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1 **TANGENCY PROPERTY AND PRIOR-SATURATION POINTS IN**  
2 **MINIMAL TIME PROBLEMS IN THE PLANE\***

3 T. BAYEN<sup>†</sup> AND O. COTS<sup>‡</sup>

4 **Abstract.** In this paper, we consider minimal time problems governed by control-affine-systems  
5 in the plane, and we focus on the synthesis problem in presence of a singular locus that involves  
6 a saturation point for the singular control. After giving sufficient conditions on the data ensuring  
7 occurrence of a prior-saturation point and a switching curve, we show that the bridge (*i.e.*, the optimal  
8 bang arc issued from the singular locus at this point) is tangent to the switching curve at the prior-  
9 saturation point. This property is proved using the Pontryagin Maximum Principle that also provides  
10 a set of non-linear equations that can be used to compute the prior-saturation point. These issues  
11 are illustrated on a fed-batch model in bioprocesses and on a Magnetic Resonance Imaging (MRI)  
12 model for which minimal time syntheses for the point-to-point problem are discussed.

13 **Key words.** Geometric optimal control, Minimum time problems, Singular arcs,

14 **AMS subject classifications.** 49J15, 49N35, 70G45

**1. Introduction.** In this paper, we consider minimal time problems governed  
by single-input control-affine-systems in the plane

$$\dot{x}(t) = f(x(t)) + u(t)g(x(t)), \quad |u(t)| \leq 1,$$

15 where  $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  are smooth vector fields. Syntheses for such problems have been  
16 investigated a lot in the literature (see, *e.g.*, [6, 12, 18, 23, 25, 24]). In particular, an  
17 exhaustive description of the various encountered singularities can be found in [12],  
18 as well as an algorithm leading to the determination of optimal paths. It is worth  
19 mentioning that even though many techniques exist in this setting, the computation  
20 of an optimal feedback synthesis (global) remains in general difficult because of the  
21 occurrence of geometric loci such as singular arcs, switching curves, cut-loci...

22 Our aim in this work is to focus on the notion of *singular arc* which appears in  
23 the synthesis when the switching function (the scalar product between the adjoint  
24 vector and the controlled vector field  $g$ ) vanishes over a time interval. In that case,  
25 the corresponding singular control  $u_s$  (which allows the associated trajectory to stay  
26 on the singular locus) can be expressed in feedback form  $x \mapsto u_s[x]$ . However, it  
27 may happen that  $u_s$  becomes non admissible, *i.e.*,  $x \mapsto u_s[x]$  takes values above the  
28 maximal value for the control (namely 1 here). Such a situation naturally appears  
29 in several application models, see, *e.g.*, [2, 3, 15, 20]. In that case, we say that  
30 a *saturation phenomenon* occurs. The occurrence of such a phenomenon implies the  
31 following (non-intuitive) property that, if a singular arc is optimal, then it should leave  
32 the singular locus at a so-called *prior-saturation point* before reaching the saturation  
33 point. This property has been studied in the literature in various situations such as  
34 for control-affine systems in dimension 2 and 4 (see, *e.g.*, [21, 22, 10, 3] and references  
35 herein).

36 Our main goal in this paper is to provide new qualitative properties on the min-  
37 imum time synthesis in presence of a saturation point. More precisely, our objective  
38 is twofold:

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- We first give a set of conditions on the system that ensure occurrence of prior-saturation showing that, under certain assumptions, the system leaves the singular arc at this point (before reaching the saturation point) with the maximal value for the control, see Proposition 3.3. This last arc is usually called *bridge* following the terminology as in [9, 7] (see also [11, 6]).
- Second, we introduce a shooting function that allows an effective computation of the prior-saturation point. This mapping is used to show our main result (Theorem 4.6) which can be stated as follows: when the system exhibits a switching curve emanating from the prior-saturation point, then this curve is tangent to the bridge (in the cotangent bundle) at this point.

The tangency property (in the state space) has been pointed out in several application models (see, *e.g.*, [3, 9]). To the best of our knowledge, this property has not been addressed previously in this general setting in the literature. It allows to better understand the construction of optimal paths locally at the prior-saturation point.

The paper is structured as follows: in Section 2, we recall classical expressions and properties of singular controls for control-affine-systems in the plane introducing the saturation phenomenon. In Section 3, we provide a set of conditions involving the target set and the system ensuring the occurrence of the prior-saturation phenomenon. In Section 4, we show the tangency property between the switching curve emanating from a prior-saturation point and the bridge, and we describe how to compute the prior-saturation point thanks to a shooting function constructed via the Hamiltonian lifts of  $f$  and  $g$ . Finally, we depict this geometrical property in Section 5 for a fed-batch model [17, 3] and MRI model [9, 7]. This allows us to illustrate the notion of bridge in various contexts: first, when it connects a component of the singular locus to another one (see the MRI-model in Section 5 and [9, 7]), and then when it connects a component of a singular locus to an extended target set (see the fed-batch model in Section 5 and [17, 3]).

**2. Saturation phenomenon.** The purpose of this section is to recall some facts about minimum time control problems in the plane that will allow us to introduce the saturation phenomenon. Throughout the paper, the standard inner product in  $\mathbb{R}^2$  is written  $a \cdot b$  for  $a, b \in \mathbb{R}^2$ , and  $a^\perp$  denotes the vector  $a^\perp := (-a_2, a_1)$  orthogonal to  $a$ . The interior of a subset  $S \subset \mathbb{R}^n$ ,  $n \geq 2$ , is denoted by  $\text{Int}(S)$ .

**2.1. Pontryagin's Principle.** We start by applying the classical optimality conditions provided by the Pontryagin Maximum Principle (PMP), see [19]. Let  $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be two vector fields of class  $C^\infty$ , and consider the controlled dynamics:

$$(2.1) \quad \dot{x}(t) = f(x(t)) + u(t)g(x(t)),$$

with admissible controls in the set

$$\mathcal{U} := \{u : [0, +\infty) \rightarrow [-1, 1] ; u \text{ meas.}\}.$$

Given an initial point  $x_0 \in \mathbb{R}^2$  and a non-empty closed subset  $\mathcal{T} \subset \mathbb{R}^2$ , we focus on the problem of driving (2.1) in minimal time from  $x_0$  to the target set  $\mathcal{T}$ :

$$(2.2) \quad \inf_{u \in \mathcal{U}} T_u \quad \text{s.t.} \quad x_u(T_u) \in \mathcal{T},$$

where  $x_u(\cdot)$  denotes the unique solution of (2.1) associated with the control  $u$  such that  $x_u(0) = x_0$ , and  $T_u \in [0, +\infty]$  is the first entry time of  $x_u(\cdot)$  into the target set  $\mathcal{T}$ . We

suppose hereafter that optimal trajectories exist<sup>1</sup> and we wish to apply the PMP on (2.2). The Hamiltonian associated with (2.2) is the function  $H : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  defined as

$$H(x, p, p^0, u) := p \cdot f(x) + u p \cdot g(x) + p^0.$$

78 If  $u$  is an optimal control and  $x_u$  is the associated trajectory steering  $x_0$  to the target  
79 set  $\mathcal{T}$  in time  $T_u \geq 0$ , the following conditions are fulfilled:

80 • There exist  $p^0 \leq 0$  and an absolutely continuous function  $p : [0, T_u] \rightarrow \mathbb{R}^2$   
81 satisfying the adjoint equation

$$(2.3) \quad \dot{p}(t) = -\nabla_x H(x_u(t), p(t), p^0, u(t)) \quad \text{a.e. } t \in [0, T_u].$$

83 • The pair  $(p^0, p(\cdot))$  is non-zero.

84 • The optimal control  $u$  satisfies the *Hamiltonian condition*

$$(2.4) \quad u(t) \in \operatorname{argmax}_{\omega \in [-1, 1]} H(x_u(t), p(t), p^0, \omega) \quad \text{a.e. } t \in [0, T_u].$$

86 • At the terminal time, the *transversality condition*<sup>2</sup>  $p(T_u) \in -N_{\mathcal{T}}(x_u(T_u))$  is  
87 fulfilled.

88 Recall that an extremal  $(x_u(\cdot), p(\cdot), p^0, u(\cdot))$  satisfying (2.1) and (2.3)-(2.4) is *abnormal*  
89 whenever  $p^0 = 0$  and *normal* whenever  $p^0 \neq 0$ . In the latter case, we take  $p^0 = -1$   
90 and the corresponding extremal is denoted by  $(x_u(\cdot), p(\cdot), u(\cdot))$  and we shall then  
91 write  $H(x, p, u)$  in place of  $H(x, p, p^0, u)$ . Since  $T_u$  is free and (2.1) is autonomous,  
92 the Hamiltonian  $H$  is zero along any extremal: for a.e.  $t \in [0, T_u]$ ,

$$(2.5) \quad H = p(t) \cdot f(x_u(t)) + u(t)p(t) \cdot g(x_u(t)) + p^0 = 0.$$

94 The *switching function*  $\phi$  is defined as

$$(2.6) \quad \phi(t) := p(t) \cdot g(x_u(t)), \quad t \in [0, T_u],$$

96 and it gives us (thanks to (2.4)) the following control law:

$$(2.7) \quad \begin{cases} \phi(t) > 0 & \Rightarrow u(t) = +1, \\ \phi(t) < 0 & \Rightarrow u(t) = -1. \end{cases}$$

A *switching time* is an instant  $t_c \in (0, T_u)$  such that the control  $u$  is discontinuous at time  $t_c$ . We say that the corresponding extremal trajectory has a *switching point* at time  $t_c$ . Of particular interest is the case when there is a time interval  $[t_1, t_2]$  such that the switching function vanishes over this interval, *i.e.*,

$$\phi(t) = p(t) \cdot g(x_u(t)) = 0, \quad t \in I.$$

We then say that the extremal trajectory has a *singular arc* over  $[t_1, t_2]$ . Note that we shall suppose such an extremal to be normal, *i.e.*,  $p^0 \neq 0$ . Indeed, recall from [12, Prop. 2 p.49] that under generic conditions, abnormal extremals are bang-bang. By differentiating  $\phi$  twice w.r.t.  $t$ , one gets

$$\dot{\phi}(t) = p(t) \cdot [f, g](x_u(t)), \quad t \in [0, T_u],$$

<sup>1</sup>If the target can be reached from  $x_0$  and if  $f, g$  have linear growth, then (2.2) admits an optimal solution, thanks to Filippov's Existence Theorem, see, *e.g.*, [26].

