NIL-ANOSOV ACTIONS

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ABSTRACT. We consider Anosov actions of a Lie group G of dimension k on a closed manifold of dimension $k + n$. We introduce the notion of Nil-Anosov action of G (which includes the case where G is nilpotent) and establishes the invariance by the entire group G of the associated stable and unstable foliations. We then prove a spectral decomposition Theorem for such an action when the group G is nilpotent. Finally, we focus on the case where G is nilpotent and the unstable bundle has codimension one. We prove that in this case the action is a Nil-extension over an Anosov action of an abelian Lie group. In particular:
- if $n \geq 3$, then the action is topologically transitive,
- if $n = 2$, then the action is a Nil-extension over an Anosov flow.

1. INTRODUCTION

A locally free action ϕ of a group G on a closed manifold M is said to be *Anosov* if there exists α in $\mathcal{G}$, the Lie algebra of G, such that $g := \phi(\exp \alpha, \cdot)$ is normally hyperbolically with respect to the orbit foliation. In our serie of papers, we focus on the case of codimension one Anosov actions, i.e. Anosov actions admitting a normally hyperbolic element for which the unstable direction has dimension one. This notion is very classical when the group G is $\mathbb{R}$, i.e. in the case of Anosov flows. Until recently, there have been many works establishing what can be the behavior of a codimension one Anosov flow in a given manifold, even aiming to describe the Anosov flow when the ambient manifold is prescribed, with emphasis in dimension 3 (see [2, 5, 6, 7, 8, 9, 13, 14, 12, 19, 20, 21, 22, 23, 25, 24, 32, 40, 45]).

In our first paper [10], we intend to extend this analysis in the case where G is still abelian, but of bigger dimension. We prove in particular that if the dimension of the ambient manifold exceeds by at least 3 the dimension of the acting abelian group, then

the action is topologically transitive. We also proposed the following conjecture, natural extension of a previous well-known conjecture:

Generalized Verjovsky Conjecture. *Let G be an abelian Lie group of dimension k. Every codimension one faithful Anosov action of G on a manifold of dimension $\geq k+3$ is topologically conjugate the suspension of an Anosov action of $\mathbb{Z}^k$ on a closed manifold.*

Meanwhile, V. Arakawa completely elucidated in his Ph. D Thesis the case of Anosov actions of $\mathbb{R}^k$ on a closed manifold of dimension $k+2$: such an action reduces through a flat bundle to an Anosov flow on a 3-manifold ([1]).

Afterwards, in our work [11] aiming to generalize Tomter's classification of *algebraic Anosov flows* ([43, 44] in the context of actions of abelian groups of higher dimension, it became clear that the natural setting was the case where the group G is nilpotent. In particular, the Generalized Verjovsky Conjecture above can be replaced by the similar statement where the group G is nilpotent, not necessarily abelian. Actually, it follows from the present paper that a nilpotent Lie group admitting a faithful Anosov action of codimension one on a closed manifold is necessarily abelian (Theorem 5).

We therefore started the analysis of Anosov actions of non-abelian Lie groups, which is the topic of the present paper. The key point one needs in order to undertake such an analysis is that the Anosov action is good in the sense of [42], i.e. that the Anosov splitting associated to a partially hyperbolic element should be preserved by the other elements of G. It is for this reason that previously most authors only considered the case where the partially hyperbolic element lies in the center of G, for which this condition is obviously satisfied. Here, we observe that this property still hold if G is only nilpotent[1]. Actually, we point out a more general phenomenon: in order to have the G-invariance of the Anosov splitting, we only need the fact that the partially hyperbolic element belongs to the nilradical of G : the Anosov action is then called *Nil-Anosov* (see Theorem 1).

However, the study of Nil-Anosov actions is limited by the fact that the density of periodic orbits in the nonwandering set may fail in this more general situation. It is for this reason that we quickly restrict ourselves to the case of nilpotent Lie groups. Nevertheless, we consider that the study of Nil-Anosov actions is interesting in itself.

