Hilbert manifold structure for asymptotically hyperbolic relativistic initial data

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HILBERT MANIFOLD STRUCTURE FOR ASYMPTOTICALLY HYPERBOLIC RELATIVISTIC INITIAL DATA

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Abstract. We provide a Hilbert manifold structure à la Bartnik for the space of asymptotically hyperbolic initial data for the vacuum constraint equations. The adaptation led us to prove new weighted Poincaré and Korn type inequalities for AH manifolds with inner boundary and weakly regular metric.

Contents

Keywords : Hilbert manifold, asymptotically hyperbolic, elliptic operators , general relativity, constraint equations, weak regularity.

1. Introduction

This work follows on from a paper of 2005 [4] in which Bartnik described a Hilbert manifold structure for the space of asymptotically flat solutions of the Einstein equations (see also [17], [18], [19]). The work is done with rather weak regularity assumptions concerning the metric involved (curvature constant only modulo weighted L^2 terms) and so can be related to the context of the bounded L^2 curvature conjecture of KRS [20]. Actually Bartnik showed in [3] that these assumptions on the regularity are the weakest possible to

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define the ADM mass of the manifold, explaining why we chose the same regularity conditions. In an undergoing work ([10], [11]), we are showing that these very assumptions allow us to properly define the mass of an asymptotically hyperbolic manifold compatible with previous definitions of the mass (see [7] , [13] or [8]) and with the Hilbert manifold structure exposed here. In order to overcome difficulties arising in the asymptotically hyperbolic case, we had to create a Hessian-type operator $\mathring{T}$ and a differential operator of order two, called $\mathring{U}$, built up with the first derivatives of the Killing operator $\mathring{S}$. In particular, we obtain Poincaré and Korn-type estimates of second order on an asymptotically hyperbolic manifold with boundary. These estimates are the key to prove triviality of the adjoint's kernel.

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2. Notations and conventions

Let $(\mathcal{M}, g)$ be a Riemannian manifold. We define $T^r_m(\mathcal{M})$ to be the bundle of tensor covariant of rank m and contravariant of rank r . For all $u \in T^r_m(\mathcal{M})$, $|u|_g$ will denote the norm of u with respect to the metric g and notation $|u|_{g,x}$ allows us to precise the point of the manifold we consider. $d\mu(g)$ is the Riemannian measure determined by g. $Riem\, g, Ric\, g$ et $R(g)$ are respectively the Riemann tensor, the Ricci tensor and the scalar curvature of the metric g. For a Riemannian metric g with connection ∇, we set the following notations concerning the Hessian and Laplacian of a function u.

$$\begin{aligned} \nabla^2_{ij} u &= \nabla_i \nabla_j u \\ \Delta u &= tr_g \nabla^2 u = g^{ij} \nabla^2_{ij} u \end{aligned}$$

w.r.t is the abbreviation of "with respect to".
The work presented in this article is done on a n-dimensional manifold as often as possible and, otherwise explicitely stated, the results presented here are valid in any dimension n. Nevertheless, Sobolev inequalities strongly constrain the dimension to $n = 3$ in several proofs ; this is clearly specified when this is the case. We chose to leave n and specify $n = 3$ in the concerned results rather than replace n by its specific value since it helps to understand where the dimension play a role.
Concerning constants in norm inequalities, the constant C will design in general a constant depending on the background metric $\mathring{g}$ and the decaying rate δ and its expression may change from line to line in a proof. The nature of the dependence of the constant C will be systematically specified because it will depend on other parameters.

2.1. **Conformally compact manifold.**

Let $(\mathcal{M}, \mathring{g})$ be a $\mathscr{C}^\infty$ n-dimensional complete non compact Riemannian manifold. The manifold $(\mathcal{M}, \mathring{g})$ is conformally compact if there exists a Riemannian metric $\mathring{h}$ such that $\mathring{g} = \rho^{-2}\mathring{h}$, where $(\mathcal{M}, \mathring{h})$ is a $\mathscr{C}^\infty$ compact Riemannian manifold with boundary $\partial\mathcal{M}$ and ρ is a function on $\bar{\mathcal{M}}$, called defining function on $\partial\mathcal{M}$:

- $\rho \in \mathscr{C}^\infty(\bar{\mathcal{M}})$
- $\rho \geqslant 0$ on $\bar{\mathcal{M}}$

- $\partial\mathcal{M} = \{x \in \mathcal{M} : \rho(x) = 0\}$
- $d\rho|_{\partial\mathcal{M}} \neq 0$

We call asymptotically hyperbolic metric on $\mathcal{M}$ every Riemannian metric $\mathring{g}$ verifying:

- $\mathring{g}$ is conformally compact.
- $|d\rho|^2_{\mathring{h},\partial\mathcal{M}} = 1$.

The existence of these two metrics on $\mathcal{M}$ can be source of confusion thereafter, so we have to fix some notations. $\mathring{\nabla}, \mathring{\Delta}, \mathring{\Gamma}_{ij}^{k}$, $|u|_{\mathring{g}}$ (resp. ${}^{(\mathring{h})}\nabla, \Delta_{\mathring{h}}$, ${}^{(\mathring{h})}\Gamma_{ij}^{k}$, $|u|_{\mathring{h}}$) will respectively denote the connection, Laplace operator, Christoffel symbols and tensor norm w.r.t $\mathring{g}$ (resp. $\mathring{h}$). There are some correspondances between these quantities.
For volume measures, in dimension n, clearly $d\mu(\mathring{g}) = \rho^{-n} d\mu(\mathring{h})$.
For Christoffel symbols, $\mathring{\Gamma}_{ij}^{k} = {}^{(\mathring{h})}\Gamma_{i\,j}^{k} - \frac{1}{\rho}\left(\delta_j^k\partial_i\rho + \delta_i^k\partial_j\rho - \mathring{h}^{kl}\mathring{h}_{ij}\partial_l\rho\right)$.

In particular, for the Hessian of ρ (e.g. equation (C.12) from [5])

$$\mathring{\nabla}^2_{ij}\rho = {}^{(\mathring{h})}\mathring{\nabla}^2_{ij}\rho + \rho\left(2\frac{\partial_i\rho}{\rho}\frac{\partial_j\rho}{\rho} - \mathring{g}_{ij}|d\rho|^2_{\mathring{h}}\right). \tag{1}$$

Taking the trace of (1), we obtain the expression of the Laplace operator of ρ

$$\mathring{\Delta}\rho = \rho^2\Delta_{\mathring{h}}\rho - (n-2)\rho|d\rho|^2_{\mathring{h}}. \tag{2}$$

More generally,

$$\forall u \in \mathscr{C}^\infty_c(\mathcal{M}) \ , \ \mathring{\Delta}u = \rho^2\left(\Delta_{\mathring{h}}u - \frac{n-2}{\rho}\, d\rho \cdot_{\mathring{h}} du\right), \tag{3}$$

where $\cdot_{\mathring{h}}$ is the scalar product w.r.t the metric $\mathring{h}$.
For tensor norms, for all $u \in T^r_m(\mathcal{M})$, $|u|_{\mathring{g}} = \rho^{m-r}|u|_{\mathring{h}}$.

Definition 1. *Cut-off function:*
$(\mathcal{M}, \mathring{g})$ is a conformally compact manifold with defining function ρ.
Let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth cut-off function such that:

- $\chi(\mathbb{R}) \subset [0,1]$
- *supp* $\chi \subset (-\infty, 2]$
- $\chi_{|(-\infty,1]} = 1$

then for R large enough, we can define a cut-off function on $\mathcal{M}$ by

$$\chi_R(x) = \chi(-\ln(\rho(x))/R).$$

Setting $\Omega_R = \{x \in \mathcal{M} : \rho(x) \geqslant e^{-2R}\}$, then χ_R verifies

$$\chi_R \ = \ \begin{cases} 1 \ sur\, \Omega_{R/2} \\ 0 \ sur\, \mathcal{M} \setminus \Omega_R \end{cases}$$

In other words, χ_R is a cut-off function near the boundary of $\mathcal{M}$.

2.2. **Sobolev and Hölder weighted spaces.**
Thanks to the background metric $\mathring{g}$, we can define the following norm

$$\forall\ 1 \leqslant p < \infty \ \text{ et } \ \delta \in \mathbb{R} \ : \quad ||u||_{p,\delta} = \left(\int_{\mathcal{M}} |u|^p_{\mathring{g}}\ \rho^{p\delta} d\mu(\mathring{g}) \right)^{1/p}$$

$$\text{For } p = \infty \ \text{ et } \ \delta \in \mathbb{R} \ : \quad ||u||_{\infty,\delta} = \sup_{\mathcal{M}} \left(\rho^\delta |u|_{\mathring{g}} \right)$$

The Lebesgue weighted space L^p_δ is now define as the space of measurable functions of $L^p_{\text{loc}}(\mathbb{R}^n)$ whose norm mentionned above is finite. The Sobolev weighted space $W^{k,p}_\delta$ is then the space of measurable functions of $W^{k,p}_{\text{loc}}$ whose following norm is finite:

$$||u||_{k,p,\delta} = \sum_{|\alpha| \leqslant k} ||\mathring{\nabla}^\alpha u||_{p,\delta},$$

where α is a multi-index of size n and $\mathring{\nabla}^\alpha u = \mathring{\nabla}^{\alpha_1}_{i_1} \dots \mathring{\nabla}^{\alpha_n}_{i_n} u$,

$$\alpha = (\alpha_1, \dots, \alpha_n) \ \text{ et } \ |\alpha| = \sum_{i=1}^n \alpha_i.$$

NB: for $\delta = 0$, we get back to norms of the classic Lebesgue et Sobolev spaces.

$W^{k,p}_\delta(T^r_m\mathcal{M})$ will refer to Sobolev spaces of sections of the (r,m)-tensor bundle over $\mathcal{M}$. We will speak of $W^{k,p}_\delta$- norm to indicate the norm of a function $u \in W^{k,p}_\delta(T^r_m\mathcal{M})$. For a domain $U \subset \mathcal{M}$, $||u||_{p,\delta;U}$ will be the restriction to U of the $W^{k,p}_\delta$- norm of u. The Hölder weighted space $C^{s,\alpha}_\delta(\mathcal{M}, g)$, with $0 < \alpha < 1$ is endowed with the norm

$$||u||_{C^{s,\alpha}_\delta} = \max_{|k| \leqslant s} ||\mathring{\nabla}^k u||_{C^{0,\alpha}_\delta}$$

with

$$||u||_{C^{0,\alpha}_\delta} = \sup_{x \in \mathcal{M}} \rho^\delta |u|_{\mathring{g}} + \sup_{x \in \mathcal{M}} \rho^\delta \left(\sup_{d_{\mathring{g}}(x,y) \leqslant 1} \frac{|\widetilde{u}(x) - \widetilde{u}(y)|_{\widetilde{g}}}{d_{\mathring{g}}(x,y)^\alpha} \right),$$

where $\widetilde{u}$ and $\widetilde{g}$ represente tensors u et g in an appropriate orthonormal basis.

2.3. **Elliptic operators.**
Here we recall classic results on elliptic operators that we may find in [1] for example. Let B_1 and B_2 be two tensor bundles over a conformally compact manifold $(\mathcal{M}, g)$ with defining function ρ and $A : \mathscr{C}^\infty(B_1) \to \mathscr{C}^\infty(B_2)$ be a partial differential linear operator of order m define by

$$A = \sum_{|\alpha| \leqslant m} a_\alpha \mathring{\nabla}^\alpha \tag{4}$$

Set $s \in \mathbb{N}$. We say operator A of the form (4) has symbol in $\mathcal{OP}^m_s$ if

$$a_\alpha \in C^{s_\alpha}_{-|\alpha|} L(B_1, B_2), \quad \text{with } \ s_\alpha = \max(s, |\alpha| - m + 1).$$

We say A is an elliptic operator if $a_\alpha \xi^\alpha : B_1 \to B_2$ is a tensor bundles isomorphism, for all α such that $|\alpha| = m$ and for all $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n} \neq 0$ and if on top of that, for all ξ^α, there exists two constants c_1 and c_2 such that

$$||a_\alpha \xi^\alpha||_g < c_1\ |\xi^\alpha|_g \qquad \text{and} \qquad ||(a_\alpha \xi^\alpha)^{-1}||_g < c_2\ |\xi^\alpha|_g.$$

Lemma 1. *Set $s_0 \in \mathbb{N}$. For an elliptic operator A with symbol in $\mathcal{OP}^m_{s_0}$, there exists a positive constant $c = c\,(\mathring{g}, \delta)$ such that the following inequality is valid for all $s \leqslant s_0$:*

$$||u||_{2,s+m,\delta} \leqslant c\,\left(||Au||_{2,s,\delta} + ||u||_{2,0,\delta}\right). \tag{5}$$

Theorem 1. *For all $R > 0$, let Ω_R be like in Definition 1. Given $\delta \in \mathbb{R}$ and A an elliptic operator with symbol in $\mathcal{OP}^m_0$. Suppose there exists R large enough so that A verify*

$$\forall u \in \mathscr{C}^\infty_c(\mathcal{M} \setminus \Omega_R) \ , \ ||u||_{2,\delta} \leqslant C\,||Au||_{2,\delta}, \tag{6}$$

where C depend on A.
Then we can choose R large enough so that the following inequality

$$||u||_{m,2,\delta} \leqslant C\,\left(||Au||_{2,\delta} + ||u||_{2,\delta;\Omega_R}\right) \tag{7}$$

is valid for all $u \in \mathscr{C}^\infty_c(\mathcal{M})$. In particular, $A : W^{m,2}_\delta \to L^2_\delta$ is semi-Fredholm, i.e. *A has finite dimensional kernel and closed range.*

3. Preliminary analysis

In this section, we introduce some useful inequalities (see [1] , th 2.3 for example):

Proposition 1. *Weighted Hölder inequalities (in any dimension):*

- *Set $\delta \in \mathbb{R}$ so that $\delta = \delta_1 + \delta_2$, let $p, q, r \in \mathbb{N}$ be such that*

$$1 \leqslant p \leqslant q \leqslant r \leqslant \infty \ \text{ and } \ \frac{1}{p} = \frac{1}{q} + \frac{1}{r}\,,$$

then

$$||uv||_{p,\delta} \leqslant ||u||_{q,\delta_1}\,||v||_{r,\delta_2}. \tag{8}$$

- *Set $\delta \in \mathbb{R}$ and $\lambda \in [0,1]$, let $p, q, r \in \mathbb{N}$ be such that*

$$1 \leqslant p \leqslant q \leqslant r \leqslant \infty \ \text{ et } \ \frac{1}{p} = \frac{\lambda}{q} + \frac{1-\lambda}{r}\,,$$

then

$$||u||_{p,\delta} \leqslant ||u||^\lambda_{q,\delta}\,||u||^{1-\lambda}_{r,\delta}. \tag{9}$$

Theorem 2. *Let $(\mathcal{M}, g)$ be a conformally compact n-dimensional manifold.*

Weighted Sobolev inclusion:
For all $1 \leqslant p \leqslant q < \infty$, for all $k \geqslant k'$, if $\delta \leqslant \delta' - \frac{n}{p}\frac{(q-p)}{q}$, then $W^{k,q}_\delta \subset W^{k',p}_{\delta'}$,
i.e. *there exists a positive constant $c = c\,(\mathring{g}, \delta, \delta', k, k', n, p, q)$ such that*

$$||u||_{k',p,\delta'} \leqslant c\,||u||_{k,q,\delta}.$$

If $q = \infty$, for all $1 \leqslant p \leqslant \infty$; for all $k \geqslant k'$, if $\delta \leqslant \delta' - \frac{n}{p}$, then $W^{k,\infty}_\delta \subset W^{k',p}_{\delta'}$,
i.e. *there exists a positive constant $c = c\,(\mathring{g}, \delta, \delta', k, k', n, p)$ such that*

$$||u||_{k',p,\delta'} \leqslant c\,||u||_{k,\infty,\delta}.$$

Weighted Hölder inclusion:
Set $k \in \mathbb{N}$, $n = 3$ et $p = 2$, then for all $0 < \alpha \leqslant \frac{1}{2}$, for all $\delta \leqslant \delta'$, there exists a positive constant $c = c\,(\mathring{g}, \delta, \delta', k, k', \alpha)$ such that

$$\forall u \in W^{2,2}_\delta(\mathcal{M}) \ , \ ||u||_{C^{0,\alpha}_{\delta'}} \leqslant c\,||u||_{2,2,\delta}. \tag{10}$$

Weighted Sobolev inequalities:
Set $1 \leqslant p < \infty$ and let k, j be integers. If $u \in W^{j+k,p}_{\delta}(\mathcal{M})$, then in each of the following cases, there exists a positive constant $c = c\,(\mathring{g}, \delta, k, n, j, p, q)$ such that

- *If $pk < n$, $||u||_{j,q,\delta} \leqslant c\,||u||_{j+k,p,\delta}$, $\forall\, p \leqslant q \leqslant \frac{np}{n-kp}$.*
- *If $pk = n$, $||u||_{j,q,\delta} \leqslant c\,||u||_{j+k,p,\delta}$, $\forall\, p \leqslant q < \infty$.*
- *If $pk > n$, $||u||_{j,q,\delta} \leqslant c\,||u||_{j+k,p,\delta}$, $\forall\, p \leqslant q \leqslant \infty$.*

Ehrling inequality:
For all $\varepsilon > 0$, for all integers j, k such that $0 < j < k$, there exists a positive constant $C(\varepsilon)$ such that

$$\forall u \in W^{k,p}_{\delta} \quad , \quad ||u||_{j,p,\delta} \leqslant \varepsilon ||u||_{k,p,\delta} + C(\varepsilon)||u||_{p,\delta}. \tag{11}$$

Rellich Theorem:
For all $k > k'$ and $\delta < \delta'$, the inclusion $W^{k,2}_{\delta} \subset W^{k',2}_{\delta'}$ is compact.

A consequence of the Sobolev inequalities (cf. [5] for example) is

Proposition 2. *In dimension $n = 3$, for all $k > \frac{3}{2}$,*

$$u \in W^{k,2}_{\delta} \Rightarrow u = o(\rho^{-\delta}). \tag{12}$$

Lemma 2. *In dimension 3, for all $\delta \in \mathbb{R}$, there exists a positive constant $c = c\,(\mathring{g}, \delta_1, \delta_2)$ such that*

$$||uv||_{2,\delta} \leqslant c\,||u||_{1,2,\delta_1}\,||v||_{1,2,\delta_2} \quad , \quad \textit{with } \delta_1 + \delta_2 = \delta. \tag{13}$$

Remark: In the particular case where δ_1, δ_2 and δ are non positive, then δ_1 and δ_2 are both superior to δ. The weighted Sobolev inclusion leads to

$$||uv||_{2,\delta} \leqslant c\,||u||_{1,2,\delta}\,||v||_{1,2,\delta}. \tag{14}$$

Lemma 3. *For each of the following inequalities, in dimension 3, there exists a positive constant $c = c\,(\mathring{g}, \delta)$ such that*

$$\forall u \in W^{2,2}_{\delta}(\mathcal{M}) \quad , \quad ||u||_{\infty,\delta} \quad \leqslant \quad \varepsilon\,||u||_{2,2,\delta} + c\varepsilon^{-3}||u||_{1,2,\delta}. \tag{15}$$

$$\forall u \in W^{1,2}_{\delta}(\mathcal{M}) \quad , \quad ||u||_{3,\delta} \quad \leqslant \quad \varepsilon\,||u||_{1,2,\delta} + c\varepsilon^{-1}||u||_{2,\delta}. \tag{16}$$

Lemma 4. *In all dimension n, we define for R real large enough,*

$$E_R := \mathcal{M} \setminus \Omega_R = \{\rho \leqslant e^{-2R}\}.$$

Then $\forall u \in L^p_{\delta}(E_R)$, $\forall\, \delta \in \mathbb{R}$,

$$||u||_{p,\eta;E_R} \leqslant e^{2R(\delta-\eta)}||u||_{p,\delta;E_R} \quad , \forall\, \delta \leqslant \eta\,, \forall\, 1 \leqslant p \leqslant \infty. \tag{17}$$

4. The Hessian type operator $\mathring{T}$

Here we give some preliminary results concerning the operator $\mathring{T}$ defined for a function N by:

$$\mathring{T} = \mathring{T}(N) = \mathring{\nabla}^2 N - N\mathring{g}. \tag{18}$$

Lemma 5. *For all $\delta \in I\!R$, there exists a positive constant $c > 0$ depending on $\mathring{g}$ such that for all $N \in W^{2,2}_{-\delta}(\mathcal{M})$,*

$$||\mathring{T}(N)||_{2,-\delta} \geqslant ||\mathring{\nabla}^2 N||_{2,-\delta} - c\,||N||_{2,-\delta}. \tag{19}$$

Proof: The result stems from the definition of $\mathring{T}$ and the Triangle inequality. □

Lemma 6. *For all $\delta \in]-(n+1)/2\,,0]$, there exists a positive constant $c = c\,(\mathring{g}, \delta)$ telle such that for all $N \in W^{2,2}_{-\delta}(\mathcal{M})$,*

$$||N||_{1,2,-\delta} \leqslant c\,||\mathring{T}(N)||_{2,-\delta}. \tag{20}$$

Proof: By density, we can suppose $N \in \mathscr{C}^\infty_c(\mathcal{M})$. We use the proof of Proposition 4 which establishes (20) if the support of N is in a neighborhood of the boundary at infinity. Here we ignore the δ restriction due to positivity of the interior boundary term since N is compactly supported. We obtain the result near the boundary for $\delta \in]-(n+1)/2\,;0]$ and conclude with kernel triviality of $\mathring{T}$ for $\delta < \frac{n+1}{2}$ (see [6]) thanks to a proof similar to the one of Theorem 1. □

Combination of Lemmas 5 and 6 gives

Proposition 3. *For all $\delta \in]-(n+1)/2\,,0]$, there exists a positive constant $c = c\,(\mathring{g}, \delta)$ such that*

$$||N||_{2,2,-\delta} \leqslant c\,||\mathring{T}(N)||_{2,-\delta}. \tag{21}$$

We will need the next three lemmas which give general equalities on $(\mathcal{M}, \mathring{g})$, a n-dimensional asymptotically hyperbolic Riemannian manifold with $\mathring{g} = \rho^{-2}\mathring{h}$. From now on, $d\sigma(\mathring{g})$ will be the measure induced by $\mathring{g}$ on the boundary $\partial\mathcal{M}$ and η is the exterior unit normal to $\partial\mathcal{M}$ in $\mathcal{M}$.

