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CONFORMALLY COVARIANT PARAMETERIZATION FOR RELATIVISTIC INITIAL DATA

ERWANN DELAY

ABSTRACT. We modify the Lichnerowicz-York method in order to obtain conformally covariant systems. This type of parametrization is certainly more natural for non constant mean curvature initial data.

Keywords : Conformal riemannian geometry, non linear elliptic PDE system, vectorial laplacian, general relativity, constraints equations.

2010 MSC : 53C21, 53A45, 53A30, 58J05, 35J61.

1. INTRODUCTION

On a smooth manifold M of dimension n , for a riemannian metric g , we denote by R_g its scalar curvature and ∇ its Levi-Civita connexion. When h is a field of symmetric covariants two tensors, we define its divergence, as the 1-form given by

$$(\operatorname{div}_g h)_i = -\nabla^k h_{ki}.$$

The vacuum initial data for the relativistic Einstein equations are given by a riemannian metric \hat{g} and a symmetric two tensor field \hat{K} , verifying the constraints equations

$$(C) \begin{cases} R_{\hat{g}} - |\hat{K}|_{\hat{g}}^2 + (\operatorname{Tr}_{\hat{g}} \hat{K})^2 = 0 \\ \operatorname{div}_{\hat{g}} \hat{K} + d(\operatorname{Tr}_{\hat{g}} \hat{K}) = 0 \end{cases}.$$

This system is highly under-determined because it contains $(n + 1)$ equations for $n(n + 1)$ unknowns . It is natural to fix some of the unknowns, and to look for the $(n + 1)$ remaining ones.

The usual conformal parameterization, where appear the 1944 Lichnerowicz equation [2], together with the 1973 York decomposition [7], was very studied by a lot of authors, on compact or non compact manifolds (asymptotic to some models : euclidean, hyperbolic, cylindrical,...).

The list of studies on the subject, being too substantial to be given here, we just mention a recent paper [5] and its references for a fleeting glimpse.

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We simply recall that after the approaches realized for constant mean curvature τ , like [1] for instance, several attempts were given in order to treat some cases where this last one is changing.

The method start with a metric g , together with a trace free and divergence free symmetric two tensor σ (a TT-tensor), and a function τ . One look for the solutions to the constraint equations of the form

$$\widehat{g} = \phi^{N-2} g, \quad \widehat{K} = \frac{\tau}{n} \widehat{g} + \phi^{-2} (\sigma + \mathring{\mathcal{L}}_g W), \quad (P)$$

where $N = \frac{2n}{n-2}$, and the unknowns are a function $\phi > 0$ and a field of one forms W and where

$$(\mathring{\mathcal{L}}_g W)_{ij} = \nabla_i W_j + \nabla_j W_i - \frac{2}{n} \nabla^k W_k g_{ij}.$$

If we denote

$$w = |\sigma + \mathring{\mathcal{L}}_g W|_g,$$

we have to solve the coupled system

$$(S) \begin{cases} P_{g,w} \phi := \frac{4(n-1)}{n-2} \nabla^* \nabla \phi + R_g \phi - w^2 \phi^{-N-1} = \frac{1-n}{n} \tau^2 \phi^{N-1} & (L) \\ \operatorname{div}_g \mathring{\mathcal{L}}_g W = \frac{1-n}{n} \phi^N d\tau & (V) \end{cases}$$

The Lichnerowicz equation (L) has a covariantly conformal property (see section 8.2 for a precise definition). Indeed, if ϕ is a solution of (L), and φ is any positive function, we may write

$$\tilde{g} = \varphi^{N-2} g, \quad \tilde{w} = \varphi^{-N} w, \quad \tilde{\phi} = \varphi^{-1} \phi$$

then

$$P_{\tilde{g}, \tilde{w}} \tilde{\phi} = \varphi^{1-N} P_{g,w} \phi = \frac{1-n}{n} \tau^2 \tilde{\phi}^{N-1}.$$

However, the vectorial equation (V) does not possess this property. Moreover, the conformal transformation of w used here on (L) does not correspond to a natural transformation of σ and W .

Let us note that some of the operators appearing in (V) are conformally covariants :

$$\operatorname{div}_{\tilde{g}} h = \varphi^{-N} \operatorname{div}_g (\varphi^2 h), \quad \mathring{\mathcal{L}}_{\tilde{g}} X = \varphi^{N-2} \mathring{\mathcal{L}}_g (\varphi^{2-N} X), \quad (C)$$

although the vector Laplacian $\operatorname{div} \circ \mathring{\mathcal{L}}$ does not be.