In section 3, we prove a spectral decomposition Theorem for Anosov actions of nilpotent Lie groups: the nonwandering set of such an action can be decomposed in a finite number of basic sets, each of them being topologically transitive (Theorem 4). Observe

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[1]In [26], M.W. Hirsch already made this observation, but never afterwards published a paper including the proofs of the results stated in [26].

that such a result is not merely an adaptation of the arguments involved in the case of diffeomorphisms or flows: the notion of nonwandering set is itself delicate and our analysis involves for example the fact that the nonwandering set of any element of G coincide with the nonwandering set of the entire group G (Remark 7).

One of the most important result in section 3 is the non-trivial fact when the Anosov action is faithful, then the holonomy representations of each leaf of the weak foliations are faithful (Proposition 6).

This fact is the principal element of the proof of Theorem 5 mentioned above: the study of codimension one actions of nilpotent Lie groups reduces to the case of abelian Lie groups. Therefore, the topological transitivity of these actions in higher dimensions follows from [10], but here we propose an alternative simpler proof of this fact (see Theorem 6). Finally, we conclude with an alternative and much simpler proof of Arakawa's Theorem: Anosov actions of a nilpotent Lie group of dimension k on a manifold of dimension $k + 2$ is a Nil-extension of an Anosov flow on a 3-manifold.

2. Preliminaries

2.1. **Definitions and notations.** Let G be a connected, simply connected Lie group of dimension k, let $\mathcal{G}$ be the Lie algebra of G, and let M be a C^∞ manifold of dimension $n + k$, endowed with a Riemannian metric $\| \cdot \|$, and let ϕ be a locally free smooth action of the simply connected Lie group G on M.

We will use the following usual simplified notations, denoting $\phi(g, x)$ by $g.x$ or gx for every x in M and every g in G. We will however denote by ϕ^g the associated diffeomorphism of M, when non evaluated on an element of M, in order to make a distinction with the element g of G. We denote by $G.x$ or $\mathcal{O}_G(x) := \{gx, g \in G\}$ the orbit of $x \in M$: in the second notation, we keep in mind that the orbit is the leaf of a foliation, the orbit foliation that we denote by $\mathcal{O}_G$. We will also use the simplified notation $\mathcal{O}$ and $\mathcal{O}(x)$ when there is no ambiguity on the group G. We denote by $\Delta_x := \{g \in G : gx = x\}$ the isotropy group of x. The action can be thought as a morphism $G \to \text{Diff}(M)$, the kernel of which is called

the *kernel of ϕ:* the kernel is the intersection of all the isotropy groups. The action is said to be *locally free* if the isotropy group of every point is discrete.

We can also see this data as an injective Lie morphism from $\mathcal{G}$ into the Lie algebra of vector fields of M. For every $\mathfrak{g}$ in $\mathcal{G}$, the flow $(t,x) \to \exp(t\mathfrak{g})x$ will be called *the flow generated by* $\mathfrak{g}$, denoted by $\phi_\mathfrak{g}$, and the vector field inducing this flow is denoted by $X_\mathfrak{g}$.

Let $\mathcal{F}$ be a continuous foliation on a manifold M. We denote the leaf that contains $p \in M$ by $\mathcal{F}(p)$. For an open subset U of M, let $\mathcal{F}|_U$ be the foliation on U such that $(\mathcal{F}|_U)(p)$ is the connected component of $\mathcal{F}(p) \cap U$ containing $p \in M$. A coordinate function $\varphi = (x_1, \cdots, x_n)$ on U is called *a foliation coordinate* of $\mathcal{F}$ if $x_{m+1}, \cdots, x_n$ are constant functions on each leaf of $\mathcal{F}|_U$, where m is the dimension of $\mathcal{F}$. A foliation is of class C^r if it is covered by C^r foliation coordinates. We denote the tangent bundle of M by TM. If $\mathcal{F}$ is a C^1 foliation, then we denote the tangent bundle of $\mathcal{F}$ by $T\mathcal{F}$.