Lemma 7. *Let $(M, \mathring{g})$ be a n-dimensional asymptotically hyperbolic manifold and $N \in \mathscr{C}^\infty(\mathcal{M})$ with a compact support near $\partial\mathcal{M}$. $\forall \delta \in I\!R$,*

$$\int_{\mathcal{M}} 2N\langle dN, \frac{d\rho}{\rho}\rangle_{\mathring{g}}\,\rho^{2\delta}\,d\mu(\mathring{g}) = -\int_{\mathcal{M}} [2\delta+1-n+o(1)]N^2\rho^{2\delta}\,d\mu(\mathring{g}) + \int_{\partial\mathcal{M}} N^2\langle\frac{d\rho}{\rho}, \eta\rangle_{\mathring{g}}\,\rho^{2\delta}\,d\sigma(\mathring{g})\,,$$

Proof: Integration by parts gives

$$\begin{aligned}\int_{\mathcal{M}} \mathring{\nabla}_i(-N^2\mathring{\nabla}^i(\rho^{-1})\rho^{2\delta+1})\,d\mu(\mathring{g}) \;&=\; \int_{\mathcal{M}} 2N\langle\, dN, \frac{d\rho}{\rho}\rangle_{\mathring{g}}\,\rho^{2\delta}\,d\mu(\mathring{g}) - \int_{\mathcal{M}} N^2\mathring{\Delta}(\rho^{-1})\rho^{2\delta+1}\,d\mu(\mathring{g}) \\ &\quad + \int_{\mathcal{M}} (2\delta+1)N^2|d\rho|^2_{\mathring{h}}\rho^{2\delta}\,d\mu(\mathring{g}).\end{aligned} \tag{22}$$

Let us compute (see [5], (D.4) for instance)

$$\mathring{\nabla}^2(\rho^{-1}) = \rho^{-1}|d\rho|^2_{\mathring{h}}\mathring{g} - \rho^{-2\,(\mathring{h})}\nabla^2\rho. \tag{23}$$

Taking the $\mathring{g}$-trace,

$$\mathring{\Delta}(\rho^{-1}) = n\rho^{-1}|d\rho|^2_{\mathring{h}} - \Delta_{\mathring{h}}\rho.$$

The metric $\mathring{h}$ being defined (and so bounded) on $\partial\mathcal{M}$ and ρ being a smooth function on $\mathcal{M}$, $\Delta_{\mathring{h}}\rho$ is a smooth function bounded on $\mathcal{M}$ and so, we can write $\Delta_{\mathring{h}}\rho = O(1) = o(\rho^{-1})$ near $\partial\mathcal{M}$. We obtain

$$\mathring{\Delta}(\rho^{-1}) = \rho^{-1}\big(n|d\rho|^2_{\mathring{h}} + o(1)\big). \tag{24}$$

According to $|d\rho|^2_{\mathring{h}} = 1+o(1)$ near the boundary on an asymptotically hyperbolic manifold, (22) become

$$\begin{aligned}\int_{\mathcal{M}} \mathring{\nabla}_i(-N^2\mathring{\nabla}^i(\rho^{-1})\rho^{2\delta+1})\, d\mu(\mathring{g}) &= \int_{\mathcal{M}} 2N\langle dN, \frac{d\rho}{\rho}\rangle_{\mathring{g}}\, \rho^{2\delta}\, d\mu(\mathring{g}) \\ &\quad + \int_{\mathcal{M}} [2\delta+1-n+o(1)]N^2\rho^{2\delta}\, d\mu(\mathring{g}). \end{aligned}\tag{25}$$

From the Divergence theorem,

$$\int_{\mathcal{M}} \mathring{\nabla}_i(-N^2\mathring{\nabla}^i(\rho^{-1})\rho^{2\delta+1})\, d\mu(\mathring{g}) = \int_{\partial\mathcal{M}} N^2\langle\frac{d\rho}{\rho}, \eta\rangle_{\mathring{g}}\, \rho^{2\delta}\, d\sigma(\mathring{g}). \tag{26}$$

We end the proof replacing the left-hand side of (25) by its expression (26). □

Lemma 8. *Let $(M, \mathring{g})$ be a n-dimensional asymptotically hyperbolic manifold and $N \in \mathscr{C}^\infty(\mathcal{M})$ with a compact support near $\partial\mathcal{M}$. $\forall\delta \in I\!R$,*

$$\begin{aligned}-2\int_{\mathcal{M}} \mathring{T}(dN, \frac{d\rho}{\rho})\rho^{2\delta}\, d\mu(\mathring{g}) &= \int_{\mathcal{M}} \{2\delta+1-n+o(1)\}|dN|^2_{\mathring{g}}\rho^{2\delta}\, d\mu(\mathring{g}) \\ &\quad - \int_{\mathcal{M}} [2\delta+1-n+o(1)]N^2\rho^{2\delta}\, d\mu(\mathring{g}) \\ &\quad + \int_{\partial\mathcal{M}} (N^2 - |dN|^2_{\mathring{g}})\langle\frac{d\rho}{\rho}, \eta\rangle_{\mathring{g}}\, \rho^{2\delta}\, d\sigma(\mathring{g}), \end{aligned}\tag{27}$$

Proof: We integrate by parts the term $\mathring{\nabla}_i(|dN|^2_{\mathring{g}}\,\mathring{\nabla}^i(\rho^{-1})\,\rho^{2\delta+1})$ and the result follows on from the Divergence theorem and Lemma 7. □

Lemma 9. *Let $(M, \mathring{g})$ be a n-dimensional asymptotically hyperbolic manifold and $N \in \mathscr{C}^\infty(\mathcal{M})$ with a compact support near $\partial\mathcal{M}$. $\forall\delta \in I\!R$,*

$$\begin{aligned}-\int_{\mathcal{M}} tr_{\mathring{g}}\mathring{T}\, N\, \rho^{2\delta}\, d\mu(\mathring{g}) &= -\int_{\mathcal{M}} [\,\delta(2\delta+1-n) - n + o(1)]N^2\rho^{2\delta}\, d\mu(\mathring{g}) \\ &\quad + \int_{\mathcal{M}} |dN|^2_{\mathring{g}}\rho^{2\delta}\, d\mu(\mathring{g}) - \int_{\partial\mathcal{M}} N\langle dN, \eta\rangle_{\mathring{g}}\, \rho^{2\delta}\, d\sigma(\mathring{g}) \\ &\quad + \int_{\partial\mathcal{M}} \delta N^2\langle dN, \eta\rangle_{\mathring{g}}\, \rho^{2\delta}\, d\sigma(\mathring{g}), \end{aligned}\tag{28}$$

Proof: We integrate by parts the term $\mathring{\nabla}_i(N\mathring{\nabla}^i N\rho^{2\delta})$ and the result follows on from the Divergence theorem and Lemma 7. □

The next proposition stems from the two previous lemmas and will play an important role to prove the adjoint kernel triviality:

Proposition 4. *For every $\varepsilon > 0$, for all $\delta \in]-(n+1)/2, -1[$, there exists $R_{\varepsilon,\delta} > 0$ such that for all $R > R_{\varepsilon,\delta}$, there exists a positive constant c such that*

$$\forall N \in \mathscr{C}_c^\infty(E_R)\ ,\quad ||N||_{2,2,-\delta;E_R} \leqslant c\,||\mathring{T}||_{2,-\delta;E_R}. \tag{29}$$

Proof: From $\mathring{T}$ and $tr_{\mathring{g}}\mathring{T}$ expressions and $\mathring{\nabla}_n N$ (resp. $\mathring{\nabla}_T N$) being the component of dN normal (resp. tangential) to $\partial\mathcal{M}$,

$$\mathring{\nabla}_n N := \langle dN, \eta\rangle_{\mathring{g}} \quad \text{and} \quad |dN|^2_{\mathring{g}} = |\mathring{\nabla}_n N|^2_{\mathring{g}} + |\mathring{\nabla}_T N|^2_{\mathring{g}}.$$

$(28) - \frac{1}{2}(27)$ give

$$\begin{aligned}
&\int_{\mathcal{M}} \mathring{T}(dN, \frac{d\rho}{\rho})\rho^{2\delta}\, d\mu(\mathring{g}) - \int_{\mathcal{M}} N tr_{\mathring{g}}\mathring{T}\rho^{2\delta}\, d\mu(\mathring{g}) \\
= &\int_{\mathcal{M}} \{\tfrac{n+1}{2} - \delta + o(1)\}|dN|^2_{\mathring{g}}\rho^{2\delta}\, d\mu(\mathring{g}) + \int_{\mathcal{M}} [-2\delta^2 + n\delta + \tfrac{n+1}{2} + o(1)]N^2\rho^{2\delta}\, d\mu(\mathring{g}) \\
&+ \int_{\partial\mathcal{M}} \{(\delta - \tfrac{1}{2})N^2 + \tfrac{1}{2}|dN|^2_{\mathring{g}}\}\langle\frac{d\rho}{\rho}, \eta\rangle_{\mathring{g}}\rho^{2\delta}\, d\sigma(\mathring{g}) - \int_{\partial\mathcal{M}} N\mathring{\nabla}_n N\,\rho^{2\delta}\, d\sigma(\mathring{g}).
\end{aligned} \tag{30}$$

Application on E_R: E_R possesses two type of boundary. An exterior boundary at infinity, noted ∂E_∞ , and an interior boundary $\partial\Omega_R = \{\rho = \mathrm{e}^{-2R}\}$. Since $N \in \mathscr{C}_c^\infty(E_R)$, N is surely null near ∂E_∞ but not necessarily on $\partial\Omega_R$ and that's the reason why boundary terms in (30) will only concern $\partial\Omega_R$. If η_R is the normal to $\partial\Omega_R$ exterior to E_R and considering that
When $R \to +\infty$, $\eta_R - \frac{d\rho}{\rho} \to 0$, so that $\langle\frac{d\rho}{\rho}, \eta_R\rangle_{\mathring{g}} = \frac{|d\rho|^2_{\mathring{g}}}{\rho^2} + o(1) = 1 + o(1)$.

$$\begin{aligned}
&\int_{E_R} \mathring{T}(dN, \frac{d\rho}{\rho})\rho^{2\delta}\, d\mu(\mathring{g}) - \int_{E_R} N tr_{\mathring{g}}\mathring{T}\rho^{2\delta}\, d\mu(\mathring{g}) \\
= &\int_{E_R} \{\tfrac{n+1}{2} - \delta + o(1)\}|dN|^2_{\mathring{g}}\rho^{2\delta}\, d\mu(\mathring{g}) + \int_{E_R} [-2\delta^2 + n\delta + \tfrac{n+1}{2} + o(1)]N^2\rho^{2\delta}\, d\mu(\mathring{g}) \\
&+ \int_{\partial E_R} \{(\delta - \tfrac{1}{2} + o(1))N^2 + \tfrac{1}{2}|dN|^2_{\mathring{g}} - N\mathring{\nabla}_n N + o(1)\}\rho^{2\delta}\, d\sigma(\mathring{g}).
\end{aligned}$$

According to the following inequalities

$$\begin{cases}
|tr_{\mathring{g}}\mathring{T}|^2_{\mathring{g}} \leqslant n\,|\mathring{T}|^2_{\mathring{g}}. \\
\mathring{T}(dN, \frac{d\rho}{\rho}) \leqslant \frac{a}{2}|\mathring{T}|^2_{\mathring{g}} + \frac{1}{2a}|dN|^2_{\mathring{g}}|d\rho|^2_{\mathring{h}} \quad , \forall a > 0\,,\ a >> 1. \\
-N tr_{\mathring{g}}\mathring{T} \leqslant \frac{b}{2}|tr_{\mathring{g}}\mathring{T}|^2_{\mathring{g}} + \frac{1}{2b}N^2|d\rho|^2_{\mathring{h}} \quad , \forall b > 0\,,\ b >> 1.
\end{cases}$$

$\forall \varepsilon > 0, \exists R_\varepsilon > 0$ such that $\forall R > R_\varepsilon$,

$$\begin{aligned}\Big(\frac{a+bn}{2}\Big)\int_{E_R} |\mathring{T}|^2_{\mathring{g}}\rho^{2\delta}\, d\mu(\mathring{g}) \;\;\geqslant\;\; & \int_{E_R} \{\tfrac{n+1}{2} - \delta - \varepsilon\}|dN|^2_{\mathring{g}}\rho^{2\delta}\, d\mu(\mathring{g}) \\ & + \int_{E_R} [-2\delta^2 + n\delta + \tfrac{n+1}{2} - \varepsilon]N^2\rho^{2\delta}\, d\mu(\mathring{g}) \\ & + \int_{\partial E_R} (\delta - 1 - \varepsilon)N^2\rho^{2\delta}\, d\sigma(\mathring{g}) \\ & + \int_{\partial E_R} \Big\{\tfrac{1}{2}|\mathring{\nabla}_T N|^2_{\mathring{g}} + \tfrac{1}{2}(N - \mathring{\nabla}_n N)^2\Big\}\rho^{2\delta}\, d\sigma(\mathring{g}).\end{aligned}$$

- The N^2 interior term is non negative if $\delta \in]-\frac{1}{2}\,;\frac{n+1}{2}[$.
- The $|dN|^2_{\mathring{g}}$ interior term is non negative if $\delta < (n+1)/2$.
- The boundary term is non negative if $\delta > 1$.

Moreover, a quick calculation shows that on the interval $[0\,;\frac{n+1}{2}[$,

$$\tfrac{n+1}{2} - \delta \leqslant -2\delta^2 + n\delta + \tfrac{n+1}{2}.$$

Consequently, for $\delta \in]1\,;\frac{n+1}{2}[$,
$\forall \varepsilon > 0, \exists R_\varepsilon > 0$ such that $\forall R > R_\varepsilon$,

$$\Big(\frac{a+bn}{2}\Big)\int_{E_R} |\mathring{T}|^2_{\mathring{g}}\rho^{2\delta}\, d\mu(\mathring{g}) \;\;\geqslant\;\; \int_{E_R} \{\tfrac{n+1}{2} - \delta - \varepsilon\}\Big(N^2 + |dN|^2_{\mathring{g}}\Big)\rho^{2\delta}\, d\mu(\mathring{g}).$$

namely

$$||N||_{1,2,\delta;E_R} \leqslant c\,||\mathring{T}||_{2,\delta;E_R}.$$

Combining this inequality with Lemma 5 valid in particular on E_R, we end the proof of (29). □

5. Killing operator $\mathring{S}$

Let $\mathring{S}$ be the Killing operator defined on 1-forms by

$$\mathring{S}(Y)_{ij} = \frac{1}{2}(\mathring{\nabla}_i Y_j + \mathring{\nabla}_j Y_i) = \mathring{\nabla}_{(i}Y_{j)}. \tag{31}$$

The trace of this operator is

$$tr_{\mathring{g}}\mathring{S}(Y) = \mathring{g}^{ij}\mathring{S}(Y)_{ij} = \mathring{\nabla}^i Y_i = divY. \tag{32}$$

The next three lemmas are respectively versions of Lemma D.1 and Propositions D.2 and D.3 from [5] with boundary:

Lemma 10. *Let V be a vector field et Y a $1-$form both compactly supported on $\mathcal{M}$. Then,*

$$\begin{aligned}&\int_{\mathcal{M}}(\mathring{S}(Y)+\tfrac{1}{2}tr_{\mathring{g}}(\mathring{S}(Y))\mathring{g})(Y,V)\,d\mu(\mathring{g})\\&=-\frac{1}{2}\int_{\mathcal{M}}\left\{\mathring{\nabla}V(Y,Y)+\tfrac{1}{2}divV|Y|^2_{\mathring{g}}\right\}d\mu(\mathring{g})+\frac{1}{2}\int_{\partial\mathcal{M}}\langle Y,V\rangle_{\mathring{g}}\,\langle Y,\eta\rangle_{\mathring{g}}\,d\sigma(\mathring{g})\\&+\frac{1}{4}\int_{\partial\mathcal{M}}|Y|^2_{\mathring{g}}\,\langle V,\eta\rangle_{\mathring{g}}\,d\sigma(\mathring{g}),\end{aligned}$$

Lemma 11. *Let u be a function, V a vector field et Y a $1-$form all compactly supported on $\mathcal{M}$. Then,*

$$\begin{aligned}&\int_{\mathcal{M}}\mathrm{e}^{2u}(\mathring{S}(Y)+\tfrac{1}{2}tr_{\mathring{g}}(\mathring{S}(Y))\mathring{g})(Y,V)\,d\mu(\mathring{g})\\&=-\frac{1}{2}\int_{\mathcal{M}}\mathrm{e}^{2u}\Big\{\mathring{\nabla}V(Y,Y)+\tfrac{1}{2}divV|Y|^2_{\mathring{g}}\Big\}d\mu(\mathring{g})+\frac{1}{2}\int_{\partial\mathcal{M}}\mathrm{e}^{2u}\langle Y,V\rangle_{\mathring{g}}\,\langle Y,\eta\rangle_{\mathring{g}}\,d\sigma(\mathring{g})\\&-\frac{1}{2}\int_{\mathcal{M}}\mathrm{e}^{2u}\Big\{2\langle du,Y\rangle_{\mathring{g}}\,\langle Y,V\rangle_{\mathring{g}}+\langle du,V\rangle_{\mathring{g}}|Y|^2_{\mathring{g}}\Big\}d\mu(\mathring{g})+\frac{1}{4}\int_{\partial\mathcal{M}}\mathrm{e}^{2u}|Y|^2_{\mathring{g}}\,\langle V,\eta\rangle_{\mathring{g}}\,d\sigma(\mathring{g}),\end{aligned}$$

Lemma 12. *Let Y be a $1-$form compactly supported $\mathcal{M}$ and $u,v\in\mathscr{C}^\infty(\mathcal{M})$ two functions defined in a neighborhhod of Y support. Then,*

$$\begin{aligned}&-2\int_{\mathcal{M}}v\,\mathrm{e}^{2u}\mathring{S}(Y)(dv,dv)\langle dv,Y\rangle_{\mathring{g}}\,d\mu(\mathring{g})\\&=\int_{\mathcal{M}}\mathrm{e}^{2u}\langle dv,Y\rangle_{\mathring{g}}\Big\{\,\langle dv,Y\rangle_{\mathring{g}}\left[|dv|^2_{\mathring{g}}+v\mathring{\Delta}v+2v\langle dv,du\rangle_{\mathring{g}}\right]+2v\,\mathring{\nabla}^2v(Y,dv)\,\Big\}d\mu(\mathring{g})\\&-\int_{\partial\mathcal{M}}v\,\mathrm{e}^{2u}\langle dv,Y\rangle^2_{\mathring{g}}\langle dv,\eta\rangle_{\mathring{g}}\,d\sigma(\mathring{g}),\end{aligned}$$

Respective versions of Corollaries D.4 et D.5 de [5] with boundary come out from these previous lemmas:

Corollary 1. *Let $(M,\mathring{g})$ be an asymptotically hyperbolic manifold and Y a $1-$form compactly supported on $\partial\mathcal{M}$. $\forall\delta\in I\!R$,*

$$\begin{aligned}&2\int_{\mathcal{M}}\rho^{2\delta}(\mathring{S}(Y)+\tfrac{1}{2}tr_{\mathring{g}}(\mathring{S}(Y))\mathring{g})(Y,\frac{d\rho}{\rho})\,d\mu(\mathring{g})\\&=\int_{\mathcal{M}}\rho^{2\delta}\Big\{(\tfrac{n+1}{2}-\delta+o(1))|Y|^2_{\mathring{g}}-(2\delta+1)\langle\frac{d\rho}{\rho},Y\rangle^2_{\mathring{g}}\Big\}d\mu(\mathring{g})\\&+\frac{1}{2}\int_{\partial\mathcal{M}}\rho^{2\delta}|Y|^2_{\mathring{g}}\,\langle\frac{d\rho}{\rho},\eta\rangle_{\mathring{g}}\,d\sigma(\mathring{g})+\int_{\partial\mathcal{M}}\rho^{2\delta}\langle Y,\frac{d\rho}{\rho}\rangle_{\mathring{g}}\,\langle Y,\eta\rangle_{\mathring{g}}\,d\sigma(\mathring{g}),\end{aligned}\tag{33}$$

Proof: We apply lemma 11 with

$$\begin{cases}V=d(\rho^{-1})=-\rho^{-2}d\rho\\u=(\delta+\frac{1}{2})\ln\rho\end{cases}\quad\text{and}\quad\begin{cases}\mathring{\nabla}V=\mathring{\nabla}^2(\rho^{-1})=\rho^{-1}|d\rho|^2_{\mathring{h}}\mathring{g}-\rho^{-2}\,{}^{(\mathring{h})}\nabla^2\rho\ \text{ from (23)}\\divV=\mathring{\Delta}(\rho^{-1})=\rho^{-1}(n|d\rho|^2_{\mathring{h}}+o(1))\ \text{ from (24)}\end{cases}.$$

$$du=(\delta+\tfrac{1}{2})\frac{d\rho}{\rho}\ \text{ and }\ \mathrm{e}^{2u}=\rho^{2\delta+1}.$$

The metric $\mathring{h}$ being defined (and so bounded) on $\partial\mathcal{M}$ and ρ being a smooth function on $\mathcal{M}$, ${}^{(\mathring{h})}\nabla^2\rho$ is a smooth function bounded on $\mathcal{M}$ and so, we can write ${}^{(\mathring{h})}\nabla^2\rho = o(\rho^{-1})$ near $\partial\mathcal{M}$. Thus,

$$\begin{cases} \mathring{\nabla}V(Y,Y) = \rho^{-1}(|d\rho|^2_{\mathring{h}}|Y|^2_{\mathring{g}} + o(1)|Y|^2_{\mathring{g}}) \\ divV|Y|^2_{\mathring{g}} = \rho^{-1}(n|d\rho|^2_{\mathring{h}} + o(1))|Y|^2_{\mathring{g}} \end{cases}.$$

We obtain

$$\begin{aligned} &\int_{\mathcal{M}} \rho^{2\delta}(\mathring{S}(Y) + \tfrac{1}{2}tr_{\mathring{g}}(\mathring{S}(Y))\mathring{g})(Y, \frac{d\rho}{\rho})\, d\mu(\mathring{g}) \\ &= \frac{1}{2}\int_{\mathcal{M}} \rho^{2\delta}\Big\{ \left[1 - (\delta + \tfrac{1}{2}) + \tfrac{n}{2}\right] |d\rho|^2_{\mathring{h}}|Y|^2_{\mathring{g}} - (2\delta+1)\langle\frac{d\rho}{\rho}, Y\rangle^2_{\mathring{g}} + o(1)|Y|^2_{\mathring{g}}\Big\}\, d\mu(\mathring{g}) \\ &+\frac{1}{4}\int_{\partial\mathcal{M}} \rho^{2\delta}|Y|^2_{\mathring{g}}\, \langle\frac{d\rho}{\rho}, \eta\rangle_{\mathring{g}}\, d\sigma(\mathring{g}) + \frac{1}{2}\int_{\partial\mathcal{M}} \rho^{2\delta}\langle Y, \frac{d\rho}{\rho}\rangle_{\mathring{g}}\, \langle Y, \eta\rangle_{\mathring{g}}\, d\sigma(\mathring{g}). \end{aligned}$$