Finally, we would point out the York decomposition [7] valid for instance if M is compact and g has no conformal Killing fields (ie. $\ker \mathring{\mathcal{L}}_g$ is trivial). Any field of covariant symmetric trace free two tensor h split in a unique way as

$$h = \sigma + \mathring{\mathcal{L}}_g W, \quad (Y)$$

where σ is a TT-tensor and W a field of 1-forms.

We propose in the section 2 a parameterization ensuing the York decomposition relative to \widehat{g} and not to g . This parameterization give rise to a conformally covariant system.

This last one leads to the emergence of a vectorial laplacian, self adjoint for a weighted mesure, it will be studied in section 3.

In the section 4, we test some other parameterizations and compare them.

In the section 5, a linear version of the obtained system is computed. This last one suggest to study another system, it is given in section 6 and leads to a final parameterization who is certainly the more natural.

Finally, in section 7, we emphasise some comments related to the evident continuation to this work.

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2. A FIRST CONFORMALLY COVARIANT PARAMETERIZATION

Let us come back to (C) but using the York decomposition relative to \widehat{g} ,

$$\widehat{K} - \frac{\tau}{n}\widehat{g} = \widehat{\sigma} + \mathring{\mathcal{L}}_{\widehat{g}}\widehat{W}.$$

It follows that (C) is equivalent to

$$(C') \begin{cases} R_{\widehat{g}} - |\widehat{\sigma} + \mathring{\mathcal{L}}_{\widehat{g}}\widehat{W}|_{\widehat{g}}^2 = \frac{1-n}{n}\tau^2 \\ \operatorname{div}_{\widehat{g}} \mathring{\mathcal{L}}_{\widehat{g}}\widehat{W} = \frac{1-n}{n}d\tau \\ \operatorname{div}_{\widehat{g}} \widehat{\sigma} = 0 \end{cases}$$

If we are looking for a solution in a conformal class $\widehat{g} = \phi^{N-2}g$, from (C), it is natural to introduce $\widehat{\sigma} = \phi^{-2}\sigma$, where σ is a TT-tensor. The third equation is then automatically satisfied. But again using (C), if we translate the second equation in term of g , it came spontaneously $\widehat{W} := \phi^{N-2}W$.

Thus, already with the same fixed g, σ, τ , we can parametrize the solutions of the constraint (C) by

$$(P') \begin{cases} \widehat{g} = \phi^{N-2}g, \\ \widehat{K} = \frac{\tau}{n}\widehat{g} + \phi^{-2}(\sigma + \phi^N \mathring{\mathcal{L}}_g W) \end{cases}$$

With this parameterization in (C), if we define

$$\omega = \omega(\sigma, \phi, W, g) := |\sigma + \phi^N \mathring{\mathcal{L}}_g W|_g,$$

we obtain the new system

$$(S') \begin{cases} P_{g,\omega}\phi = \frac{1-n}{n}\tau^2\phi^{N-1} & (L') \\ \Delta_{g,\phi}W := \phi^{-N} \operatorname{div}_g(\phi^N \mathring{\mathcal{L}}_g W) = \frac{1-n}{n}d\tau & (V') \end{cases}$$

We now go through the conformal changes:

$$\tilde{g} = \varphi^{N-2}g, \quad \tilde{\phi} = \varphi^{-1}\phi, \quad \tilde{\sigma} = \varphi^{-2}\sigma, \quad \tilde{W} = \varphi^{N-2}W.$$

Note that by (\mathcal{C}) , the tensor $\tilde{\sigma}$ is still TT for \tilde{g} . This time round

$$\tilde{\omega} := \omega(\tilde{\sigma}, \tilde{\phi}, \tilde{W}, \tilde{g}) = \varphi^{-N}\omega$$

so the scalar equation (L') behave like above. For the new vectorial equation (V') , because from (\mathcal{C}) ,

$$\Delta_{\tilde{g}, \tilde{\phi}} \tilde{W} = \Delta_{g, \phi} W,$$

it became conformally covariant.

REMARKS :

- When $\varphi = \phi$ we fall on $\Delta_{g, \phi} W = \Delta_{\hat{g}, 1} \widehat{W}$.