We fix once for all the Riemannian metric $\|\cdot\|$, and denote by d the associated distance map in M. We also fix a metric $|\cdot|$ on the Lie algebra $\mathcal{G}$.

2.2. **Anosov actions.** Let $T\mathcal{O}$ the k-dimensional subbundle of TM that is tangent to the orbits of ϕ.

Definition 1.

   (1) A *G-splitting of TM* is a pair $\xi = (E_1, E_2)$ of subbundles of TM such that:

$$TM = E_1 \oplus T\mathcal{O} \oplus E_2$$

   (2) Two G-splittings $\xi = (E_1, E_2)$ and $\xi' = (E_1', E_2')$ are *transverse one to the other* if (E_1, E_2') and (E_1', E_2) are also G-splittings.

Clearly, if $\xi = (E_1, E_2)$ and $\xi' = (E_1', E_2')$ are transverse, then $\dim E_1 = \dim E_1'$ and $\dim E_2 = \dim E_2'$. Observe that we don't require the splitting to be G-invariant.

Definition 2.

   (1) We say that $\mathfrak{a} \in \mathcal{G}$ is an *Anosov element* for ϕ if $g = \phi^{\exp \mathfrak{a}}$ acts normally hyperbolically with respect to the orbit foliation. That is, there exists real numbers $\lambda > 0$, $C > 0$ and a continuous Dg-invariant G-splitting $\xi = (E^{ss}, E^{uu})$ such that if, m denotes the co-norm operator, then:

$$\|Dg^n|_{E^{ss}}\|, \quad \|Dg^{-n}|_{E^{uu}}\|, \quad \frac{\|Dg^n|_{E^{ss}}\|}{m(Dg^n|_{T\mathcal{O}})}, \quad \frac{\|Dg^n|_{T\mathcal{O}}\|}{m(Dg^n|_{E^{uu}})} \leq Ce^{-\lambda n}, \quad \text{for all} \ \ n \geq 0.$$

   In the terminology of [27], it means that g is 1-normally hyperbolic to $\mathcal{O}$. In this case, we say that the element $\mathfrak{a}$ of $\mathcal{G}$ and the element $\exp \mathfrak{a}$ of G are *ξ-Anosov.* The

splitting ξ is called a *g-hyperbolic splitting.* Observe that given an Anosov element g, the g-hyperbolic splitting is unique.

(2) Call ϕ an *Anosov action* if some $\mathfrak{a} \in \mathcal{G}$ is an Anosov element for ϕ.

(3) The action ϕ is a *codimension-one* Anosov action if E^{uu} is one-dimensional for some Anosov element $\mathfrak{a} \in \mathcal{G}$.

Note that the splitting ξ depends on the Anosov element $\mathfrak{a}$. By Hirsch-Pugh-Shub theory in [27] we obtain that the ξ is Hölder continuous and the subbundles E^{ss}, E^{uu}, $T\mathcal{O} \oplus E^{ss}$, $T\mathcal{O} \oplus E^{uu}$ are integrable. The corresponding foliations, $\mathcal{F}^{ss}[\xi]$, $\mathcal{F}^{uu}[\xi]$, $\mathcal{F}^{s}[\xi]$, $\mathcal{F}^{u}[\xi]$, are called *the strong stable foliation, the strong unstable foliation, the weak stable foliation,* and *the weak unstable foliation,* respectively. The orbits of ϕ are the leaves of a central foliation $\mathcal{O}$ that we call *orbit foliation.*

We insist on the fact that these foliations are not necessarily G-invariant: consider for example the case the affine group acting on the left on the quotient $\mathrm{SL}(2,\mathbb{R})/\Gamma$ where Γ is a cocompact lattice of $\mathrm{SL}(2,\mathbb{R})$ and G the group Aff of upper triangular matrices. In this case, any non-trivial diagonal matrix is an Anosov element g of Aff for which, let's say, the unstable bundle E^{uu} is trivial. Then the stable bundle E^{ss} is g-invariant, but not Aff-invariant.