We end with $|d\rho|^2_{\mathring{h}} = 1 + o(1)$ on an asymptotically hyperbolic manifold. □

Corollary 2. *Let $(M, \mathring{g})$ be an asymptotically hyperbolic manifold and Y a $1-$form compactly supported on $\partial\mathcal{M}$. Then,*

$$\begin{aligned} &2\int_{\mathcal{M}} \rho^{2\delta}\mathring{S}(Y)(\frac{d\rho}{\rho}, \frac{d\rho}{\rho})\, \langle\frac{d\rho}{\rho}, Y\rangle_{\mathring{g}}\, d\mu(\mathring{g}) \\ &= \int_{\mathcal{M}} \rho^{2\delta}\, (n - 1 - 2\delta + o(1))\, \langle\frac{d\rho}{\rho}, Y\rangle^2_{\mathring{g}}\, d\mu(\mathring{g}) + \int_{\partial\mathcal{M}} \rho^{2\delta}\, \langle\frac{d\rho}{\rho}, Y\rangle^2_{\mathring{g}}\langle\frac{d\rho}{\rho}, \eta\rangle_{\mathring{g}}\, d\sigma(\mathring{g}), \end{aligned} \tag{34}$$

Proof: We apply lemma 12 with

$$\begin{cases} v = \rho^{-1} \\ u = (\delta+2)\ln\rho \end{cases}, \quad \begin{cases} dv = d(\rho^{-1}) = -\rho^{-1}\frac{d\rho}{\rho} \\ du = (\delta+2)\frac{d\rho}{\rho} \end{cases} \quad \text{and} \quad \begin{cases} \mathrm{e}^{2u} = \rho^{2\delta+4} \\ |dv|^2_{\mathring{g}} = \rho^{-2}|d\rho|^2_{\mathring{h}} \end{cases}$$

together with

$$\begin{cases} \mathring{\nabla}^2 v = \mathring{\nabla}^2(\rho^{-1}) = \rho^{-1}\left(|d\rho|^2_{\mathring{h}}\mathring{g} - \rho^{-1}\,{}^{(\mathring{h})}\nabla^2\rho\right) \ \text{ from (23)} \\ \mathring{\Delta}v = \mathring{\Delta}(\rho^{-1}) = \rho^{-1}\left(n|d\rho|^2_{\mathring{h}} + o(1)\right) \ \text{ from (24)} \end{cases}.$$

Since $|d\rho|^2_{\mathring{h}} = 1 + o(1)$ on an asymptotically hyperbolic manifold, we end up with

$$\begin{aligned} 2\textstyle\int_{\mathcal{M}} \rho^{2\delta}\mathring{S}(Y)(\frac{d\rho}{\rho}, \frac{d\rho}{\rho})\, \langle\frac{d\rho}{\rho}, Y\rangle_{\mathring{g}}\, d\mu(\mathring{g}) \ &= \int_{\mathcal{M}} \rho^{2\delta}\Big\{ n + 3 - 2(\delta+2) + o(1)\Big\}\langle\frac{d\rho}{\rho}, Y\rangle^2_{\mathring{g}}\, d\mu(\mathring{g}) \\ &+ \int_{\partial\mathcal{M}} \rho^{2\delta}\, \langle\frac{d\rho}{\rho}, Y\rangle^2_{\mathring{g}}\langle\frac{d\rho}{\rho}, \eta\rangle_{\mathring{g}}\, d\sigma(\mathring{g}). \end{aligned}$$ □

The following lemma establishes a Korn-type inequality for the Killing operator $\mathring{S}$:

Lemma 13. *For all $\delta > -(n+1)/2$ and $\delta \neq -(n-1)/2$, there exists a positive constant $c = c\,(\mathring{g}, \delta)$ such that for all 1-form $Y \in W^{1,2}_{-\delta}(T^*\mathcal{M})$,*

$$||Y||_{1,2,-\delta} \leqslant c\,||\mathring{S}(Y)||_{2,-\delta}.$$

Proof: Here again we base ourselves on the proof of Theorem 1 but for the operator $\mathring{S}$. Lemma 2.8 from [5] (for $\mathring{g}$ with $N = 0$) replaces Lemma 1, in order to get a Korn-type inequality

$$\begin{aligned} ||Y||_{1,2,-\delta} &\leqslant c\,(||\mathring{S}(Y) - tr_{\mathring{g}}(S(Y))\mathring{g}||_{2,-\delta} + ||Y||_{2,-\delta}) \\ &\leqslant c\,(||\mathring{S}(Y)||_{2,-\delta} + ||Y||_{2,-\delta})\,, \end{aligned}$$

where $c = c\,(\mathring{g}, \delta)$ is a positive constant.
We now use Proposition $D12$ from [5]: Let $(\mathcal{M}, \mathring{g})$ be a conformally compact manifold with $\mathring{g} = \rho^{-2}\mathring{h}$. For all $\delta \neq -(n+1)/2$ and $\delta \neq -(n-1)/2$, there exists two constants $c_\delta > 0$ and $\rho_{\varepsilon,\delta} > 0$ such that for all differentiable vector field Y compactly supported in $\{\rho < \rho_{\varepsilon,\delta}\}$:

$$||Y||_{2,-\delta} \leqslant c_\delta\, ||\mathring{S}(Y)||_{2,-\delta}. \tag{35}$$

Defining R such that $\rho_{\varepsilon,\delta} = e^{-2R}$, we set Ω_R as in Definition 1 and we have (35) on $\mathcal{M}\setminus\Omega_R$, as in hypothesis of Theorem 1. The rest of the proof is analogous to the one of Theorem 1 and we can choose R large enough so that for all $Y \in \mathscr{C}^\infty_c(T^*\mathcal{M})$,

$$||Y||_{1,2,-\delta} \leqslant c\,\left(||\mathring{S}(Y)||_{2,-\delta} + ||Y||_{2,-\delta;\Omega_R}\right).$$

Hence operator $\mathring{S} : W^{1,2}_{-\delta}(T^*\mathcal{M}) \to L^2_{-\delta}(T^*\mathcal{M})$ has a finite dimensional kernel. So we can write $W^{1,2}_{-\delta}(T^*\mathcal{M}) = \ker\mathring{S} \oplus (\ker\mathring{S})^\perp$. From then, there exists a positive constant c such that for all $Y \in (\ker\mathring{S})^\perp$,

$$||Y||_{1,2,-\delta} \leqslant c\,||\mathring{S}(Y)||_{2,-\delta}.$$

It remains to show that $\ker\,\mathring{S} \cap W^{1,2}_{-\delta} = \{0\}$, $\forall\delta > -(n+1)/2$.
For that matter, we use the coordinate system $(x^1 = \rho, x^2, \ldots, x^n) = (\rho, \theta)$ on a neighborhood of the boundary $[0, \varepsilon] \times \partial\mathcal{M}$ that we may find in [5]. From the expression of the defining function ρ, the metric $\mathring{g}$ can be written:

$$\mathring{g} = \rho^{-2}\mathring{h} = \rho^{-2}(d\rho^2 + \hat{g}(\rho)) \quad \text{with} \quad \hat{g}(\rho)(\partial_\rho, .) = 0.$$

The same conventions and notations as in the paper are in order: the index ρ will be the radial coordinate one whereas indices relative to tangential coordinates will be designated by latin capital indices. Finally lower case latin indices designate any components. Christoffel symbols, in this coordinate system, are given in [5], just as the equation

$$\mathring{\nabla}_i X_j + \mathring{\nabla}_j X_i = 0 \tag{36}$$

which become the system:

$$\partial_\rho X_\rho + \rho^{-1} X_\rho = 0 \tag{37}$$

$$\partial_\rho X_A + \partial_A X_\rho + 2\rho^{-1} X_A - \hat{g}^{CD}(\rho)\hat{g}'_{DA}(\rho) X_C = 0 \tag{38}$$

$$\partial_A X_B + \partial_B X_A - 2\hat{\Gamma}^{\ C}_{A\ B}(\rho) X_C + (\hat{g}'_{AB}(\rho) - 2\rho^{-1}\hat{g}_{AB}(\rho)) X_\rho = 0$$

where $f' := \partial_\rho f$.
Solving equation (37) give

$$X_\rho = \rho^{-1} K(\theta).$$

As the metric $\mathring{g}$ is polyhomogeneous , $\hat{g}$ can be written as a development of powers of ρ and $\ln\rho$, and the first terms only contain powers of ρ. We set

$$\hat{g}^{CD}\hat{g}'_{DA} := T^C_A,$$

where T is and order two tensor whose development is

$$T^C_A(\rho,\theta) = \rho^1 T^C_A(\theta) + o(1).$$

Andersson and Chruściel have shown in [2] that the solution X of

$$(\Delta - Ric\,\mathring{g})X = 0$$

is also polyhomogeneous

$$X_A(\rho,\theta) = \rho^s Z_A(\theta) + o(\rho^s).$$

Replacing in equation (38), we find $s=-2$ et $X_A = \rho^{-2}Z_A(\theta) + o(\rho^{-2})$.
We obtain the form of the solution of (36) near the boundary

$$X = \rho^{-1}K(\theta)\,d\rho + [\rho^{-2}Z_A(\theta) + o(\rho^{-2})]\,dx^A.$$

Moreover, $X \in W^{1,2}_{-\delta}$, leading to $Z=0$.
Indeed, suppose $Z\neq 0$. The $\mathring{g}$-norm of X is

$$|X|^2_{\mathring{g}} \simeq \rho^{-2}|K|^2_{\mathring{g}} + \rho^{-4}|Z|^2_{\mathring{g}},$$

with $|K|^2_{\mathring{g}} \simeq |Z|^2_{\mathring{g}} \simeq \rho^2\,O(1)$.

$$|X|^2_{\mathring{g}} \simeq \rho^{-2}\,O(1).$$

Consequently

$$\begin{aligned}
||X||_{2,-\delta} < \infty \quad &\Leftrightarrow \quad \int_0^\varepsilon |X|^2_{\mathring{g}}\rho^{-2\delta}\rho^{-n}d\rho < \infty \\
&\Leftrightarrow \quad \int_0^\varepsilon \rho^{-2}\rho^{-2\delta}\rho^{-n}d\rho < \infty \\
&\Leftrightarrow \quad -2\delta - n - 2 > -1 \\
&\Leftrightarrow \quad \delta < -(n+1)/2.
\end{aligned}$$

Given that $\delta > -(n+1)/2$, then $X \notin W^{1,2}_{-\delta}$. Hence, Z is necessarily null near the boundary. Analysing the X development coefficients, we realize $Z=0$ lead to $|X|_{\mathring{g}} = O(\rho^\infty)$. We conclude the proof using the unique continuation theorem from [16]. □

6. The contraint operator $\mathbf{\Phi}$

Let $\mathcal{M}$ be a n-dimensional connected non compact oriented manifold. We consider $\mathcal{M}$ as a spacelike totally geodesic hypersurface of a $(n+1)$-dimensional Loretzian manifold $(\mathcal{N},\gamma)$, from now on refered to as spacetime. We will distinguish the two manifolds by different indices: latin indices will take values from 1 to n and are spatial indices whereas greec indices will take values from 0 to n and are spacetime indices. K is the second fondamental form of $\mathcal{M}$ in $\mathcal{N}$ defined by

$$K(X,Y) = \gamma(X, {}^{(\gamma)}\nabla_Y \vec{n}), \tag{39}$$

where ${}^{(\gamma)}\nabla$ is the spacetime connection on $T\mathcal{N}$, $X, Y \in T\mathcal{M}$ and $\vec{n}$ is the future-directed unit normal to $\mathcal{M}$ in $\mathcal{N}$. It is convenient to consider the conjugate momentum π as a reparametrisation of K

$$\pi^{ij} = \widetilde{\pi}^{ij}\sqrt{g} \quad \text{with } \widetilde{\pi}^{ij} = K^{ij} - tr_g K g^{ij}. \tag{40}$$

where $\sqrt{g}$ is the volume measure of the metric g.

$$\sqrt{g} = \frac{\sqrt{det(g)}}{\sqrt{det(\mathring{g})}}\, d\mu(\mathring{g}).$$

Remark: $\nabla(\sqrt{g}) = 0$ since the covariant derivative of the volume form is null.
$\widetilde{\pi}$ is a section of the bundle $S^2T\mathcal{M}$ whereas π is a section of the bundle $\widetilde{S} = S^2T\mathcal{M} \otimes \Lambda^3 T^*\mathcal{M}$. We consider a smooth and polyhomogeneous asymptotically hyperbolic metric $\mathring{g}$ asymptotically hyperbolic on $\mathcal{M}$ as model. We ask $\mathring{g}$ to satisfy the following integrability condition

$$Riem\, \mathring{g}_{ijkl} - \mathring{g}_{il}\mathring{g}_{jk} + \mathring{g}_{ik}\mathring{g}_{jl} \in L^2_\delta. \tag{41}$$

In particular,

$$Ric\, \mathring{g}_{jl} + (n-1)\mathring{g}_{jl} \in L^2_\delta. \tag{42}$$

For any sufficiently regular Riemannian metric g on $\mathcal{M}$, we define the constraint operator $\mathbf{\Phi} = (\mathbf{\Phi_0}, \mathbf{\Phi_i}) = \mathbf{\Phi}(g, \pi)$ as follows:

$$\begin{aligned} \mathbf{\Phi_0}(g,\pi) &:= \left(R(g) - 2\Lambda - |K|^2_g + (tr_g K)^2\right)\sqrt{g} \\ &= \left(R(g) - 2\Lambda\right)\sqrt{g} - \left(|\pi|^2_g - \tfrac{1}{n-1}(tr_g\pi)^2\right)/\sqrt{g}. \qquad (43) \\ \mathbf{\Phi_i}(g,\pi) &:= 2(\nabla^j K_{ij} - \nabla_i(tr_g K))\sqrt{g} \\ &= 2g_{ij}\nabla_k \pi^{jk} = 2g_{ij}\nabla_k \widetilde{\pi}^{jk}\sqrt{g}. \qquad (44) \end{aligned}$$

We set

$$\mathring{K} = \tau \mathring{g}\,, \tag{45}$$

where τ is a real parameter.
The cosmological constant Λ is normalized here in dimension n by

$$2\Lambda = n(n-1)(\tau^2 - 1), \tag{46}$$

so that $\mathbf{\Phi}(\mathring{g}, \mathring{K}) = 0$ at infinity. Taking the $\mathring{g}$-trace of (42) considering (46), we end up with the intégrability condition

$$R(\mathring{g}) - 2\Lambda + n(n-1)\tau^2 \in L^2_\delta. \tag{47}$$

The conjugate momentum $\mathring{\pi}$ is then

$$\mathring{\pi}^{ij} = (\mathring{K}^{ij} - tr_{\mathring{g}}\mathring{K}\mathring{g}^{ij})\, d\mu(\mathring{g}) = \tau(1-n)\mathring{g}^{ij}\, d\mu(\mathring{g}). \tag{48}$$

Remark : $\mathring{\nabla}\mathring{\pi} = \mathring{\nabla}\mathring{K} = 0$.
If spacetime satisfies Einstein's equations, the normalisation chosen insures the constraint operator and energy-momentum tensor are related by

$$\mathbf{\Phi}_\alpha = 16\pi G T_{\vec{n}\alpha}\sqrt{g}\,,$$

where G is Newton's gravitationnal constant. $\xi = (N, X^i)$ is the *lapse-shift* associated to the spacetime foliation. We study the constraint operator $\mathbf{\Phi}$ for Riemannian metrics of

the form $g = \mathring{g} + h$. g is asymptotic to $\mathring{g}$, *i.e.* $|g - \mathring{g}|_{\mathring{g}} = |h|_{\mathring{g}} \underset{\infty}{\longrightarrow} 0$.

$\mathcal{S} := S^2T^*\mathcal{M}$ is the bundle of symmetric bilinear forms on $\mathcal{M}$. $\widetilde{\mathcal{S}} := S^2T\mathcal{M} \otimes \Lambda^3T^*\mathcal{M}$ is the bundle of symmetric 2-tensors-valued densities (3-forms) on $\mathcal{M}$. $\mathcal{T} := T\mathcal{N}$ is the spacetime tangent bundle. The following spaces will be of particular interest in the sequel:

$$\begin{array}{rcl} \mathcal{G} & := & W^{2,2}_{\delta}(\mathcal{S}). \\ \mathcal{K} & := & \{\pi : \pi - \mathring{\pi} \in W^{1,2}_{\delta}(\widetilde{\mathcal{S}})\}. \\ \mathcal{G}^+ & := & \{g : g - \mathring{g} \in \mathcal{G}, g > 0\}. \\ \mathcal{G}^+_\lambda & := & \{g \in \mathcal{G}^+ : \lambda\mathring{g} < g < \lambda^{-1}\mathring{g}\}\ ,\ 0 < \lambda < 1. \\ \mathcal{L}^* & := & L^2_{\delta}(\mathcal{T}^* \otimes \Lambda^3T^*\mathcal{M}) \text{ is the dual space of } \mathcal{L} := L^2_{-\delta}(\mathcal{T}). \end{array}$$

From (10), tensors in $\mathcal{G}$ are Hölder-continuous and thus, matrices inequalities in spaces $\mathcal{G}^+$ et $\mathcal{G}^+_\lambda$ are satisfied pointwise. In particular, for all metric $g \in \mathcal{G}^+_\lambda$, metrics g et $\mathring{g}$ are equivalent in the following sense

$$\lambda\mathring{g}_{ij}(x)\, v^iv^j < g_{ij}(x)\, v^iv^j < \lambda^{-1}\mathring{g}_{ij}(x)\, v^iv^j\ , \quad \forall x \in \mathcal{M}\,, \forall v \in T\mathcal{M} \tag{49}$$

So $|g|_{\mathring{g}} \simeq c\,|\mathring{g}|_{\mathring{g}} \simeq c\,|g|_g \simeq \sqrt{n}$.
$\mathcal{F} = \mathcal{G}^+ \times \mathcal{K}$ will be the phase space of the contraint operator $\mathbf{\Phi}$. We will use (g, π) as well as (g, K) to express coordinates on $\mathcal{F}$.

Let $\mathring{\Gamma}$ and $\mathring{\nabla}$ (resp. Γ and ∇) be the Christoffel symbols and the Levi-Civita connection for $\mathring{g}$ (resp. g). We define

$$A_{i\,j}^{\ k} = \Gamma_{i\,j}^{\ k} - \mathring{\Gamma}_{i\,j}^{\ k} \tag{50}$$

Remark: A is a symmetric tensor, arising from symmetry of the Christoffel symbols. We easily show

$$A_{i\,j}^{\ k} = g^{kl}A_{ilj} = \frac{1}{2}g^{kl}(\mathring{\nabla}_i g_{jl} + \mathring{\nabla}_j g_{il} - \mathring{\nabla}_l g_{ij}) \tag{51}$$

The scalar curvature of g can be express with $\mathring{\nabla}$ and $A_{i\,k}^{\ j}$ (see eq. (21) of [4]):

Lemma 14.

$$\begin{array}{rcl} R(g) & = & g^{jk}Ric\,\mathring{g}_{jk} + g^{jk}(\mathring{\nabla}_i A_{j\,k}^{\ i} - \mathring{\nabla}_j A_{i\,k}^{\ i} + A_{j\,k}^{\ l}A_{i\,l}^{\ i} - A_{j\,l}^{\ i}A_{k\,i}^{\ l}) \\ & = & g^{jk}Ric\,\mathring{g}_{jk} + Q(g^{-1}, \mathring{\nabla}g) + g^{ik}g^{jl}(\mathring{\nabla}^2_{ij}g_{kl} - \mathring{\nabla}^2_{ik}g_{jl}) \end{array} \tag{52}$$

where Q is a sum of quadratic terms in $g^{-1}, \mathring{\nabla}g$.

This result relies on the following fact:

Lemma 15.

$$Ric\,g_{jk} - Ric\,\mathring{g}_{jk} = \mathring{\nabla}_i A_{j\,k}^{\ i} - \mathring{\nabla}_j A_{i\,k}^{\ i} + A_{j\,k}^{\ \mu}A_{i\,\mu}^{\ i} - A_{j\,\mu}^{\ i}A_{k\,i}^{\ \mu} \tag{53}$$

Proof:

$$\begin{array}{rcl} Ric\,g_{jk} - Ric\,\mathring{g}_{jk} & = & \partial_i A_{j\,k}^{\ i} - \partial_j A_{i\,k}^{\ i} + [\Gamma_{l\,i}^{\ i}\Gamma_{j\,k}^{\ l} - \mathring{\Gamma}_{l\,i}^{\ i}\mathring{\Gamma}_{j\,k}^{\ l}] - [\Gamma_{k\,i}^{\ l}\Gamma_{j\,l}^{\ i} - \mathring{\Gamma}_{k\,i}^{\ l}\mathring{\Gamma}_{j\,l}^{\ i}] \\ & = & \partial_i A_{j\,k}^{\ i} - \partial_j A_{i\,k}^{\ i} + [A_{l\,i}^{\ i}A_{j\,k}^{\ l} + \mathring{\Gamma}_{l\,i}^{\ i}A_{j\,k}^{\ l} + \mathring{\Gamma}_{j\,k}^{\ l}A_{l\,i}^{\ i}] \\ & & -[A_{k\,i}^{\ l}A_{j\,l}^{\ i} + \mathring{\Gamma}_{k\,i}^{\ l}A_{j\,l}^{\ i} + \mathring{\Gamma}_{j\,l}^{\ i}A_{k\,i}^{\ l}] \end{array}$$

We end the proof adding and substracting $\mathring{\Gamma}_{j\,i}^{\ l}A_{l\,k}^{\ i}$. □

Here we show $\mathbf{\Phi}$ is a well-defined mapping between the Hilbert spaces $\mathcal{F}$ et $\mathcal{L}^*$.