- As ω depend on ϕ , the following version of (L') is more explicit :

$$\begin{aligned} \frac{4(n-1)}{n-2} \nabla^* \nabla \phi + R_g \phi - |\sigma|^2 \phi^{-N-1} - 2\langle \sigma, \mathring{\mathcal{L}}_g W \rangle \phi^{-1} - |\mathring{\mathcal{L}}_g W|_g^2 \phi^{N-1} \\ = \frac{1-n}{n} \tau^2 \phi^{N-1} \quad (L') \end{aligned}$$

- Let us consider the operator given by

$$\mathcal{P}_g \begin{pmatrix} \phi \\ W \\ \sigma \end{pmatrix} := \begin{pmatrix} \phi^{1-N} P_{g, \omega} \phi \\ \Delta_{g, \phi} W \\ \phi^{-N} \operatorname{div}_g \sigma \end{pmatrix},$$

in such a way that \mathcal{P}_g is conformally covariant :

$$\mathcal{P}_{\tilde{g}} \begin{pmatrix} \tilde{\phi} \\ \tilde{W} \\ \tilde{\sigma} \end{pmatrix} = \mathcal{P}_g \begin{pmatrix} \phi \\ W \\ \sigma \end{pmatrix}.$$

The system (S') (modulo adding the third equation) is simply written

$$\mathcal{P}_g \begin{pmatrix} \phi \\ W \\ \sigma \end{pmatrix} = \frac{1-n}{n} \begin{pmatrix} \tau^2 \\ d\tau \\ 0 \end{pmatrix}.$$

- There was been some others tentative to modify the Lichnerowicz-York method, like the ‘‘Conformal Thin-Sandwich’’ (see for instance [5] for a comparison, see also [6]). In particular, there already exist some parameterizations where

$$\widehat{K} = \frac{\tau}{n} \widehat{g} + \phi^{-2}(\sigma + f \mathring{\mathcal{L}}_g W)$$

but the function f seems always fixed at first, and not a function of ϕ , but $f = \phi^N$ is the simplest one to make the system conformally covariant (see sections 4 and 6 for some others possibilities).

- More generally, for any covariantly conform differential linear operator of order k , we could interpose a function between itself and its formal adjoint, in order to produce an operator of order $2k$, covariantly conform, positive and self adjoint for a weighted measure.

3. THE VECTORIAL EQUATION

We consider the vectorial laplacian, of Witten type, obtained previously :

$$\Delta_{g,\phi}W = \phi^{-N} \operatorname{div}_g \phi^N \mathring{\mathcal{L}}_g W = \operatorname{div}_g \mathring{\mathcal{L}}_g W - N \mathring{\mathcal{L}}_g W (\nabla \ln \phi, \cdot).$$

We choose to work here on a compact manifold. Because $2 \operatorname{div}_g$ is the formal $L^2(d\mu_g)$ adjoint of $\mathring{\mathcal{L}}_g$, we have

$$\int_M \langle \Delta_{g,\phi}W, V \rangle_g \phi^N d\mu_g = \frac{1}{2} \int_M \langle \mathring{\mathcal{L}}_g W, \mathring{\mathcal{L}}_g V \rangle_g \phi^N d\mu_g$$

This Laplacian is then $L^2(\phi^N d\mu_g)$ self adjoint and its kernel is reduced to conformal killing 1-forms.

We want to solve (with for simplification g, ϕ, τ smooth, we could also choose $\phi \in L^\infty$),

$$\Delta_{g,\phi}W = \frac{1-n}{n} d\tau. \quad (V')$$

By the Fredholm alternative, a necessary and sufficient condition for existence is orthogonality to the kernel:

$$\int_M \langle X, d\tau \rangle_g \phi^N d\mu_g = 0, \quad (\perp)$$

for all X in $\ker \mathring{\mathcal{L}}_g$. The uniqueness occurring up to the addition of an element of $\ker \mathring{\mathcal{L}}_g$. Note that if $\tilde{X} = \varphi^{N-2} X$,

$$\langle \tilde{X}, Y \rangle_{\tilde{g}} \tilde{\phi}^N d\mu_{\tilde{g}} = \langle X, Y \rangle_g \phi^N d\mu_g,$$

then because of (C), the condition (\perp) is conformally covariant. It can be written

$$\int_M \tau d_g^*(\phi^N X) d\mu_g = 0. \quad (\perp^*)$$

Of course, if g does not possess some conformal Killing, the equation (V') always has a unique solution whatever the (positive) function ϕ is.