<sup>2</sup>Here,  $N_{\mathcal{T}}(x)$  stands for the (Mordukovitch) limiting normal cone to  $\mathcal{T}$  at point  $x \in \mathcal{T}$ , see [26]. It coincides with the normal cone in the sense of convex analysis when  $\mathcal{T}$  is convex.

where  $[f, g](x)$  is the Lie bracket of  $f$  and  $g$  at point  $x$ , and

$$\ddot{\phi}(t) = p(t) \cdot [f, [f, g]](x_u(t)) + u(t) p(t) \cdot [g, [f, g]](x_u(t)) \quad \text{a.e. } t \in [0, T_u].$$

98 The *singular locus*  $\Delta_{SA}$  (in the state space) is defined as the (possibly empty) subset  
99 of  $\mathbb{R}^2$

$$100 \quad (2.8) \quad \Delta_{SA} := \{x \in \mathbb{R}^2 ; \det(g(x), [f, g](x)) = 0\}.$$

101 For future reference, we set  $\delta_{SA}(x) := \det(g(x), [f, g](x))$  for  $x \in \mathbb{R}^2$ . Note that if  
102 an extremal is singular over a time interval  $[t_1, t_2]$ , then one has  $x_u(t) \in \Delta_{SA}$  for  
103 any  $t \in [t_1, t_2]$  because  $p(\cdot)$  must be non-zero and orthogonal to the vector space  
104  $\text{span}\{g(x_u(t)), [f, g](x_u(t))\}$  over  $[t_1, t_2]$ . The *singular control*  $u_s$  is then the value of  
105 the control for which the trajectory stays on the singular locus  $\Delta_{SA}$ . Supposing then  
106 that  $\phi(t) = \dot{\phi}(t) = 0$  over  $[t_1, t_2]$  gives:

$$107 \quad (2.9) \quad u_s(t) := -\frac{p(t) \cdot [f, [f, g]](x_u(t))}{p(t) \cdot [g, [f, g]](x_u(t))}, \quad t \in [0, T_u],$$

108 provided that  $p(t) \cdot [g, [f, g]](x_u(t))$  is non zero for  $t \in [t_1, t_2]$ . This expression of the  
109 singular control does not guarantee that  $u_s$  is admissible, that is,  $u_s(t) \in [-1, 1]$ :

- 110 • When we have  $u_s(t) \in [-1, 1]$ , the point  $x_u(t)$  is said *hyperbolic* if  $p(t) \cdot$   
111  $[g, [f, g]](x_u(t)) > 0$ , and *elliptic* if  $p(t) \cdot [g, [f, g]](x_u(t)) < 0$  (see [11, 6]).
- 112 • When we have  $|u_s(t)| > 1$  for some instant  $t$ , we say that a *saturation phe-*  
113 *nomenon* occurs and that the corresponding points of the singular locus are  
114 *parabolic* (see [11, 6]).

115 Our purpose in what follows is precisely to investigate properties of the synthesis of  
116 optimal paths when saturation occurs. To this end, we suppose in the rest of the  
117 paper that extremals are normal, *i.e.*,  $p^0 \neq 0$  (we take hereafter  $p^0 = -1$ ).

118 **2.2. Singular control and saturation phenomenon.** In this part, we derive  
119 classical expressions of the singular control in terms of feedback that will allow us to  
120 introduce saturation points (in terms of the data defining the system). The *collinearity*  
121 *set* associated with (2.1) is the (possibly empty) subset of  $\mathbb{R}^2$  defined as

$$122 \quad (2.10) \quad \Delta_0 := \{x \in \mathbb{R}^2 ; \det(f(x), g(x)) = 0\}.$$

123 Define two functions  $\delta_0, \psi : \mathbb{R}^2 \rightarrow \mathbb{R}$  as  $\delta_0(x) := \det(f(x), g(x))$ ,  $x \in \mathbb{R}^2$ , and

$$124 \quad (2.11) \quad \psi(x) := -\frac{\det(g(x), [f, [f, g]](x))}{\det(g(x), [g, [f, g]](x))}, \quad x \in \mathbb{R}^2.$$

125

126 **LEMMA 2.1.** *Suppose that  $\Delta_{SA} \neq \emptyset$ , that  $x \mapsto \det(g(x), [g, [f, g]](x))$  is non-zero*  
127 *over  $\Delta_{SA}$ , and consider a singular arc defined over an interval  $[t_1, t_2]$ . Then, one has:*

$$128 \quad (2.12) \quad u_s(t) = \psi(x(t)), \quad t \in [t_1, t_2],$$

129 where  $x(\cdot)$  is the corresponding singular trajectory such that  $x(t) \in \Delta_{SA}$  for  $t \in [t_1, t_2]$ .

*Proof.* Since we supposed extremals to be normal, for every  $t \in [t_1, t_2]$ , the family  
 $\{f(x(t)), g(x(t))\}$  is a basis of  $\mathbb{R}^2$  (using that  $H = 0$ ). Using that  $H = 0$  and  $p^0 = -1$ ,  
we obtain that  $p(t) \cdot f(x(t)) = 1$  for  $t \in [t_1, t_2]$  which gives  $p(t) = -\frac{g^\perp(x(t))}{\delta_0(x(t))}$ ,  $t \in [t_1, t_2]$ .

We also deduce (decomposing  $[f, [f, g]](x(t))$  and  $[g, [f, g]](x(t))$  on  $\{f(x(t)), g(x(t))\}$ ) that

$$\begin{aligned} p(t) \cdot [f, [f, g]](x(t)) &= \det(g(x(t)), [f, [f, g]](x(t)))\Lambda(x(t)), \\ p(t) \cdot [g, [f, g]](x(t)) &= \det(g(x(t)), [g, [f, g]](x(t)))\Lambda(x(t)), \end{aligned}$$

130 where  $\Lambda(x) := \frac{g^\perp(x) \cdot f(x)}{\delta_0(x)^2}$ ,  $x \notin \Delta_0$ . Moreover,  $g^\perp(x(t)) \cdot f(x(t)) = -\delta_0(x(t))$  and thus,  
131 this scalar product is non-zero because extremals are normal ( $p^0 \neq 0$ ). This ends the  
132 proof using (2.9).  $\square$

133 *Remark 2.2. Steady-state singular points* are defined as the points  $x^* \in \Delta_{SA} \cap \Delta_0$   
134 such that  $g(x^*) \neq 0$ , see [12, 4] (if  $\Delta_{SA} \cap \Delta_0 \neq \emptyset$ ). Such points are equilibria of (2.1)  
135 with  $u = \psi(x)$ . A singular arc defined over a time interval  $[t_1, t_2]$  does not contain  
136 such a point because  $f(x(t))$  and  $g(x(t))$  must be linearly independent over  $[t_1, t_2]$ .  
137 But, it can contain points  $x^* \in \Delta_{SA} \cap \Delta_0$  such that  $g(x^*) = 0$ .

To introduce the notion of saturation point, it is convenient to consider a parametrization of  $\Delta_{SA}$  as follows. When  $\Delta_{SA} \cap \Delta_0$  is non-empty,  $\Delta_{SA} \setminus \Delta_0$  can be divided into several subsets (called components hereafter). Hence, we write this set as

$$\Delta_{SA} \setminus \Delta_0 = \bigcup_{k \in K} \gamma_k,$$

138 where  $K$  is an index set.

139 **LEMMA 2.3.** *Suppose that  $\Delta_{SA}$  is non-empty and that  $x \mapsto \det(g(x), [g, [f, g]](x))$*   
140 *is non-zero over  $\Delta_{SA}$ . Then, each component  $\gamma_k$  of  $\Delta_{SA} \setminus \Delta_0$  can be parametrized by*  
141 *a one-to-one parametrization  $\zeta : J \rightarrow \gamma_k$ ,  $\tau \mapsto \zeta(\tau)$  of class  $C^1$ , where  $J$  is an interval*  
142 *of  $\mathbb{R}$ .*

143 *Proof.* For  $x \notin \Delta_0$ , one has  $\text{span}\{f(x), g(x)\} = \mathbb{R}^2$ , hence, there exist  $\alpha(x), \beta(x) \in$   
144  $\mathbb{R}$  such that

$$145 \quad (2.13) \quad [f, g](x) = \alpha(x)f(x) + \beta(x)g(x).$$

By taking the determinant, we find that for  $x \notin \Delta_0$ ,

$$\alpha(x) = -\frac{\det(g(x), [f, g](x))}{\delta_0(x)} \quad \text{and} \quad \beta(x) = \frac{\det(f(x), [f, g](x))}{\delta_0(x)}.$$

Consider now a component  $\gamma_k$  of  $\Delta_{SA} \setminus \Delta_0$  and  $x \in \gamma_k$ . By computing  $[f, [f, g]](x)$  thanks to (2.13), we get

$$\det(g(x), [g, [f, g]](x)) = -\delta_0(x)\nabla\alpha(x) \cdot g(x), \quad x \in \gamma_k.$$

146 Since  $x \mapsto \det(g(x), [g, [f, g]](x))$  is non-zero over  $\Delta_{SA}$ , the preceding equality implies  
147 that the scalar product  $\nabla\alpha(x) \cdot g(x)$  is non-zero. On the other hand,  $\gamma_k$  is defined by  
148 the implicit equation  $\delta_{SA}(x) = 0$ . Observe that for  $x \notin \Delta_0$ ,  $\delta_{SA}(x) = -\alpha(x)\delta_0(x)$ .  
149 By taking the derivative, we find that for  $x \notin \Delta_0$ , one has  $\nabla\delta_{SA}(x) = -\delta_0(x)\nabla\alpha(x) -$   
150  $\alpha(x)\nabla\delta_0(x)$ . Therefore, for  $x \in \gamma_k$ , we obtain  $\nabla\delta_{SA}(x) = -\delta_0(x)\nabla\alpha(x)$ . We can  
151 conclude that for any point  $x \in \gamma_k$ , the partial derivative  $\partial_1\alpha(x)$  (w.r.t.  $x_1$ ) or  $\partial_2\alpha(x)$   
152 (w.r.t.  $x_2$ ) is non-zero. We are then in a position to apply the implicit function theorem  
153 to  $\delta_{SA}$  locally at each point  $x \in \gamma_k$ , which then implies the desired property.  $\square$

Under the assumptions of Lemma 2.3, given a component  $\gamma_k$  of  $\Delta_{SA}$ , there is a parametrization  $\zeta$  such that

$$\gamma_k := \{\zeta(\tau) ; \tau \in J\},$$

154 where  $\zeta : J \rightarrow \mathbb{R}^2$  is  $C^1$ -mapping (injective) and  $J$  is an interval.

155 DEFINITION 2.4. A point  $x^* := \zeta(\tau^*)$  with  $\tau^* \in \text{Int}(J)$  is called *saturation point*  
 156 if  $\psi(x^*) = 1$ ,  $\psi(\zeta(\tau)) \in (-1, 1)$  for any  $\tau \in J$  such that  $\tau < \tau^*$ , and  $\psi(\zeta(\tau)) > 1$  for  
 157 any  $\tau \in J$  such that  $\tau > \tau^*$ .