Proposition 5. *Set $(g,\pi)\in\mathcal{G}^+_\lambda\times\mathcal{K}$, with $0<\lambda<1$. Then in dimension $n=3$, for all $\delta\leqslant 0$, there exists a positive constant $c=c\,(\lambda,\mathring{g},\delta)$ such that*

$$||\mathbf{\Phi_0}(g,\pi)||_{2,\delta} \leqslant c\left(1+||g-\mathring{g}||^2_{2,2,\delta}+||\pi-\mathring{\pi}||^2_{1,2,\delta}\right) \tag{54}$$

$$||\mathbf{\Phi_i}(g,\pi)||_{2,\delta} \leqslant c\left(||\mathring{\nabla}(\pi-\mathring{\pi})||_{2,\delta}+||\mathring{\nabla}g||_{1,2,\delta}(1+||\pi-\mathring{\pi}||_{1,2,\delta})\right) \tag{55}$$

Proof: From $R(g)$ expression (52), we can write

$$\begin{aligned}
\mathbf{\Phi_0}(g,\pi) &= (R(g)-2\Lambda)\sqrt{g}-(|\pi|^2_g-\tfrac{1}{n-1}(tr_g\pi)^2]/\sqrt{g}\\
&= [R(g)-\mathbf{R}(\mathring{\mathbf{g}})+\mathbf{R}(\mathring{\mathbf{g}})-2\Lambda+\mathbf{n(n-1)}\tau^{\mathbf{2}}-\mathbf{n(n-1)}\tau^{\mathbf{2}}]\sqrt{g}\\
&\quad -[|\pi-\mathring{\pi}|^2_g-|\mathring{\pi}|^2_g+2(\pi-\mathring{\pi})_{ij}\mathring{\pi}^{ij}+2\,|\mathring{\pi}|^2_g]/\sqrt{g}\\
&\quad +\tfrac{1}{n-1}[(tr_g(\pi-\mathring{\pi}))^2+(tr_g\mathring{\pi})^2+2\,tr_g(\pi-\mathring{\pi})\,tr_g\mathring{\pi}]/\sqrt{g}\\
&= [R(g)-\mathbf{R}(\mathring{\mathbf{g}})+\mathbf{R}(\mathring{\mathbf{g}})-2\Lambda+\mathbf{n(n-1)}\tau^{\mathbf{2}}-\mathbf{n(n-1)}\tau^{\mathbf{2}}]\sqrt{g}\\
&\quad -[|\pi-\mathring{\pi}|^2_g+2(\pi-\mathring{\pi})_{ij}\mathring{\pi}^{ij}+|\mathring{\pi}|^2_g-\tfrac{1}{n-1}(tr_{\mathring{g}}\mathring{\pi})^2]/\sqrt{g}\\
&\quad +\tfrac{1}{n-1}[(tr_g(\pi-\mathring{\pi}))^2+((g-\mathring{g})_{ij}\mathring{\pi}^{ij})^2+2(g-\mathring{g})_{ij}\mathring{\pi}^{ij}tr_{\mathring{g}}\mathring{\pi}+2\,tr_g(\pi-\mathring{\pi})\,tr_g\mathring{\pi}]/\sqrt{g}\\
&= [R(g)-R(\mathring{g})+R(\mathring{g})-2\Lambda+n(n-1)\tau^2]\sqrt{g}-[|\pi-\mathring{\pi}|^2_g+2(\pi-\mathring{\pi})_{ij}\mathring{\pi}^{ij}]/\sqrt{g}\\
&\quad +\tfrac{1}{n-1}[(tr_g(\pi-\mathring{\pi}))^2+((g-\mathring{g})_{ij}\mathring{\pi}^{ij})^2+2(g-\mathring{g})_{ij}\mathring{\pi}^{ij}tr_{\mathring{g}}\mathring{\pi}+2\,tr_g(\pi-\mathring{\pi})\,tr_g\mathring{\pi}]/\sqrt{g}.
\end{aligned}$$

Since $g\in\mathcal{G}^+_\lambda$, we can use (49) and from Cauchy-Schwarz inequality and $2ab\leqslant a^2+b^2$

$$\begin{aligned}
|\mathbf{\Phi_0}(g,\pi)|_{\mathring{g}} &\leqslant [|R(g)-R(\mathring{g})|_{\mathring{g}}+|R(\mathring{g})-2\Lambda+n(n-1)\tau^2|_{\mathring{g}}]\sqrt{g}\\
&\quad +c\,[1+|\pi-\mathring{\pi}|^2_g+|g-\mathring{g}|^2_g]/\sqrt{g}.
\end{aligned}$$

From (53), $Ric\,g-Ric\,\mathring{g}\simeq\mathring{\nabla}A+A^2\simeq(\mathring{\nabla}g)^2+g\mathring{\nabla}^2g+g^{-2}(\mathring{\nabla}g)^2$.
Using (13),

$$\begin{aligned}
||Ric\,g-Ric\,\mathring{g}||_{2,\delta} &\leqslant (c\,||\mathring{\nabla}g||^2_{1,2,\delta}+c\,||\mathring{\nabla}^2g||_{2,\delta})\\
&\leqslant c\,||g-\mathring{g}||_{2,2,-\delta}.
\end{aligned}$$

In particular, we have the following integrability conditions

$$Ric\,g-Ric\,\mathring{g}\in L^2_\delta. \tag{56}$$

$$R(g)-R(\mathring{g})\in L^2_\delta. \tag{57}$$

Thanks to (57),(47) and (14),

$$\begin{aligned}
||\mathbf{\Phi_0}(g,\pi)||_{2,\delta} &\leqslant c\left(1+||(\pi-\mathring{\pi})^2||_{2,\delta}+||(g-\mathring{g})^2||_{2,\delta}\right)\\
&\leqslant c\left(1+||\pi-\mathring{\pi}||^2_{1,2,\delta}+||g-\mathring{g}||^2_{2,2,\delta}\right).
\end{aligned}$$

Hence, $\mathbf{\Phi_0}(g,\pi)\in\mathcal{L}^*$.
Regarding $\mathbf{\Phi_i}(g,\pi)$, using (50),

$$\mathbf{\Phi_i}(g,\pi)=2g_{ij}(\mathring{\nabla}_k(\pi-\mathring{\pi})^{jk}+A^{\,j}_{k\,l}\,(\pi-\mathring{\pi})^{kl})+A^{\,j}_{k\,l}\,\mathring{\pi}^{kl}).$$

Considering (51), $\mathbf{\Phi_i}(g,\pi)$ is of the form

$$\mathbf{\Phi_i}(g,\pi)\simeq g(\mathring{\nabla}(\pi-\mathring{\pi})+g^{-1}\mathring{\nabla}g\,(\pi-\mathring{\pi})+g^{-1}\mathring{\nabla}g\,\mathring{\pi}). \tag{58}$$

$$\begin{aligned}||\mathbf{\Phi_i}(g,\pi)||_{2,\delta} &\leqslant c\,(||\mathring{\nabla}(\pi-\mathring{\pi})||_{2,\delta}+||\mathring{\nabla}g\,(\pi-\mathring{\pi})||_{2,\delta}+||\mathring{\nabla}g\,\mathring{\pi}||_{2,\delta})\\ &\leqslant c\,(||\mathring{\nabla}(\pi-\mathring{\pi})||_{2,\delta}+||\mathring{\nabla}g||_{1,2,\delta}||\pi-\mathring{\pi}||_{1,2,\delta}+||\mathring{\nabla}g||_{2,\delta}||\mathring{\pi}||_{\infty,0})\\ &\leqslant c\,\big(||\mathring{\nabla}(\pi-\mathring{\pi})||_{2,\delta}+||\mathring{\nabla}g||_{1,2,\delta}(1+||\pi-\mathring{\pi}||_{1,2,\delta})\big).\end{aligned}$$ □

Proposition 6. *Let $(g,\pi)\in\mathcal{F}$. Then in dimension $n=3$, for all $\delta\leqslant 0$, $\mathbf{\Phi}:\mathcal{F}\to\mathcal{L}^*$ is a smooth map between Hilbert spaces.*

Proof: We recall the proof of [4] for completeness. From Proposition 5, $||\mathbf{\Phi}(g,\pi)||_{\mathcal{L}^*}\leqslant c(1+||g-\mathring{g}||^2_{\mathcal{G}}+||\pi-\mathring{\pi}||^2_{\mathcal{K}})$, *i.e.* $\mathbf{\Phi}$ is locally bounded on $\mathcal{F}$. The polynomial structure of the constraint operator allows us to show $\mathbf{\Phi}$ is smooth, *i.e.* indefinitely differentiable in a Fréchet sense. From the expression (52) of scalar curvature and given (58) , $\mathbf{\Phi}$ can be expressed as

$$\mathbf{\Phi}(g,\pi)=F(g,g^{-1},\sqrt{g},1/\sqrt{g},\mathring{\nabla}g,\mathring{\nabla}^2g,\pi,\mathring{\nabla}\pi)\,,$$

where $F=F(a_1,\dots,a_8)$ is a polynomial function quadratic in a_5 et a_7 and linear in the remaining parameters. The map $g\mapsto(g,g^{-1},\sqrt{g},1/\sqrt{g})$ is analytic on the space of positive definite matrices and the maps $g\mapsto\mathring{\nabla}g$, $g\mapsto\mathring{\nabla}^2g$ and $\pi\mapsto\mathring{\nabla}\pi$ are bounded linear, thus smooth, from $\mathcal{F}$ to $\mathcal{L}^*$, which are Hilbert spaces. A result from Hille [14] on locally bounded polynomial functionals shows $\mathbf{\Phi}$ admit continuous Fréchet-derivatives of all orders. □

The set $\mathcal{C}=\{(g,\pi)\in\mathcal{G}^+\times\mathcal{K}:\mathbf{\Phi}(g,\pi)=0\}:=\mathbf{\Phi}^{-1}(\{0\})\subset\mathcal{F}$ is the set of initial data for the vacuum Einstein's equations. To prove $\mathcal{C}$ is a submanifold of $\mathcal{F}$, we show 0 is a regular value of $\mathbf{\Phi}$, so we take an interest in the surjectivity of the differential of $\mathbf{\Phi}$.

7. Expressions of the linearization of $\mathbf{\Phi}$ and its adjoint

In this section we recall the expression of the linearization of $\mathbf{\Phi}$ and its adjoint that we may find in [4] or [9] for example.

Proposition 7.

$$\begin{aligned}D\mathbf{\Phi_0}(g,\pi).(h,p) &= (\nabla^i\nabla^jh_{ij}-\Delta_g tr_g h)\sqrt{g}-h_{ij}[R^{ij}-\tfrac{1}{2}(R(g)-2\Lambda)g^{ij}]\sqrt{g}\\ &\quad+h_{ij}\big(\tfrac{2}{n-1}tr_g\pi\pi^{ij}-2\pi^i_{\ k}\pi^{kj}+\tfrac{1}{2}|\pi|^2_g\,g^{ij}-\tfrac{1}{2(n-1)}(tr_g\pi)^2g^{ij}\big)/\sqrt{g}\\ &\quad+p^{ij}(\tfrac{2}{n-1}tr_g\pi g_{ij}-2\pi_{ij})/\sqrt{g}. \qquad (59)\\ D\mathbf{\Phi_i}(g,\pi).(h,p) &= \pi^{jk}(2\nabla_kh_{ij}-\nabla_ih_{jk})+2h_{ij}\nabla_k\pi^{jk}+2g_{ik}\nabla_jp^{jk}. \qquad (60)\end{aligned}$$

Using notations of [4] ,

$$\begin{aligned}\delta_g\delta_gh &= \nabla^i\nabla^jh_{ij}.\\ E^{ij} &= R^{ij}-\tfrac{1}{2}(R(g)-2\Lambda)g^{ij}.\\ \Pi^{ij} &= \big(\tfrac{2}{n-1}tr_g\pi\pi^{ij}-2\pi^i_{\ k}\pi^{kj}+\tfrac{1}{2}|\pi|^2_g\,g^{ij}-\tfrac{1}{2(n-1)}(tr_g\pi)^2g^{ij}\big)/(\sqrt{g})^2.\end{aligned}$$

We can express $D\mathbf{\Phi}$ in the matricial following form

$$D\mathbf{\Phi}(g,\pi).(h,p)=\begin{bmatrix}\sqrt{g}(\delta_g\delta_g-\Delta_gtr_g+\Pi-E) & -2K\\ \hat{\pi}\nabla+2\delta_g\pi & 2\delta_g\end{bmatrix}\begin{bmatrix}h\\ p\end{bmatrix}, \qquad (61)$$

with $\hat{\pi}\nabla h = \hat{\pi}_i^{jkl}\nabla_j h_{kl} = (\pi^{jk}\delta_i^l + \pi^{jl}\delta_i^k - \pi^{kl}\delta_i^j)\nabla_j h_{kl}$.

To prove surjectivity of the differential of $\mathbf{\Phi}$, we investigate injectivity of the adjoint operator. Integrating by parts and ignoring boundary terms leads (cf. [9] for example) to the expression of the formal $L^2(d\mu(\mathring{g}))$-adjoint of $D\mathbf{\Phi}(g,\pi)$:

$$\int_{\mathcal{M}} D\mathbf{\Phi}(g,\pi).(h,p)\,(N,X^i) = \int_{\mathcal{M}} (h,p) \bullet D\mathbf{\Phi}(g,\pi)^*(N,X^i).$$

Proposition 8.

$$\begin{aligned}
(h,p) \bullet D\mathbf{\Phi_0}(g,\pi)^* N &= h_{ij}[\nabla^i\nabla^j N - g^{ij}\Delta_g N - [R^{ij} - \tfrac{1}{2}(R(g)-2\Lambda)g^{ij}]N]\sqrt{g} \\
&\quad + N h_{ij}\big(\tfrac{2}{n-1} tr_g\pi \pi^{ij} - 2\pi^i_{\ k}\pi^{kj} + \tfrac{1}{2}|\pi|^2_g\, g^{ij} - \tfrac{1}{2(n-1)}(tr_g\pi)^2 g^{ij}\big)/\sqrt{g} \\
&\quad + N p^{ij}(\tfrac{2}{n-1} tr_g\pi\, g_{ij} - 2\pi_{ij})/\sqrt{g}. \\
(h,p) \bullet D\mathbf{\Phi_i}(g,\pi)^* X^i &= h_{ij}(X^k\nabla_k\pi^{ij} + \nabla_k X^k \pi^{ij} - 2\nabla_k X^{(i}\pi^{j)k}) - 2p^{ij}\nabla_{(i}X_{j)}.
\end{aligned}$$

Then we can put $D\mathbf{\Phi}^*$ in the matricial form

$$D\mathbf{\Phi}(g,\pi)^*.(N,X) = \begin{bmatrix} \sqrt{g}(\nabla^2 - g\Delta_g + \Pi - E) & \nabla\pi - \hat{\pi}\nabla \\ -2K & -\mathcal{L}_g \end{bmatrix} \begin{bmatrix} N \\ X \end{bmatrix}, \tag{62}$$

with

$$\begin{aligned}
(\nabla\pi - \hat{\pi}\nabla)X &= \mathcal{L}_X\pi = \nabla_X\pi^{ij} - \hat{\pi}_l^{kij}\nabla_k X^l. \\
\mathcal{L}_g(X) &= \mathcal{L}_X g = 2\,\nabla_{(i}X_{j)} = 2\,S(X).
\end{aligned}$$

Denoting $D\mathbf{\Phi}(g,\pi)^*_1.\xi$ and $D\mathbf{\Phi}(g,\pi)^*_2.\xi$ the two components of $D\mathbf{\Phi}(g,\pi)^*$ in (62).
$L^2_\delta\xi$ (resp. $W^{1,2}_\delta\mathring{\nabla}\xi$) is the set of terms of the form $u\,\xi$ (resp. $u\,\mathring{\nabla}\xi$) such that $||u||_{2,\delta} \leqslant C$ (resp. $||u||_{1,2,\delta} \leqslant C$), where C is a constant depending on $\mathring{g}, \delta$ et $||(g,\pi)||_{\mathcal{F}}$.

$$\begin{aligned}
D\mathbf{\Phi}(g,\pi)^*_1.\xi &= [\nabla_i\nabla_j N - g_{ij}\Delta_g N + (\Pi_{ij} - E_{ij})]N]\sqrt{g} + (\nabla\pi - \hat{\pi}\nabla)X. \\
(\nabla\pi - \hat{\pi}\nabla)X &= X^k\nabla_k\pi_{ij} - (\pi^k_i\delta_{lj} + \pi^k_j\delta_{li} - \pi_{ij}\delta^k_l)\nabla_k X^l \\
&= L^2_\delta X + W^{1,2}_\delta\mathring{\nabla}X + (n-1)\tau(2\mathring{S}(X) - \mathring{g}tr_{\mathring{g}}\mathring{S}(X)). \\
\Pi(g,\pi)N &= L^2_\delta N + \Pi(\mathring{g},\mathring{\pi})N \\
&= L^2_\delta N + -\tfrac{1}{2}(n-1)(n-4)\tau^2\mathring{g}N.
\end{aligned}$$

So we have the integrability condition

$$\Pi(g,\pi) + \tfrac{1}{2}(n-1)(n-4)\tau^2\mathring{g} \in L^2_\delta. \tag{63}$$

Taking into account (42) and (47),

$$E + (n-1)\mathring{g} - \tfrac{1}{2}n(n-1)\tau^2\mathring{g} \in L^2_\delta. \tag{64}$$

On one hand,

$$\begin{aligned} D\mathbf{\Phi}(g,\pi)^*_1.\xi/\sqrt{g} &= \nabla^2 N - g\Delta_g N + (n-1)\mathring{g} + [\Pi + \tfrac{1}{2}(n-1)(n-4)\tau^2\mathring{g}]N \\ &\quad +(n-1)\tau(2\mathring{S}(X) - \mathring{g}tr_{\mathring{g}}\mathring{S}(X)) - [E + (n-1)\mathring{g} - \tfrac{1}{2}n(n-1)\tau^2\mathring{g}]N] \\ &\quad -(n-1)(n-2)\tau^2\mathring{g}N + L^2_\delta\xi + W^{1,2}_\delta\mathring{\nabla}X \\ &= \nabla^2 N - g\Delta_g N + (n-1)g + (n-1)\tau(2\mathring{S}(X) - \mathring{g}tr_{\mathring{g}}\mathring{S}(X)) \\ &\quad -(n-1)(n-2)\tau^2\mathring{g}N + L^2_\delta\xi + W^{1,2}_\delta\mathring{\nabla}X. \end{aligned}$$

On the other hand,

$$\begin{aligned} D\mathbf{\Phi}(g,\pi)^*_2.\xi &= -2KN - 2S(X) \\ &= -2(\mathring{S}(X) + \tau\mathring{g}N) + W^{1,2}_\delta\xi. \end{aligned}$$

From the definition of the operator $T = \nabla^2 N - Ng$ and the expression of $\mathring{S}$ as a function of $D\mathbf{\Phi}(g,\pi)^*_2.\xi$, we obtain

$$\begin{aligned} D\mathbf{\Phi}(g,\pi)^*_1.\xi/\sqrt{g} &= T - gtr_g T + (n-1)\tau(2\mathring{S}(X) - \mathring{g}tr_{\mathring{g}}\mathring{S}(X)) \\ &\quad -(n-1)(n-2)\tau^2\mathring{g}N + L^2_\delta\xi + W^{1,2}_\delta\mathring{\nabla}X. \qquad (65) \\ D\mathbf{\Phi}(g,\pi)^*_2.\xi &= -2(\mathring{S}(X) + \tau\mathring{g}N) + W^{1,2}_\delta\xi. \qquad (66) \end{aligned}$$

It is useful to restructure $D\mathbf{\Phi}^*$ into the operator P^* defined by

$$\begin{aligned} P^*(\xi) = P^*_{(g,\pi)}(\xi) &= \begin{bmatrix} g^{1/4}\left(\nabla^i\nabla_j N - \delta^i_j\Delta_g N + (S^i_j - E^i_j)N\right) + g^{-1/4}\mathcal{L}_X\pi^i_j \\ -2g^{-1/4}\nabla_l(K^i_j N + S(X)^i_j) \end{bmatrix} \\ &= \zeta \circ \begin{bmatrix} 1 & 0 \\ 0 & \nabla \end{bmatrix} \circ D\mathbf{\Phi}(g,\pi)^*\xi\,, \qquad (67) \end{aligned}$$

where $g^{1/4} = (det(g)/det(\mathring{g}))^{1/4}\, d\mu(\mathring{g})$ is a density of weight $\frac{1}{2}$ and

$$\zeta = \zeta(g) = \begin{bmatrix} g^{-1/4}g_{jk} & 0 \\ 0 & g^{1/4}g^{ik} \end{bmatrix}. \qquad (68)$$

Finally, we can put $P^*_{(g,\pi)}(\xi)$ into the form

$$P^*_{(g,\pi)}(\xi) = \begin{pmatrix} g^{-1/4}\, D\mathbf{\Phi}(g,\pi)^*_1.\xi \\ g^{1/4}\,\nabla D\mathbf{\Phi}(g,\pi)^*_2.\xi \end{pmatrix}. \qquad (69)$$

Expression (67) of P^* allows us to rewrite the $L^2(d\mu(\mathring{g}))$-adjoint of P^* as follows

$$P_{(g,\pi)} = D\mathbf{\Phi}(g,\pi) \circ \begin{bmatrix} 1 & 0 \\ 0 & -\delta_g \end{bmatrix} \circ \zeta\,, \qquad (70)$$

with $\delta_g q = \nabla^l(q^{ij}_l)$ so that $P(f^i_j, q^j_{li}) = D\mathbf{\Phi}(f_{ij}, q^{ij}_l)$ and so the composition PP^* is well defined.

8. Elliptic estimates relative to the adjoint

In this section, we gather elliptic estimates satisfied by the adjoint operator of $D\mathbf{\Phi}$.