If $\mathcal{K} = \ker \mathring{\mathcal{L}}$ is not trivial, we may define $\Pi_{g,\phi}$ the orthogonal projection on \mathcal{K} for the weighted scalar product :

$$(X, Y)_{g,\phi} = \int_M \langle X, Y \rangle_g \phi^N d\mu_g.$$

We can then solve

$$\Delta_{g,\phi}W = \frac{1-n}{n}(Id - \Pi_{g,\phi})d\tau, \quad (V'_{\text{II}})$$

W being unique if it is chosen orthogonal to \mathcal{K} (for the weighted scalar product).

REMARK : By analogy with the Witten laplacian, we could consider the similar laplacian $\phi^{-\frac{N}{2}} \operatorname{div}(\phi^{\frac{N}{2}} \phi^{\frac{N}{2}} \mathring{\mathcal{L}}\phi^{-\frac{N}{2}})$.

4. SOME OTHERS PARAMETERIZATIONS AND MATCHING

Let us consider another conformal metric $\tilde{g} = \varphi^{N-2}g$. The York decomposition (Y), related to \tilde{g} , of the tensor

$$\phi^2\varphi^{-2}(\widehat{K} - \frac{\tau}{n}\widehat{g}),$$

culminated to the parameterization

$$(P_\varphi) \begin{cases} \widehat{g} = \phi^{N-2}g, \\ \widehat{K} = \frac{\tau}{n}\widehat{g} + \phi^{-2}(\sigma + \varphi^N \mathring{\mathcal{L}}_g W). \end{cases}$$

With this parameterization in (C), if we denote by

$$\omega_\varphi = \omega(\sigma, \varphi, W, g) := |\sigma + \varphi^N \mathring{\mathcal{L}}_g W|_g,$$

we obtain the new system

$$(S_\varphi) \begin{cases} P_{g,\omega_\varphi}\phi = \frac{1-n}{n}\tau^2\phi^{N-1} & (L_\varphi) \\ \phi^{-N} \operatorname{div}_g(\varphi^N \mathring{\mathcal{L}}_g W) = \frac{1-n}{n}d\tau & (V_\varphi) \end{cases}$$

If g has no conformal killing, by (Y), for all W and all function φ , there exist a unique field of 1-forms V and a TT-tensor σ' such that

$$\sigma' + \mathring{\mathcal{L}}_g V = \varphi^N \mathring{\mathcal{L}}_g W.$$

It follows that the solutions of any system (S_φ) match, but for different σ .

REMARK: We may take φ depending of ϕ and eventually of some other parameters in the (then abusively denoted) system (P_φ) . For instance $\varphi^N = \phi^N F^2(|\mathring{\mathcal{L}}W|_g)$, for a function F , will give again a conformally covariant system of order of derivation 2. The parameterization $\varphi^N = |\tau|\phi^N$ seems also produce an interesting system.

5. A LINEAR VERSION

In order to a better understanding of some future studies of (S') , we will linearise here the operator \mathcal{P}_g introduced in section 2. We denote by (ψ, V, h) the variations of (ϕ, W, σ) , v the variation of τ , we keep here the metric g fixed. Because of the conformally covariant properties

of \mathcal{P}_g , the computation of the linearisation can be done at $\phi = 1$. We then find

$$D\mathcal{P}_g^{(1,W,\sigma)} \begin{pmatrix} \psi \\ V \\ h \end{pmatrix} = \begin{pmatrix} P_{g,\sigma,W} & -2\langle \mathring{K}, \mathring{\mathcal{L}} \cdot \rangle, & -2\langle \mathring{K}, \cdot \rangle \\ -N\mathring{\mathcal{L}}_g W(\nabla_{g,\cdot}, \cdot) & \operatorname{div}_g \mathring{\mathcal{L}}_g & 0 \\ 0 & 0 & \operatorname{div}_g \end{pmatrix} \begin{pmatrix} \psi \\ V \\ h \end{pmatrix}$$

where

$$\mathring{K} = \sigma + \mathring{\mathcal{L}}_g W,$$

and

$$\begin{aligned} P_{g,\sigma,W} &= \frac{4}{n-2} \left((n-1)\nabla^* \nabla - R_g + n|\sigma|^2 + n\langle \sigma, \mathring{\mathcal{L}}_g W \rangle \right) \\ &= \frac{4}{n-2} \left((n-1)\nabla^* \nabla - R_g + \frac{n}{2} \left(|\sigma|^2 + |\mathring{K}|^2 - |\mathring{\mathcal{L}}_g W|^2 \right) \right). \end{aligned}$$

