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NON-SINGULAR SPACETIMES WITH A NEGATIVE COSMOLOGICAL CONSTANT: II. STATIC SOLUTIONS OF THE EINSTEIN-MAXWELL EQUATIONS

PIOTR T. CHRUŚCIEL AND ERWANN DELAY

Abstract. We construct infinite-dimensional families of non-singular static space times, solutions of the vacuum Einstein-Maxwell equations with a negative cosmological constant. The families include an infinite-dimensional family of solutions with the usual AdS conformal structure at conformal infinity.

Contents

1. Introduction

A class of spacetimes of interest is that of metrics satisfying the Einstein-Maxwell equations with a negative cosmological constant and admitting a smooth conformal completion at infinity. It is natural to seek for static solutions with this property. In this paper we show that a large class of such fields can be constructed by prescribing the conformal class of a static Lorentzian metric and the asymptotic behaviour of the electric field on the conformal boundary $\partial\mathscr{M}$, provided that the boundary class is sufficiently close to, e.g., that of anti-de Sitter spacetime and the freely prescribable leading coefficient in an asymptotic expansion of the electric potential is sufficiently small. This complements our previous work on vacuum metrics in [7], which we think of as paper I in this series.

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The key new difference of the current result, as compared to [3, 4, 7], is that we can drive the solution with the electric field maintaining, if desired, a conformally flat structure at the conformal boundary at infinity.

We thus seek to construct Lorentzian metrics $\mathbf{g}$, solutions of Einstein-Maxwell equations with a negative cosmological constant Λ, in any space-dimension $n \geq 3$. More precisely, we consider the following field equations for a metric

$$\mathbf{g} = g_{\mu\nu} dx^\mu dx^\nu$$

and a two-form field F in space-time dimension $n+1$, $n \geq 3$ generalising the Einstein-Maxwell equations in dimension $3+1$ as

$$\mathrm{Ric}(\mathbf{g}) - \frac{\mathrm{Tr}\,\mathrm{Ric}(\mathbf{g})}{2}\mathbf{g} + \Lambda \mathbf{g} = F \circ F - \frac{1}{4}\langle F, F\rangle_{\mathbf{g}}\mathbf{g}\,, \tag{1.1}$$

where Tr denotes a trace with the relevant metric (which should be obvious from the context), $(F \circ F)_{\alpha\beta} := g^{\mu\nu} F_{\alpha\mu} F_{\beta\nu}$, and

$$\langle F, F\rangle \equiv g^{\alpha\beta} g^{\mu\nu} F_{\alpha\mu} F_{\beta\nu} =: |F|^2\,.$$

This is complemented with the Maxwell equations

$$\mathrm{div}_{\mathbf{g}} F = 0 = dF\,. \tag{1.2}$$

We further assume existence of a hypersurface-orthogonal globally timelike Killing vector $X = \partial/\partial t$. In adapted coordinates the space-time metric can be written as

$$\mathbf{g} = -V^2 dt^2 + \underbrace{g_{ij} dx^i dx^j}_{=g}\,, \tag{1.3}$$

$$\partial_t V = \partial_t g = 0\,. \tag{1.4}$$

We further assume that the Maxwell field takes the form

$$F = d(U dt)\,, \quad \partial_t U = 0\,. \tag{1.5}$$

Our main result reads as follows (see Section 2 below for the definition of non-degeneracy; the function ρ in (1.7) is a coordinate near the conformal boundary at infinity ∂M that vanishes at ∂M):

THEOREM 1.1. *Let $n = \dim M \geq 3$, $k \in \mathbb{N} \setminus \{0\}$, $\alpha \in (0,1)$, and consider an Einstein metric $\mathring{\mathbf{g}}$ as in (1.3)-(1.4) with strictly positive $V = \mathring{V}$, $g = \mathring{g}$, such that the associated Riemannian metric $\mathring{\mathfrak{g}} = \mathring{V}^2 d\varphi^2 + \mathring{g}$ on $\mathbb{S}^1 \times M$ is C^2-compactifiable and non-degenerate, with smooth conformal infinity. For every smooth $\widehat{U}$, sufficiently close to zero in $C^{k+2,\alpha}(\partial M)$, there exists a unique, modulo diffeomorphisms which are the identity at the boundary, solution of the static Einstein-Maxwell equations of the form (1.3)-(1.5) such that, in local coordinates near the conformal boundary ∂M,*

$$V - \mathring{V} = O(\rho)\,, \quad g_{ij} - \mathring{g}_{ij} = O(1)\,, \tag{1.6}$$

$$U = \widehat{U} + \begin{cases} O(\rho), & n = 3; \\ O(\rho^2 \ln \rho), & n = 4; \\ O(\rho^2), & n \geq 5. \end{cases} \tag{1.7}$$

REMARK 1.2. We have emphasised the freedom to choose the leading-order behaviour $\widehat{U}$ of U. As such, the leading-order behaviour of both $\mathring{V}$ and $\mathring{g}$ is, similarly, freely prescribable near a non-degenerate solution [4, 7]. Here one can, if desired, proceed in two steps: first, find the static solution with new fields $\mathring{V}$ and $\mathring{g}$ near a non-degenerate solution; small such perturbations will preserve non-degeneracy. One can then use Theorem 1.1 at the perturbed static solution to obtain a new $\widehat{U}$. □

The $(n+1)$-dimensional anti-de Sitter metric is non-degenerate in the sense above (see eg. [4] appendix D), so Theorem 1.1 provides in particular an infinite dimensional family of solutions near that metric. We note existence of further large classes of Einstein metrics satisfying the non-degeneracy condition [1, 2, 4, 13].

The requirement of strict positivity of $\mathring{V}$ excludes black hole solutions, we are hoping to return to this question in a near future. In fact, this work was prompted by [12], where static Einstein-Maxwell black holes solutions driven by the asymptotics of the electric field have been discovered numerically.