158 As well, we can define saturation points  $x^*$  such that  $\psi(x^*) = -1$ , that is, when  
 159 the lower bound of the admissible control set is saturated. Our next aim is to study  
 160 the optimality of singular arcs in presence of a saturation point.

161 **3. Existence of a prior-saturation point.** In this section, we show that a  
 162 prior-saturation phenomenon can occur whenever the system exhibits a saturation  
 163 point. We start by introducing our main assumptions.

164 ASSUMPTION 3.1. The system (2.1) satisfies the following hypotheses:

- 165 (i) One has  $\Delta_0 = \emptyset$  and  $\delta_0(x) < 0$  for all  $x \in \mathbb{R}^2$ .  
 166 (ii) The set  $\Delta_{SA}$  is non-empty, simply connected, and has exactly one saturation  
 167 point  $x^*$  with  $\psi(x^*) = 1$ .  
 168 (iii) Along the singular locus, the strict Legendre-Clebsch optimality condition is  
 169 satisfied, that is, any singular extremal  $(x_u(\cdot), p(\cdot), u(\cdot))$  defined over  $[t_1, t_2]$   
 170 satisfies:

$$171 \quad (3.1) \quad \frac{\partial}{\partial u} \frac{d^2}{dt^2} \frac{\partial H}{\partial u}(x_u(t), p(t), u(t)) > 0, \quad \forall t \in [t_1, t_2].$$

- 172 (iv) If  $\Gamma_-$  is the forward semi-orbit of (2.1) with  $u = -1$  with the initial condition  
 173  $x^*$  at time 0, then

$$174 \quad (3.2) \quad \mathcal{T} \cap \Gamma_- = \emptyset.$$

- 175 (v) The target  $\mathcal{T}$  is reachable from every point  $x_0 \in \mathbb{R}^2$ .

Remark 3.1. (i) The hypothesis  $\Delta_0 = \emptyset$  is not restrictive since we could restrict  
 our analysis to a component  $\gamma$  of  $\Delta_{SA}$  in place of  $\Delta_{SA}$ .

(ii) By the previous computations, we can observe that (3.1) is equivalent to

$$\det(g(x), [g, [f, g]](x)) > 0, \quad \forall x \in \Delta_{SA}.$$

176 Recall that, under the strict Legendre-Clebsch condition, the singular arc is a *turnpike*,  
 177 i.e., it is time-minimizing in every neighborhood of a hyperbolic point of  $\Delta_{SA}$ , [11].  
 178 This property can be retrieved by the clock form argument [13].

Under Assumption 3.1, the singular locus  $\Delta_{SA}$  is written  $\Delta_{SA} := \zeta(J)$  where  
 $J \subset \mathbb{R}$  is an interval and  $\zeta : J \rightarrow \Delta_{SA}$  is a  $C^1$ -mapping. In addition,  $\Delta_{SA}$  partitions  
 the state space into two simply connected (open) subsets  $\Delta_{SA}^\pm$ :

$$\begin{aligned} \Delta_{SA}^+ &:= \{x \in \mathbb{R}^2 ; \det(g(x), [f, g](x)) > 0\}, \\ \Delta_{SA}^- &:= \{x \in \mathbb{R}^2 ; \det(g(x), [f, g](x)) < 0\}. \end{aligned}$$

Given a normal extremal  $(x_u(\cdot), p(\cdot), u(\cdot))$ , the function

$$t \mapsto \gamma_u(t) := \beta(x_u(t)) - \alpha(x_u(t))u(t), \quad t \in [0, T_u],$$

179 is well-defined since  $\Delta_0 = \emptyset$ .

180 LEMMA 3.2. Along a normal extremal  $(x_u(\cdot), p(\cdot), u(\cdot))$ , the switching function  $\phi$   
 181 satisfies the ODE

$$182 \quad (3.3) \quad \dot{\phi}(t) = \gamma_u(t)\phi(t) + \alpha(x_u(t)) \quad \text{a.e. } t \in [0, T_u].$$

183 *Proof.* The proof follows using the expression of  $\dot{\phi}$  and the fact that the Hamil-  
184 tonian  $H$  is constant equal to zero.  $\square$

185 The next proposition shows that an extremal trajectory containing a singular arc  
186 until the point  $x^*$  is not optimal.

187 **PROPOSITION 3.3.** *Suppose that Assumption 3.1 holds true, and consider an op-  
188 timal trajectory steering  $x_0$  to the target  $\mathcal{T}$  in time  $T_u$ . Then, the corresponding  
189 extremal  $(x_u(\cdot), p(\cdot), u(\cdot))$  does not contain a singular arc defined over a time interval  
190  $[t_1, t_2]$  such that  $x_u(t_2) = x^*$ .*

191 *Proof.* Suppose by contradiction that there is a time interval  $[t_1, t_2]$  such that the  
192 trajectory is singular over  $[t_1, t_2]$  with  $x_u(t_2) = x^*$ . We claim that, at time  $t_2$ , the  
193 vector  $f(x_u(t_2)) + g(x_u(t_2))$  is tangent to  $\Delta_{SA}$ . Indeed, it is enough to check that the  
194 vector  $f(x^*) + g(x^*)$  is orthogonal to  $\nabla\delta_{SA}(x^*) = -\delta_0(x^*)\nabla\alpha(x^*)$ . As we have seen  
195 in the proof of Lemma 2.3, one has for  $x \in \Delta_{SA}$ :

$$196 \quad (3.4) \quad \begin{aligned} \det(g(x), [g, [f, g]](x)) &= -\delta_0(x)\nabla\alpha(x) \cdot g(x), \\ \det(g(x), [f, [f, g]](x)) &= -\delta_0(x)\nabla\alpha(x) \cdot f(x). \end{aligned}$$

These equalities imply that

$$-\delta_0(x^*)\nabla\alpha(x^*) \cdot (f(x^*) + g(x^*)) = \det(g(x^*), [g, [f, g]](x^*)) + \det(g(x^*), [f, [f, g]](x^*)).$$

197 Since  $\psi(x^*) = 1$ , the right member of the above equality is zero which shows the  
198 claim. In addition, for  $x \in \Delta_{SA}$ , (3.4) implies the equalities

$$199 \quad \begin{aligned} \nabla\delta_{SA}(x) \cdot (f(x) + g(x)) &= \det(g(x), [g, [f, g]](x))(1 - \psi(x)), \\ \nabla\delta_{SA}(x) \cdot (f(x) - g(x)) &= \det(g(x), [g, [f, g]](x))(-1 - \psi(x)). \end{aligned}$$

202 Consider now the unique solution  $x_-$  of (2.1) with  $u = -1$  starting from  $x^*$  at time  
203  $t_2$ . This trajectory enters into the set  $\Delta_{SA}^-$  for  $t > t_2$ ,  $t$  close to  $t_2$ , because one has  
204  $\nabla\delta_{SA}(x^*) \cdot (f(x^*) - g(x^*)) < 0$ . Going back to the optimal trajectory, there are now  
205 two possibilities for  $x_u(\cdot)$ . In a right neighborhood of  $t = t_2$ , either  $x_u(\cdot)$  enters into  
206  $\Delta_{SA}^+$  or into  $\Delta_{SA}^-$  (because the singular control becomes non admissible).

207 Suppose first that  $x_u(\cdot)$  enters into  $\Delta_{SA}^+$ . Then, there is  $\varepsilon > 0$  such that one has  
208  $\alpha(x_u(t)) > 0$  for  $t \in (t_2, t_2 + \varepsilon]$ . It follows from (3.3) that one has  $u = +1$  on this  
209 interval. But the velocity set being convex, we obtain a contradiction with the non-  
210 admissibility of the singular control at  $x^*$  (because  $x_-$  enters into  $\Delta_{SA}^-$ ). It follows  
211 that the optimal trajectory necessarily enters into the set  $\Delta_{SA}^-$ . But then, since  $\alpha < 0$   
212 in  $\Delta_{SA}^-$ , (3.3) implies that  $u = -1$  in some time interval  $(t_2, t_2 + \varepsilon]$ .

213 From Assumption 3.1, the forward semi-orbit with  $u = -1$  starting from  $x^*$  does  
214 not reach the target set. Hence,  $x_u(\cdot)$  must have a switching point to  $u = +1$  in  $\Delta_{SA}^-$   
215 or it must reach  $\Delta_{SA}$  with the control  $u = -1$ . We see from (3.3) that the first case  
216 is not possible because at a switching time  $t_c$  such that  $x_u(t_c) \in \Delta_{SA}^-$ , we would have  
217  $\dot{\phi}(t_c) \geq 0$  in contradiction with  $\alpha(x_u(t_c)) < 0$ .

218 Suppose now that  $x_u(\cdot)$  reaches  $\Delta_{SA}$  at some point  $x := \zeta(\tau)$  with  $\tau < \tau^*$ . Then,  
219 we obtain  $\nabla\delta_{SA}(x) \cdot (f(x) - g(x)) < 0$  since  $\psi(x) > -1$ . But, as  $x_u(\cdot)$  reaches  $\Delta_{SA}$   
220 with  $u = -1$  at point  $x$ , the trajectory enters into the set  $\Delta_{SA} \cup \Delta_{SA}^+$  and we must  
221 have  $\nabla\delta_{SA}(x) \cdot (f(x) - g(x)) \geq 0$  ( $\nabla\delta_{SA}(x)$  is collinear to the outward normal vector  
222 to  $\Delta_{SA}$  at point  $x$ ). This gives a contradiction. In the same way, the trajectory  
223 cannot reach a point  $x \in \Delta_{SA}$  such that  $x = \zeta(\tau)$  with  $\tau > \tau^*$ .



224 We can conclude that for any time  $t \geq t_2$ , one has  $u(t) = -1$ , but then, the  
 225 optimal trajectory cannot reach the target set which is a contradiction (Assumption  
 226 3.1 (iv)). This concludes the proof.  $\square$

227 As an example, if  $x_0 := \zeta(\tau_0)$  belongs to the singular locus with  $\tau_0 < \tau^*$ , and  
 228 if an optimal trajectory starting from  $x_0$  contains a singular arc, then the trajectory  
 229 should leave the singular locus before reaching  $x^*$ . Let us insist on the fact that this  
 230 property of leaving the singular locus before reaching  $x^*$  relies on the fact that the  
 231 optimal trajectory should contain a singular arc. In the fed-batch model presented in  
 232 Section 5.1, this property can be easily verified (see [3]).