Proposition 9. *Set $\delta \in]-(n+1)/2\,,0]$, with $n=3$, and $\delta \neq -(n-1)/2$. There exists a positive constant $C = C\,(\mathring{g}, \lambda, \delta, ||g||_{\mathcal{F}})$ such that the following elliptic estimate is satisfied: $\forall \xi \in W^{2,2}_{-\delta}(\mathcal{T})$,*

$$||\xi||_{2,2,-\delta} \leqslant c\left(||D\mathbf{\Phi}(g,\pi)^*_1.\xi||_{2,-\delta} + ||D\mathbf{\Phi}(g,\pi)^*_2.\xi||_{1,2,-\delta}\right) + C\,||\xi||_{1,2,-2\delta}\,, \tag{71}$$

Proof: Considering expression (66) of $\mathring{S}$ as a function of $D\mathbf{\Phi}(g,\pi)^*_2.\xi$,

$$\begin{aligned} T - gtr_g T &= D\mathbf{\Phi}(g,\pi)^*_1.\xi/\sqrt{g} + (n-1)\tau\big(D\mathbf{\Phi}(g,\pi)^*_2.\xi - \tfrac{1}{2}\mathring{g}tr_{\mathring{g}} D\mathbf{\Phi}(g,\pi)^*_2.\xi\big) \\ &\quad + L^2_\delta \xi + W^{1,2}_\delta \mathring{\nabla} X \end{aligned} \tag{72}$$

From Proposition 3 and (8),

$$\begin{aligned} ||N||_{2,2,-\delta} &\leqslant c\left(||D\mathbf{\Phi}(g,\pi)^*_1.\xi||_{2,-\delta} + (n-1)\tau\,(1+\tfrac{n^2}{4})^{\frac{1}{2}}||D\mathbf{\Phi}(g,\pi)^*_2.\xi||_{2,-\delta}\right) \\ &\quad + C\,(||\xi||_{\infty,-2\delta} + ||\mathring{\nabla}\xi||_{3,-2\delta}). \end{aligned}$$

Using (15), (16) and Sobolev inclusion ($\delta \leqslant 0$), there exists a positive constant $C = C\,(\mathring{g}, \lambda, \delta, ||g||_{\mathcal{F}})$ such that

$$\begin{aligned} ||N||_{2,2,-\delta} &\leqslant c\left(||D\mathbf{\Phi}(g,\pi)^*_1.\xi||_{2,-\delta} + (n-1)\tau\,(1+\tfrac{n^2}{4})^{\frac{1}{2}}||D\mathbf{\Phi}(g,\pi)^*_2.\xi||_{2,-\delta}\right) \\ &\quad + \varepsilon\,||\xi||_{2,2,-\delta} + C\,||\xi||_{1,2,-2\delta}. \end{aligned} \tag{73}$$

Besides, for every sufficiently regular 1-form X on $\mathcal{M}$, we have the following identity for the metric $\mathring{g}$ (cf equation (29) of [4] for example)

$$\mathring{\nabla}^2_{kj} X_i := \mathring{\nabla}_k \mathring{\nabla}_j X_i = Riem\,\mathring{g}_{ijkl} X^l + \mathring{\nabla}_k \mathring{S}(X)_{ij} + \mathring{\nabla}_j \mathring{S}(X)_{ik} - \mathring{\nabla}_i \mathring{S}(X)_{jk}, \tag{74}$$

Hence

$$\begin{aligned} ||\mathring{\nabla}^2 X||_{2,-\delta} &\leqslant ||Riem\,\mathring{g}\;X||_{2,-\delta} + c\,||\mathring{\nabla}\mathring{S}(X)||_{2,-\delta} \\ &\leqslant c\,||X||_{2,-\delta} + c\,||\mathring{\nabla}\mathring{S}(X)||_{2,-\delta}. \end{aligned} \tag{75}$$

A consequence of Lemma 13 is

$$||X||_{2,-\delta} \leqslant ||X||_{1,2,-\delta} \leqslant c\,||\mathring{S}(X)||_{2,-\delta}, \tag{76}$$

which imply, with (75)

$$||\mathring{\nabla}^2 X||_{2,-\delta} \leqslant c\,||\mathring{S}(X)||_{2,-\delta} + c\,||\mathring{\nabla}\mathring{S}(X)||_{2,-\delta},$$

and considering (76),

$$||X||_{2,2,-\delta} \leqslant c_1\,||\mathring{S}(X)||_{1,2,-\delta}.$$

From (66) and (15),

$$||\mathring{S}(X)||_{1,2,-\delta} \leqslant \tfrac{1}{4}||D\mathbf{\Phi}(g,\pi)^*_2.\xi||_{1,2,-\delta} + n\tau\,||N||_{1,2,-\delta} + \varepsilon||\xi||_{2,2,-\delta} + C\,||\xi||_{1,2,-2\delta}.$$

Thus there exists a constant C depending on $\mathring{g}, \lambda, \varepsilon, \delta$ et $||(g,\pi)||_{\mathcal{F}}$ such that

$$||X||_{2,2,-\delta} - nc_1\,\tau\,||N||_{1,2,-\delta} \leqslant \tfrac{c_1}{4}\,||D\mathbf{\Phi}(g,\pi)^*_2.\xi||_{1,2,-\delta} + \varepsilon||\xi||_{2,2,-\delta} + C\,||\xi||_{1,2,-2\delta}. \tag{77}$$

We can choose $\varepsilon_0 << 1$ so that (73) + ε_0(77) combine to yield (71). □

Remark: it may be possible to extend this result to $\delta = -(n-1)/2$ using operator $\mathring{U}$ introduced later on.

Combining Proposition 9 and Ehrling inequality (11), we get

Corollary 3. *Let $\delta \in]-(n+1)/2\,,0]$, with $n=3$ and $\delta \neq -(n-1)/2$.*
Then the following estimate is verified: $\forall \xi \in W^{2,2}_{-\delta}(\mathcal{T})$,

$$||\xi||_{2,2,-\delta} \leqslant c\left(||D\mathbf{\Phi}(g,\pi)^*_1.\xi||_{2,-\delta} + ||D\mathbf{\Phi}(g,\pi)^*_2.\xi||_{1,2,-\delta}\right) + C\,||\xi||_{2,-2\delta}\,, \tag{78}$$

where C depends on $\mathring{g}, \lambda, \delta$ and $||(g,\pi)||_{\mathcal{F}}$

The next lemma will be very useful during the proof of proposition 10 since it is the Time-symmetric version.

Lemma 16. *In dimension $n=3$, let $\delta \leqslant 0$, then the operator*
$D\mathbf{\Phi}(g,0)^*_0(.,0) : W^{2,2}_{-\delta}(\mathcal{M}) \longrightarrow L^2_{-\delta}(\widetilde{\mathcal{S}})$ is bounded and depends on g in a Lipschitz way,

$$\left|\left|\left[D\mathbf{\Phi}(g,0)^*_0 - D\mathbf{\Phi}(\tilde{g},0)^*_0\right](N,0)\right|\right|_{2,-\delta} \leqslant C||g-\tilde{g}||_{\mathcal{F}}\,||N||_{2,2,-\delta}\,, \tag{79}$$

where constant C depends on $\mathring{g}, \delta, ||g||_{\mathcal{F}}$ et $||\tilde{g}||_{\mathcal{F}}$.

Proof: Let us recall the expression of $D\mathbf{\Phi}(g,0)^*_0$

$$D\mathbf{\Phi}(g,0)^*_0.(N,0) = [\nabla_i\nabla_j N - g_{ij}\Delta_g N - [R_{ij} - \tfrac{1}{2}(R(g)-2\Lambda)g_{ij}]N]\sqrt{g}. \tag{80}$$

Let us begin by showing $D\mathbf{\Phi}(g,0)^*_0$ is bounded.
We set $O = \nabla^2 N - g\,\Delta_g N = L(\nabla^2 N)$ where L is a linear invertible operator, so

$$\begin{aligned} ||O||_{2,-\delta} &\leqslant c\,||\nabla^2 N||_{2,-\delta} = c\,\left(||\mathring{\nabla}^2 N||_{2,-\delta} + ||A\,dN||_{2,-\delta}\right) \\ &\leqslant C\,||N||_{2,2,-\delta}. \end{aligned}$$

Indeed, $A\,dN \simeq g^{-1}\mathring{\nabla}g\;dN$. Using Hölder inequality (8) , (13) and Sobolev inclusion,

$$\begin{aligned} ||A\,dN||_{2,-\delta} &\leqslant ||g^{-1}||_{\infty,0}||\mathring{\nabla}g\,dN||_{2,-\delta} \\ &\leqslant c\,||\mathring{\nabla}g||_{1,2,\delta}||dN||_{1,2,-2\delta} \\ &\leqslant C\,||N||_{2,2,-\delta}. \end{aligned} \tag{81}$$

$$\begin{aligned} ||D\mathbf{\Phi}(g,0)^*_0.(N,0)/\sqrt{g}||_{2,-\delta} \leqslant\ & ||O||_{2,-\delta} + ||(Ric\,g - Ric\,\mathring{g})\,N||_{2,-\delta} + ||(n-1)\mathring{g}N||_{2,-\delta} \\ & + ||[Ric\,\mathring{g} + (n-1)\mathring{g}]\,N||_{2,-\delta} + ||\tfrac{1}{2}n(n-1)\tau^2\,g\,N||_{2,-\delta} \\ & + \tfrac{1}{2}\left|\left|\left[R(g) - 2\Lambda + n(n-1)\tau^2\right]\,g\,N\right|\right|_{2,-\delta}. \end{aligned}$$

Considering (56), (42),

$$\begin{aligned} ||(Ric\,g - Ric\,\mathring{g})N||_{2,-\delta} &\leqslant C\,||N||_{2,2,-\delta} \\ ||[Ric\,\mathring{g} + (n-1)\mathring{g}]N||_{2,-\delta} &\leqslant c\,||N||_{2,2,-\delta}. \end{aligned}$$

For the scalar curvature term, using Hölder inequality (8) and Sobolev inclusion together with (47) and (57),

$$\begin{aligned} ||(R(g) - 2\Lambda + n(n-1)\tau^2)\,g\,N||_{2,-\delta} &\leqslant ||(R(\mathring{g}) - 2\Lambda + n(n-1)\tau^2)\,g\,N||_{2,-\delta} \\ &\quad + ||(R(g) - R(\mathring{g}))\,g\,N||_{2,-\delta} \\ &\leqslant C\,||N||_{2,2,-\delta}. \end{aligned}$$

$$\begin{aligned} ||(n-1)\mathring{g}N||_{2,-\delta} &\leqslant (n-1)||\mathring{g}||_{\infty,0}||N||_{2,-\delta} \\ &\leqslant c\,||N||_{2,2,-\delta}. \end{aligned}$$

Similarly,

$$||\tfrac{1}{2}n(n-1)\tau^2 gN||_{2,-\delta} \leqslant c\,||N||_{2,2,-\delta}.$$

We end up with

$$||D\mathbf{\Phi}(g,0)^*_0.(N,0)/\sqrt{g}||_{2,-\delta} \leqslant C\,||N||_{2,2,-\delta}$$

and finally

$$||D\mathbf{\Phi}(g,0)^*_0.(N,0)||_{2,-\delta} \leqslant C\,||\sqrt{g}||_{\infty,0}||N||_{2,2,-\delta} \leqslant C\,||N||_{2,2,-\delta}, \tag{82}$$

where C is a constant depending upon $\mathring{g}, \delta$ and $||g||_{\mathcal{F}}$.

Proof of (79): $\widetilde{\nabla}$, $\widetilde{\Delta}$, $Ric(\tilde{g})$ and $R(\tilde{g})$ will denote The Levi-Civita connection, the Laplacian, the Ricci tensor and the scalar curvature of the Riemannian metric $\tilde{g}$.
In order to lighten notations, we set

$$D\mathbf{\Phi_0}(g)^*N := D\mathbf{\Phi}(g,0)^*_0(N,0) \text{ and } D\mathbf{\Phi_0}(\tilde{g})^*N := D\mathbf{\Phi}(\tilde{g},0)^*_0(N,0)$$

$$[D\mathbf{\Phi_0}(g)^* - D\mathbf{\Phi_0}(\tilde{g})^*]N = (\sqrt{g}-\sqrt{\tilde{g}})\,\frac{D\mathbf{\Phi_0}(g)^*N}{\sqrt{g}} + \sqrt{\tilde{g}}\left[\frac{D\mathbf{\Phi_0}(g)^*N}{\sqrt{g}} - \frac{D\mathbf{\Phi_0}(\tilde{g})^*N}{\sqrt{\tilde{g}}}\right].$$

$$\begin{aligned} \left\|[D\mathbf{\Phi_0}(g)^* - D\mathbf{\Phi_0}(\tilde{g})^*]N\right\|_{2,-\delta} &\leqslant ||g-\tilde{g}||_{\mathcal{F}}\left\|\frac{D\mathbf{\Phi_0}(g)^*N}{\sqrt{g}}\right\|_{2,-2\delta} \\ &\quad + c\left\|\frac{D\mathbf{\Phi_0}(g)^*N}{\sqrt{g}} - \frac{D\mathbf{\Phi_0}(\tilde{g})^*N}{\sqrt{\tilde{g}}}\right\|_{2,-\delta}. \end{aligned} \tag{83}$$

$$\begin{aligned} \Big(\frac{D\mathbf{\Phi_0}(g)^*N}{\sqrt{g}} - \frac{D\mathbf{\Phi_0}(\tilde{g})^*N}{\sqrt{\tilde{g}}}\Big) &= (\nabla-\widetilde{\nabla})\,dN + g\,\Delta_g N - \tilde{g}\,\widetilde{\Delta}N - [Ric(g) - Ric(\tilde{g})]N \\ &\quad + \tfrac{1}{2}\left[(R(g)-2\Lambda)g - (R(\tilde{g})-2\Lambda)\tilde{g}\right]N. \end{aligned}$$

$$\begin{aligned} \left\|\frac{D\mathbf{\Phi_0}(g)^*N}{\sqrt{g}} - \frac{D\mathbf{\Phi_0}(\tilde{g})^*N}{\sqrt{\tilde{g}}}\right\|_{2,-\delta} &\leqslant ||(\nabla-\widetilde{\nabla})\,dN||_{2,-\delta} + ||g\,\Delta_g N - \tilde{g}\,\widetilde{\Delta}N||_{2,-\delta} \\ &\quad + \tfrac{1}{2}\left\|\left[(R(g)-2\Lambda)g - (R(\tilde{g})-2\Lambda)\tilde{g}\right]N\right\|_{2,-\delta} \\ &\quad - ||[Ric(g) - Ric(\tilde{g})]N||_{2,-\delta}. \end{aligned}$$

- For the Hessian:

$$\nabla - \widetilde{\nabla} = (g^{-1} - \tilde{g}^{-1})\mathring{\nabla}g + \tilde{g}^{-1}\mathring{\nabla}(g-\tilde{g}). \tag{84}$$

Using Hölder and Sobolev weighted inequalities, Sobolev inclusion ($\delta \leqslant 0$) and (13),

$$||(\nabla-\widetilde{\nabla})\,dN||_{2,-\delta} \leqslant C\,||g-\tilde{g}||_{2,2,\delta}||N||_{2,2,-\delta}. \tag{85}$$

- For the Laplacian:

$$\begin{aligned} g\,\Delta_g N - \tilde{g}\,\widetilde{\Delta} N &= g\,\Delta_g N - \tilde{g}\,\Delta_g N + \tilde{g}\,\Delta_g N - \tilde{g}\,\widetilde{\Delta} N \\ &= (g-\tilde{g})\Delta_g N + \tilde{g}(\Delta_g N - \widetilde{\Delta} N) \\ &= (g-\tilde{g})g^{-1}\nabla dN + \tilde{g}(g^{-1}-\tilde{g}^{-1})\nabla dN + \tilde{g}\tilde{g}^{-1}(\nabla - \widetilde{\nabla})dN. \end{aligned}$$

Using Hölder inequality (8), Sobolev inclusion ($\delta \leqslant 0$) and Sobolev inequality,

$$||g\,\Delta_g N - \tilde{g}\,\widetilde{\Delta} N||_{2,-\delta} \leqslant c\,||g-\tilde{g}||_{2,2,\delta}||\nabla dN||_{2,-\delta} + c\,||(\nabla - \widetilde{\nabla})dN||_{2,-\delta}.$$

Considering (81) and given that $\nabla \simeq A + \mathring{\nabla}$,

$$||\nabla dN||_{2,-\delta} \leqslant C\,||N||_{2,2,-\delta}\,, \tag{86}$$

Using (85) et (86) , we get

$$||g\,\Delta_g N - \tilde{g}\,\widetilde{\Delta} N||_{2,\delta} \leqslant C\,||g-\tilde{g}||_{2,2,\delta}||N||_{2,2,-\delta}.$$

- For the Ricci tensor:
We define $\widetilde{A}_{i\,j}^{\,k} = \widetilde{\Gamma}_{i\,j}^{\,k} - \mathring{\Gamma}_{i\,j}^{\,k}$.
Set $T := \widetilde{\nabla} - \nabla = \tilde{g}^{-1}\mathring{\nabla}\tilde{g} - g^{-1}\mathring{\nabla}g = (g^{-1}-\tilde{g}^{-1})\mathring{\nabla}g + \tilde{g}^{-1}\mathring{\nabla}(g-\tilde{g})$.
Using Hölder inequality (8), Sobolev inclusion ($\delta \leqslant 0$) and Sobolev inequality,

$$||T||_{1,2,\delta} \leqslant C||g-\tilde{g}||_{2,2,\delta}. \tag{87}$$

We can show, adding and substracting $Ric(\mathring{g})$ and using (53), that

$$[Ric(g) - Ric(\tilde{g})]N \simeq (\mathring{\nabla}T + \widetilde{A}T + T^2)N\,,$$

which leads to

$$||[Ric(g) - Ric(\tilde{g})]N||_{2,-\delta} \leqslant ||\mathring{\nabla}TN||_{2,-\delta} + ||\widetilde{A}TN||_{2,-\delta} + ||T^2N||_{2,-\delta}. \tag{88}$$

Using Hölder and Sobolev inequalities along with l'inclusion de Sobolev ($\delta \leqslant 0$) and (87)

$$||\mathring{\nabla}TN||_{2,-\delta} \leqslant C\,||g-\tilde{g}||_{2,2,\delta}||N||_{2,2,-\delta}$$

The same method for the term $\widetilde{A}TN$ gives , considering (13)

$$||\widetilde{A}TN||_{2,-\delta} \leqslant C\,||g-\tilde{g}||_{2,2,\delta}||N||_{2,2,-\delta}$$

Using (8) , (14) , Sobolev inclusion ($\delta \leqslant 0$) and Sobolev inequality together with (87) ,

$$||T^2N||_{2,-\delta} \leqslant C\,||g-\tilde{g}||^2_{2,2,\delta}||N||_{2,2,-\delta}.$$

Replacing in (88), we obtain

$$||[Ric(g) - Ric(\tilde{g})]N||_{2,-\delta} \leqslant C\,||g-\tilde{g}||_{2,2,\delta}||N||_{2,2,-\delta}. \tag{89}$$

- For the scalar curvature,

$$\begin{aligned}(R(g)-2\Lambda)g\,-(R(\tilde g)-2\Lambda)\tilde g &= (g-\tilde g)(R(g)-2\Lambda)+\tilde g\tilde g^{-1}(Ric\,g-Ric\,\tilde g)\\ &\quad+\tilde g(g^{-1}-\tilde g^{-1})Ric\,g\\ &= (g-\tilde g)\left[R(g)-2\Lambda+n(n-1)\tau^2\right]\\ &\quad-n(n-1)\tau^2(g-\tilde g)+\tilde g\tilde g^{-1}(Ric\,g-Ric\,\tilde g)\\ &\quad+\tilde g(g^{-1}-\tilde g^{-1})\big\{Ric\,g-Ric\,\mathring g\big\}.\end{aligned}$$

Hölder and Sobolev inequalities as well as Sobolev inclusion ($\delta\leqslant 0$) and (89) yield

$$||[(R(g)-2\Lambda)g\,-(R(\tilde g)-2\Lambda)\tilde g]N||_{2,-\delta}\leqslant C\,||g-\tilde g||_{2,2,\delta}||N||_{2,2,\delta}\,,$$

given that $\forall u\in L^\infty_0,\forall v\in L^2_\delta$ such that $||vN||_{2,-\delta}\leqslant C\,||N||_{2,2,-\delta}$,

$$\begin{aligned}||(g-\tilde g)\,u\,v\,N||_{2,-\delta} &\leqslant ||g-\tilde g||_{\infty,0}||u||_{\infty,0}||vN||_{2,-\delta}\\ &\leqslant C\,||g-\tilde g||_{2,2,\delta}||v||_{2,\delta}||N||_{2,2,-\delta},\end{aligned}$$

where C is a positive constant depending on $\mathring g,\delta$ et $||g||_{\mathcal F}$.

Putting the pieces all together in (83) and taking (82) into account lead to

$$||[D\mathbf{\Phi_0}(g)^*-D\mathbf{\Phi_0}(\tilde g)^*]N||_{2,-\delta}\leqslant C\,||g-\tilde g||_{2,2,\delta}||N||_{2,2,\delta}.\ \square$$

The dependence in (g,π) of P^* is controled as follows :

Proposition 10. *Let $\delta\leqslant 0$, then in dimension 3, operator $P^*:W^{2,2}_{-\delta}(\mathcal T)\longrightarrow L^2_{-\delta}$ is bounded and satisfy*

$$||\xi||_{2,2,-\delta}\leqslant c\,||P^*\xi||_{2,-\delta}+C\,||\xi||_{1,2,-2\delta}\,, \tag{90}$$

where C depends on $\mathring g,\delta$ and $||(g,\pi)||_{\mathcal F}$.
Moreover, $P^*_{(g,\pi)}$ depends on $(g,\pi)\in\mathcal F$ in a Lipschitz way,

$$||(P^*_{(g,\pi)}-P^*_{(\tilde g,\tilde\pi)})\,\xi||_{2,-\delta}\leqslant C_1||(g-\tilde g,\pi-\tilde\pi)||_{\mathcal F}\,||\xi||_{2,2,-\delta}\,, \tag{91}$$

where constant C_1 depends on $\mathring g,\delta,||(g,\pi)||_{\mathcal F}$ et $||(\tilde g,\tilde\pi)||_{\mathcal F}$.