Similarly to [12], for generic $\widehat{U}$ the resulting space-time metric will have no isometries other than time-translations.

When the free boundary data $\widehat{U}$ are smooth and when $\mathring{\mathfrak{g}}$ is conformally smooth, the solutions constructed here will have a polyhomogeneous expansion at the conformal boundary at infinity. The proof of this is an immediate repetition of the argument presented in [7, Section 7], where the reader can also find the definition of polyhomogeneity. The decay rates in (1.6)-(1.7) have to be compared with the local-coordinates leading-order behavior ρ^{-2} both for $\mathring{V}^2$ and $\mathring{g}_{ij}$. A more precise version of (1.6)-(1.7) in terms of weighted function spaces (we follow the notation in [7, 13]) reads, in all dimensions,

$$(V-\mathring{V}) \in C_1^{k+2,\alpha}(M)\,,\ (g-\mathring{g}) \in C_2^{k+2,\alpha}(M,\mathcal{S}_2)\,, \tag{1.8}$$

$$U-\widehat{U} \in C_1^{k+2,\alpha}(M)\,, \tag{1.9}$$

and the norms of the differences above are small in those spaces. To this one can add, in dimensions $n \geq 4$,

$$U-\widehat{U}-U_{\ln}\,\rho^2\ln\rho \in C_2^{k+2,\alpha}(M)\,, \tag{1.10}$$

with the function $U_{\ln} \in C^\infty(\mathbb{S}^1 \times \overline{M})$ given by (5.4) below when $n=4$, and $U_{\ln} \equiv 0$ in dimension $n \geq 5$. Interestingly enough, in dimension $n=4$ the function $U_{\ln}$ vanishes only when $dU \equiv 0$, hence the space-time is vacuum, see Remark 5.3 below.

The proof of Theorem 1.1 can be found at the end of Section 5. It follows closely [7], and proceeds through an implicit-function argument, with the key isomorphism properties of the associated linearised operator borrowed from [7, 13], and presented in Section 3.

We note that a constant $\widehat{U}$ implies vacuum, by uniqueness of solutions. Hence the energy density of the Maxwell field behaves as ρ^4 for small ρ if not identically zero (cf. (A.14) below), which leads to finite-total-energy configurations in space-dimensions $n=3$ and 4, but infinite matter energy in higher dimensions.

2. Definitions, notations and conventions

Our definitions and conventions are identical to those in [7, Section 2]. Here we simply recall that the Lichnerowicz Laplacian acting on a symmetric two-tensor field is defined as [6, § 1.143]

$$\Delta_L h_{ij} = -\nabla^k\nabla_k h_{ij} + R_{ik}h^k{}_j + R_{jk}h^k{}_i - 2R_{ikjl}h^{kl}\,.$$

The operator $\Delta_L + 2n$ arises naturally when linearising the equation

$$\mathrm{Ric}(\mathfrak{g}) = -n\mathfrak{g}\,, \tag{2.1}$$

where $\mathrm{Ric}(\mathfrak{g})$ is the Ricci curvature of $\mathfrak{g}$, at a solution. We will say that a metric is *non-degenerate* if $\Delta_L + 2n$ has no L^2-kernel.

3. Isomorphism theorems

In this section we recall two isomorphism results from [7, Section 3] and prove an isomorphism theorem on scalars, as needed in the remainder of this work. The reader might also want to consult the introductory comments in [7, Section 3], which remain valid in the current context.

3.1. **An isomorphism on two-tensors.** We will need [7, Corollary 3.2]:

Corollary 3.1. *Let $(\mathbb{S}^1 \times M, V^2 d\varphi^2 + g)$ be an asymptotically hyperbolic non-degenerate Riemannian manifold with $\partial_\varphi V = 0 = \partial_\varphi g$. Consider the map*

$$(W, h) \mapsto (l(W, h), L(W, h))\,,$$

where

$$\begin{aligned} l(W,h) = V\big[&(\nabla^*\nabla + 2n + V^{-1}\nabla^*\nabla V + V^{-2}|dV|^2)W \\ &+V^{-1}\nabla_j V\nabla^j W - V^{-1}\nabla^j V\nabla^k V h_{kj} + \langle \mathrm{Hess}_g V, h\rangle_g\big]\,, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} L_{ij}(W,h) \;=\;& \frac{1}{2}\Delta_L h_{ij} + nh_{ij} - \frac{1}{2}V^{-1}\nabla^k V\nabla_k h_{ij} \\ &+\frac{1}{2}V^{-2}(\nabla_i V\nabla^k V h_{kj} + \nabla_j V\nabla^k V h_{ki}) \\ &-\frac{1}{2}V^{-1}(\nabla_i\nabla^k V h_{kj} + \nabla_j\nabla^k V h_{ki}) \\ &+2V^{-2}W(\mathrm{Hess}_g V)_{ij} - 2V^{-3}\nabla_i V\nabla_j V W\,. \end{aligned} \tag{3.2}$$

Then (l, L) is an isomorphism from $C^{k+2,\alpha}_{\delta-1}(M) \times C^{k+2,\alpha}_{\delta}(M, \mathcal{S}_2)$ to $C^{k,\alpha}_{\delta-2}(M) \times C^{k,\alpha}_{\delta}(M, \mathcal{S}_2)$ when $\delta \in (0, n)$.