233 We now introduce the following definition (in line with [15, 21, 22]). Hereafter,  
 234 the notation  $\mathcal{S}_{[\tau'_0, \tau_0]}$  denotes a singular arc passing through the points  $\zeta(\tau'_0)$  and  $\zeta(\tau_0)$   
 235 with  $\tau'_0 \leq \tau_0 < \tau^*$ .

236 **DEFINITION 3.4.** *Let  $\tau_0 < \tau^*$ . A point  $x_e := \zeta(\tau_e) \in \Delta_{SA}$  with  $\tau_0 < \tau_e < \tau^*$   
 237 is called a prior-saturation point if the singular arc  $\mathcal{S}_{[\tau_0, \tau]}$  ceases to be optimal for  
 238  $\tau \geq \tau_e$ .*

239 This definition makes sense only for initial conditions  $\zeta(\tau_0)$  with  $\tau_0 < \tau^*$  because  
 240 for  $\tau_0 \geq \tau^*$ , optimal controls are not singular (since the singular control is non-  
 241 admissible). We highlight the dependency of  $x_e$  w.r.t. initial conditions  $\zeta(\tau_0) \in \Delta_{SA}$   
 242 as follows.

243 **PROPOSITION 3.5.** *Suppose that Assumption 3.1 holds true and that there are  
 244  $\tau_1, \tau_2 \in J$  with  $\tau_1 < \tau_2 < \tau^*$  such that any optimal trajectory starting from  $\zeta(\tau_0)$   
 245 with  $\tau_0 \in [\tau_1, \tau_2)$  contains a singular arc  $\mathcal{S}_{[\tau_0, \tau_2]}$ . Then, for any initial condition  
 246  $\tau_0 \in [\tau_1, \tau_2)$ , one has  $x_e = \zeta(\tau_e)$  with*

$$247 \quad (3.5) \quad \tau_e := \sup\{\tau \in J \ ; \ \mathcal{S}_{[\tau_1, \tau]} \text{ is optimal}\} \in [\tau_2, \tau^*).$$

248 *Moreover, for any  $\tau_0 \in [\tau_e, \tau^*]$  an optimal trajectory starting at  $\zeta(\tau_0)$  leaves the sin-  
 249 gular locus at  $\zeta(\tau_0)$ .*

250 *Proof.* Let  $E := \{\tau \in J \ ; \ \mathcal{S}_{[\tau_1, \tau]} \text{ is optimal}\}$  and  $F := \{\tau \in J \ ; \ \mathcal{S}_{[\tau_0, \tau]} \text{ is optimal}\}$   
 251 where  $\tau_0 \in [\tau_1, \tau_2)$  is fixed. Take a point  $\tau \in F$ . Then, from our assumption,  $\mathcal{S}_{[\tau_1, \tau]}$   
 252 is also optimal (by concatenation) which shows that  $\tau \in E$ . On the other hand, if  
 253  $\tau \in E$ , then  $\mathcal{S}_{[\tau_0, \tau]}$  remains optimal (as a sub-arc). It follows that  $E = F$  and, in  
 254 addition, since  $x_e$  is defined as the point such that  $\mathcal{S}_{[\tau_1, \tau]}$  ceases to be optimal, we  
 255 obtain (3.5). From Proposition 3.3, we have  $\tau_e < \tau^*$ .

256 Finally, for every  $\tau_0 \in [\tau_e, \tau^*)$ , a singular arc  $\mathcal{S}_{[\tau_0, \tau'_0]}$  with  $\tau_0 < \tau'_0 < \tau^*$  cannot  
 257 be optimal, since otherwise, this would contradict the definition of  $\tau_e$ . It follows that  
 258 for every  $\tau_0 \in [\tau_e, \tau^*)$ , an optimal path cannot contain a singular arc, implying the  
 259 desired property.  $\square$

260 This property implies in particular that for every initial conditions  $x_0 := \zeta(\tau_0) \in$   
 261  $\Delta_{SA}$  such that  $\tau_0 \in [\tau_1, \tau_2]$ , then the corresponding optimal path has a singular arc  
 262 until the point  $x_e$  and a switching point at this point.

263 **Remark 3.6.** In addition to Assumption 3.1 (in particular (3.2)), if we suppose  
 264 that  $\mathcal{T}$  is not reachable with the constant control  $u = -1$  from those points of  $\Delta_{SA}$   
 265 located between  $x_e$  and  $x^*$  (i.e. corresponding to  $\tau \in [\tau_e, \tau^*]$ ), then the maximal value  
 266 for the control  $u = +1$  is locally optimal from the prior-saturation point  $x_e$ . In other  
 267 words, the *bridge* (the last arc leaving  $\Delta_{SA}$ ) corresponds to  $u = +1$ . This can be  
 268 proved by using similar arguments as for proving Proposition 3.5. Since the singular  
 269 arc is a turnpike, this additional hypothesis also implies the existence of a switching

270 curve emanating from  $x_e$ . Our next aim is precisely to investigate more into details  
 271 geometric properties of optimal paths at the point  $x_e$ .

272 **4. Tangency property and prior-saturation phenomenon.** The aim of this  
 273 section is to prove the tangency property as stated in Theorem 4.6.

274 **4.1. Introduction to prior-saturation lift and tangency property.** In this  
 275 section, we first introduce the concept of *prior-saturation lift* and discuss its local  
 276 uniqueness. We also provide a set of nonlinear equations allowing to compute prior-  
 277 saturation lifts given by the PMP. We end this section with an introduction to the  
 278 tangency property on an example.

279 **DEFINITION 4.1.** *Let  $x_e$  be a prior-saturation point. Any point  $z_e$  in the cotangent*  
 280 *space at  $x_e$  is called a prior-saturation lift of  $x_e$ .*

To introduce the computation of prior-saturation lifts given by the PMP, let us  
 start with an example. Consider a target set  $\mathcal{T} := \{x_f\}$ ,  $x_f \in \mathbb{R}^2$ , with an optimal  
 trajectory of the form  $\sigma_- \sigma_s \sigma_+$ , where  $\sigma_-$ ,  $\sigma_+$  and  $\sigma_s$  are arcs, respectively, with  
 control  $u = -1$ ,  $u = +1$  and  $u = u_s$ , where  $u_s$  is the singular control. Assume that  
 the optimal trajectory is unique and that the switching point between the singular  
 arc  $\sigma_s$  (supposed to be non-empty) and the positive bang arc  $\sigma_+$  is a prior-saturation  
 point. The PMP<sup>3</sup> gives necessary optimality conditions satisfied by this extremal  
 trajectory that we can write as a system of nonlinear equations, the so-called *shooting*  
*equations*. We introduce some notation to define this set of shooting equations. We  
 define the Hamiltonian lifts associated with  $f$  and  $g$  as

$$H_f(z) := p \cdot f(x) ; H_g(z) := p \cdot g(x),$$

281 where  $z := (x, p)$  belongs to the cotangent bundle. All the others Hamiltonian lifts in  
 282 the rest of the paper are defined like this. Define also the Hamiltonians  $H_{\pm} := H_f \pm H_g$   
 283 and  $H_s := H_f + u_s H_g$ , where  $u_s$  is viewed here as a function of  $z$ :

$$284 \quad (4.1) \quad u_s(z) := -\frac{p \cdot [f, [f, g]](x)}{p \cdot [g, [f, g]](x)} = -\frac{H_{[f, [f, g]]}(z)}{H_{[g, [f, g]]}(z)}.$$

285 For any Hamiltonian  $H$  we define the *Hamiltonian system*  $\vec{H} := (\partial_p H, -\partial_x H)$ , and  
 286 finally, we introduce the *exponential mapping*  $\exp(t\varphi)(z_0)$  as the solution at time  $t$  of  
 287 the differential equation  $\dot{z}(s) = \varphi(z(s))$  with initial condition  $z(0) = z_0$ , where  $\varphi$  is  
 288 supposed to be smooth. The shooting equations are then given by

$$289 \quad S(y) = 0, \quad y := (p_0, t_1, t_2, t_f, z_1, z_2) \in \mathbb{R}^{n+3+(2n) \times 2}, \quad n := 2,$$

290 where the *shooting function* is defined by

$$291 \quad (4.2) \quad S(y) := \begin{pmatrix} H_g(z_1) \\ H_{[f, g]}(z_1) \\ H_+(\exp((t_f - t_2)\vec{H}_+)(z_2)) + p^0 \\ \pi(\exp((t_f - t_2)\vec{H}_+)(z_2)) - x_f \\ z_1 - \exp(t_1\vec{H}_-)(x_0, p_0) \\ z_2 - \exp((t_2 - t_1)\vec{H}_s)(z_1) \end{pmatrix},$$

292 where  $\pi(x, p) := x$ , and where  $x_0 \in \mathbb{R}^2$  is given and  $p^0 = -1$  in the normal case.  
 293 The two first equations mean that the trajectory is entering the singular locus at  $z_1$ .

<sup>3</sup>Since  $\mathcal{T} := \{x_f\}$  is a point, there is no transversality condition at the terminal time.

294 Hence, the second arc is a singular arc. The third equation takes into account the  
 295 free terminal time. It could be replaced by  $H_-(x_0, p_0) + p^0 = 0$  since the maximized  
 296 Hamiltonian is constant along the extremal. The fourth equation implies that the last  
 297 bang arc reaches the target  $\mathcal{T} = \{x_f\}$  at the final time  $t_f$ , and the last two equations  
 298 are the so-called *matching conditions* (which are not required but improve numerical  
 299 stability). Given a solution  $y^* := (p_0, t_1, t_2, t_f, z_1, z_2)$  to  $S(y) = 0$  associated with the  
 300 unique optimal trajectory, the point  $\pi(z_2)$  is then a prior-saturation point thanks to  
 301 our hypotheses, and so,  $z_2$  is a prior-saturation lift.

Let us now discuss the uniqueness of the prior-saturation lift, considering for  
 instance, a smooth and local one-parameter family of initial conditions  $x_0(\alpha)$ ,  $\alpha \in$   
 $(-\varepsilon, \varepsilon)$ ,  $\varepsilon > 0$ , in relation with the construction of optimal syntheses (see section 5)  
 and in relation with Proposition 3.5. Let us assume that for any  $\alpha \in (-\varepsilon, \varepsilon)$ , the  
 unique optimal trajectory is of the form  $\sigma_- \sigma_s \sigma_+$  and denote by

$$y^*(\alpha) := (p_0(\alpha), t_1(\alpha), t_2(\alpha), t_f(\alpha), z_1(\alpha), z_2(\alpha))$$

the corresponding solution to the equation

$$S(y, \alpha) = 0,$$

302 where  $S(\cdot, \alpha)$  is defined as (4.2) but with the initial condition  $x_0(\alpha)$  in place of  $x_0$ .  
 303 In addition, suppose that the lengths  $t_1$ ,  $t_2 - t_1$  and  $t_f - t_2$  are positive, that is, each  
 304 arc is defined on a time interval of positive length. In this setting, for any  $\alpha$ , we have  
 305  $x_e := \pi(z_2(0)) = \pi(z_2(\alpha))$ , that is, the prior-saturation point  $x_e$  is locally unique.  
 306 This is related to Proposition 3.5 and illustrated on Figure 1. Besides, whenever the  
 307 prior-saturation lift  $z_e := z_2(0)$  is also locally unique, see Proposition 4.3 and remark  
 308 4.2, we have  $z_2(0) = z_2(\alpha)$  for any  $\alpha \in (-\varepsilon, \varepsilon)$ .