This last operator, evaluated at a solution of (S') , can be written using the Yamabe laplacian, because

$$R_g = |\mathring{K}|^2 + \frac{1-n}{n} \tau^2.$$

The linear version of the system becomes

$$D\mathcal{P}_g^{(1,W,\sigma)} \begin{pmatrix} \psi \\ V \\ h \end{pmatrix} = \frac{1-n}{n} \begin{pmatrix} 2\tau v \\ dv \\ 0 \end{pmatrix}.$$

If σ is fixed, its impose $h = 0$, if not, the global form of the system suggest to look about some others possibilities.

6. AN OTHER CONFORMALLY COVARIANT PARAMETERIZATION

In the parameterization of the section 2, we choose a fixed TT -tenseur σ and defined $\hat{\sigma} = \phi^{-2}\sigma$ in order to verify immediately the last equation of the system (C') . Without this particular choice, (C') has $2n+1$ equations. Rather than that, we can fix a trace free tensor $\mathring{\sigma}$ an look for the unique TT-tensor $\hat{\sigma}$ for \hat{g} , such that

$$\phi^{-2}\mathring{\sigma} = \hat{\sigma} - \mathring{\mathcal{L}}_{\hat{g}} \hat{Y}.$$

It is then convenient to define

$$\hat{\sigma} = \phi^{-2}(\mathring{\sigma} + \phi^N \mathring{\mathcal{L}}_g Y), \quad \varpi = |\mathring{\sigma} + \phi^N \mathring{\mathcal{L}}_g Y + \phi^N \mathring{\mathcal{L}}_g W|$$

It result the conformally covariant system

$$(S'') \begin{cases} \phi^{1-N} P_{g,\varpi} \phi = \frac{1-n}{n} \tau^2 & (L'') \\ \Delta_{g,\phi} W = \frac{1-n}{n} d\tau & (V_1'') \\ \Delta_{g,\phi} Y = -\phi^{-N} \operatorname{div}_g \mathring{\sigma}. & (V_2'') \end{cases}$$

The matching linear operator has the form :

$$\begin{pmatrix} \phi^{1-N} P_{g,\varpi} \phi \\ \Delta_{g,\phi} \\ \Delta_{g,\phi} \end{pmatrix}' = \begin{pmatrix} P_{g,\dot{\sigma},W,Y} & -2\langle \dot{K}, \dot{\mathcal{L}} \cdot \rangle, & -2\langle \dot{K}, \dot{\mathcal{L}} \cdot \rangle \\ -N \dot{\mathcal{L}}_g W(\nabla_{g,\cdot}, \cdot) & \Delta_{g,1} & 0 \\ -N \dot{\mathcal{L}}_g Y(\nabla_{g,\cdot}, \cdot) & 0 & \Delta_{g,1} \end{pmatrix}$$

where

$$\dot{K} = \dot{\sigma} + \dot{\mathcal{L}}_g Y + \dot{\mathcal{L}}_g W,$$

and

$$\begin{aligned} P_{g,\dot{\sigma},W,Y} &= \frac{4}{n-2} \left((n-1) \nabla^* \nabla - R_g + n |\dot{\sigma}|^2 + n \langle \dot{\sigma}, \dot{\mathcal{L}}_g(W+Y) \rangle \right) \\ &= \frac{4}{n-2} \left((n-1) \nabla^* \nabla - R_g + \frac{n}{2} \left(|\dot{\sigma}|^2 + |\dot{K}|^2 - |\dot{\mathcal{L}}_g(W+Y)|^2 \right) \right). \end{aligned}$$

This operator evaluated at a solution of (S'') , can again be written with the Yamabe laplacian, because then

$$R_g = |\dot{K}|^2 + \frac{1-n}{n} \tau^2.$$

A retrospection about this last system shows that only $X = Y + W$ is important and the natural system to solve seems to be

$$(S''') \begin{cases} \phi^{1-N} P_{g,\varpi} \phi = \frac{1-n}{n} \tau^2 & (L''') \\ \Delta_{g,\phi} X = \frac{1-n}{n} d\tau - \phi^{-N} \operatorname{div}_g \dot{\sigma} & (V''') \end{cases}$$

where we recall that the function τ , and the trace free symmetric two tensor $\dot{\sigma}$ are given. The parameterization for the solutions of (C) being then

$$\widehat{g} = \phi^{N-2} g, \quad \widehat{K} = \phi^{-2} (\dot{\sigma} + \phi^N \dot{\mathcal{L}}_g X) \quad (P''')$$

REMARK : We come back to the system (S') if we choose a divergence free $\dot{\sigma}$.