3.2. **An isomorphism on one-forms.** We will also need [7, Corollary 3.5]:

Corollary 3.2. *Let $k \in \mathbb{N}$, $\alpha \in (0, 1)$. Under the hypotheses of Corollary 3.1, suppose moreover that the Ricci tensor of $V^2 d\varphi^2 + g$ is negative. Consider the operator*

$$\Omega_i \mapsto B(\Omega)_i + R_{ij}\Omega^j - V^{-1}\nabla_i\nabla^j V\Omega_j =: \mathcal{B}(\Omega)_i\,,$$

where

$$B(\Omega)_i := \nabla^k\nabla_k\Omega_i + V^{-1}\nabla^k V\nabla_k\Omega_i - V^{-2}\nabla_i V\nabla^k V\Omega_k\,.$$

Then $\mathcal{B}$ is an isomorphism from $C^{k+2,\alpha}_\delta(M,\mathcal{T}_1)$ to $C^{k,\alpha}_\delta(M,\mathcal{T}_1)$ when $|\delta-\frac{n}{2}|<\sqrt{\frac{n^2}{4}+1}$.

3.3. **An isomorphism on functions.** If we assume that $V^2d\varphi^2+g$ is a static asymptotically hyperbolic metric on $\mathbb{S}^1\times M$, then it is easy to check that

$$V^{-2}|dV|^2\to 1 \text{ and } V^{-1}\nabla^i\nabla_i V\to n \tag{3.3}$$

as the conformal boundary is approached. We will need an isomorphism property for the following operator acting on functions with $s=-1$; in [7] the result has already been established with $s=-3$ and $s=3$, so for future reference it appears useful to consider all values of s:

$$\sigma\mapsto \mathcal{T}_s\sigma := V^{-s}\nabla^i(V^s\nabla_i\sigma)=\nabla^i\nabla_i\sigma+sV^{-1}\nabla^iV\nabla_i\sigma\,.$$

THEOREM 3.3. *Let (M,g) be an n-dimensional Riemannian manifold with an asymptotically hyperbolic metric with $V>0$ and assume that* (3.3) *holds. Let $s\neq 1-n$ and suppose that*

$$\frac{s+n-1-|s+n-1|}{2}<\delta<\frac{s+n-1+|s+n-1|}{2}\,.$$

If $s\geq -\frac{n-1}{2}$ then $\mathcal{T}_s$ is an isomorphism from $C^{k+2,\alpha}_\delta(M)$ to $C^{k,\alpha}_\delta(M)$. If $s<-\frac{n-1}{2}$, then $\mathcal{T}_s$ is an isomorphism from $C^{k+2,\alpha}_\delta(M)/\mathbb{R}$ to

$$\left\{\sigma\in C^{k,\alpha}_\delta(M):\int_M V^s\sigma=0\right\}. \tag{3.4}$$

Proof. When $s+n-1>0$, we can use [5, Theorem 7.2.1 (ii) and Remark (i), p. 77] to conclude. As we want the result for any $s\neq 1-n$, for the remaining cases we appeal to the results of Lee [13]. For this we need a formally self-adjoint operator, so we set $\sigma=V^{-\frac{s}{2}}f$, thus

$$\mathcal{T}_s\sigma=V^{-\frac{s}{2}}\left[\nabla^i\nabla_i f-\frac{s}{2}\left((\frac{s}{2}-1)V^{-2}|dV|^2+V^{-1}\nabla^i\nabla_iV\right)f\right]=:V^{-\frac{s}{2}}T_sf\,. \tag{3.5}$$

By assumption $V^{-2}|dV|^2\to 1$ and $V^{-1}\nabla^i\nabla_iV\to n$ at the conformal boundary, leading to the following indicial exponents for T_s:

$$\delta=\frac{n-1\pm|s+n-1|}{2}\,.$$

We want to show that T_s satisfies condition (1.4) of [13],

$$\|u\|_{L^2}\leq C\|T_su\|_{L^2}\,, \tag{3.6}$$

for smooth u compactly supported in a sufficiently small open set $\mathcal{U}\subset M$ such that $\overline{\mathcal{U}}$ is a neighborhood of ∂M. Let us recall the following, well known result (see eg. [7, Lemma 3.8]):

LEMMA 3.4. *On an asympotically hyperbolic manifold (M,g) with boundary definining function ρ we have, for all compactly supported C^2 functions,*

$$\int u\nabla^*\nabla u\geq\left(\frac{n-1}{2}\right)^2\int(1+O(\rho))u^2.$$

Lemma 3.4 combined with the hypothesis (3.3) shows that

$$\|u\|_{L^2}\|T_s u\|_{L^2} \geq -\int uT_s u \geq \int \Big(\frac{(n-1)^2}{4} + \frac{s}{2}(\frac{s}{2}+n-1) + o(1)\Big)u^2\,,$$

from which it follows that T_s satisfies indeed (3.6) with

$$C^{-1} = \frac{(s+n-1)^2}{4}\,.$$

We recall that the critical fall-off for a function to belong to L^2 is $O(\rho^{\frac{n-1}{2}})$. This can be used to show that the L^2-kernel of T_s equals

$$\ker T_s = V^{\frac{s}{2}}\mathbb{R} \cap L^2 = \begin{cases} \{0\} \text{ if } s \geq -\frac{n-1}{2}\,, \\ V^{\frac{s}{2}}\mathbb{R} \text{ if } s < -\frac{n-1}{2}\,. \end{cases}$$

Indeed, assume that f is in the L^2-kernel of T_s. By elliptic regularity (see [13, Lemma 4.8] for instance) f is H^2 on M. This implies in particular that no boundary terms arise in the following integration by parts:

$$\begin{aligned} 0 &= -\int fT_s f = -\int fV^{-\frac{s}{2}}\nabla^i[V^s\nabla_i(V^{-\frac{s}{2}}f)] \\ &= \int V^s|\nabla(V^{-\frac{s}{2}}f)|^2. \end{aligned}$$

So $V^{-\frac{s}{2}}f$ is a constant. Using [13], Theorem C(c), we find:

- If $s \geq \frac{n-1}{2}$ then the kernel (and then the cokernel) of T_s is trivial so T_s is an isomorphism from $C^{k+2,\alpha}_{\delta-\frac{s}{2}}(M)$ to $C^{k,\alpha}_{\delta-\frac{s}{2}}(M)$.
- If $s < -\frac{n-1}{2}$, then T_s is an isomorphism from $C^{k+2,\alpha}_{\delta-\frac{s}{2}}(M)/V^{s/2}\mathbb{R}$ to the orthogonal to the kernel:

$$\Big\{f \in C^{k,\alpha}_{\delta-\frac{s}{2}}(M) : \int_M V^{s/2}f = 0\Big\}\,.$$

The conclusion follows for $\mathcal{T}_s$ if we recall that $\sigma = V^{-\frac{s}{2}}f$ is in $C^{k,\alpha}_{\delta}(M)$ iff $f \in C^{k,\alpha}_{\delta-\frac{s}{2}}(M)$. □

4. The equations

Rescaling the metric to achieve a convenient normalisation of the cosmological constant,

$$\Lambda = -\frac{n(n-1)}{2}\,, \tag{4.1}$$

the vacuum Einstein-Maxwell equations for a metric satisfying (1.3)-(1.4) read (see Appendix A)

$$\begin{cases} V(\nabla^*\nabla V + nV) = -\frac{n-2}{n-1}|dU|^2_g\,, \\ \mathrm{Ric}(g) + ng - V^{-1}\,\mathrm{Hess}_g V = V^{-2}\left(-dU\otimes dU + \frac{1}{n-1}|dU|^2_g g\right)\,, \\ \mathrm{div}_g(V^{-1}\nabla U) = 0\,. \end{cases} \tag{4.2}$$

4.1. **The linearised equation.** We use the symbol Tr, or Tr_g when the metric could be ambiguous, to denote the trace. As in [7] we set

$$\mathrm{grav}\, h = h - \frac{1}{2}\,\mathrm{Tr}_g\, hg,\ \ (\mathrm{div}\, h)_i = -\nabla^k h_{ik},\ \ (\mathrm{div}^*\, w)_{ij} = \frac{1}{2}(\nabla_i w_j + \nabla_j w_i)\,,$$

(note the geometers' convention, in which we have a negative sign in the definition of divergence). We consider the operator from the set of functions times symmetric two tensor fields to itself, defined as

$$\left(\begin{array}{c} V \\ g \end{array}\right) \mapsto \left(\begin{array}{c} V(\nabla^*\nabla V + nV) \\ \mathrm{Ric}(g) + ng - V^{-1}\,\mathrm{Hess}_g\, V \end{array}\right).$$

(To avoid ambiguities, $\nabla^*\nabla \equiv -\nabla^i\nabla_i =: -\Delta$.) The two components of its linearisation at (V, g) are

$$\begin{aligned} p(W,h) &= V\Big[(\nabla^*\nabla + 2n + V^{-1}\nabla^*\nabla V)W + \langle \mathrm{Hess}_g\, V, h\rangle_g \\ &\quad -\langle \mathrm{div}\,\mathrm{grav}\, h, dV\rangle_g\Big]\,, \\ P_{ij}(W,h) &= \frac{1}{2}\Delta_L h_{ij} + nh_{ij} + \frac{1}{2}V^{-1}\nabla^k V(\nabla_i h_{kj} + \nabla_j h_{kj} - \nabla_k h_{ij}) \\ &\quad -(\mathrm{div}^*\,\mathrm{div}\,\mathrm{grav}\, h)_{ij} + V^{-2}W(\mathrm{Hess}_g\, V)_{ij} - V^{-1}(\mathrm{Hess}_g\, W)_{ij}\,. \end{aligned}$$

It turns out to be convenient to introduce the one-form

$$w_j = V^{-1}\nabla^k V h_{kj} + \nabla^k h_{kj} - \frac{1}{2}\nabla_j(\mathrm{Tr}\, h) - V^{-1}\nabla_j W - V^{-2}\nabla_j V W\,,$$

which allows us to rewrite $P(W,h)$ as

$$P(W,h) \;=\; L(W,h) + \mathrm{div}^*\, w\,,$$

where L is as in Corollary 3.1. Similarly, $p(W,h)$ can be rewritten as

$$p(W,h) \;=\; l(W,h) + V\langle w, dV\rangle_g\,,$$

with l given by (3.1).

4.2. **The modified equation.** In Section 5 we will use the implicit function theorem to construct our solutions, using the observation of [10], how to obtain a well-behaved equation by adding "gauge fixing terms". We choose those terms as in [7], appealing to harmonic maps for the vacuum Einstein equations in one dimension higher.

Indeed, we will be solving the following system of equations

$$\left\{\begin{array}{l} V(\nabla^*\nabla V + nV + \langle \Omega, dV\rangle) + \frac{n-2}{n-1}|dU|_g^2 = 0\,, \\ \mathrm{Ric}(g) + ng - V^{-1}\,\mathrm{Hess}_g\, V + \mathrm{div}^*\,\Omega \\ \qquad\qquad -V^{-2}\left(-dU\otimes dU + \frac{1}{n-1}|dU|_g^2 g\right) = 0\,, \\ \mathrm{div}_g(V^{-1}\nabla U) = 0\,, \end{array}\right. \tag{4.3}$$

where

$$\begin{aligned} -\Omega_j &\equiv -\Omega(V, g, \mathring{V}, \mathring{g})_j \\ &:= \mathfrak{g}_{j\mu}\mathfrak{g}^{\alpha\beta}(\Gamma(\mathfrak{g})^\mu_{\alpha\beta} - \Gamma(\mathring{\mathfrak{g}})^\mu_{\alpha\beta}) \\ &= g^{\ell m}(\mathring{\nabla}_m g_{j\ell} - \frac{1}{2}\mathring{\nabla}_j g_{\ell m}) + V^{-2}g_{jk}(\mathring{V}\mathring{\nabla}^k\mathring{V} - V\nabla^k V)\,, \end{aligned} \tag{4.4}$$

where $\mathring{\nabla}$-derivatives are relative to the metric $\mathring{g}$, with Christoffel symbols $\mathring{\Gamma}^{\alpha}_{\beta\gamma}$, latin indices run from 0 to n, and $\mathfrak{g} := V^2(dx^0)^2 + g$ with Christoffel symbols $\Gamma(\mathfrak{g})^{\alpha}_{\beta\gamma}$, while the $\Gamma(\mathring{\mathfrak{g}})^{\alpha}_{\beta\gamma}$'s are the Christoffel symbols of the metric $\mathring{\mathfrak{g}}$.