Proof: Let us begin by showing P^* is bounded, *i.e.*

$$||P^*\,\xi||_{2,-\delta}\leqslant C\,||\xi||_{2,2,-\delta}. \tag{92}$$

Set

$$\begin{cases}P^*=P^*_{(g,\pi)}\\ D\mathbf{\Phi}^*_1=D\mathbf{\Phi}(g,\pi)^*_1\\ D\mathbf{\Phi}^*_2=D\mathbf{\Phi}(g,\pi)^*_2\end{cases}.$$

From (69),

$$\begin{aligned}||P^*\,\xi||_{2,-\delta} &\leqslant c\,(||D\mathbf{\Phi}^*_1.\xi||_{2,-\delta}+||\nabla D\mathbf{\Phi}^*_2.\xi||_{2,-\delta})\\ &\leqslant c\,(||D\mathbf{\Phi}^*_1.\xi||_{2,-\delta}+||\mathring\nabla D\mathbf{\Phi}^*_2.\xi||_{2,-\delta}+||AD\mathbf{\Phi}^*_2.\xi||_{2,-\delta}).\end{aligned} \tag{93}$$

From (65),(8), (13), Sobolev inequality and inclusion ($\delta\leqslant 0$)

$$\begin{aligned}||D\mathbf{\Phi}^*_1.\xi||_{2,-\delta} &\leqslant c\left(||T||_{2,-\delta}+||\mathring S(X)||_{2,-\delta}+||N||_{2,-\delta}\right)+C\left(||\xi||_{\infty,-2\delta}+||\mathring\nabla X||_{1,2,-2\delta}\right)\\ &\leqslant c\left(||N||_{2,2,-\delta}+||X||_{1,2,-\delta}+||N||_{2,-\delta}\right)+C\left(||\xi||_{2,-\delta}+||\mathring\nabla X||_{1,2,-\delta}\right)\\ &\leqslant C\,||\xi||_{2,2,-\delta}.\end{aligned} \tag{94}$$

From (66) along with (8), (13), Sobolev inequality and inclusion ($\delta \leqslant 0$)

$$||D\mathbf{\Phi}_2^*.\xi||_{2,-\delta} \leqslant c\left(||\mathring{S}(X)||_{2,-\delta} + ||N||_{2,-\delta}\right) + C\,||\xi||_{1,2,-2\delta}. \tag{95}$$

$$\begin{aligned} ||AD\mathbf{\Phi}_2^*.\xi||_{2,-\delta} &\leqslant c\left(||A\mathring{S}(X)||_{2,-\delta} + ||AN||_{2,-\delta}\right) + ||\xi||_{\infty,-2\delta} \\ &\leqslant C\left(||\mathring{S}(X)||_{1,2,-\delta} + ||N||_{1,2,-\delta}\right) + ||\xi||_{2,2,-2\delta} \\ ||\mathring{\nabla}D\mathbf{\Phi}_2^*.\xi||_{2,-\delta} &\leqslant c\left(||\mathring{\nabla}\mathring{S}(X)||_{2,-\delta} + ||\mathring{\nabla}N||_{2,-\delta}\right) + ||\xi||_{\infty,-2\delta} \\ &\leqslant C\,||\xi||_{2,2,-\delta}. \end{aligned}$$

Consequently,

$$||D\mathbf{\Phi}_2^*.\xi||_{2,-\delta} \leqslant ||D\mathbf{\Phi}_2^*.\xi||_{1,2,-\delta} \leqslant C\,||\xi||_{2,2,-\delta}. \tag{96}$$

Every term of (93) is controled by $||\xi||_{2,2,-2\delta}$ leading to (92).
Estimate (90) verified by P^* directly comes from (71). We now look into the Lipschitz behaviour of P^*:
Set

$$\begin{cases} \widetilde{P}^* = P^*_{(\tilde{g},\tilde{\pi})} \\ D\widetilde{\mathbf{\Phi}}_1^* = D\mathbf{\Phi}(\tilde{g},\tilde{\pi})_1^* \\ D\widetilde{\mathbf{\Phi}}_2^* = D\mathbf{\Phi}(\tilde{g},\tilde{\pi})_2^* \end{cases}.$$

$$(P^* - \widetilde{P}^*)\,\xi = \begin{pmatrix} g^{-1/4}\,D\mathbf{\Phi}_1^*.\xi - \tilde{g}^{-1/4}\,D\widetilde{\mathbf{\Phi}}_1^*.\xi \\ g^{1/4}\,\nabla D\mathbf{\Phi}_2^*.\xi - \tilde{g}^{1/4}\,\tilde{\nabla}D\widetilde{\mathbf{\Phi}}_2^*.\xi \end{pmatrix} = \begin{pmatrix} E \\ F \end{pmatrix}.$$

So

$$||(P^* - \widetilde{P}^*)\,\xi||_{2,-\delta} \leqslant ||E||_{2,-\delta} + ||F||_{2,-\delta}. \tag{97}$$

$$\begin{aligned} E &= g^{-1/4}\,D\mathbf{\Phi}_1^*.\xi - \tilde{g}^{-1/4}\,D\widetilde{\mathbf{\Phi}}_1^*.\xi \\ &= (g^{-1/4} - \tilde{g}^{-1/4})\,D\mathbf{\Phi}_1^*.\xi + \tilde{g}^{-1/4}\,(D\mathbf{\Phi}_1^*.\xi - D\widetilde{\mathbf{\Phi}}_1^*.\xi). \end{aligned}$$

Using (8), Sobolev inequality and inclusion ($\delta \leqslant 0$)

$$\begin{aligned} ||E||_{2,-\delta} &\leqslant ||(g^{-1/4} - \tilde{g}^{-1/4})\,D\mathbf{\Phi}_1^*.\xi||_{2,-\delta} + ||\tilde{g}^{-1/4}\,(D\mathbf{\Phi}_1^*.\xi - D\widetilde{\mathbf{\Phi}}_1^*.\xi)||_{2,-\delta} \\ &\leqslant c\,||g - \tilde{g}||_{\mathcal{F}}||D\mathbf{\Phi}_1^*.\xi||_{2,-\delta} + c\,||D\mathbf{\Phi}_1^*.\xi - D\widetilde{\mathbf{\Phi}}_1^*.\xi||_{2,-\delta}. \end{aligned}$$

From (65),

$$\begin{aligned} D\mathbf{\Phi}_1^*.\xi - D\widetilde{\mathbf{\Phi}}_1^*.\xi &= [D\mathbf{\Phi}(g,0)_0^* - D\mathbf{\Phi}(\tilde{g},0)_0^*]\,(N,0) + (S\sqrt{g} - \tilde{S}\sqrt{\tilde{g}})N + X\mathring{\nabla}(\pi - \tilde{\pi}) \\ &\quad + (\pi - \tilde{\pi})\mathring{\nabla}X + AX(\pi - \tilde{\pi}). \end{aligned}$$

Using (8), (13), Sobolev inequality and inclusion ($\delta \leqslant 0$)

$$\begin{aligned} ||(\pi - \tilde{\pi})\mathring{\nabla}X||_{2,-\delta} + ||X\mathring{\nabla}(\pi - \tilde{\pi})||_{2,-\delta} &\leqslant ||\pi - \tilde{\pi}||_{1,2,\delta}||\mathring{\nabla}X||_{1,2,-2\delta} \\ &\quad + ||\mathring{\nabla}(\pi - \tilde{\pi})||_{2,\delta}||X||_{\infty,-2\delta} \\ &\leqslant c\,||\pi - \tilde{\pi}||_{1,2,\delta}\,||X||_{2,2,-\delta}. \end{aligned}$$

$$\begin{aligned} ||AX(\pi - \tilde{\pi})||_{2,-\delta} &\leqslant ||A(\pi - \tilde{\pi})||_{2,\delta}||X||_{\infty,-2\delta} \\ &\leqslant C\,||\pi - \tilde{\pi}||_{1,2,\delta}\,||X||_{2,2,-\delta}. \end{aligned}$$

$$\begin{aligned} S\sqrt{g}-\tilde{S}\sqrt{\tilde{g}} &\sim \frac{1}{\sqrt{g}}g^{-1}\pi^2-\frac{1}{\sqrt{\tilde{g}}}\tilde{g}^{-1}\tilde{\pi}^2 \\ &\sim \frac{1}{\sqrt{g}}(g^{-1}-\tilde{g}^{-1})\pi^2+\frac{1}{\sqrt{g}}\tilde{g}^{-1}(\pi^2-\tilde{\pi}^2)+(\frac{1}{\sqrt{g}}-\frac{1}{\sqrt{\tilde{g}}})\tilde{g}^{-1}\tilde{\pi}^2. \end{aligned}$$

$$||(S\sqrt{g}-\tilde{S}\sqrt{\tilde{g}})N||_{2,-\delta} \leqslant C\,||(g-\tilde{g},\pi-\tilde{\pi})||_{\mathcal{F}}||N||_{2,2,-\delta}.$$

Given (79),

$$||D\mathbf{\Phi}_1^*.\xi-D\widetilde{\mathbf{\Phi}}_1^*.\xi||_{2,-\delta}\leqslant C\,||(g-\tilde{g},\pi-\tilde{\pi})||_{\mathcal{F}}||\xi||_{2,2,-\delta}$$

and taking (94) into account,

$$||E||_{2,-\delta}\leqslant C\,||(g-\tilde{g},\pi-\tilde{\pi})||_{\mathcal{F}}||\xi||_{2,2,-\delta}. \tag{98}$$

$$\begin{aligned} F &= g^{1/4}\,\nabla D\mathbf{\Phi}_2^*.\xi-\tilde{g}^{1/4}\,\tilde{\nabla}D\widetilde{\mathbf{\Phi}}_2^*.\xi \\ &= g^{1/4}\,(\nabla-\tilde{\nabla})D\mathbf{\Phi}_2^*.\xi+(g^{1/4}-\tilde{g}^{1/4})\,\tilde{\nabla}D\widetilde{\mathbf{\Phi}}_2^*.\xi+g^{1/4}\,\tilde{\nabla}(D\mathbf{\Phi}_2^*.\xi-D\widetilde{\mathbf{\Phi}}_2^*.\xi). \end{aligned}$$

Using (84),(8), (13), Sobolev inequality and inclusion ($\delta\leqslant 0$)

$$\begin{aligned} ||F||_{2,-\delta} &\leqslant c\,||\nabla-\tilde{\nabla}||_{1,2,\delta}||D\mathbf{\Phi}_2^*.\xi||_{1,2,-2\delta}+||g^{1/4}-\tilde{g}^{1/4}||_{\infty,-\delta}||\tilde{\nabla}D\widetilde{\mathbf{\Phi}}_2^*.\xi||_{2,-2\delta} \\ &\quad +c\,||\mathring{\tilde{\nabla}}(D\mathbf{\Phi}_2^*.\xi-D\widetilde{\mathbf{\Phi}}_2^*.\xi)||_{2,-\delta}+c\,||A(D\mathbf{\Phi}_2^*.\xi-D\widetilde{\mathbf{\Phi}}_2^*.\xi)||_{2,-\delta} \\ &\leqslant C\,||g-\tilde{g}||_{\mathcal{F}}||D\mathbf{\Phi}_2^*.\xi||_{1,2,-2\delta}+c\,||g-\tilde{g}||_{\mathcal{F}}||\tilde{\nabla}D\widetilde{\mathbf{\Phi}}_2^*.\xi||_{2,-2\delta} \\ &\quad +c\,||\mathring{\tilde{\nabla}}(D\mathbf{\Phi}_2^*.\xi-D\widetilde{\mathbf{\Phi}}_2^*.\xi)||_{2,-\delta}+c\,||A(D\mathbf{\Phi}_2^*.\xi-D\widetilde{\mathbf{\Phi}}_2^*.\xi)||_{2,-\delta}. \end{aligned}$$

Considering (95) and (96),

$$\begin{aligned} ||F||_{2,-\delta} &\leqslant C\,||g-\tilde{g}||_{\mathcal{F}}||\xi||_{2,2,-\delta}+c\,||\mathring{\tilde{\nabla}}(D\mathbf{\Phi}_2^*.\xi-D\widetilde{\mathbf{\Phi}}_2^*.\xi)||_{2,-\delta} \\ &\quad +c\,||A(D\mathbf{\Phi}_2^*.\xi-D\widetilde{\mathbf{\Phi}}_2^*.\xi)||_{2,-\delta}. \end{aligned} \tag{99}$$

$$\begin{aligned} D\mathbf{\Phi}_2^*.\xi-D\widetilde{\mathbf{\Phi}}_2^*.\xi &\sim (K-\tilde{K})N+(A-\tilde{A})X \\ &\sim (\pi-\tilde{\pi})N+(\nabla-\tilde{\nabla})X. \end{aligned}$$

Using (84),(8), (13), Sobolev inequality and inclusion ($\delta\leqslant 0$)

$$\begin{aligned} ||\mathring{\tilde{\nabla}}(D\mathbf{\Phi}_2^*.\xi-D\widetilde{\mathbf{\Phi}}_2^*.\xi)||_{2,-\delta} &\leqslant c\,||\mathring{\tilde{\nabla}}(\pi-\tilde{\pi})||_{2,\delta}||N||_{\infty,-2\delta}+c\,||\pi-\tilde{\pi}||_{1,2,\delta}||\mathring{\tilde{\nabla}}N||_{1,2,-2\delta} \\ &\quad +||\mathring{\tilde{\nabla}}(\nabla-\tilde{\nabla})||_{2,\delta}||X||_{\infty,-2\delta}+||\nabla-\tilde{\nabla}||_{1,2,\delta}||\mathring{\tilde{\nabla}}X||_{1,2,-2\delta} \\ &\leqslant c\,||\pi-\tilde{\pi}||_{1,2,\delta}||N||_{2,2,-\delta}+c\,||\nabla-\tilde{\nabla}||_{1,2,\delta}||X||_{2,2,-\delta} \\ &\leqslant C\,||(g-\tilde{g},\pi-\tilde{\pi})||_{\mathcal{F}}||\xi||_{2,2,-\delta}. \end{aligned}$$

In the same way,

$$\begin{aligned} ||A(D\mathbf{\Phi}_2^*.\xi-D\widetilde{\mathbf{\Phi}}_2^*.\xi)||_{2,-\delta} &\leqslant c\,||A(\pi-\tilde{\pi})||_{2,\delta}||N||_{\infty,-2\delta}+||A(\nabla-\tilde{\nabla})||_{2,\delta}||X||_{\infty,-2\delta} \\ &\leqslant C\,||(g-\tilde{g},\pi-\tilde{\pi})||_{\mathcal{F}}||\xi||_{2,2,-\delta}. \end{aligned}$$

We deduce from (99)

$$||F||_{2,-\delta}\leqslant C\,||(g-\tilde{g},\pi-\tilde{\pi})||_{\mathcal{F}}||\xi||_{2,2,-\delta}. \tag{100}$$

and résult (91) arises from (97), considering (98) and (100). □

We show in the following proposition that the estimate (71) of Proposition 9 is also verified by weak solutions ξ only in $L^2_{-\delta}(\mathcal{T})$. We say that $\xi \in \mathcal{L}$ is a weak solution of $D\mathbf{\Phi}(g,\pi)^*\xi = (f_1,f_2)$, with $(f_1,f_2) \in L^2_{-\delta}(\widetilde{\mathcal{S}}) \times W^{1,2}_{-\delta}(\mathcal{S})$ when

$$\int_{\mathcal{M}} \langle \xi, D\mathbf{\Phi}(g,\pi).(h,p)\rangle_{\mathring{g}} = \int_{\mathcal{M}} \langle (f_1,f_2),(h,p)\rangle_{\mathring{g}} \ , \ \forall (h,p) \in \mathcal{G}\times\mathcal{K}.$$

It suffices to test with $(h,p) \in \mathscr{C}^\infty_c(S \times \widetilde{\mathcal{S}})$ since this place is dense in $\mathcal{G}\times\mathcal{K}$.

Proposition 11. *Let $\delta \in]-(n+1)/2\,,0] \setminus \{-(n-1)/2\}$ with $n=3$, $(g,\pi) \in \mathcal{G}^+ \times \mathcal{K}$, $(f_1,f_2) \in L^2_{-\delta}(\widetilde{\mathcal{S}}) \times W^{1,2}_{-\delta}(\mathcal{S})$. Let $\xi \in \mathcal{L}$ be a weak solution of $D\mathbf{\Phi}(g,\pi)^*\xi = (f_1,f_2)$. Then $\xi \in W^{2,2}_{-\delta}(\mathcal{T})$ is a strong solution and satisfies (71).*

Proof: In [4], Bartnik shows that $\xi \in W^{2,2}_{\text{loc}}$.
From (78), we can write

$$\begin{aligned} ||\xi||_{2,2,-\delta} &\leqslant c\left(||D\mathbf{\Phi}(g,\pi)^*_1.\xi||_{2,-\delta} + ||D\mathbf{\Phi}(g,\pi)^*_2.\xi||_{1,2,-\delta}\right) + C\,||\xi||_{2,-2\delta} \\ &\leqslant c\left(||f_1||_{2,-\delta} + ||f_2||_{1,2,-\delta}\right) + C\,||\xi||_{2,-\delta}. \end{aligned}$$

So $\xi \in W^{2,2}_{-\delta}(\mathcal{T})$ and ξ verifies (71). □

9. The operator $\mathring{U}$

Here we introduce an operator $\mathring{U}$ inspired by the formula (74). It will allow us to control the $W^{2,2}_\delta$-norm of X with the L^2_δ-norms of $\mathring{S}$ and $\mathring{U}$, that's to say with the $W^{1,2}_\delta$-norm of $\mathring{S}$. The key estimate will arise from a succession of lemmas.
Let $\mathring{U}$ be the operator defined on 1-forms by

$$\mathring{U}_{kji}(X) = \mathring{\nabla}^2_{kj}X_i - \mathring{g}_{jk}X_i + \mathring{g}_{ik}X_j. \tag{101}$$

This readily imply

$$||\mathring{\nabla}^2 X||_{2,-\delta} - c\,||X||_{2,-\delta} \leqslant ||\mathring{U}(X)||_{2,-\delta}. \tag{102}$$

The next four lemmas are established on an asymptotically hyperbolic manifold $(M,\mathring{g})$, with $\mathring{g} = \rho^{-2}\mathring{h}$ and $|d\rho|^2_{\mathring{h}} = 1 + o(1)$ near the boundary.

Lemma 17. *Let $(M,\mathring{g})$ be an asymptotically hyperbolic manifold and $X \in \mathscr{C}^\infty(T^*\mathcal{M})$ compactly supported on a neighborhood of $\partial\mathcal{M}$. $\forall \delta \in \mathbb{R}$,*

$$\begin{aligned} \int_{\mathcal{M}} \mathring{\nabla}X(\frac{d\rho}{\rho},X)\rho^{2\delta}\,d\mu(\mathring{g}) &= \int_{\mathcal{M}} \rho^{2\delta}\big(\tfrac{n-1}{2} - \delta + o(1)\big)|X|^2_{\mathring{g}}\,d\mu(\mathring{g}) \\ &\quad + \frac{1}{2}\int_{\partial\mathcal{M}} |X|^2_{\mathring{g}}\langle\frac{d\rho}{\rho},\eta\rangle_{\mathring{g}}\,\rho^{2\delta}\,d\sigma(\mathring{g}) \qquad (103) \\ \int_{\mathcal{M}} \mathring{\nabla}X(X,\frac{d\rho}{\rho})\rho^{2\delta}\,d\mu(\mathring{g}) &= -\int_{\mathcal{M}} \rho^{2\delta}\left(divX\langle X,\frac{d\rho}{\rho}\rangle + (2\delta+1)\langle X,\frac{d\rho}{\rho}\rangle^2\right)d\mu(\mathring{g}) \\ &\quad + \int_{\mathcal{M}} \rho^{2\delta}\big(1+o(1)\big)|X|^2_{\mathring{g}}\,d\mu(\mathring{g}) \\ &\quad + \int_{\partial\mathcal{M}} \rho^{2\delta}\,\langle X,\frac{d\rho}{\rho}\rangle\langle X,\eta\rangle_{\mathring{g}}\,d\sigma(\mathring{g})\,, \qquad (104) \end{aligned}$$

Proof: For (103), we integrate by parts the term $\mathring{\nabla}_i(|X|^2_{\mathring{g}}\,\mathring{\nabla}^i\rho\,\rho^{2\delta-1})$ and the result follows from the Divergence theorem , along with the definition (101) of $\mathring{U}$ and (1). For (103), we integrate by parts the term $\mathring{\nabla}_i\big(X^i\,\langle X, d\rho\rangle\,\rho^{2\delta-1}\big)$ and the result follows on from the Divergence theorem , along with the definition (101). □

Lemma 18. *Let $(M,\mathring{g})$ be an asymptotically hyperbolic manifold and $X\in\mathscr{C}^\infty(T^*\mathcal{M})$ compactly supported on a neighborhood of $\partial\mathcal{M}$. $\forall\delta\in\mathbb{R}$,*

$$\begin{aligned}\int_{\mathcal{M}}\mathring{U}_{kji}(X)\mathring{g}^{kj}X^i\rho^{2\delta}\,d\mu(\mathring{g}) &= -\int_{\mathcal{M}}\rho^{2\delta}\,|\mathring{\nabla}X|^2_{\mathring{g}}\,d\mu(\mathring{g})\\ &\quad+\int_{\mathcal{M}}\rho^{2\delta}\,[2\delta^2-\delta(n-1)-(n-1)+o(1)]|X|^2_{\mathring{g}}\,d\mu(\mathring{g})\\ &\quad+\int_{\partial\mathcal{M}}\rho^{2\delta}\,\big(\mathring{\nabla}X(\eta,X)-\delta|X|^2_{\mathring{g}}\langle\frac{d\rho}{\rho},\eta\rangle_{\mathring{g}}\big)\,d\sigma(\mathring{g}).\end{aligned}\tag{105}$$

Proof: From (101), $\mathring{U}_{kji}(X)\,\mathring{g}^{kj}X^i=\langle X,\mathring{\Delta}X\rangle-(n-1)|X|^2_{\mathring{g}}$.
We integrate the term $\mathring{\nabla}_k(\mathring{\nabla}^kX_iX^i\rho^{2\delta})$ and the lemma stems from the Divergence theorem and Lemma 17. □

Lemma 19. *Let $(M,\mathring{g})$ be an asymptotically hyperbolic manifold and $N\in\mathscr{C}^\infty(\mathcal{M})$ compactly supported on a neighborhood of $\partial\mathcal{M}$. $\forall\delta\in\mathbb{R}$,*

$$\begin{aligned}&\int_{\mathcal{M}}\rho^{2\delta}\,\big(\mathring{U}_{kji}(X)\mathring{\nabla}^jX^i\frac{\mathring{\nabla}^k\rho}{\rho}+divX\langle X,\frac{d\rho}{\rho}\rangle\big)\,d\mu(\mathring{g})=\\ &\int_{\mathcal{M}}\rho^{2\delta}\,\big(\tfrac{n-1}{2}-\delta+o(1)\big)[|\mathring{\nabla}X|^2_{\mathring{g}}-|X|^2_{\mathring{g}}]\,d\mu(\mathring{g})-\int_{\mathcal{M}}\rho^{2\delta}\,[(2\delta+1)\langle X,\frac{d\rho}{\rho}\rangle^2-|X|^2_{\mathring{g}}]\,d\mu(\mathring{g})\\ &+\frac{1}{2}\int_{\partial\mathcal{M}}\rho^{2\delta}\,\big(|\mathring{\nabla}X|^2_{\mathring{g}}\langle\frac{d\rho}{\rho},\eta\rangle_{\mathring{g}}-|X|^2_{\mathring{g}}\langle\frac{d\rho}{\rho},\eta\rangle_{\mathring{g}}\big)\,d\sigma(\mathring{g})+\int_{\partial\mathcal{M}}\rho^{2\delta}\,\langle X,\frac{d\rho}{\rho}\rangle\langle X,\eta\rangle\,d\sigma(\mathring{g}).\end{aligned}\tag{106}$$

Proof: From (101),

$$\mathring{U}_{kji}(X)\mathring{\nabla}^jX^i\frac{\mathring{\nabla}^k\rho}{\rho}=\mathring{\nabla}^2_{kj}X_i\mathring{\nabla}^jX^i\frac{\mathring{\nabla}^k\rho}{\rho}-\mathring{\nabla}X(\frac{d\rho}{\rho},X)+\mathring{\nabla}X(X,\frac{d\rho}{\rho}).$$

We integrate the term $\mathring{\nabla}_k(|\mathring{\nabla}X|^2_{\mathring{g}}\frac{\mathring{\nabla}^k\rho}{\rho}\rho^{2\delta})$ and the lemma stems from the Divergence theorem and Lemma 17. □

Considering also the equality $2\,|\mathring{S}(X)|^2_{\mathring{g}}=|\mathring{\nabla}X|^2_{\mathring{g}}+\mathring{\nabla}_jX_k\mathring{\nabla}^kX^j$, we get

Lemma 20. *Let $(M,\mathring{g})$ be an asymptotically hyperbolic manifold and $X \in \mathscr{C}^\infty(T^*\mathcal{M})$ compactly supported on a neighborhood of $\partial\mathcal{M}$. $\forall \delta \in \mathbb{R}$,*

$$\begin{aligned}&\int_{\mathcal{M}} \rho^{2\delta}\big(\mathring{U}_{kji}(X)\mathring{g}^{ik}X^j + 2\,|\mathring{S}(X)|^2_{\mathring{g}} - 2\delta\, div X\langle X, \frac{d\rho}{\rho}\rangle\big)\, d\mu(\mathring{g}) = \\ &+ \int_{\mathcal{M}} \rho^{2\delta}\,[n-1-2\delta+o(1)]|X|^2_{\mathring{g}}\, d\mu(\mathring{g}) + \int_{\mathcal{M}} \rho^{2\delta}\,|\mathring{\nabla}X|^2_{\mathring{g}}\, d\mu(\mathring{g}) \\ &+ \int_{\mathcal{M}} \rho^{2\delta}\, 2\delta(2\delta+1)\langle X, \frac{d\rho}{\rho}\rangle^2_{\mathring{g}}\, d\mu(\mathring{g}) + \int_{\partial\mathcal{M}} \rho^{2\delta}\,\big(\mathring{\nabla}X(X,\eta) - 2\delta\,\langle X, \frac{d\rho}{\rho}\rangle_{\mathring{g}}\langle X,\eta\rangle_{\mathring{g}}\big)\, d\sigma(\mathring{g})\,,\end{aligned} \tag{107}$$

Proof: From (101), $\mathring{U}_{kji}(X)\,\mathring{g}^{ik}X^j = \mathring{\nabla}^2_{kj}X^kX^j + (n-1)|X|^2_{\mathring{g}}$.
We integrate by parts the term $\mathring{\nabla}_k(\mathring{\nabla}_j X^k X^j \rho^{2\delta})$ and the result follows on from the Divergence theorem along with (104) and the equality $2\,|\mathring{S}(X)|^2_{\mathring{g}} = |\mathring{\nabla}X|^2_{\mathring{g}} + \mathring{\nabla}_j X_k \mathring{\nabla}^k X^j$. □

We can now prove the following proposition, crucial in the demonstration of the adjoint kernel triviality.