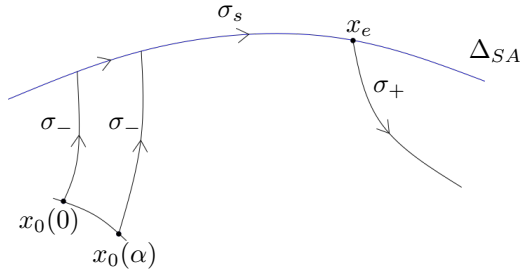


FIG. 1. Local uniqueness of the prior-saturation point  $x_e$ .

309 Assuming that the prior-saturation lift  $z_e$  is locally unique, we can compute it  
 310 with a set of equations excerpt from the shooting equations but with some minor  
 311 modifications. Roughly speaking, the main idea is to consider the particular case  
 312 where the initial condition is the prior-saturation point, that is such that  $x_0 = x_e$ .  
 313 In this case, we have  $t_1 = t_2 = 0$  and  $z_1 = z_2 = (x_e, p_0) = z_e$ . This emphasizes the  
 314 fact that what happens before the prior-saturation lift is useless to compute it. With  
 315 these considerations in mind we introduce

$$316 \quad F_{\text{ex}}(t_b, z_b) := \begin{pmatrix} H_g(\exp(-t_b \overrightarrow{H_+})(z_b)) \\ H_{[f,g]}(\exp(-t_b \overrightarrow{H_+})(z_b)) \\ H_+(z_b) + p^0 \\ \pi(z_b) - x_f \end{pmatrix},$$

317 where  $F_{\text{ex}} : \mathbb{R}^5 \rightarrow \mathbb{R}^5$ , and where we use the notation  $t_b, z_b$  ( $b$  stands for bridge) in  
 318 relation with the concept of *bridge* defined in [9] and detailed in the MRI example  
 319 in Section 5.2. Note that the exponential mapping is here computed by backward  
 320 integration. Hence, with the preceding notation, we have  $z_b = \exp((t_f - t_2)\overrightarrow{H_+})(z_2) =$   
 321  $\exp(t_f\overrightarrow{H_+})(z_e)$  and  $t_b = t_f - t_2 = t_f$ . At the end, the prior-saturation lift is simply  
 322 given by

$$323 \quad z_e = \exp(-t_b\overrightarrow{H_+})(z_b),$$

324 for a couple  $(t_b, z_b)$  solution of  $F_{\text{ex}} = 0$ .

*Tangency property.* We end this section with an introduction to the tangency  
 property. Let us start with solutions of the form  $\sigma_- \sigma_s \sigma_+$ , considering a smooth  
 and local one-parameter family of initial conditions  $x_0(\alpha)$ ,  $\alpha \in (-\varepsilon, \varepsilon)$ ,  $\varepsilon > 0$ , but  
 assuming that for  $\alpha = 0$ , the optimal solution is of the form  $\sigma_- \sigma_s \sigma_+$  with  $\sigma_s$  reduced  
 to a single point, that is,  $t_2(0) - t_1(0) = 0$ , with

$$y^*(0) := (p_0(0), t_1(0), t_2(0), t_f(0), z_1(0), z_2(0)),$$

325 the solution to the associated shooting equations, still denoted  $S(y, \alpha) = 0$ . Assume  
 326 also that for  $\alpha > 0$ , we are in the previous case, that is one has  $t_2(\alpha) - t_1(\alpha) > 0$   
 327 with  $y^*(\alpha) := (p_0(\alpha), t_1(\alpha), t_2(\alpha), t_f(\alpha), z_1(\alpha), z_2(\alpha))$  the corresponding solution of  
 328  $S(\cdot, \alpha) = 0$ . The prior-saturation lift is thus given by  $z_e = z_2(\alpha)$  for  $\alpha \in [0, \varepsilon)$ .

329 The idea is now to consider the case where there is a bifurcation in the structure  
 330 of the optimal trajectories when  $\alpha = 0$ . We thus assume that for  $\alpha \in (-\varepsilon, 0)$ , the  
 331 solutions are of the form  $\sigma_- \sigma_+$  and we denote by  $z_1(\alpha)$  the switching point (in the  
 332 cotangent bundle) between the two arcs. In this setting, there exists a switching locus  
 333 in the optimal synthesis denoted  $\Sigma_- \cup \Sigma_0$ , where

$$334 \quad \Sigma_- := \{z_1(\alpha); \alpha \in (-\varepsilon, 0]\} \quad \text{and} \quad \Sigma_0 := \{z_1(0) = z_e\}.$$

335 The aim of the next section is to prove that the semi-orbit  $\Gamma_+$  of  $\dot{z} = \overrightarrow{H_+}(z)$   
 336 starting from  $z_e$  is tangent to the switching curve  $\Sigma_- \cup \Sigma_0$  at the prior-saturation lift  
 $z_e$  in a general frame. This is precisely the *tangency property* (see Fig. 2).

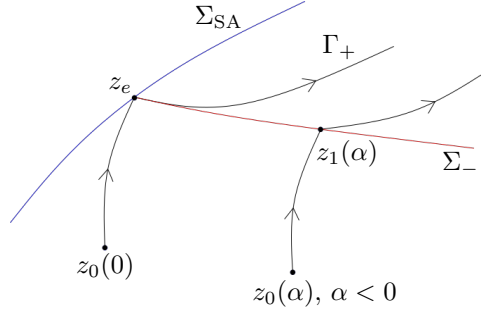


FIG. 2. Illustration of the tangency property between  $\Gamma_+$  and  $\Sigma_- \cup \Sigma_0$  at the prior-saturation lift  $z_e$ . The singular locus in the cotangent bundle is  $\Sigma_{\text{SA}} := \{z \in \mathbb{R}^{2n} ; H_g(z) = H_{[f,g]}(z) = 0\}$ .

337

338 **4.2. Proof of the tangency property.** From a general point of view, we shall  
 339 assume that the prior-saturation lift is given by solving a set of nonlinear equations  
 340 of the following form:

$$341 \quad (4.3) \quad F(t_b, z_b, \lambda) := \begin{pmatrix} H_{[f,g]}(\exp(-t_b\overrightarrow{H_+})(z_b)) \\ G(t_b, z_b, \lambda) \end{pmatrix},$$

342 where  $\lambda \in \mathbb{R}^k$  is a vector of  $k \in \mathbb{N}$  parameters, where  $F$  is a function from  $\mathbb{R}^{5+k}$  to  
 343  $\mathbb{R}^{5+k}$  and where  $G : \mathbb{R}^{5+k} \rightarrow \mathbb{R}^{4+k}$  is defined by

$$344 \quad (4.4) \quad G(t_b, z_b, \lambda) := \begin{pmatrix} H_g(\exp(-t_b \overrightarrow{H_+})(z_b)) \\ H_+(z_b) + p^0 \\ \Psi(z_b, \lambda) \end{pmatrix},$$

345 with  $\Psi : \mathbb{R}^{4+k} \rightarrow \mathbb{R}^{2+k}$  a given function and  $p^0 = -1$  considering the normal case. We  
 346 assume that all the functions  $F$ ,  $G$  and  $\Psi$  are smooth. It is important to notice that  
 347 the mapping  $\Psi$  does not depend on  $t_b$  and that we can replace  $H_+$  by  $H_-$  without any  
 348 loss of generality. In the previous example from section 4.1, we have (with a slight  
 349 abuse of notation)  $\Psi(z_b) = \pi(z_b) - x_f$  which corresponds to the simplest case where  
 350 there are no transversality conditions and no additional parameters, that is  $k = 0$ .  
 351 For a more complex structure of the form  $\sigma_- \sigma_s \sigma_+ \sigma_-$ , the parameter  $\lambda$  would be the  
 352 last switching time between the  $\sigma_+$  and  $\sigma_-$  arcs. In this case,  $\Psi$  would contain the  
 353 additional switching condition  $H_g = 0$  at this time.

354 Let  $(t_b^*, z_b^*, \lambda^*) \in \mathbb{R}^{5+k}$  be a solution to the equation  $F = 0$  and define

$$355 \quad (4.5) \quad z_e := \exp(-t_b^* \overrightarrow{H_+})(z_b^*) \in \Sigma_{\text{SA}} := \{z \in \mathbb{R}^{2n} ; H_g(z) = H_{[f,g]}(z) = 0\}.$$

356 We introduce the following assumptions at the point  $z_e$ .

357 ASSUMPTION 4.1. We have  $u_s(z_e) < 1$  with  $u_s$  the singular control given by (4.1).

358 ASSUMPTION 4.2. The matrix

$$359 \quad \begin{bmatrix} \frac{\partial G}{\partial z_b}(t_b^*, z_b^*, \lambda^*) & \frac{\partial G}{\partial \lambda}(t_b^*, z_b^*, \lambda^*) \end{bmatrix} \in \text{GL}_{4+k}(\mathbb{R}),$$

360 i.e., it is invertible in  $\mathbb{R}^{(4+k) \times (4+k)}$ .

361 Remark 4.2. Assumption 4.1 is related to the prior-saturation phenomenon while  
 362 in combination with Assumption 4.2, it is related to the well-posedness of the shooting  
 363 system  $F = 0$ . Besides, the point  $z_e$  is locally unique under these assumptions,  
 364 according to the following result.