The derivative of Ω with respect to (V, g) at $(\mathring{V}, \mathring{g})$ is

$$D_{(V,g)}\Omega(\mathring{V}, \mathring{g}, \mathring{V}, \mathring{g})(W, h) = -w,$$

where w is the one-form defined in Section 4.1 with (V, g) replaced with $(\mathring{V}, \mathring{g})$. Thus, the linearisation of (q, Q) at $(\mathring{V}, \mathring{g})$ is

$$D(q, Q)(\mathring{V}, \mathring{g}) = (l, L)\,,$$

where (l, L) is the operator defined in Section 4.1 with (V, g) replaced with $(\mathring{V}, \mathring{g})$. We will show that, under reasonable conditions, solutions of (4.3) are solutions of (4.2)).

PROPOSITION 4.1. *If (U, V, g) solves (4.3) then Ω is in the kernel of $\mathcal{B}$. If moreover $\Omega \in C^{2,\alpha}_{\delta}$ for some $\delta > n/2 - \sqrt{1 + n^2/4}$, then $\Omega \equiv 0$, so that (U, V, g) solves (4.2).*

Proof. A standard adaptation of the argument in [7] gives the result; we sketch this for completeness. Set

$$a := -\tfrac{n-2}{n-1}V^{-2}|dU|^2\,,$$
$$A := V^{-2}(-dU \otimes dU + \tfrac{1}{n-1}|dU|^2 g)\,.$$

With this notation, the first two equations in (4.3) take the form

$$\begin{cases} \nabla^*\nabla V + nV + \langle \Omega, dV\rangle = Va\,, \\ \mathrm{Ric}(g) + ng - V^{-1}\,\mathrm{Hess}_g V + \mathrm{div}^*\,\Omega = A\,, \end{cases} \tag{4.5}$$

The tensor field $E(g) = \mathrm{grav}_g \mathrm{Ric}(g)$, has vanishing divergence, which provides the equation $\mathcal{B}(\Omega) = 0$ for Ω. Indeed, the operator $\mathcal{B}(\Omega)$ defined above has been constructed in [7] so that it vanishes when the modified equation for the metric holds and when the Lorentzian $(n+1)$-dimensional divergence of the $(n + 1)$-dimensional energy-momentum tensor vanishes. The latter condition is satisfied when the $(n + 1)$-dimensional matter field equations hold; in the current case, this coincides with the last equation in (4.3).

An alternative, n-dimensional argument proceeds as follows: The calculation following [7, Equation (4.6)] shows that for a solution to the modified equation, the divergence of $E(g)$ equals

$$0 \equiv \mathrm{div}\, E(g) = \frac{1}{2}\mathcal{B}(\Omega) + \beta\,,$$

where

$$\beta := V^{-1}d(Va) - V^{-1}A(\nabla V, .) + \mathrm{div}(\mathrm{grav}_g A + \frac{a}{2}g)\,. \tag{4.6}$$

The vanishing of β can be checked by a somewhat lengthy calculation using (4.3).

Either way, the Bianchi identity $\mathrm{div}\, E(g) = 0$ shows that Ω is in the kernel of $\mathcal{B}$. It follows from Corollary 3.2 that the only solution of this equation which decays as described is zero. □

For future reference we consider the static equations, modified as in (4.5), with a general energy-momentum tensor:

$$(4.7)\ -V^{-1}\nabla^i\nabla_i V - \frac{2\Lambda}{n-1} + V^{-1}\nabla^i\Omega\nabla_i V = \underbrace{-T_{\alpha\beta}N^\alpha N^\beta - \frac{g^{\alpha\beta}T_{\alpha\beta}}{n-1}}_{=:a},$$

$$(4.8)\ R_{ij} - V^{-1}\nabla_i\nabla_j V - \frac{2\Lambda}{n-1}g_{ij} + \frac{1}{2}(\nabla_i\Omega_j + \nabla_j\Omega_i) = \underbrace{T_{ij} - \frac{g^{\alpha\beta}T_{\alpha\beta}}{n-1}g_{ij}}_{=:A_{ij}},$$

where $N^\alpha\partial_\alpha$ is the unit timelike normal to the initial data surface (compare (A.8)-(A.9), Appendix A). We have:

PROPOSITION 4.2. *Let $T_{\mu\nu}$ be divergence-free with respect to the time-independent metric*

$$\mathbf{g} = -V^2dt^2 + g$$

as a consequence of the matter field equations, with $\partial_t T_{\mu\nu} = T_{0i} = 0$. Set

$$(4.9)\qquad a := -T_{\alpha\beta}N^\alpha N^\beta - \frac{g^{\alpha\beta}T_{\alpha\beta}}{n-1}, \quad A_{ij} := T_{ij} - \frac{g^{\alpha\beta}T_{\alpha\beta}}{n-1}g_{ij}.$$