Proposition 12. *Set $E_R := \mathcal{M} \setminus \Omega_R$, for an open set Ω_R. For all $\varepsilon > 0$, for all $\delta \in]-2\,,-1[$, there exists $R_{\varepsilon,\delta} > 0$ such that for all $R > R_{\varepsilon,\delta}$, there exists a constant $c > 0$ such that*

$$\forall X \in \mathscr{C}^\infty_c(E_R)\ \ ,\ \ ||X||_{1,2,-\delta;E_R} \leqslant c\,(||\mathring{U}(X)||_{2,-\delta;E_R} + ||\mathring{S}(X)||_{2,-\delta;E_R}). \tag{108}$$

Proof: The linear combination (106) $-\frac{1}{2}$(105) $+\frac{1}{2}$(107) + (33) $-\frac{1}{2}$(34) yield

$$\begin{aligned}&\int_{\mathcal{M}} \rho^{2\delta}\,\mathring{U}_{kji}(X)\Big(\mathring{\nabla}^j X^i \frac{\mathring{\nabla}^k\rho}{\rho} - \tfrac{1}{2}\mathring{g}^{kj}X^i + \tfrac{1}{2}\mathring{g}^{ik}X^j\Big)\, d\mu(\mathring{g}) \\ &+ \int_{\mathcal{M}} \rho^{2\delta}\,\Big(|\mathring{S}(X)|^2_{\mathring{g}} - \mathring{S}(X)(\frac{d\rho}{\rho},\frac{d\rho}{\rho})\,\langle X,\frac{d\rho}{\rho}\rangle_{\mathring{g}} + (2-\delta) div X\langle X,\frac{d\rho}{\rho}\rangle\Big)\, d\mu(\mathring{g}) \\ = &\int_{\mathcal{M}} \{-\delta^2 + (\tfrac{n-3}{2})\delta + n + 1 + o(1)\}|X|^2_{\mathring{g}}\rho^{2\delta}\, d\mu(\mathring{g}) \\ &+ \int_{\mathcal{M}} \rho^{2\delta}\,\big\{2\delta^2 - 2\delta - (\tfrac{n+3}{2}) + o(1)\big\}\,\langle X,\frac{d\rho}{\rho}\rangle^2_{\mathring{g}}\, d\mu(\mathring{g}) \\ &+ \int_{\mathcal{M}} \big[\tfrac{n+1}{2} - \delta + o(1)\big]\,|\mathring{\nabla}X|^2_{\mathring{g}}\rho^{2\delta}\, d\mu(\mathring{g}) + \int_{\partial\mathcal{M}} \rho^{2\delta}\,\tfrac{1}{2}(\delta-1)|X|^2_{\mathring{g}}\langle\frac{d\rho}{\rho},\eta\rangle_{\mathring{g}}\, d\sigma(\mathring{g}) \\ &+ \int_{\partial\mathcal{M}} \Big(\tfrac{1}{2}|\mathring{\nabla}X|^2_{\mathring{g}}\langle\frac{d\rho}{\rho},\eta\rangle_{\mathring{g}} + \tfrac{1}{2}|X|^2_{\mathring{g}}\langle\frac{d\rho}{\rho},\eta\rangle_{\mathring{g}} + \tfrac{1}{2}\mathring{\nabla}X(X,\frac{d\rho}{\rho}) - \tfrac{1}{2}\mathring{\nabla}X(\frac{d\rho}{\rho},X)\Big)\, d\sigma(\mathring{g}) \\ &+ \int_{\partial\mathcal{M}} \rho^{2\delta}\,\Big((2-\delta)\langle X,\frac{d\rho}{\rho}\rangle_{\mathring{g}}\langle X,\eta\rangle_{\mathring{g}} - \tfrac{1}{2}\langle X,\frac{d\rho}{\rho}\rangle^2_{\mathring{g}}\langle\frac{d\rho}{\rho},\eta\rangle_{\mathring{g}}\Big)\, d\sigma(\mathring{g}).\end{aligned} \tag{109}$$

Application on E_R: E_R possesses two disjoint boundary components. An exterior boundary at infinity, noted ∂E_∞ , and an interior boundary $\partial\Omega_R = \{\rho = \mathrm{e}^{-2R}\}$. Since $X \in \mathscr{C}^\infty_c(E_R)$, X is surely null near ∂E_∞ but not necessarily on $\partial\Omega_R$ and that's the reason why boundary terms in (30) will only concern $\partial\Omega_R$. If η_R is the normal to $\partial\Omega_R$ exterior to E_R and considering that

When $R \to +\infty$, $\eta_R - \frac{d\rho}{\rho} \to 0$, so that $\langle \frac{d\rho}{\rho}, \eta_R \rangle_{\mathring{g}} = \frac{|d\rho|^2_{\mathring{g}}}{\rho^2} + o(1) = 1 + o(1)$.
X_n (resp. X_T) being the component of X normal (resp. tangential) to $\partial\mathcal{M}$,

$$X_n := \langle X, \eta \rangle_{\mathring{g}} \quad \text{and} \quad |X|^2_{\mathring{g}} = X_n^2 + X_T^2.$$

$$\begin{aligned}
&\int_{E_R} \rho^{2\delta}\, \mathring{U}_{kji}(X)\Big(\mathring{\nabla}^j X^i \frac{\mathring{\nabla}^k \rho}{\rho} - \tfrac{1}{2}\mathring{g}^{kj} X^i + \tfrac{1}{2}\mathring{g}^{ik} X^j\Big)\, d\mu(\mathring{g}) \\
&+ \int_{E_R} \rho^{2\delta} \left(|\mathring{S}(X)|^2_{\mathring{g}} - \mathring{S}(X)(\eta_R, \eta_R)\, X_n + (2-\delta) div X X_n\right) d\mu(\mathring{g}) \\
= &\int_{E_R} \rho^{2\delta} \left\{-\delta^2 + (\tfrac{n-3}{2})\delta + n + 1 + o(1)\right\} X_T^2\, d\mu(\mathring{g}) \\
&+ \int_{E_R} \rho^{2\delta} \left\{ \left[\tfrac{n+1}{2} - \delta + o(1)\right] |\mathring{\nabla} X|^2_{\mathring{g}} + \left[\delta^2 + (\tfrac{n-7}{2})\delta + \tfrac{n-1}{2} + o(1)\right] X_n^2 \right\} d\mu(\mathring{g}) \\
&+ \int_{\partial E_R} \rho^{2\delta}\, \tfrac{1}{2}(\delta - 1)\big(X_T^2 + o(1)\big)\, d\sigma(\mathring{g}) + \int_{\partial E_R} \rho^{2\delta}\, \tfrac{1}{2}(2-\delta)\big(X_n^2 + o(1)\big)\, d\sigma(\mathring{g}) \\
&+ \int_{\partial E_R} \Big(\tfrac{1}{2}|\mathring{\nabla} X|^2_{\mathring{g}} |\eta|^2_{\mathring{g}} + \tfrac{1}{2}|X|^2_{\mathring{g}} + \tfrac{1}{2}\mathring{\nabla} X(X, \eta) - \tfrac{1}{2}\mathring{\nabla} X(\eta, X)\Big)\, d\sigma(\mathring{g}).
\end{aligned}$$

Given the following equalities

$$\begin{aligned}
0 \leqslant \tfrac{1}{4}|\mathring{\nabla}_i X_j \eta^i - X_j|^2_{\mathring{g}} &= \tfrac{1}{4}|\mathring{\nabla} X|^2_{\mathring{g}} |\eta|^2_{\mathring{g}} + \tfrac{1}{4}|X|^2_{\mathring{g}} - \tfrac{1}{2}\mathring{\nabla} X(\eta, X) \\
0 \leqslant \tfrac{1}{4}|\mathring{\nabla}_i X_j \eta^j + X_i|^2_{\mathring{g}} &= \tfrac{1}{4}|\mathring{\nabla} X|^2_{\mathring{g}} |\eta|^2_{\mathring{g}} + \tfrac{1}{4}|X|^2_{\mathring{g}} + \tfrac{1}{2}\mathring{\nabla} X(X, \eta)
\end{aligned}$$

we get that for all $\varepsilon > 0$, there exists $R_\varepsilon > 0$ and $c_\varepsilon >> 1$ such that $\forall R > R_\varepsilon$,

$$\begin{aligned}
c_\varepsilon \int_{E_R} \rho^{2\delta} \left(|\mathring{U}(X)|^2_{\mathring{g}} + |\mathring{S}(X)|^2_{\mathring{g}}\right) d\mu(\mathring{g}) \;\geqslant\; &\int_{E_R} \rho^{2\delta} \left\{-\delta^2 + (\tfrac{n-3}{2})\delta + n + 1 - \varepsilon\right\} X_T^2\, d\mu(\mathring{g}) \\
&\int_{E_R} \rho^{2\delta} \left[\tfrac{n+1}{2} - \delta - \varepsilon\right] |\mathring{\nabla} X|^2_{\mathring{g}}\, d\mu(\mathring{g}) \\
&+ \int_{E_R} \rho^{2\delta} \left\{\delta^2 + (\tfrac{n-7}{2})\delta + \tfrac{n-1}{2} - \varepsilon\right\} X_n^2\, d\mu(\mathring{g}) \\
&+ \frac{1}{2} \int_{\partial E_R} \rho^{2\delta}\, (2 - \delta - \varepsilon) X_n^2\, d\sigma(\mathring{g}) \\
&+ \frac{1}{2} \int_{\partial E_R} \rho^{2\delta}\, (\delta - 1 - \varepsilon) X_T^2\, d\sigma(\mathring{g}). \qquad (110)
\end{aligned}$$

- The X_T^2 term is non negative if $\delta \in]-2\,;\frac{n+1}{2}[$.
- The X_n^2 term is non negative $\forall n \geqslant 3$, $\forall \delta \in]\frac{n-1}{2}\,;\frac{n+1}{2}[$.
- The $|\mathring{\nabla} X|^2_{\mathring{g}}$ term is non negative if $\delta < (n+1)/2$.
- The boundary term is non negative if $\delta \in]1, 2[$.

Hence, $\forall \delta \in]1\,;2[$, $\forall \varepsilon > 0, \exists R_\varepsilon > 0$ and $c_\varepsilon >> 1$ such that $\forall R > R_\varepsilon$,

$$c_\varepsilon \int_{E_R} \rho^{2\delta} \left(|\mathring{U}(X)|^2_{\mathring{g}} + |\mathring{S}(X)|^2_{\mathring{g}}\right) d\mu(\mathring{g}) \geqslant \int_{E_R} \rho^{2\delta} \left[\tfrac{n+1}{2} - \delta - \varepsilon\right] |\mathring{\nabla} X|^2_{\mathring{g}}\, d\mu(\mathring{g})$$
$$+ \int_{E_R} \rho^{2\delta} \left(\{-\delta^2 + (\tfrac{n-3}{2})\delta + n + 1 - \varepsilon\} X_T^2 + \{\delta^2 + (\tfrac{n-7}{2})\delta + \tfrac{n-1}{2} - \varepsilon\} X_n^2\, d\mu(\mathring{g}) \right) d\mu(\mathring{g}). \quad \square$$

10. The adjoint kernel triviality

In this section we show that the kernel of $D\mathbf{\Phi}(g,\pi)^*$ is trivial. We will need the following lemmas and propositions in the demonstration of Theorem 3.

Lemma 21. *Let $\delta \in I\!R$ and $\xi \in W^{2,2}_{-\delta}(\mathcal{T})$ be a solution of $D\mathbf{\Phi_0}(g,\pi)^*\xi = 0$. Then $\xi = (N, X)$ satisfies the system*

$$\begin{cases} \mathring{T}(N) = b_0\xi + b_1\mathring{\nabla}\xi \\ \mathring{S}(X) = -\mathring{g}\tau N + b_2\xi \end{cases}, \tag{111}$$

with $b_0 \in L^2_\delta$ et $b_1, b_2 \in W^{1,2}_\delta$.

Proof: From (72) and (66) , $D\mathbf{\Phi_0}(g,\pi)^*\xi = 0$ lead to

$$\begin{aligned} T - (tr_g T)g &= L^2_\delta\xi + W^{1,2}_\delta\mathring{\nabla}X. \\ \mathring{S}(X) &= -\mathring{g}\tau N + W^{1,2}_\delta\xi. \end{aligned} \tag{112}$$

Taking the trace of (112),

$$\begin{aligned} \mathring{T}(N) &= L^2_\delta\xi + W^{1,2}_\delta\mathring{\nabla}\xi. \\ \mathring{S}(X) &= -\mathring{g}\tau N + W^{1,2}_\delta\xi. \end{aligned} \tag{113}$$

$\square$

Theorem 3. *Triviality of ker $D\mathbf{\Phi_0}(g)^*$.*

Let $\Omega \subset \mathcal{M}$ be a connected open set such that $E_R \subset \Omega$. We fix $(g,\pi) \in \mathcal{F}$. Set $\delta \in]-(n+1)/2\,, -1]$ with $n = 3$. Suppose $\xi \in L^2_{-\delta}(\mathcal{M})$ verifies $D\mathbf{\Phi_0}(g,\pi)^*\xi = 0$ on Ω. Then $\xi \equiv 0$ on Ω.

Proof: From Proposition 11 , $\xi \in W^{2,2}_{-\delta}(\mathcal{M})$. According to the Lemma 21 and since ξ is a solution of $D\mathbf{\Phi_0}(g)^*\xi = 0$, ξ satisfies

$$\begin{cases} \mathring{T}(N) = b_0\xi + b_1\mathring{\nabla}\xi \\ \mathring{S}(X) = -\mathring{g}\tau N + b_2\xi \end{cases},$$

with $b_0 \in L^2_\delta$ et $b_1, b_2 \in W^{1,2}_\delta$.

From (41),

$$\mathring{U}_{kji}(X) = \mathring{\nabla}^2_{kj}X_i - Riem\,\mathring{g}_{ijkl} + L^2_\delta X. \tag{114}$$

Regarding (74)

$$\begin{aligned} \mathring{U}(X) &= c_1\mathring{\nabla}\mathring{S}(X) + L^2_\delta X \\ &= -c_1\tau\mathring{g}\mathring{\nabla}N + L^2_\delta\xi + W^{1,2}_\delta\mathring{\nabla}\xi. \end{aligned} \tag{115}$$

We must show that a solution ξ of (111) such that $\xi = o(\rho^\delta)$ (from (12)) vanishes. Before pursuing the proof of the theorem, let's recall Proposition 3.9 of [4] :

Proposition 13. *In dimension $n = 3$, set $\delta \leqslant 0$ and $(g, \pi) \in \mathcal{F}$. Let Ω be a connected subset of $\mathcal{M}$. Let ξ satisfy $D\mathbf{\Phi}(g,\pi)^*\xi = 0$ sur Ω. If in addition $\xi \equiv 0$ on an open set $U \subset \Omega$, then $\xi \equiv 0$ on Ω.*

Given the previous proposition, it remains to show ξ vanishes near infinity. The boundary version of (102) gives

$$||\mathring{\nabla}^2 X||_{2,-\delta;E_R} - c\,||X||_{2,-\delta;E_R} \leqslant ||\mathring{U}(X)||_{2,-\delta;E_R}. \tag{116}$$

Combine to (108), together with (115), Sobolev inequality and (8), we obtain

$$\begin{aligned} ||X||_{2,2,-\delta;E_R} &\leqslant c\,(||\mathring{U}(X)||_{2,-\delta;E_R} + ||\mathring{S}(X)||_{2,-\delta;E_R}) \\ &\leqslant c\,||N||_{1,2,-\delta;E_R} + C\,(||\xi||_{\infty,-2\delta;E_R} + ||\mathring{\nabla}\xi||_{3,-2\delta;E_R}). \end{aligned} \tag{117}$$

Using Proposition 4 , there exists $\varepsilon_0 << 1$ such that $(29) + \varepsilon_0(117)$ give

$$||\xi||_{2,2,-\delta;E_R} \leqslant c\,||\mathring{T}||_{2,-\delta;E_R} + C\,(||\xi||_{\infty,-2\delta;E_R} + ||\mathring{\nabla}\xi||_{3,-2\delta;E_R}).$$

Considering Lemma 21 along with Sobolev inequality and (17) with $\delta \leqslant 0$,

$$\begin{aligned} ||\xi||_{2,2,-\delta;E_R} &\leqslant C\,||\xi||_{2,2,-2\delta;E_R} \\ &\leqslant C\,e^{4R\delta}\,||\xi||_{2,2,-\delta;E_R}. \end{aligned}$$

We end up with $\forall\,\delta \in]-(n+1)/2\,,-1]$,

$$||\xi||_{2,2,-\delta;E_R} \leqslant C\,e^{4R\delta}\,||\xi||_{2,2,-\delta;E_R}.$$

For R large enough, $||\xi||_{2,2,-\delta;E_R} = 0$ and thanks to Sobolev inequality, ξ vanishes on E_R, for $R >> 1$. From Prop.13, $\xi \equiv 0$ on Ω, because Ω is connected by assumption. This ends the proof of Theorem 3. □

Corollary 4. *Set $\delta \in]-(n+1)/2\,,0] \setminus \{-(n-1)/2\}$, with $n = 3$. There exists a constant $C > 0$ depending on $||(g,\pi)||_{\mathcal{F}}$ such that for $\xi \in W^{2,2}_{-\delta}(\mathcal{T})$,*

$$||\xi||_{2,2,-\delta} \leqslant C\,||P^*\xi||_{2,-\delta}. \tag{118}$$

Proof: In order to show that the kernel of P^* is finite dimensional, we apply Riesz theorem showing every bounded subset of ker P^* is $||\ ||_{2,2,-\delta}-$ compact. Let $\{\xi_k\}$ be a sequence of ker P^* such that $||\xi_k||_{2,2,-\delta} = 1$. Rellich theorem tells us we can extract from $\{\xi_k\}$ a sub-sequence, also noted $\{\xi_k\}$, converging in $W^{1,2}_{-2\delta}$ to a limit $\bar{\xi}$. Hence, $\{\xi_k\}$ is a Cauchy sequence in $W^{1,2}_{-2\delta}$. From (71), considering that $\{\xi_k\} \in \ker P^*$, $\{\xi_k\}$ is a Cauchy sequence in $W^{2,2}_{-\delta}$, and so converges to $\bar{\xi}$ in $W^{2,2}_{-\delta}$, from the limit uniqueness. This ends the proof of the finite dimension of ker P^*. ker P^* is thus a closed subspace of the Hilbert vector space $W^{2,2}_{-\delta}$. Being a finite dimensional closed subspace of a normed vector space , it splits and if we set W to be the closed complement of ker P^*,

$$W^{2,2}_{-\delta} = \ker P^* \oplus W.$$

From the same argument as in the proof of Theorem 1 , there exists a constant $C > 0$ depending on $||(g,\pi)||_{\mathcal{F}}$ such that for all $\xi \in W$,

$$||\xi||_{2,2,-\delta} \leqslant C\,||P^*\xi||_{2,-\delta} \tag{119}$$

We conclude thanks to the triviality of $\ker P^*$ from Theorem 3. □

11. Submanifold structure

Lemma 22. *Let X, Y be two Banach spaces and T a linear operator with closed range.*

$$\begin{aligned} T : X &\to Y \\ T^* : Y^* &\to X^* \end{aligned}$$

then $(CokerT)^* \simeq kerT^*$, where $CokerT = {}^{Y}\!/_{ImT}$

Proof: We define

$$\begin{aligned} \psi\colon \mathrm{ker}T^* &\to (\mathrm{Coker}T)^* = \mathcal{L}\big({}^{Y}\!/_{ImT}, \mathbb{R}\big) \\ \rho &\mapsto (\lambda\colon y + TX \mapsto \rho(y)) \end{aligned}$$

The map λ is well defined because $\forall x \in X\,, \rho(Tx) = T^*(\rho)(x) = 0$.
The map ψ is invertible and

$$\begin{aligned} \psi^{-1}\colon (\mathrm{Coker}T)^* &\to \mathrm{ker}T^* \\ \lambda &\mapsto \lambda(y + TX) := \rho(y) \end{aligned}$$

$\rho \in \mathrm{ker}T^*$ because $T^*(\rho)(x) = \rho(Tx) = \lambda(\bar{0}) = 0$. □
Remark: The closed range of T imply $\mathrm{Coker}T$ is a Banach space.