365 PROPOSITION 4.3. Suppose that Assumptions 4.1 and 4.2 hold true. Then,

$$366 \quad F'(t_b^*, z_b^*, \lambda^*) \in \text{GL}_{5+k}(\mathbb{R}).$$

367 Proof. The Jacobian of the mapping  $F$  at the point  $(t_b^*, z_b^*, \lambda^*)$  is given by:

$$368 \quad F'(t_b^*, z_b^*, \lambda^*) = \begin{bmatrix} -a & * & * \\ -b & \frac{\partial G}{\partial z_b}(t_b^*, z_b^*, \lambda^*) & \frac{\partial G}{\partial \lambda}(t_b^*, z_b^*, \lambda^*) \end{bmatrix},$$

369 where  $a := H_{[f,[f,g]]}(z_e) + H_{[g,[f,g]]}(z_e)$  and  $b := (H_{[f,g]}(z_e), 0, 0)$ . Observe now that  
 370  $b = 0$  since  $F(t_b^*, z_b^*, \lambda^*) = 0$  and that  $a \neq 0$  since  $u_s(z_e) < 1$ , which ends the proof.  $\square$

371 LEMMA 4.4. Suppose that Assumption 4.2 holds true. Then, there exists  $\varepsilon > 0$   
 372 and a  $C^1$ -map  $t_b \mapsto \sigma(t_b) := (z_b(t_b), \lambda(t_b)) \in \mathbb{R}^{4+k}$  defined over  $I_\varepsilon := (t_b^* - \varepsilon, t_b^* + \varepsilon)$ ,  
 373 that satisfies

$$374 \quad (4.6) \quad \forall t_b \in I_\varepsilon, \quad G(t_b, \sigma(t_b)) = 0.$$

375 In addition, one has  $\sigma(t_b^*) = (z_b^*, \lambda^*)$  and  $\sigma'(t_b^*) = 0_{\mathbb{R}^{4+k}}$ .

376 *Proof.* The existence of  $\sigma$  follows from the implicit function theorem applied to  
 377 the mapping  $G$  at  $(t_b^*, z_b^*, \lambda^*)$  which also gives (4.6). The derivative of  $\sigma$  is then  
 378 obtained from (4.6):

$$379 \quad \sigma'(t_b) = - \left[ \frac{\partial G}{\partial z_b}[t_b] \quad \frac{\partial G}{\partial \lambda}[t_b] \right]^{-1} \cdot \frac{\partial G}{\partial t_b}[t_b], \quad t_b \in I_\varepsilon,$$

380 where  $[t_b]$  stands for  $(t_b, \sigma(t_b))$ . Since

$$381 \quad \frac{\partial G}{\partial t_b}[t_b^*] = (H_{[f,g]}(z_e), 0_{\mathbb{R}^{3+k}}) = 0_{\mathbb{R}^{4+k}},$$

382 the result follows.  $\square$

383 Let us introduce the mapping  $\varphi(t_b) := \exp(-t_b \overrightarrow{H_+})(z_b(t_b))$  for  $t_b \in I_\varepsilon$  and define

$$384 \quad (4.7) \quad \Sigma := \{\varphi(t_b) ; t_b \in I_\varepsilon\}.$$

385 *Remark 4.5.* The curve  $\Sigma$  is a switching curve in the cotangent bundle since one  
 386 has  $H_g(\varphi(t_b)) = 0$  by definition of  $G$ . However, this switching curve is not necessarily  
 387 optimal, that is, the optimal synthesis, with respect to the initial condition, may not  
 388 contain  $\Sigma$ . Let us stratify  $\Sigma$  according to  $\Sigma = \Sigma_- \cup \Sigma_0 \cup \Sigma_+$ , with

$$389 \quad \begin{aligned} \Sigma_- &:= \{\varphi(t_b) ; t_b \in (t_b^* - \varepsilon, t_b^*)\}, \\ \Sigma_0 &:= \{\varphi(t_b^*)\} = \{z_e\}, \\ \Sigma_+ &:= \{\varphi(t_b) ; t_b \in (t_b^*, t_b^* + \varepsilon)\}. \end{aligned}$$

390 A typical situation is when  $\Sigma_- \cup \Sigma_0$  is contained in the optimal synthesis while  $\Sigma_+$  is  
 391 not optimal for local and/or global optimality reasons. See the end of Section 4.1  
 392 for an example of this typical situation.

393 Our first main result is given by Proposition 3.5 which states the existence of a  
 394 prior-saturation point  $x_e$  in the state space under Assumption 3.1. Our second main  
 395 result is the following.

396 **THEOREM 4.6.** *Suppose the existence of a triple  $(t_b^*, z_b^*, \lambda^*) \in \mathbb{R}^{5+k}$  such that*  
 397  *$F(t_b^*, z_b^*, \lambda^*) = 0$ , with  $F$  defined by (4.3) and set  $z_e := \exp(-t_b^* \overrightarrow{H_+})(z_b^*)$ . Suppose*  
 398 *also that Assumption 4.2 holds true. Then, the switching curve  $\Sigma$  given by (4.7) is*  
 399 *tangent at  $z_e$  to the forward semi-orbit  $\Gamma_+$  of  $\dot{z} = \overrightarrow{H_+}(z)$  starting from  $z_e$ .*

400 *Proof.* From Assumption 4.2 and by lemma 4.4, one can define the switching  
 401 curve  $\Sigma$  by (4.7). To prove the tangency property, we have to show that  $\varphi'(t_b^*)$  is  
 402 collinear to  $\overrightarrow{H_+}(z_e)$ . For any  $t_b \in I_\varepsilon$ , we have

$$403 \quad \varphi'(t_b) = -\overrightarrow{H_+}(\varphi(t_b)) + \Phi(t_b, z_b(t_b)) z_b'(t_b),$$

where  $\Phi(t, z_0)$  is defined as the solution at time  $t$  of the Cauchy problem

$$\dot{X}(s) = A(s, z_0)X(s), \quad X(0) = I_{2n},$$

404 with  $A(s, z_0) := -\overrightarrow{H_+}'(\exp(-s \overrightarrow{H_+})(z_0))$ . By lemma 4.4, one has  $\sigma'(t_b^*) = 0$  thus  
 405  $z_b'(t_b^*) = 0$  and we get  $\varphi'(t_b^*) = -\overrightarrow{H_+}(\varphi(t_b^*)) = -\overrightarrow{H_+}(z_e)$ , which concludes the proof.  $\square$

406 *Remark 4.7.* It is worth to mention that the tangency property is proved in the  
 407 cotangent bundle, and thus it is also true in the state space at a prior saturation point  
 408 (under the assumptions of Proposition 3.3).

409 Setting  $\xi(z) := (H_g(z), H_{[f,g]}(z))$  the singular locus  $\Sigma_{\text{SA}}$  can be written  $\Sigma_{\text{SA}} =$   
 410  $\xi^{-1}(\{0_{\mathbb{R}^2}\})$ , and we have the following relation between the singular locus and the  
 411 switching curve.

412 **COROLLARY 4.8.** *Suppose that  $\xi$  is a submersion at  $z_e$  and that Assumptions 4.1*  
 413 *and 4.2 hold true. Then the switching curve  $\Sigma$  is transverse to the singular locus  $\Sigma_{\text{SA}}$*   
 414 *at  $z_e$ .*

415 *Proof.* Since  $\xi$  is a submersion at  $z_e$ , the singular locus  $\Sigma_{\text{SA}}$  is locally a regular  
 416 submanifold of codimension two near  $z_e$ . Its tangent space at  $z_e$  is given by the kernel  
 417 of the matrix  $\xi'(z_e)$ . But,

$$\begin{aligned} \xi'(z_e) \varphi'(t_b^*) &= -\xi'(z_e) \overrightarrow{H_+}(z_e) \quad (\text{see the proof of Theorem 4.6}) \\ &= - \begin{pmatrix} \partial_x H_g(z_e) & \partial_p H_g(z_e) \\ \partial_x H_{[f,g]}(z_e) & \partial_p H_{[f,g]}(z_e) \end{pmatrix} \begin{pmatrix} \partial_p H_+(z_e) \\ -\partial_x H_+(z_e) \end{pmatrix} \\ &= - \begin{pmatrix} H_{[f,g]}(z_e) \\ H_{[f,[f,g]]}(z_e) + H_{[g,[f,g]]}(z_e) \end{pmatrix} \\ &\neq 0_{\mathbb{R}^2} \quad (\text{by Assumption 4.1}), \end{aligned}$$

419 recalling that  $\varphi$  is given from lemma 4.4 by Assumption 4.2. □

420 *Remark 4.9.* From Theorem 4.6, the tangency property holds even if the singular  
 421 control at  $z_e$  is saturating. The main reason of the tangency property comes from the  
 422 fact that  $z_e$  belongs to the singular locus  $\Sigma_{\text{SA}}$ . However, if the singular control at  $z_e$   
 423 is not saturating, for instance if  $z_e$  is a prior-saturation lift, then the switching curve  
 424  $\Sigma$  is transverse to the singular locus  $\Sigma_{\text{SA}}$  at  $z_e$  according to Corollary 4.8.

425 **5. Illustration of the prior-saturation phenomenon.** The aim of this section  
 426 is to develop two examples arising in the field of bioprocesses and magnetic  
 427 resonance imaging respectively, that will highlight the various concepts introduced in  
 428 Sections 3-4. For the related minimal time problems, we shall also briefly discuss the  
 429 corresponding optimal syntheses that exhibit prior-saturation points and bridges.

430 **5.1. The fed-batch model.** A bioreactor operated in fed-batch is described by  
 431 the controlled dynamics (see [17]):

$$432 \quad (5.1) \quad \begin{cases} \dot{s} &= -\mu(s) \left( \frac{M}{v} + s_{in} - s \right) + \frac{Q_{max}(1+u)}{2v} (s_{in} - s), \\ \dot{v} &= \frac{Q_{max}}{2} (1+u), \end{cases}$$

where  $s_{in}$  and  $s$  denote respectively the input substrate and substrate concentrations,  
 and  $v$  is the volume of the reactor<sup>4</sup>. The parameter  $Q_{max} > 0$  is the maximal speed  
 of the input pump so that  $\frac{Q_{max}}{2}(1+u)$  represents the input flow rate,  $u(\cdot)$  being the  
 control variable with values in  $[-1, 1]$ . Finally,  $M \in \mathbb{R}$  depends on the initial value of  
 micro-organism concentration<sup>5</sup>. As in many engineering applications (see, *e.g.*, [5]),  
 the kinetics  $\mu$  of the reaction is of Haldane type, *i.e.*,

$$\mu(s) := \frac{\mu_h s}{K + s + \frac{s^2}{K_I}},$$

<sup>4</sup>In contrast with the previous sections in which state variables are  $(x_1, x_2)$ , we chose to adopt  
 the notation  $(s, v)$  that is commonly used in the bioprocesses literature for fed-batch operations.

<sup>5</sup>Micro-organism concentration  $X > 0$  can be expressed as a simple function of  $(s, v)$ , namely  
 $X := M/v + s_{in} - s$ , thus (5.1) is enough to describe a bioreactor operated in fed-batch mode.