Then

$$(4.10)\qquad \beta \equiv 0.$$

Proof. Equation (4.6) can be rewritten as

$$\begin{aligned}(4.11)\ -\beta_j &= \nabla_i\big(A^i{}_j - \frac{1}{2}(A^k{}_k + a)\delta^i_j\big) + V^{-1}\big(A_{ij} - ag_{ij}\big)\nabla V^i \\ &= \nabla_i\Big(T^i{}_j - \frac{1}{2}(\frac{2g^{\alpha\beta}T_{\alpha\beta}}{n-1} + A^k{}_k + a)\delta^i_j\Big) \\ &\quad + V^{-1}\Big(T_{ij} - (\frac{g^{\alpha\beta}T_{\alpha\beta}}{n-1} - T_{\alpha\beta}N^\alpha N^\beta - \frac{g^{\alpha\beta}T_{\alpha\beta}}{n-1})g_{ij}\Big)\nabla V^i.\end{aligned}$$

We have

$$\begin{aligned}&\frac{2g^{\alpha\beta}T_{\alpha\beta}}{n-1} + A^k{}_k + a \\ &\quad = \frac{2g^{\alpha\beta}T_{\alpha\beta}}{n-1} + g^{ij}T_{ij} - \frac{n}{n-1}g^{\alpha\beta}T_{\alpha\beta} - T_{\alpha\beta}N^\alpha N^\beta - \frac{g^{\alpha\beta}T_{\alpha\beta}}{n-1} = 0.\end{aligned}$$

The $(n+1)$-decomposition of the equation $\nabla^\mu T_{\mu\nu} = 0$, where ∇_μ is the space-time covariant derivative associated with $\mathbf{g}$, using the formulae for the Christoffel symbols in Appendix A gives

$$(4.12)\qquad \nabla^i T_{ij} + VT_{\alpha\beta}N^\alpha N^\beta\nabla_j V + V^{-1}\nabla^j V T_{ij} = 0,$$

where ∇^i is the covariant derivative operator of g. Comparing with (4.11) gives the result. □

5. The construction

We consider an asymptotically hyperbolic Einstein static metric $\mathring{\mathfrak{g}} = \mathring{V}^2 d\varphi^2 + \mathring{g}$ on $\mathbb{S}^1 \times M$. We apply Theorem 3.3 with $s = -1$, g — a Riemannian metric on M close to $\mathring{g}$ in $C^{k+2,\alpha}_0(M, \mathcal{S}_2)$, and V — a function on M close to $\mathring{V}$ in $C^{k+2,\alpha}_{-1}(M)$. It is convenient to choose some

$$\delta \in (0,1) \text{ when } n = 3 \text{ and } \delta = 1 \text{ if } n > 3. \tag{5.1}$$

We conclude that for any $\widehat{U} \in C^{k+2,\alpha}(\partial M)$, there exists a unique solution

$$U = U(\widehat{U}, V, g) \in C^{k+2,\alpha}_0(M)$$

to

$$\left\{ \begin{array}{l} \nabla^*(V^{-1}\nabla U) = 0, \\ U - \widehat{U} \in C^{k+2,\alpha}_\delta(M)\,. \end{array} \right.$$

Moreover, the map $(\widehat{U}, V, g) \mapsto U - \widehat{U} \in C^{k+2,\alpha}_\delta(M)$ is smooth.

We define a new map F, defined on the set of functions on the conformal boundary at infinity $\partial_\infty M$ times functions on M times symmetric two-tensor fields on M, mapping to functions on M times symmetric two-tensor fields on M, which to $(\widehat{U}, V, g)$ associates

$$\left(\begin{array}{c} V(\nabla^*\nabla V + nV + \langle \Omega(V, g, \mathring{V}, \mathring{g}), dV\rangle) + \frac{n-2}{n-1}|dU|^2 \\ \mathrm{Ric}(g) + ng - V^{-1}\nabla_i\nabla_j V + \mathrm{div}^*\,\Omega(V, g, \mathring{V}, \mathring{g}) + V^{-2}(dU \otimes dU - \frac{1}{n-1}|dU|^2 g) \end{array} \right).$$

Proposition 5.1. *Let $\mathring{\mathfrak{g}} = \mathring{V}^2 d\varphi^2 + \mathring{g}$ be an asymptotically hyperbolic static Einstein metric on $\mathbb{S}^1 \times M$, $k \in \mathbb{N}$, $\alpha \in (0,1)$. The map $\mathcal{F}$ defined as*

$$\begin{array}{ccc} C^{k+2,\alpha}(\partial M) \times C^{k+2,\alpha}_1(M) \times C^{k+2,\alpha}_2(M, \mathcal{S}_2) & \longrightarrow & C^{k,\alpha}_0(M) \times C^{k,\alpha}_2(M, \mathcal{S}_2) \\ (\widehat{U}, W, h) & \longmapsto & F(\widehat{U}, \mathring{V} + W, \mathring{g} + h) \end{array}$$

is smooth in a neighborhood of zero.

Proof. Let δ be as in (5.1). We have, by direct estimations,

$$V^{-2}(dU \otimes dU - \frac{1}{n-1}|dU|^2 g) \in C^{k,\alpha}_{2+2\delta}(M, \mathcal{S}_2) \subset C^{k,\alpha}_2(M, \mathcal{S}_2)\,.$$

The remaining arguments of the proof of [7, Proposition 5.2] apply with trivial modifications: The function $\mathring{V} \in C^{k+2,\alpha}_{-1}(M)$ is strictly positive, so the same is true for $\mathring{V} + W$ if W is sufficiently small in $C^{k+2,\alpha}_1(M) \subset C^{k+2,\alpha}_{-1}(M)$. Similarly, the symmetric two-tensor field $\mathring{g} + h \in C^{k+2,\alpha}_0(M, \mathcal{S}_2)$ is positive definite when h is small in $C^{k+2,\alpha}_2(M, \mathcal{S}_2) \subset C^{k+2,\alpha}_0(M, \mathcal{S}_2)$. The map $(\widehat{U}, V, g) \mapsto U$ is smooth. The fact that the remaining terms in $F(\widehat{U}, \mathring{V} + W, \mathring{g} + h)$ are in the space claimed, and that the map is smooth, follows from standard calculations (see [11, proof of Theorem 4.1] or [9, Appendix] for associated detailed calculations). □