We are now in possession of all the tools necessary to demonstrate the manifold structure of $\mathcal{C}$ as a sooth submanifold of $\mathcal{F}$. In fact, all level sets of $\mathbf{\Phi}(g,\pi)$ are smooth submanifolds of F.

Theorem 4. *Let $\mathbf{\Phi} : \mathcal{F} \to \mathcal{L}^*$ be the constraint operator in dimension $n = 3$.*
For every $\varepsilon \in \mathcal{L}^*$, for all $\delta \in]-2\,, -1]$, the set of solutions of the constraint equations

$$\mathcal{C}(\varepsilon) := \{(g,\pi) \in \mathcal{F} : \mathbf{\Phi}(g,\pi) = \varepsilon\}$$

is a submanifold of $\mathcal{F}$. In particular, the space of solutions of the vacuum constraint equations $\mathcal{C} = \mathcal{C}(0)$ has a Hilbert submanifold structure.

Demonstration: In order to use the implicit function theorem, we want to show:

- $\ker\, D\mathbf{\Phi}(g,\pi)$ splits.
- $D\mathbf{\Phi}(g,\pi)$ is surjective.

Given that the kernel of $D\mathbf{\Phi}(g,\pi)$ is finite dimensional, we show that $\ker\, D\mathbf{\Phi}(g,\pi)$ is closed and hence splits. $D\mathbf{\Phi}(g,\pi)$ being a bounded operator, its kernel is closed by continuity.
As we established the triviality of $\ker\, D\mathbf{\Phi}(g,\pi)^*$ in Theorem 3, leading to

$$(\mathrm{ker}D\mathbf{\Phi}(g,\pi)^*)^{\perp} = \mathcal{L}^*.$$

Using the classic relation

$$(\ker D\mathbf{\Phi}(g,\pi)^*)^\perp = \overline{\mathrm{Im} D\mathbf{\Phi}(g,\pi)},$$

we get

$$\overline{\mathrm{Im} D\mathbf{\Phi}(g,\pi)} = \mathcal{L}^*.$$

So in order to have the surjectivity of $D\mathbf{\Phi_0}(g)$, il suffices to prove $D\mathbf{\Phi}(g,\pi)$ has closed range. To do so, we consider particular variations (h,p) of (g,π) of the form

$$\begin{cases} h_{ij} = 2\, y\, g_{ij} \\ p^{ij} = \left(2\mathcal{S}(Y)^{ij} - g^{ij} tr_g \mathcal{S}(Y) - (n-1)(n-2)\tau y\, g^{ij}\right)\sqrt{g} \end{cases} , \tag{120}$$

determined from fields (y, Y^i). We define the operator

$$F(y,Y^i) = [F_0(y,Y^i), F_i(y,Y^i)] = [D\mathbf{\Phi_0}(g,\pi)(h,p), D\mathbf{\Phi_i}(g,\pi)(h,p)]. \tag{121}$$

From (59) and (60),

$$\begin{cases} F_0(y,Y^i) &= 2(n-1)\sqrt{g}\,[-\Delta y + ny] + (4-n)\,\mathbf{\Phi_0}(g,\pi)\, y + 2(n-2)\tau div Y\sqrt{g} \\ & \quad + L^2_\delta\,[y+Y] + W^{1,2}_\delta\, \mathring{\nabla} Y. \\ F_i(y,Y^i) &= -2\sqrt{g}\,[-\Delta Y^i + (n-1)Y^i] + 2\,\mathbf{\Phi_i}(g,\pi)\, y + W^{1,2}_\delta\, \mathring{\nabla} y + L^2_\delta\,[y+Y]. \end{cases}$$

Definition 2. *Operator asymptotic to* $\mathring{\Delta}$.
We say an operator P *of the form*

$$Pu = a^{ij}(x)\mathring{\nabla}^2_{ij} u + b^i(x)\partial_i u + c(x)u$$

is asymptotic to $\mathring{\Delta}$ *with a decaying rate* τ *if there exists* $n < q < \infty$, $\tau \leqslant 0$ *and two positive constants* C_1, λ *such that*

$$\lambda|\xi|^2_{\mathring{g}} \leqslant a^{ij}(x)\xi_i\xi_j \leqslant \lambda^{-1}|\xi|^2_{\mathring{g}}, \forall x \in \mathcal{M}\ ,\ \xi \in T\mathcal{M}.$$
$$||a^{ij} - \mathring{g}^{ij}||_{1,q,\tau} + ||b^i||_{q,\tau} + ||c||_{q/2,\tau} \leqslant C_1.$$

Proposition 14. *Let* $g \in \mathcal{G}^+$ *with* $\delta \leqslant 0$. *Then* Δ *is asymptotic to* $\mathring{\Delta}$ *with a decaying rate* δ.

Proof:

$$\begin{aligned} \Delta = g^{ij}\nabla^2_{ij} &= g^{ij}\mathring{\nabla}^2_{ij} + g^{ij}(\nabla_i - \mathring{\nabla}_i)\mathring{\nabla}_j \\ &= g^{ij}\mathring{\nabla}^2_{ij} - g^{ij} A_{i\,j}^{\ k}\, \mathring{\nabla}_k. \end{aligned} \tag{122}$$

The metrics g et $\mathring{g}$ being equivalent, equation (49) directly gives

$$\lambda|\xi|^2_{\mathring{g}} \leqslant g^{ij}(x)\xi_i\xi_j \leqslant \lambda^{-1}|\xi|^2_{\mathring{g}}, \forall x \in \mathcal{M}\ ,\ \xi \in T\mathcal{M}.$$

Setting

$$b^k = g^{ij} A_{i\,j}^{\ k},$$

then $b \in W^{1,2}_\delta$ from (51). Given the Sobolev inequality, there exists a constant $C_1 > 0$ such that

$$||g^{ij} - \mathring{g}^{ij}||_{1,6,\delta} + ||b^k||_{6,\delta} \leqslant c\,(||g^{ij} - \mathring{g}^{ij}||_{2,2,\delta} + ||b^k||_{1,2,\delta}) \leqslant C_1. \quad \square$$

We will justify later on the terminology "asymptotic" used in this definition.
The operator $\mathcal{A} = -\Delta + n$, acting on functions, will be of great interest.

Proposition 15. *Let $g \in \mathcal{G}^+$ with $\delta \leqslant 0$ and $\mathcal{A} = -\Delta + n$. Let $s \in \mathbb{R}$. There exists a constant $C = C(n, p, q, s, \delta, C_1, \lambda)$ such that if $u \in L^2_s$ and $\mathcal{A}u \in L^2_s$, then $u \in W^{2,2}_s$ et*

$$||u||_{2,2,s} \leqslant C\ (||\mathcal{A}u||_{2,s} + ||u||_{2,s})\,.$$

Proof: By elliptic regularity, $u \in W^{2,2}_{loc}$ and the estimate arises from interior estimates (see [12] for example) and scaling. □

Theorem 5. *Let $g \in \mathcal{G}^+$ with $\delta \leqslant 0$ so that Δ is asymptotic to $\mathring{\Delta}$.*
Set $\mathcal{A} = -\Delta + n$, with $n = 3$.
Let $|s| < (n+1)/2$. Then $\mathcal{A} : W^{2,2}_s(\mathcal{M}) \to L^2_s(\mathcal{T}^* \otimes \Lambda^3 T^*\mathcal{M})$ is bounded.
Moreover, it satisfy the following elliptic estimate

$$||u||_{2,2,s} \leqslant C\ (||\mathcal{A}u||_{2,s} + ||u||_{2,s;\Omega_R})\,. \tag{123}$$

In particular, $\mathcal{A}$ is a semi-Fredholm operator , i.e. *$\mathcal{A}$ has finite dimensional kernel and closed range.*

Proof: We define the following operator norm:

$$||\Delta - \mathring{\Delta}||_{op} = \sup_{\mathcal{M}}\{||(\Delta - \mathring{\Delta})u||_{2,s} : u \in W^{2,2}_s, ||u||_{2,2,s} = 1\}$$

and $||\bullet||_{op,R}$ denotes the same norm restricted to functions supported in $E_R = \mathcal{M} \setminus \Omega_R$.
If $supp(u) \subset E_R$, then from the expression (122) of Δ ,

$$\begin{aligned} ||(\Delta - \mathring{\Delta})u||_{2,s} &\leqslant ||(g^{ij} - \mathring{g}^{ij})\mathring{\nabla}^2_{ij}u||_{2,s} + ||b^k\mathring{\nabla}_k u||_{2,s} \\ &\leqslant \sup_{E_R}\{g^{ij} - \mathring{g}^{ij}\}||\mathring{\nabla}^2_{ij}u||_{2,s} + ||b^k\mathring{\nabla}_k u||_{2,s} \\ &\leqslant c\,||g - \mathring{g}||_{\infty,0;E_R}||\mathring{\nabla}^2 u||_{2,s} + ||b\,\mathring{\nabla}u||_{2,s}. \end{aligned}$$

Using (8), Sobolev inequality and inclusion ($\delta \leqslant 0$),

$$||(\Delta - \mathring{\Delta})u||_{2,s} \leqslant c\left(||g - \mathring{g}||_{2,2,\delta;E_R} + ||b||_{1,2,\delta;E_R}\right)||u||_{2,2,s}.$$

Recalling that $||g - \mathring{g}||_{2,2,\delta} + ||b||_{1,2,\delta}$ is bounded because $g \in \mathcal{G}^+$,

$$||\Delta - \mathring{\Delta}||_{op,R} = o(1) \ \text{ quand } \ R \to +\infty. \tag{124}$$

This justifies *a posteriori* terminology used in the definition 2.

Let χ_R be a cut-off function as in Definition 1

$$\chi_R \ = \ \begin{cases} 1 \text{ on } \Omega_{R/2} \\ 0 \text{ on } \mathcal{M} \setminus \Omega_R \end{cases}$$

Then we can decompose $u = u_0 + u_\infty$, with $u_\infty = (1 - \chi_R)u$.
We look into the operator $\mathring{\mathcal{A}} = -\mathring{\Delta} + n$ acting on functions. Using Corollary 3.13 of [1] with $\lambda = n$, we obtain that $\forall |s| < (n+1)/2$, $\mathring{\mathcal{A}} : W^{2,2}_s \to L^2_s$ is a Fredholm operator and an isomorphism. So there exists a positive constant $C = C(n, s)$ such that

$$||u||_{2,2,s} \leqslant C\,||\mathring{\mathcal{A}}u||_{2,s}. \tag{125}$$

Applying (125) to u_∞ ,

$$\begin{aligned} ||u_\infty||_{2,2,s} &\leqslant C\,||\mathring{\mathcal{A}}u_\infty||_{2,s} \\ &\leqslant C\,||\mathcal{A}u_\infty||_{2,s} + ||\Delta - \mathring{\Delta}||_{op,R}||u_\infty||_{2,2,s}. \end{aligned} \tag{126}$$

Yet $\mathcal{A}u_\infty = \mathcal{A}u - \mathcal{A}u_0 = \mathcal{A}u - \chi_{\mathbf{R}}\mathcal{A}\mathbf{u} + \chi_{\mathbf{R}}\mathcal{A}\mathbf{u} - \mathcal{A}u_0$.
Thus,

$$\begin{aligned} ||\mathcal{A}u_\infty||_{2,s} &\leqslant ||\mathcal{A}u||_{2,s} + ||\chi_R\mathcal{A}u||_{2,s} + ||\chi_R\mathcal{A}u - \mathcal{A}u_0||_{2,s} \\ &\leqslant C\,||\mathcal{A}u||_{2,s} + ||\chi_R\mathcal{A}u - \mathcal{A}u_0||_{2,s;\Omega_R}. \end{aligned}$$

From the expression (122) of Δ,

$$\begin{aligned} \chi_R\mathcal{A}u - \mathcal{A}u_0 &= -u\mathcal{A}\chi_R + n\chi_R u + 2g^{ij}\partial_i u\partial_j\chi_R \\ &= 2g^{ij}\partial_i u\,\partial_j\chi_R + (g^{ij}\mathring{\nabla}^2_{ij}\chi_R + b^i\partial_i\chi_R)u\,, \end{aligned}$$

leading to

$$||\chi_R\mathcal{A}u - \mathcal{A}u_0||_{2,s;\Omega_R} \leqslant c\,||u||_{1,2,s;\Omega_R}.$$

Finally

$$||\mathcal{A}u_\infty||_{2,s} \leqslant C\,\left(||\mathcal{A}u||_{2,s} + ||u||_{1,2,s;\Omega_R}\right).$$

Replacing (126) and considering (124), we obtain for R large enough

$$||u_\infty||_{2,\delta} \leqslant ||u_\infty||_{2,2,s} \leqslant C\,\left(||\mathcal{A}u||_{2,s} + ||u||_{1,2,s;\Omega_R}\right). \tag{127}$$

Using (127) and the fact that on $\Omega_R, |u_0|_{\mathring{g}} \leqslant |u|_{\mathring{g}}$

$$\begin{aligned} ||u||_{2,s} &\leqslant ||u_\infty||_{2,s} + ||u_0||_{2,s} \\ &\leqslant C\,\left(||\mathcal{A}u||_{2,s} + ||u||_{1,2,s;\Omega_R}\right) + ||u||_{2,s;\Omega_R}. \end{aligned}$$

Thanks to Ehrling inequality (11),

$$\begin{aligned} ||u||_{2,s} &\leqslant C\,\left(||\mathcal{A}u||_{2,s} + ||u||_{2,s;\Omega_R}\right) + \varepsilon||u||_{2,2,s;\Omega_R} \\ &\leqslant C\,\left(||\mathcal{A}u||_{2,s} + ||u||_{2,s;\Omega_R}\right) + \varepsilon||u||_{2,2,s}. \end{aligned}$$

and we conclude with Proposition 15. □

Let $\mathring{B} = -\mathring{\Delta} + n - 1$ be an operator acting on 1-forms.

Theorem 6. *Let $\delta \leqslant 0$ and $g \in \mathcal{G}^+$. Setting $B = -\Delta + n - 1$ and $|s| < (n+1)/2$. Then $B : W^{2,2}_s(T^*\mathcal{M}) \to L^2_s(\mathcal{T}^* \otimes \Lambda^3 T^*\mathcal{M})$ is bounded. Furthermore, it satisfies*

$$||Y||_{2,2,s} \leqslant C\,\left(||BY||_{2,s} + ||Y||_{2,s;\Omega_R}\right). \tag{128}$$

In particular, B is a semi-Fredholm operator , i.e. *B has finite dimensional kernel and closed range.*

Proof: From Proposition E of [15], the indicial radius of $\mathring{B}$ is $(n+1)/2$ and by Theorem C of [15], $\forall |s| < (n+1)/2$, $\mathring{B} : W^{2,2}_s \to L^2_s$ is a Fredholm operator. By Corollary 3.13 of [1] , $\mathring{B}$ is an isomorphism for the same span of s. So there exists a positive constant $C = C(n,s)$ such that

$$||Y||_{2,2,s} \leqslant C\,||\mathring{B}Y||_{2,s}. \tag{129}$$

The proof is nearly identical to the one of Theorem 5 with (129) replacing (125). □

Theorem 7. *Let $\delta \in]-(n+1)/2, 0]$ with $n = 3$. Then the operator $F : W^{2,2}_{\delta}(\mathcal{T}) \to L^2_{\delta}(\mathcal{T}^* \otimes \Lambda^3 T^*\mathcal{M}) := \mathcal{L}^*$ is bounded. Furthermore, it verifies*

$$||(y,Y)||_{2,2,\delta} \leqslant C\ (||F(y,Y)||_{2,\delta} + ||(y,Y)||_{2,0} + ||(y,Y)||_{2,\delta;\Omega_R})\,. \tag{130}$$

In particular, F is a semi Fredholm operator, i.e. *F has finite dimensional kernel and closed range.*

Proof: Plugging the expression of $F_0(y, Y^i)$ in (123) and using Hölder inequality (8) , (15) ,Ehrling inequality (11) along with Sobolev inclusion (with $\delta \leqslant 0$) and $\mathbf{\Phi_0}(g, \pi) \in L^2_{\delta}$,

$$\begin{aligned} ||y||_{2,2,\delta} &\leqslant C\ (||-\Delta y + ny||_{2,\delta} + ||y||_{2,\delta;\Omega_R}) \\ &\leqslant C\ (||F_0(y,Y)||_{2,\delta} + ||(y,Y)||_{2,0} + ||Y||_{2,2,\delta} + ||y||_{2,\delta;\Omega_R})\,. \end{aligned} \tag{131}$$

Plugging the expression of $F_i(y, Y^i)$ in (128) and using Hölder inequality (8) , (15) ,Ehrling inequality (11) along with Sobolev inclusion (with $\delta \leqslant 0$) and $\mathbf{\Phi_i}(g, \pi) \in L^2_{\delta}$,

$$\begin{aligned} ||Y||_{2,2,\delta} &\leqslant C\ (||-\Delta Y + (n-1)Y||_{2,\delta} + ||Y||_{2,\delta;\Omega_R}) \\ &\leqslant C\ (||F_i(y,Y)||_{2,\delta} + ||(y,Y)||_{2,0} + ||Y||_{2,\delta;\Omega_R})\,. \end{aligned} \tag{132}$$

Finally, combination of (131) and (132) gives (130).
Starting from the definition of F, the Triangle inequality together with (54) and the Sobolev inclusion (with $\delta \leqslant 0$) directly yield

$$||F(y)||_{2,\delta} \leqslant C\,||y||_{2,2,\delta},$$

where C is a constant depending on $\mathring{g}$ and $||g||_{\mathcal{F}}$. Hence F is a bounded (continuous) operator. For all $\delta \in]-(n+1)/2, 0]$, the estimate (130) verified by F est analoguous to the one of Theorem 1 and by a similar proof, we show F is semi-Fredholm, *i.e.* F has finite diensional kernel and closed range. □

Now F and its adjoint F^* have similar structure (F is formally self-adjoint)

$$F^* : L^2_{-\delta}(\mathcal{T}) \to W^{-2,2}_{-\delta}(\mathcal{T}^* \otimes \Lambda^3 T^*\mathcal{M}).$$

Let $\widetilde{F^*}$ be the restriction of F^* defined as follows

$$\widetilde{F^*} : W^{2,2}_{-\delta}(\mathcal{T}) \to L^2_{-\delta}(\mathcal{T}^* \otimes \Lambda^3 T^*\mathcal{M}).$$

We can apply Theorem 7 to $\widetilde{F^*}$:

Theorem 8. *Let $\delta \in]-(n+1)/2, 0]$ with $n = 3$. Then the operator $\widetilde{F^*} : W^{2,2}_{-\delta}(\mathcal{T}) \to L^2_{-\delta}(\mathcal{T}^* \otimes \Lambda^3 T^*\mathcal{M})$ is bounded. Furthermore, it satisfies*

$$||(y,Y)||_{2,2,-\delta} \leqslant C\ \left(||\widetilde{F^*}(y,Y)||_{2,-\delta} + ||(y,Y)||_{2,-2\delta} + ||(y,Y)||_{2,-\delta;\Omega_R}\right). \tag{133}$$

In particular, $\widetilde{F^*}$ is a semi-Fredholm operator , i.e. *$\widetilde{F^*}$ has finite dimensional kernel and closed range.*

Proof: Plugging the expression of $\widetilde{F^*_0}(y, Y^i)$ (formally identical to $F_0(y, Y^i)$) in (123) and using Hölder inequality (8) , (15) ,Ehrling inequality (11) along with Sobolev inclusion

(with $\delta \leqslant 0$) and $\mathbf{\Phi_0}(g,\pi) \in L^2_\delta$,

$$\begin{aligned} ||(y,Y)||_{2,2,-\delta} &\leqslant C\ (||-\Delta y + ny||_{2,-\delta} + ||N||_{2,-\delta;\Omega_R}) \\ &\leqslant C\ \left(||\widetilde{F_0^*}(y,Y)||_{2,\delta} + ||(y,Y)||_{2,-2\delta} + ||Y||_{2,2,-\delta} + ||y||_{2,-\delta;\Omega_R}\right) \end{aligned} \quad (134)$$

Plugging the expression of $\widetilde{F_i^*}(y,Y^i)$ (formally identical to $F_i(y,Y^i)$) in (128) and using Hölder inequality (8) , (15) ,Ehrling inequality (11) along with the Sobolev inclusion (with $\delta \leqslant 0$) and $\mathbf{\Phi_i}(g,\pi) \in L^2_\delta$,

$$\begin{aligned} ||Y||_{2,2,-\delta} &\leqslant C\ (||-\Delta Y + (n-1)Y||_{2,-\delta} + ||Y||_{2,-\delta;\Omega_R}) \\ &\leqslant C\ \left(||\widetilde{F_i^*}(y,Y)||_{2,-\delta} + ||(y,Y)||_{2,-2\delta} + ||Y||_{2,-\delta;\Omega_R}\right). \end{aligned} \quad (135)$$

Finally, combination of (134) and (135) gives (133).
From the definition of $\widetilde{F^*}$, the Triangle inequality together with (54) and the Sobolev inclusion with $\delta \leqslant 0$ directly yield

$$||\widetilde{F^*}(y)||_{2,-\delta} \leqslant C\,||y||_{2,2,-\delta}$$

and for all $\delta \in]-(n+1)/2, 0]$, $\widetilde{F^*}$ is a semi-Fredholm operator. By elliptic regularity, $\mathrm{ker}F^* = \mathrm{ker}\widetilde{F^*}$ is also finite dimensional. If we apply Lemma 22 to F , we get

$$(\mathrm{Coker}F)^* \simeq \mathrm{ker}F^*.$$

So $(\mathrm{Coker}F)^*$ is finite dimensional and $(\mathrm{Coker}F)^* \simeq \mathrm{Coker}F$. Thus we have the isomorphism

$$\mathrm{Coker}F = \mathcal{L}^*_{\diagup_{\mathrm{Im}F}} \simeq \mathrm{ker}F^*.$$

The operator F satisfies

$$\mathrm{Im}F \subset \mathrm{Im}D\mathbf{\Phi}(g,\pi) \subset \mathcal{L}^*.$$

Let π be the canonic projection:

$$\pi : \mathcal{L}^* \to \mathcal{L}^*_{\diagup_{\mathrm{Im}F}}.$$

$\pi(\mathrm{Im}D\phi)$ is closed as subspace of a finite dimensional vector space. $\mathrm{Im}(D\phi)$ is closed, being the inverse image of a closed set by a continuous map. □

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