7. COMMENTS AND PROSPECTS

We give here some comments together with the first fruits of a future study of conformally covariants systems, starting with the compact manifolds.

- OTHER YORK DECOMPOSITIONS : Conformally covariant or not, some other choices of $\varphi = \varphi(\phi, W, \sigma, \tau, t, f, \dots)$ in (S_φ) may be judicious.

- CONSTRAINTS WITH RIGHT HAND SIDE : The parameterization works for some other stress energy tensors like for a scalar field for instance.

- MATCHING OF PARAMETERIZATIONS : Because of the matching between the different parameterizations, the solutions of (S) already existing produce some solutions of (S') for example, even if it means to change the parameters. If we consider the set of solutions of (C) as a manifold, it is probable that the conformal method give a local chart, but a

conformally covariant method produce a more global chart, and allows a larger choice of (τ, σ) (or $(\tau, \tilde{\sigma})$ for (S''')).

- **VARIATIONAL STUDY** : The systems we obtains are not triangular like for the clasical conformal method. However, a variational study will be particularly adjusted.

- **FIXED POINT** : As for a lot of studies of (S) , a classical method to solve a system like (S') is to consider a map $T : E \rightarrow E$, for an appropriate space E , defined like this. If there is no conformal killing, for ψ in E , we consider the solution W of (V') with ϕ replaced by ψ . We then define $\omega := \omega(\sigma, \psi, W, g)$ and we solve (L') (see for instance [4]), The solution define $T(\psi) := \phi$. We the have to look for a fixed point $\phi > 0$ of T . If necessary, we could fit some ϕ together with some ψ in the choice of ω (and/or in (L')) and solve another scalar equation, similar to (L') on ϕ , for example linear relatively to ϕ . If there exist some conformal killing field, we can in the process, replace the solving of $d(V')$ by the one of (V'_{Π}) , if under appropriate conditions, the projection disappear at the limit .

- **THE LIMIT EQUATION** : In [3] a (family of) limit equation(s) is given for (S) . It has the upsetting property to guaranty the resolvability of (S) , if itself has zero as unique solution. We have to look if this limit equation measure the asymmetry of the old parameterization or if a similar equation exist for the system (S''') .

8. APPENDICE

8.1. Vector laplacian.

$$\begin{aligned} \operatorname{div}_g \mathring{\mathcal{L}}_g &= \nabla_g^* \nabla_g - \operatorname{Ric}_g + \frac{n-2}{2} dd_g^* \\ &= \Delta_{\text{Hodge}} - 2 \operatorname{Ric}_g + \frac{n-2}{2} dd_g^* \\ &= d_g^* d + \frac{n}{2} dd_g^* - 2 \operatorname{Ric}_g \end{aligned}$$

8.2. **Conformal covariance.** Let us consider tree products of tensor bundles over M ,

$$E = E_1 \times \dots \times E_k, \quad F = F_1 \times \dots \times F_l, \quad G = G_1 \times \dots \times G_m,$$

and a differential operator acting on the sections :

$$P_g : \Gamma(E) \longrightarrow \Gamma(F),$$

with coefficients determined by $g = (g_1, \dots, g_m) \in G$. We will said that P_g is conformally covariant if there exist $a = (a_1, \dots, a_k) \in \mathbb{R}^k$, $b = (b_1, \dots, b_l) \in \mathbb{R}^l$ and $c = (c_1, \dots, c_m) \in \mathbb{R}^m$ such that for all smooth section e of E , and any smooth function φ on M , we have

$$\varphi^b \odot P_{\varphi^c \odot g}(\varphi^a \odot e) = P_g(e),$$

where we have defined for instance

$$\varphi^a \odot e = (\varphi^{a_1} e_1, \dots, \varphi^{a_k} e_k).$$

A differential system will be said conformally covariant if it can be written of the form $P_g(e) = f$, for a conformally covariant operator P_g .

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