from z_e .

400 *Proof.* From Assumption 4.2 and by lemma 4.4, one can define the switching 401 curve Σ by (4.7). To prove the tangency property, we have to show that $\varphi'(t_b^*)$ is 402 collinear to $\overline{H}_+(z_e)$. For any $t_b \in I_{\varepsilon}$, we have

$$\varphi'(t_b) = -\overrightarrow{H_+}(\varphi(t_b)) + \Phi(t_b, z_b(t_b)) \, z'_b(t_b)$$

where $\Phi(t, z_0)$ is defined as the solution at time t of the Cauchy problem

$$X(s) = A(s, z_0)X(s), \quad X(0) = I_{2n},$$

404 with $A(s, z_0) := -\overrightarrow{H_+}'(\exp(-s\overrightarrow{H_+})(z_0))$. By lemma 4.4, one has $\sigma'(t_b^*) = 0$ thus 405 $z_b'(t_b^*) = 0$ and we get $\varphi'(t_b^*) = -\overrightarrow{H_+}(\varphi(t_b^*)) = -\overrightarrow{H_+}(z_e)$, which concludes the proof. \Box

Remark 4.7. It is worth to mention that the tangency property is proved in the cotangent bundle, and thus it is also true in the state space at a prior saturation point (under the assumptions of Proposition 3.3).

409 Setting $\xi(z) \coloneqq (H_g(z), H_{[f,g]}(z))$ the singular locus Σ_{SA} can be written $\Sigma_{SA} =$ 410 $\xi^{-1}(\{0_{\mathbb{R}^2}\})$, and we have the following relation between the singular locus and the 411 switching curve.

412 COROLLARY 4.8. Suppose that ξ is a submersion at z_e and that Assumptions 4.1 413 and 4.2 hold true. Then the switching curve Σ is transverse to the singular locus Σ_{SA} 414 at z_e .

415 Proof. Since ξ is a submersion at z_e , the singular locus Σ_{SA} is locally a regular 416 submanifold of codimension two near z_e . Its tangent space at z_e is given by the kernel 417 of the matrix $\xi'(z_e)$. But,

$$\begin{aligned} \xi'(z_e) \,\varphi'(t_b^*) &= -\xi'(z_e) \overrightarrow{H_+}(z_e) \quad \text{(see the proof of Theorem 4.6)} \\ &= -\begin{pmatrix} \partial_x H_g(z_e) & \partial_p H_g(z_e) \\ \partial_x H_{[f,g]}(z_e) & \partial_p H_{[f,g]}(z_e) \end{pmatrix} \begin{pmatrix} \partial_p H_+(z_e) \\ -\partial_x H_+(z_e) \end{pmatrix} \\ &= -\begin{pmatrix} H_{[f,g]}(z_e) \\ H_{[f,[f,g]]}(z_e) + H_{[g,[f,g]]}(z_e) \end{pmatrix} \\ &\neq 0_{\mathbb{R}^2} \quad \text{(by Assumption 4.1),} \end{aligned}$$

419 recalling that φ is given from lemma 4.4 by Assumption 4.2.

420 Remark 4.9. From Theorem 4.6, the tangency property holds even if the singular 421 control at z_e is saturating. The main reason of the tangency property comes from the 422 fact that z_e belongs to the singular locus Σ_{SA} . However, if the singular control at z_e 423 is not saturating, for instance if z_e is a prior-saturation lift, then the switching curve 424 Σ is transverse to the singular locus Σ_{SA} at z_e according to Corollary 4.8.

5. Illustration of the prior-saturation phenomenon. The aim of this section is to develop two examples arising in the field of bioprocesses and magnetic resonance imaging respectively, that will highlight the various concepts introduced in Sections 3-4. For the related minimal time problems, we shall also briefly discuss the corresponding optimal syntheses that exhibit prior-saturation points and bridges.