433 with a unique maximum  $s^* := \sqrt{KK_I} \in (0, s_{in})$  (parameters  $\mu_h, K, K_I$  are positive).  
 434 This type of growth function models inhibition by substrate (microbial growth is  
 435 limited when  $s$  is too large w.r.t.  $s^*$ ). It is worth mentioning that  $\mathcal{D} := (0, s_{in}] \times \mathbb{R}_+^*$   
 436 is invariant by (5.1). For waste water treatment purpose, the problem of interest is:

$$437 \quad (5.2) \quad \inf_{u \in \mathcal{U}} T_u \quad \text{s.t.} \quad (s(T_u), v(T_u)) \in \mathcal{T},$$

438 where  $\mathcal{T} := (0, s_{ref}] \times \{v_{max}\}$  is the target set,  $s_{ref} \ll s_{in}$  is a given threshold, and  
 439  $v_{max} > 0$  denotes the maximal volume of the bioreactor. From a practical point of  
 440 view, the goal is to treat a volume  $v_{max}$  of wasted water in minimal time. For more  
 441 details about this system, we refer to [17, 3].

It appears that Problem (5.2) may exhibit a saturation phenomenon. Indeed, by using the PMP, we can check that there is a singular locus that is the line segment

$$\Delta_{SA} := \{s^*\} \times (0, v_{max}],$$

and that the singular control can be expressed in feedback form as

$$u_s[v] := \frac{\mu(s^*)[M + v(s_{in} - s^*)]}{(s_{in} - s^*)Q_{max}} - 1,$$

(writing  $\dot{s} = 0$  along  $s = s^*$ ). It follows that there exists a unique saturation point

$$x_{sat} := (s^*, v^*),$$

442 with  $v^* := \frac{2Q_{max}}{\mu(s^*)} - \frac{M}{s_{in} - s^*}$  and  $u_s[v^*] = 1$  if the following condition is fulfilled

$$443 \quad (5.3) \quad 0 < v^* < v_{max}.$$

444 This typically happens when  $v_{max}$  (the volume of water to be treated) is too high,  
 445 see [3]. Next, we suppose that (5.3) holds true.

At this step, we wish to know if prior-saturation occurs (according to Propositions 3.3 and 3.5). Doing so, let us check Assumption 3.1. One gets

$$\delta_0(s, v) = -\mu(s)(M/v + s_{in} - s)Q_m/2 = -\mu(s)XQ_m/2 < 0,$$

446 hence  $\Delta_0 \cap \mathcal{D} = \emptyset$  and  $\delta_0 < 0$  in  $\mathcal{D}$ . Now, the singular arc is of turnpike type and  
 447 Legendre-Clebsch's optimality condition holds true because  $\mu$  has a unique maximum  
 448 for  $s = s^*$ , see [1], or a clock form argumentation in [17]. In addition, observe that,  
 449 in the  $(s, v)$ -plane, trajectories of (5.1) with  $u = -1$  are horizontal, hence, every arc  
 450 with  $u = -1$  and starting at a volume value  $v_0 < v_{max}$  never reaches the target set  
 451  $\mathcal{T}$ . Finally,  $\mathcal{T}$  is reachable from  $\mathcal{D}$  taking the control  $u = +1$  until reaching  $v = v_{max}$   
 452 and then  $u = -1$  until reaching  $s_{ref}$ .

Second, let us verify the hypotheses of Proposition 3.5. Doing so, let  $v \mapsto \hat{s}(v)$  be the unique solution to the Cauchy problem

$$\frac{ds}{dv} = - \left( - \frac{\mu(s)}{Q_{max}} \left[ \frac{M}{v} + s_{in} - s \right] + \frac{s_{in} - s}{v} \right), \quad s(v_{max}) = s^*,$$

453 (the solution of (5.1) with  $u = 1$  backward in time from  $(s^*, v_{max})$ ). From [3], if  
 454 there exists  $v_* \in (0, v^*)$  such that  $\hat{s}(v_*) = s^*$ , then optimal paths starting at a volume  
 455 value sufficiently small necessarily contain a singular arc (this actually follows using



456 the PMP). Now, by using Cauchy-Lipschitz's Theorem, the existence of  $v_*$  is easy  
 457 to verify when  $M = 0$ , and thus, it is also verified for small values of the parameter  
 458  $M$  (by a continuity argumentation). To pursue our analysis, we suppose next the  
 459 existence of  $v_* \in (0, v^*)$ . We are then in a position to apply Propositions 3.3 and  
 460 3.5. It follows that there is a unique volume value  $v_e \in (0, v^*)$  such that any singular  
 461 arc starting at a volume value  $v_0 < v_e$  will be optimal only until  $v_e$ . In addition,  
 462 combining this result with a study of extremals using the PMP, we obtain that

- 463 • if the initial condition is  $(s^*, v_0)$  with  $v_0 < v_e$ , then the optimal path is of the  
 464 form  $\sigma_s \sigma_+^b \sigma_-$  (see below for the definition of  $\sigma_+^b$ );
- 465 • if the initial condition is  $(s^*, v_0)$  with  $v_0 \geq v_e$ , then the optimal path is of the  
 466 form  $\sigma_+ \sigma_-$  ;
- 467 • for any initial condition  $(s_{in}, v_0)$  with  $v_e \leq v_0 < v_{max}$ , the optimal path is  
 468 of the form  $\sigma_- \sigma_+ \sigma_-$  where the first switching time appears on a switching  
 469 curve emanating from  $(s^*, v_e)$ .

470 To determine the prior-saturation point  $x_e := (s^*, v_e)$  numerically, we proceed as in  
 471 Section 4. For this application model, it is convenient to introduce an *extended target*  
 472 *set* as  $\overline{\mathcal{T}} := (0, s_{in}] \times \{v_{max}\}$  (observe that for initial conditions on  $\overline{\mathcal{T}}$ , optimal paths  
 473 are  $\sigma_-$  arcs). In this context, a *bridge* is defined as an arc  $\sigma_+$  (denoted by  $\sigma_+^b$ ) on  
 474  $[0, t_b]$  such that

$$475 \quad \phi(0) = \dot{\phi}(0) = \phi(t_b) = 0 \quad \text{and} \quad v(t_b) = v_{max},$$

476 where  $\phi$  is the switching function defined by (2.6) and  $t_b$  is the time to steer  $x_e$  at  
 477 time 0 to the extended target set  $\overline{\mathcal{T}}$  with  $u = +1$ . To compute  $x_e$ , we need to compute  
 478 the extremities of the bridge together with its length. Denoting by  $t_b^*$  the length of  
 479 the bridge and by  $z_b^*$  its extremity in the cotangent bundle whose projection on the  
 480 state space belongs to  $\overline{\mathcal{T}}$ , the point  $(t_b^*, z_b^*)$  is then a solution of the equation  $F_{\text{bio}} = 0$   
 481 with

$$482 \quad (5.4) \quad F_{\text{bio}}(t_b, z_b) := \begin{pmatrix} H_{[f,g]}(\exp(-t_b \overrightarrow{H_+})(z_b)) \\ H_g(\exp(-t_b \overrightarrow{H_+})(z_b)) \\ H_+(z_b) + p^0 \\ H_g(z_b) \\ v_b - v_{max} \end{pmatrix},$$

483 where  $(s_b, v_b)$  is the projection of  $z_b$  on the state space. From Theorem 4.6, the bridge  
 484 is then tangent to the switching curve at  $x_e$  (the projection of  $\Sigma$  given by (4.7) onto  
 485 the state space). To conclude this part, let us comment Fig. 3 on which the optimal  
 486 synthesis is plotted in a neighborhood of the prior-saturation point:

- 487 • In black, the switching curve  $\Sigma^\pi$  emanates from the prior-saturation point.  
 488 It is computed using the shooting functions  $F = 0$  adapted to (5.2).
- 489 • The synthesis is such that trajectories are horizontal ( $u = -1$ ) until reaching  
 490  $\Delta_{SA}$  or the switching curve. For initial conditions with a substrate concen-  
 491 tration less than  $s^*$  and  $v_0 \geq v_e$ , then  $u = 1$  is optimal until reaching  $\overline{\mathcal{T}}$ .

492 **5.2. The MRI model.** In Nuclear Magnetic Resonance (NMR) *saturating* one  
 493 chemical species consists in driving the magnetization vector representing the state  
 494 to zero. In Magnetic Resonance Imaging (MRI) a challenging problem is to maximize  
 495 the *contrast* between two observed species (for instance, healthy tissues and tumors)  
 496 saturating one species. For the model, we consider an ensemble of spin-1/2 particles,  
 497 excited by a radio-frequency (RF) field which is ideally assumed homogeneous, each

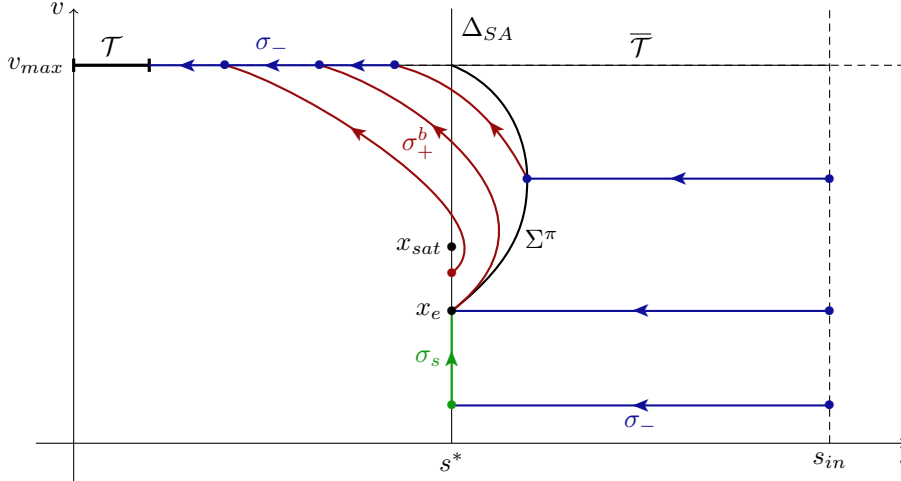


FIG. 3. Minimal time synthesis for (5.2): the target set  $\mathcal{T} = (0, s_{ref}] \times \{v_{max}\}$  is in black (left). The switching curve  $\Sigma^\pi$  (in black) is tangent to the bridge  $\sigma_+^b$  (in red) at  $x_e$ . Arcs with  $u = +1$  (resp.  $u = -1$ ) are depicted in red (resp. in blue).