We are ready to formulate now:

Theorem 5.2. *Let $\dim M = n \geq 3$ and let $\mathring{V}^2 d\varphi^2 + \mathring{g}$ be a non-degenerate asymptotically hyperbolic static Einstein metric on $\mathbb{S}^1 \times M$, $k \in \mathbb{N}$, $\alpha \in (0,1)$,*

$\delta \in (0,1)$ in $n=3$ and $\delta = 1$ if $n>3$. For all $\widehat{U}$ close to zero in $C^{k+2,\alpha}(\partial M)$ there exists a unique solution

$$(U,V,g) = (\widehat{U}+u, \mathring{V}+W, \mathring{g}+h)$$

to (4.2) *with*

$$(u,W,h) \in C^{k+2,\alpha}_{\delta}(M) \times C^{k+2,\alpha}_{1}(M) \times C^{k+2,\alpha}_{2}(M, \mathcal{S}_2)\,,$$

close to zero, satisfying the gauge condition $\Omega = 0$. Moreover, the map $\widehat{U} \mapsto (u,W,h)$ is a smooth map of Banach spaces near zero.

Proof. As already pointed out, the function $U = U(\widehat{U}, V, g)$ exists and is unique when W and h are small. From Proposition 5.1 we know that the map $\mathcal{F}$ is smooth. The linearisation of $\mathcal{F}$ at zero is

$$D_{(W,h)}\mathcal{F}(0,0,0) = D_{(V,g)}F(0,\mathring{V},\mathring{g}) = (l,L)\,.$$

From Corollary 3.1, with $\delta = 2$, we obtain that $D_{(W,h)}\mathcal{F}(0,0,0)$ is an isomorphism. The implicit function theorem shows that the conclusion of Theorem 5.2 remains valid *for the modified equation* (4.3). Returning to Section 4.2, we see that $\Omega = \Omega(V,g,\mathring{V},\mathring{g}) \in C^{k+1,\alpha}_2(M,T_1)$. Corollary 3.2 gives $\Omega = 0$, and we have obtained a solution to (4.2). □

Proof of Theorem 1.1: Existence and uniqueness of a solution with

$$V - \mathring{V} = O(\rho)\,, \quad U = \widehat{U} + O(\rho^{\delta}), \quad g - \mathring{g} = O_{\mathring{g}}(\rho^2) = O_{\rho^2\mathring{g}}(1)\,, \tag{5.2}$$

follows from Theorem 5.2. Here and elsewhere we write $u = O(\rho^\sigma)$ for a tensor u if the coordinate components of u in local coordinates near the boundary are $O(\rho^\sigma)$, and $u = O_{\mathring{g}}(\rho^\sigma)$ if the norm $|u|_{\mathring{g}}$ of u with respect to the metric $\mathring{g}$ is $O(\rho^\sigma)$. Standard arguments show that all the fields V, U and g are polyhomogeneous (cf. e.g. [8], compare [7, Section 7]).

To justify (1.7), we plug-in a polyhomogeneous asymptotics as in (5.2) in the equations. Since the trace-free part of the Ricci tensor of $\mathbf{g} = -V^2dt^2 + g$ decays in $\mathring{\mathfrak{g}}$-norm as ρ^4 near $\rho = 0$, the metric $\mathbf{g} = -V^2dt^2 + g$ is vacuum up to this order, which implies that the usual Fefferman-Graham expansion holds this metric up to terms $O_{\mathring{\mathfrak{g}}}(\rho^4 \ln \rho)$. This leads to the following form of the metric near the conformal boundary

$$\mathring{g} = \rho^{-2}(d\rho^2 + \mathring{h})\,, \quad \mathring{h} = \breve{h} + O_{\breve{h}}(\rho^2)\,, \quad \mathring{V} = \breve{V}\rho^{-1} + O(\rho)\,, \tag{5.3}$$

where $\mathring{h}$ is a family of Riemannian metrics on the boundary depending upon ρ, with

$$g = \mathring{g} + O_{\mathring{g}}(\rho^2)\,, \quad V = \mathring{V} + O(\rho)\,.$$

The equation $\nabla_i(V^{-1}\nabla^i U) = 0$ in local coordinates (ρ, x^A) as in (5.3) takes the form

$$\begin{aligned}
&\breve{V}^{-1}\sqrt{\det \breve{h}}\;\partial_\rho\left[\rho^{3-n}(1+O(\rho^2))\partial_\rho U\right] \\
&= -\rho^{3-n}\partial_A\left[\left(\breve{V}^{-1}\sqrt{\det\breve{h}} + O(\rho^2)\right)(\breve{h}^{AB} + O^{AB}(\rho^2))\partial_B U\right] \\
&\qquad + \partial_\rho\left(\rho^{5-n}O^A(1)\partial_A U\right) + \partial_A\left(\rho^{5-n}O^A(1)\partial_\rho U\right).
\end{aligned}$$

Let U a polyhomogenous solution with $U = \widehat{U} + O(\rho^\delta)$. When $n = 3$, we can see that no logarithmic term $O(\rho \ln \rho)$ is needed in U so that $U = \widehat{U} + O(\rho)$. When $n = 4$, we find

$$U = \widehat{U} \underbrace{-\frac{1}{2}\breve{V}\breve{\nabla}_A(\breve{V}^{-1}\breve{\nabla}^A\widehat{U})}_{=:U_{\ln}} \rho^2 \ln \rho + O(\rho^2)\,. \tag{5.4}$$

(In dimensions $n \geq 5$ on expects in general a logarithmic term $O(\rho^{n-2} \ln \rho)$ in an asymptotic expansion of U.) □

Remark 5.3. The equality

$$2\int_{\partial M} \breve{V}^{-1}\widehat{U}U_{\ln} = \int_{\partial M} \breve{V}^{-1}|\breve{\nabla}\widehat{U}|^2,$$

proves that $U_{\ln} = 0$ if and only if $\widehat{U}$ is constant. When $\widehat{U}$ is constant the function $U := \widehat{U}$ solves the equations and satisfies the right-boundary values, so the associated solution is vacuum.