430 **5.1. The fed-batch model.** A bioreactor operated in fed-batch is described by 431 the controlled dynamics (see [17]):

432 (5.1)
$$\begin{cases} \dot{s} = -\mu(s)\left(\frac{M}{v} + s_{in} - s\right) + \frac{Q_{max}(1+u)}{2v}(s_{in} - s), \\ \dot{v} = \frac{Q_{max}}{2}(1+u), \end{cases}$$

where s_{in} and s denote respectively the input substrate and substrate concentrations, and v is the volume of the reactor⁴. The parameter $Q_{max} > 0$ is the maximal speed of the input pump so that $\frac{Q_{max}}{2}(1+u)$ represents the input flow rate, $u(\cdot)$ being the control variable with values in [-1, 1]. Finally, $M \in \mathbb{R}$ depends on the initial value of micro-organism concentration⁵. As in many engineering applications (see, *e.g.*, [5]), the kinetics μ of the reaction is of Haldane type, *i.e.*,

$$\mu(s) \coloneqq \frac{\mu_h s}{K + s + \frac{s^2}{K_I}}$$

14

⁴In contrast with the previous sections in which state variables are (x_1, x_2) , we chose to adopt the notation (s, v) that is commonly used in the bioprocesses literature for fed-batch operations.

⁵Micro-organism concentration X > 0 can be expressed as a simple function of (s, v), namely $X := M/v + s_{in} - s$, thus (5.1) is enough to describe a bioreactor operated in fed-batch mode.

with a unique maximum $s^* \coloneqq \sqrt{KK_I} \in (0, s_{in})$ (parameters μ_h, K, K_I are positive). This type of growth function models inhibition by substrate (microbial growth is limited when s is too large w.r.t. s^*). It is worth mentioning that $\mathcal{D} := (0, s_{in}] \times \mathbb{R}^*_+$ is invariant by (5.1). For waste water treatment purpose, the problem of interest is:

437 (5.2)
$$\inf_{u \in \mathcal{U}} T_u \quad \text{s.t.} \ (s(T_u), v(T_u)) \in \mathcal{T},$$

438 where $\mathcal{T} \coloneqq (0, s_{ref}] \times \{v_{max}\}$ is the target set, $s_{ref} \ll s_{in}$ is a given threshold, and 439 $v_{max} > 0$ denotes the maximal volume of the bioreactor. From a practical point of 440 view, the goal is to treat a volume v_{max} of wasted water in minimal time. For more 441 details about this system, we refer to [17, 3].

It appears that Problem (5.2) may exhibit a saturation phenomenon. Indeed, by using the PMP, we can check that there is a singular locus that is the line segment

$$\Delta_{SA} := \{s^*\} \times (0, v_{max}],$$

and that the singular control can be expressed in feedback form as

$$u_s[v] \coloneqq \frac{\mu(s^*) \left[M + v(s_{in} - s^*)\right]}{(s_{in} - s^*)Q_{max}} - 1,$$

(writing $\dot{s} = 0$ along $s = s^*$). It follows that there exists a unique saturation point

$$x_{sat} := (s^*, v^*),$$

442 with $v^* \coloneqq \frac{2Q_{max}}{\mu(s^*)} - \frac{M}{s_{in}-s^*}$ and $u_s[v^*] = 1$ if the following condition is fulfilled

443 (5.3)
$$0 < v^* < v_{max}$$

This typically happens when v_{max} (the volume of water to be treated) is too high, see [3]. Next, we suppose that (5.3) holds true.

At this step, we wish to know if prior-saturation occurs (according to Propositions 3.3 and 3.5). Doing so, let us check Assumption 3.1. One gets

$$\delta_0(s,v) = -\mu(s)(M/v + s_{in} - s)Q_m/2 = -\mu(s)XQ_m/2 < 0,$$

hence $\Delta_0 \cap \mathcal{D} = \emptyset$ and $\delta_0 < 0$ in \mathcal{D} . Now, the singular arc is of turnpike type and Legendre-Clebsch's optimality condition holds true because μ has a unique maximum for $s = s^*$, see [1], or a clock form argumentation in [17]. In addition, observe that, in the (s, v)-plane, trajectories of (5.1) with u = -1 are horizontal, hence, every arc with u = -1 and starting at a volume value $v_0 < v_{max}$ never reaches the target set \mathcal{T} . Finally, \mathcal{T} is reachable from \mathcal{D} taking the control u = +1 until reaching $v = v_{max}$ and then u = -1 until reaching s_{ref} .