498 spin of this ensemble being described by its magnetization vector whose dynamics is  
 499 governed in a specific rotating frame, after some normalizations and considering the  
 500 2-dimensional case, by the *Bloch equation* [16]:

$$501 \quad (5.5) \quad \begin{cases} \dot{x}_1 = -\Gamma x_1 - u x_2, \\ \dot{x}_2 = \gamma(1 - x_2) + u x_1, \end{cases}$$

502 where  $x := (x_1, x_2)$  is the normalized magnetization vector, where  $(\gamma, \Gamma)$  is a couple  
 503 of parameters satisfying the physical constraint  $0 < \gamma \leq 2\Gamma$  and depending on the  
 504 longitudinal and transversal relaxation constants specific to the observed species, and  
 505 where  $u$  is the RF-field which plays the role of the control. The time-minimal problem  
 506 of interest here is the following:

$$507 \quad (5.6) \quad \inf_{u \in \mathcal{U}} T_u \quad \text{s.t.} \quad x_u(T_u, x_0) = O := (0, 0),$$

508 where the initial condition  $x_0$  belongs to the set  $B := \{(x_1, x_2) \in \mathbb{R}^2 ; x_1^2 + x_2^2 \leq 1\}$   
 509 called the *Bloch ball* and where  $x_u(\cdot, x_0)$  is the unique solution of (5.5) such that  
 510  $x_u(0, x_0) = x_0$ .

511 *Remark 5.1.* The problem of saturation in MRI is the problem (5.6) with  $x_0 = N$ ,  
 512 where  $N := (0, 1)$  is the North pole of the Bloch ball. We refer to [8, 7] for more details  
 513 about the saturation and contrast problems in MRI. In [8], the following optimal  
 514 synthesis is constructed: the authors give the optimal paths to go from  $N$  to any  
 515 reachable point of the Bloch ball. Hence, the initial point is fixed to the North pole  
 516 while the final point may be seen as a parameter. Here, we are interested in the  
 517 converse problem, that is, the parameter is the initial condition and we want to steer  
 518 the system to a given target, which is the origin  $O$ . The common problem in these  
 519 two cases is the problem of saturation where the initial condition is  $N$  and where the  
 520 target is  $O$ .

521 In this MRI application [8], the singular locus has a singularity at the intersection

522 of two lines. Setting  $\delta := \gamma - \Gamma$ , the singular locus is described by

$$523 \quad \Delta_{SA} = \Delta_{SA}^h \cup \Delta_{SA}^v,$$

where  $\Delta_{SA}^h := \{x_2 = \gamma/(2\delta)\}$  is a horizontal line and where  $\Delta_{SA}^v := \{x_1 = 0\}$  is the vertical axis. On the vertical axis, the singular control is zero while on the horizontal line, the singular control is given in feedback form by

$$u_s[x_1, x_2] := \gamma(2\Gamma - \gamma)/(2\delta x_1).$$

524 Considering only the half space  $x_1 \leq 0$  of the Bloch ball (this is possible due to a  
525 discrete symmetry) and restricting  $(\gamma, \Gamma)$  to the interesting case  $0 < 3\gamma \leq 2\Gamma$  (in this  
526 case, the horizontal line cuts the Bloch ball), there exists only one saturation point  
527 denoted by  $x_{\text{sat}} \in \Delta_{SA}$ . The point  $x_{\text{sat}}$  belongs to the set  $\Delta_{SA}^h \cap \{x_1 < 0\}$ , it satisfies  
528  $u_s(x_{\text{sat}}) = 1$ , and it is given by

$$529 \quad x_{\text{sat}} := \left( \frac{\gamma(2\Gamma - \gamma)}{2\delta}, \frac{\gamma}{2\delta} \right).$$

530 Following [8], we introduce the concept of bridge. An arc  $\sigma_+$  or  $\sigma_-$  with control  
531  $u = +1$  or  $u = -1$ , is called a *bridge* on  $[0, t_b]$  if its extremities correspond to non  
532 ordinary switching points, that is, if

$$533 \quad \phi(0) = \dot{\phi}(0) = \phi(t_b) = \dot{\phi}(t_b) = 0,$$

534 where  $\phi$  is the switching function defined by (2.6). According to [8], there exists  
535 a bridge  $\sigma_+^b$  (supposed to be unique) connecting  $\Delta_{SA}^h$  and  $\Delta_{SA}^v$ . We denote by  
536  $x_e := (x_{e,1}, x_{e,2})$  the extremity of the bridge on the horizontal line  $\Delta_{SA}^h$  and we  
537 can now restrict the analysis to the following situation. We assume that the following  
538 conditions are satisfied by the couple of parameters  $(\gamma, \Gamma)$  (see Fig. 3 of [8] and the  
539 description that comes after for details):

- 540 (i)  $x_e$  belongs to the Bloch ball  $B$  (this implies in particular that  $3\gamma \leq 2\Gamma$ ),
- 541 (ii)  $0 < \gamma$  (this comes from the physical constraint),
- 542 (iii)  $0 \leq (2\Gamma^2 - \gamma\Gamma + 1) \exp((\alpha - \gamma)t_0) - 2\delta$  (hence the origin  $O$  is reachable by a  
543 Bang-Singular sequence from  $x_{\text{sat}}$  and so also from  $x_e$ ),

544 where  $\alpha := \delta/2$  and  $t_0 := \arctan(-\beta/\alpha)/\beta$  with  $\beta := \sqrt{1 - \alpha^2}$ . In this setting, for  
545 any initial condition  $x_0 := (x_{0,1}, x_{0,2}) \in \Delta_{SA}^h \cap B$  such that  $x_{0,1} \leq x_{e,1}$ , the optimal  
546 trajectory (see [8]) is of the form  $\sigma_s \sigma_+^b \sigma_0$ , that is composed of a singular arc on  $\Delta_{SA}^h$   
547 followed by the bridge with  $u = +1$  and ending with a singular arc  $\sigma_0$  along  $\Delta_{SA}^v$   
548 with  $u = 0$ . The first singular arc reduces to a point if  $x_0 = x_e$ . At  $x_e$ , the singular  
549 control is not saturating, so, in conclusion, the point  $x_e$  is a prior-saturation point.

550 *Remark 5.2.* In the MRI application, Assumption 3.1 is not exactly satisfied since  
551 the collinearity set  $\Delta_0$  is non-empty and plays a role in the optimal synthesis, such as  
552 the singularity of the singular locus at the intersection of the two lines. However, the  
553 singular arcs are turnpikes and Legendre-Clebsch optimality condition holds. Besides,  
554 there exists a prior-saturation point and so this case is more general than the fed-  
555 batch application. We will see hereinafter that the tangency property holds at the  
556 prior-saturation point and that the switching curve is transverse to the singular locus.

557 We end this part by showing how to compute the prior-saturation point  $x_e$  and by  
558 giving the optimal synthesis near  $x_e$  for an initial condition on the horizontal singular  
559 line. To compute  $x_e$ , we need to compute the extremities of the bridge together with

560 its length. Denoting by  $t_b^*$  the length of the bridge and by  $z_b^*$  the extremity of the  
 561 bridge in the cotangent bundle whose projection on the state space belongs to  $\Delta_{SA}^v$ ,  
 562 the point  $(t_b^*, z_b^*)$  is then a solution of the equation  $F_{\text{mri}} = 0$  with

$$563 \quad (5.7) \quad F_{\text{mri}}(t_b, z_b) := \begin{pmatrix} H_{[f,g]}(\exp(-t_b \overrightarrow{H_+})(z_b)) \\ H_g(\exp(-t_b \overrightarrow{H_+})(z_b)) \\ H_+(z_b) + p^0 \\ H_{[f,g]}(z_b) \\ H_g(z_b) \end{pmatrix},$$

564 where the vector fields  $f$  and  $g$  are given by (5.5) and where the Hamiltonians, the  
 565 Hamiltonian lifts and the Hamiltonian vector field are defined in Section 4.1. We  
 566 recognize here a function of the form (4.3) without any additional parameter  $\lambda$  and  
 567 so,  $z_e := \exp(-t_b^* \overrightarrow{H_+})(z_b^*)$  is the prior-saturation lift such that  $\pi(z_e) = x_e \in \Delta_{SA}^h$ .  
 568 Finally, the optimal synthesis near  $x_e$  is given on Fig. 4. The optimal solution from  
 569 the initial condition  $x_0 \in \Delta_{SA}^h$  is of the form  $\sigma_s \sigma_+^b \sigma_0$ . The red arc  $\sigma_+^b$  is the bridge  
 570 starting from  $x_e$ , it is a part of the forward semi-orbit  $\Gamma_+$  of  $\dot{z} = \overrightarrow{H_+}(z)$  starting  
 571 from  $z_e$  projected into the state space. The black curve  $\Sigma^\pi$  is the existing part in  
 572 the optimal synthesis of the projection of the switching curve  $\Sigma$  defined by (4.7).  
 573 According to the tangency property from Theorem 4.6, the arc  $\sigma_+^b$  is tangent to  $\Sigma^\pi$   
 574 at the prior-saturation point  $x_e$ . Note also that the switching curve  $\Sigma^\pi$  is transverse  
 575 to the singular locus  $\Delta_{SA}^h$  in accordance with Corollary 4.8.

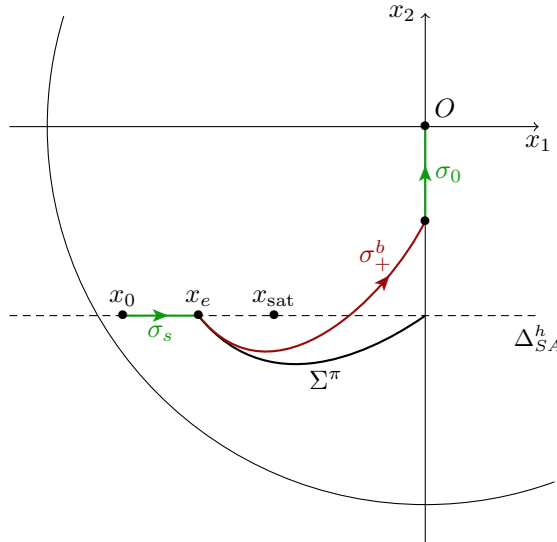


FIG. 4. Optimal synthesis near the prior-saturation point  $x_e$  in the left part of the Bloch ball.

576 **6. Conclusion.** Even though the tangency property between the bridge and the  
 577 switching curve provides useful informations on the minimum time synthesis when  
 578 prior saturation occurs (typically, under assumptions of Proposition 3.3), it remains  
 579 valid in a larger context (under the hypotheses of Theorem 4.6) and not only in the  
 580 framework of saturation and prior-saturation of the singular control for affine-control  
 581 systems in the plane. This property also appears in other settings such as in Lagrange  
 582 control problems governed by one-dimensional systems, see, *e.g.*, [14]. Future works

583 could then investigate prior-saturation phenomenon and the tangency property in  
584 other frameworks or in dimension  $n \geq 3$ .

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