Appendix A. Static Einstein-Maxwell equations in dimensions $n + 1 \geq 4$

Consider a Riemannian manifold (M, g), let us denote by ∇ the Levi-Civita derivative operator associated with g. Let $V : M \to \mathbb{R}$, and set

$$(\mathscr{M} = \mathbb{R} \times M\,,\, \widetilde{g} = \varepsilon V^2 dt^2 + g)\,, \quad \varepsilon = \pm 1\,,$$

and $(x^\mu) = (x^0 = t, x^i = (x^1, \ldots, x^n))$. We have

$$\widetilde{\Gamma}^0_{00} = \widetilde{\Gamma}^0_{ij} = \widetilde{\Gamma}^k_{i0} = 0,\ \ \widetilde{\Gamma}^0_{i0} = V^{-1}\partial_i V,\ \ \widetilde{\Gamma}^k_{00} = -\varepsilon V \nabla^k V,\ \ \widetilde{\Gamma}^k_{ij} = \Gamma^k_{ij}\,, \tag{A.1}$$

$$\widetilde{R}^l{}_{ijk} = R^l{}_{ijk},\ \ \widetilde{R}^l{}_{0j0} = -\varepsilon V \nabla_j \nabla^l V,\ \ \widetilde{R}^0{}_{ij0} = V^{-1}\nabla_j\nabla_i V\,, \tag{A.2}$$

$$\widetilde{R}^0{}_{ijk} = \widetilde{R}^l{}_{ij0} = \widetilde{R}^l{}_{0jk} = \widetilde{R}^0{}_{0jk} = \widetilde{R}^0{}_{0j0} = 0\,, \tag{A.3}$$

$$\widetilde{R}_{ik} = R_{ik} - V^{-1}\nabla_k\nabla_i V\,,\ \ \widetilde{R}_{0k} = 0\,,\ \ \widetilde{R}_{00} = -\varepsilon V \nabla^i \nabla_i V\,, \tag{A.4}$$

$$\widetilde{R} = R - 2V^{-1}\nabla^i\nabla_i V\,. \tag{A.5}$$

Consider the $(n + 1)$-dimensional Einstein equations

$$\tilde{R}_{\alpha\beta} - \frac{1}{2}\tilde{R}\tilde{g}_{\alpha\beta} + \Lambda\tilde{g}_{\alpha\beta} = T_{\alpha\beta}\,. \tag{A.6}$$

Equations (A.6) with $\tilde{g} = -V^2 dt^2 + g$ lead to

$$\widetilde{R}_{\alpha\beta} = \frac{2\Lambda - \mathrm{Tr}_{\tilde{g}}\, T}{n-1}\widetilde{g}_{\alpha\beta} + T_{\alpha\beta}\,, \tag{A.7}$$

$$R_{ij} = V^{-1}\nabla_i\nabla_j V + \frac{2\Lambda}{n-1}g_{ij} + T_{ij} - \frac{\mathrm{Tr}_{\tilde{g}}\, T}{n-1}g_{ij}\,, \tag{A.8}$$

$$V\nabla^i\nabla_i V = V^2\left(T_{\alpha\beta}N^\alpha N^\beta + \frac{\mathrm{Tr}_{\tilde{g}}\, T - 2\Lambda}{n-1}\right)\,, \tag{A.9}$$

where $N^\alpha\partial_\alpha$ is the unit timelike normal to the level sets of t.

For a static electric field $F = d(Udt)$, $\partial_t U = 0$, and $T_{\mu\nu}$ as in (1.1) we find

$$|F|^2 = -2V^{-2}|dU|^2\,, \tag{A.10}$$

$$\tilde{g}^{\mu\nu}T_{\mu\nu} \equiv \mathrm{Tr}_{\tilde{g}}\, T = \frac{(3-n)}{4}|F|^2 = -\frac{(3-n)}{2}V^{-2}|dU|^2\,, \tag{A.11}$$

$$T_{ij} = -V^{-2}\nabla_i U\nabla_j U + \frac{1}{2}V^{-2}|dU|^2 g_{ij}\,, \tag{A.12}$$

$$g^{ij}T_{ij} = \frac{(n-2)}{2}V^{-2}|dU|^2 \tag{A.13}$$

$$T_{\widehat{0}\widehat{0}} := T_{\mu\nu}N^\mu N^\nu = \frac{1}{2}V^{-2}|dU|^2\,. \tag{A.14}$$

Choosing Λ as in (4.1), Equation (A.9) gives

$$V\Delta V = \frac{n-2}{(n-1)}|dU|^2 - \frac{2\Lambda V^2}{n-1} = \frac{n-2}{(n-1)}|dU|^2 + nV^2\,. \tag{A.15}$$

When $\widetilde{g} = \varepsilon V^2 dt^2 + g$ is static and

$$T = T_{00}(dx^0)^2 + T_{ij}dx^i dx^j\,, \quad \partial_0 T_{00} = \partial_0 T_{ij} = 0\,, \tag{A.16}$$

then

$$\tilde{\nabla}^\alpha T_{\alpha\beta}dx^\beta = \big(\nabla^i T_{ij} + V^{-1}\nabla^i V T_{ij} - \varepsilon V^{-3}\nabla_j V T_{00}\big)dx^j\,. \tag{A.17}$$

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