Second, let us verify the hypotheses of Proposition 3.5. Doing so, let $v \mapsto \hat{s}(v)$ be the unique solution to the Cauchy problem

$$\frac{ds}{dv} = -\left(-\frac{\mu(s)}{Q_{max}}\left[\frac{M}{v} + s_{in} - s\right] + \frac{s_{in} - s}{v}\right), \quad s(v_{max}) = s^*,$$

453 (the solution of (5.1) with u = 1 backward in time from (s^*, v_{max})). From [3], if

there exists $v_* \in (0, v^*)$ such that $\hat{s}(v_*) = s^*$, then optimal paths starting at a volume value sufficiently small necessarily contain a singular arc (this actually follows using

the PMP). Now, by using Cauchy-Lipschitz's Theorem, the existence of v_* is easy to verify when M = 0, and thus, it is also verified for small values of the parameter M (by a continuity argumentation). To pursue our analysis, we suppose next the existence of $v_* \in (0, v^*)$. We are then in a position to apply Propositions 3.3 and 3.5. It follows that there is a unique volume value $v_e \in (0, v^*)$ such that any singular arc starting at a volume value $v_0 < v_e$ will be optimal only until v_e . In addition, combining this result with a study of extremals using the PMP, we obtain that

- if the initial condition is (s^*, v_0) with $v_0 < v_e$, then the optimal path is of the form $\sigma_s \sigma^b_+ \sigma_-$ (see below for the definition of σ^b_+);
- if the initial condition is (s^*, v_0) with $v_0 \ge v_e$, then the optimal path is of the form $\sigma_+\sigma_-$;
 - for any initial condition (s_{in}, v_0) with $v_e \leq v_0 < v_{max}$, the optimal path is of the form $\sigma_{-}\sigma_{+}\sigma_{-}$ where the first switching time appears on a switching curve emanating from (s^*, v_e) .

To determine the prior-saturation point $x_e := (s^*, v_e)$ numerically, we proceed as in Section 4. For this application model, it is convenient to introduce an *extended target* set as $\overline{\mathcal{T}} := (0, s_{in}] \times \{v_{max}\}$ (observe that for initial conditions on $\overline{\mathcal{T}}$, optimal paths are σ_{-} arcs). In this context, a *bridge* is defined as an arc σ_{+} (denoted by σ_{+}^{b}) on $[0, t_b]$ such that

475
$$\phi(0) = \phi(0) = \phi(t_b) = 0$$
 and $v(t_b) = v_{max}$,

476 where ϕ is the switching function defined by (2.6) and t_b is the time to steer x_e at 477 time 0 to the extended target set $\overline{\mathcal{T}}$ with u = +1. To compute x_e , we need to compute 478 the extremities of the bridge together with its length. Denoting by t_b^* the length of 479 the bridge and by z_b^* its extremity in the cotangent bundle whose projection on the 480 state space belongs to $\overline{\mathcal{T}}$, the point (t_b^*, z_b^*) is then a solution of the equation $F_{\text{bio}} = 0$ 481 with

482 (5.4)
$$F_{\text{bio}}(t_b, z_b) \coloneqq \begin{pmatrix} H_{[f,g]}(\exp(-t_b \overrightarrow{H_+})(z_b)) \\ H_g(\exp(-t_b \overrightarrow{H_+})(z_b)) \\ H_+(z_b) + p^0 \\ H_g(z_b) \\ v_b - v_{max} \end{pmatrix},$$

where (s_b, v_b) is the projection of z_b on the state space. From Theorem 4.6, the bridge is then tangent to the switching curve at x_e (the projection of Σ given by (4.7) onto the state space). To conclude this part, let us comment Fig. 3 on which the optimal synthesis is plotted in a neighborhood of the prior-saturation point:

- In black, the switching curve Σ^{π} emanates from the prior-saturation point. It is computed using the shooting functions F = 0 adapted to (5.2).
- The synthesis is such that trajectories are horizontal (u = -1) until reaching 490 Δ_{SA} or the switching curve. For initial conditions with a substrate concen-491 tration less than s^* and $v_0 \geq v_e$, then u = 1 is optimal until reaching $\overline{\mathcal{T}}$.

5.2. The MRI model. In Nuclear Magnetic Resonance (NMR) *saturating* one chemical species consists in driving the magnetization vector representing the state to zero. In Magnetic Resonance Imaging (MRI) a challenging problem is to maximize the *constrast* between two observed species (for instance, healthy tissues and tumors) saturating one species. For the model, we consider an ensemble of spin-1/2 particles, excited by a radio-frequency (RF) field which is ideally assumed homogeneous, each

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FIG. 3. Minimal time synthesis for (5.2): the target set $\mathcal{T} = (0, s_{ref}] \times \{v_{max}\}$ is in black (left). The switching curve Σ^{π} (in black) is tangent to the bridge σ^b_+ (in red) at x_e . Arcs with u = +1 (resp. u = -1) are depicted in red (resp. in blue).

spin of this ensemble being described by its magnetization vector whose dynamics is governed in a specific rotating frame, after some normalizations and considering the 2-dimensional case, by the *Bloch equation* [16]:

501 (5.5)
$$\begin{cases} \dot{x}_1 = -\Gamma x_1 - u \, x_2, \\ \dot{x}_2 = \gamma (1 - x_2) + u \, x_1, \end{cases}$$

where $x := (x_1, x_2)$ is the normalized magnetization vector, where (γ, Γ) is a couple of parameters satisfying the physical constraint $0 < \gamma \leq 2\Gamma$ and depending on the longitudinal and transveral relaxation constants specific to the observed species, and where u is the RF-field which plays the role of the control. The time-minimal problem of interest here is the following:

507 (5.6)
$$\inf_{u \in \mathcal{U}} T_u \quad \text{s.t.} \quad x_u(T_u, x_0) = O \coloneqq (0, 0),$$

where the initial condition x_0 belongs to the set $B := \{(x_1, x_2) \in \mathbb{R}^2 ; x_1^2 + x_2^2 \le 1\}$ called the *Bloch ball* and where $x_u(\cdot, x_0)$ is the unique solution of (5.5) such that $x_u(0, x_0) = x_0$.

511 *Remark* 5.1. The problem of saturation in MRI is the problem (5.6) with $x_0 = N$, where N := (0, 1) is the North pole of the Bloch ball. We refer to [8, 7] for more details 512about the saturation and contrast problems in MRI. In [8], the following optimal synthesis is constructed: the authors give the optimal paths to go from N to any 514reachable point of the Bloch ball. Hence, the initial point is fixed to the North pole 516while the final point may be seen as a parameter. Here, we are interested in the converse problem, that is, the parameter is the initial condition and we want to steer 517the system to a given target, which is the origin O. The common problem in these 518two cases is the problem of saturation where the initial condition is N and where the 519520 target is O.

521 In this MRI application [8], the singular locus has a singularity at the intersection

of two lines. Setting $\delta \coloneqq \gamma - \Gamma$, the singular locus is described by 522

523
$$\Delta_{SA} = \Delta^h_{SA} \cup \Delta^v_{SA},$$

where $\Delta_{SA}^h := \{x_2 = \gamma/(2\delta)\}$ is a horizontal line and where $\Delta_{SA}^v := \{x_1 = 0\}$ is the vertical axis. On the vertical axis, the singular control is zero while on the horizontal line, the singular control is given in feedback form by

$$u_s[x_1, x_2] \coloneqq \gamma(2\Gamma - \gamma)/(2\delta x_1).$$

524Considering only the half space $x_1 \leq 0$ of the Bloch ball (this is possible due to a discrete symmetry) and restricting (γ, Γ) to the interesting case $0 < 3\gamma \leq 2\Gamma$ (in this case, the horizontal line cuts the Bloch ball), there exists only one saturation point denoted by $x_{\text{sat}} \in \Delta_{SA}$. The point x_{sat} belongs to the set $\Delta_{SA}^h \cap \{x_1 < 0\}$, it satisfies 527 $u_s(x_{\text{sat}}) = 1$, and it is given by 528

529
$$x_{\text{sat}} \coloneqq \left(\frac{\gamma(2\Gamma - \gamma)}{2\delta}, \frac{\gamma}{2\delta}\right)$$

Following [8], we introduce the concept of bridge. An arc σ_+ or σ_- with control 530 u = +1 or u = -1, is called a *bridge* on $[0, t_b]$ if its extremities correspond to non ordinary switching points, that is, if

533
$$\phi(0) = \dot{\phi}(0) = \phi(t_b) = \dot{\phi}(t_b) = 0$$

where ϕ is the switching function defined by (2.6). According to [8], there exists 534a bridge σ^b_+ (supposed to be unique) connecting Δ^h_{SA} and Δ^v_{SA} . We denote by 535 $x_e \coloneqq (x_{e,1}, x_{e,2})$ the extremity of the bridge on the horizontal line Δ_{SA}^h and we 536 can now restrict the analysis to the following situation. We assume that the following conditions are satisfied by the couple of parameters (γ, Γ) (see Fig. 3 of [8] and the 538 539description that comes after for details):

(i) x_e belongs to the Bloch ball B (this implies in particular that $3\gamma \leq 2\Gamma$), 540

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(ii) $0 < \gamma$ (this comes from the physical constraint),

(iii) $0 \leq (2\Gamma^2 - \gamma\Gamma + 1) \exp((\alpha - \gamma)t_0) - 2\delta$ (hence the origin O is reachable by a 542Bang-Singular sequence from x_{sat} and so also from x_e),

where $\alpha \coloneqq \delta/2$ and $t_0 \coloneqq \arctan(-\beta/\alpha)/\beta$ with $\beta \coloneqq \sqrt{1-\alpha^2}$. In this setting, for 544any initial condition $x_0 \coloneqq (x_{0,1}, x_{0,2}) \in \Delta_{SA}^h \cap B$ such that $x_{0,1} \leq x_{e,1}$, the optimal trajectory (see [8]) is of the form $\sigma_s \sigma_+^b \sigma_0$, that is composed of a singular arc on Δ_{SA}^h 545546 followed by the bridge with u = +1 and ending with a singular arc σ_0 along Δ_{SA}^v 547with u = 0. The first singular arc reduces to a point if $x_0 = x_e$. At x_e , the singular 548control is not saturating, so, in conclusion, the point x_e is a prior-saturation point. 549

Remark 5.2. In the MRI application, Assumption 3.1 is not exactly satisfied since 550the collinearity set Δ_0 is non-empty and plays a role in the optimal synthesis, such as 551the singularity of the singular locus at the intersection of the two lines. However, the singular arcs are turnpikes and Legendre-Clebsch optimality condition holds. Besides, 554there exists a prior-saturation point and so this case is more general than the fedbatch application. We will see hereinafter that the tangency property holds at the prior-saturation point and that the switching curve is transverse to the singular locus. 556

We end this part by showing how to compute the prior-saturation point x_e and by giving the optimal synthesis near x_e for an initial condition on the horizontal singular 558line. To compute x_e , we need to compute the extremities of the bridge together with

its length. Denoting by t_b^* the length of the bridge and by z_b^* the extremity of the bridge in the cotangent bundle whose projection on the state space belongs to Δ_{SA}^v , the point (t_b^*, z_b^*) is then a solution of the equation $F_{\rm mri} = 0$ with

563 (5.7)
$$F_{\rm mri}(t_b, z_b) \coloneqq \begin{pmatrix} H_{[f,g]}(\exp(-t_b \overrightarrow{H_+})(z_b)) \\ H_g(\exp(-t_b \overrightarrow{H_+})(z_b)) \\ H_+(z_b) + p^0 \\ H_{[f,g]}(z_b) \\ H_g(z_b) \end{pmatrix}$$

where the vector fields f and q are given by (5.5) and where the Hamiltonians, the 564565 Hamiltonian lifts and the Hamiltonian vector field are defined in Section 4.1. We recognize here a function of the form (4.3) without any additional parameter λ and 566 so, $z_e := \exp(-t_b^* \overrightarrow{H_+})(z_b^*)$ is the prior-saturation lift such that $\pi(z_e) = x_e \in \Delta_{SA}^h$. Finally, the optimal synthesis near x_e is given on Fig. 4. The optimal solution from the initial condition $x_0 \in \Delta_{SA}^h$ is of the form $\sigma_s \sigma_+^b \sigma_0$. The red arc σ_{\pm}^b is the bridge 567568 569 starting from x_e , it is a part of the forward semi-orbit Γ_+ of $\dot{z} = H_+(z)$ starting from z_e projected into the state space. The black curve Σ^{π} is the existing part in 571the optimal synthesis of the projection of the switching curve Σ defined by (4.7). 572According to the tangency property from Theorem 4.6, the arc σ^b_+ is tangent to Σ^{π} 573 at the prior-saturation point x_e . Note also that the switching curve Σ^{π} is transverse 574to the singular locus Δ_{SA}^h in accordance with Corollary 4.8. 575



FIG. 4. Optimal synthesis near the prior-saturation point x_e in the left part of the Bloch ball.

6. Conclusion. Even though the tangency property between the bridge and the switching curve provides useful informations on the minimum time synthesis when prior saturation occurs (typically, under assumptions of Proposition 3.3), it remains valid in a larger context (under the hypotheses of Theorem 4.6) and not only in the framework of saturation and prior-saturation of the singular control for affine-control systems in the plane. This property also appears in other settings such as in Lagrange control problems governed by one-dimensional systems, see, *e.g.*, [14]. Future works could then investigate prior-saturation phenomenon and the tangency property in other frameworks or in dimension $n \geq 3$.

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