Remark 1. For any splitting ξ, let $\mathcal{A}_\xi$ denote the set of ξ-Anosov elements. By the Lie product formula we have, for any $\mathfrak{a}$, $\mathfrak{b}$ in $\mathcal{A}_\xi$:

$$\exp(s\mathfrak{a} + t\mathfrak{b}) = \lim_{n\to\infty} \big[\exp(\frac{s\mathfrak{a}}{n}) \exp(\frac{t\mathfrak{b}}{n}) \big]^n, \quad \text{for any positive real numbers } s,t.$$

Consequently, $s\mathfrak{a} + t\mathfrak{b}$ is an Anosov element in $\mathcal{A}_\xi$ which preserves ξ. It follows that $\mathcal{A}_\xi$ is a convex cone in $\mathcal{G}$.

Definition 3 (Anosov subcone)**.** A *subcone* $\mathcal{C}$ is an open convex cone in $\mathcal{G}$. A subcone is *Anosov* if it is made of Anosov elements. It is *strict* if any nonzero element of the closure $\overline{\mathcal{C}}$ is Anosov.

The $\mathcal{C}$-*orbit of a point* x, denoted by $\mathcal{C}_x$, is the set comprising points of the form $\exp(\mathfrak{g}_1)...\exp(\mathfrak{g}_N)x$ where the $\mathfrak{g}_i$'s are elements of $\mathcal{C}$ in arbitrary numbers.

We consider Anosov subcones as approximations of one parameter subgroups: the $\mathcal{C}$-orbit of a point x is contained in the orbit $\mathcal{O}_x$ and is made of the points of the form $c(1)$ where $c : [0,1] \to M$ is a differentiable curve such that $c(0) = x$ and at any time t, the derivative $\dot{c}(t)$ is tangent to the orbit of the flow $\phi_\mathfrak{g}$ where $\mathfrak{g}$ is an element of $\mathcal{C}$. The thiner the cone is, the closer are the $\mathcal{C}$-orbits to the trajectory of $\phi_\mathfrak{g}$ for any element $\mathfrak{g}$ of $\mathcal{G}$.

The following remark is a consequence of the structural stability of Anosov elements (see Theorems (7.1), (7.3) and the Remark after Theorem (7.3) in [27]).

Remark 2. *The set $\mathcal{A} = \mathcal{A}(\phi)$ of Anosov elements for ϕ is an open subset of $\mathcal{G}$. More precisely, the map associating to an element $\mathfrak{a}$ in $\mathcal{A}$ its Anosov splitting $\xi(\mathfrak{a})$ is continuous.*

Remark 3. *On the other hand, there is a neighborhood U of $\xi(\mathfrak{a})$ in the space of splittings (with prescribed stable/unstable dimensions) such that $\xi(\mathfrak{a})$ is the only $\mathfrak{a}$-invariant splitting in U.*

More precisely: let $\xi' = (E^1, E^2)$ be any G-splitting transverse to $\xi = (E^{ss}, E^{uu})$. Then, at every point x of M, the subspace E_x^1 is the graph of a linear endomorphism $E_x^{ss} \to E_x^{uu} \oplus T\mathcal{O}_x$, hence ξ' provides a section σ of $Hom(E^{ss}, E^{uu} \oplus T\mathcal{O})$. If ξ' is $\exp(\mathfrak{a})$-invariant, then σ is $\exp(\mathfrak{a})$-equivariant. But it follows easily from the Anosov property that the unique $\exp(\mathfrak{a})$-equivariant section of $Hom(E^{ss}, E^{uu} \oplus T\mathcal{O})$ is the null one. Therefore we have $E^1 = E^{ss}$, and, similarly, $E^2 = E^{uu}$.

2.3. **Nil-Anosov actions.** Assume that G has a nontrivial nilradical N. We call ϕ a *Nil-Anosov action* if some $\mathfrak{a} \in \mathcal{N}$, the Lie algebra of N, is an Anosov element for ϕ. In this case $\mathfrak{a}$ is call a *Nil-Anosov element* for ϕ.

Theorem 1. *If ϕ is a Nil-Anosov action, then for any Anosov element $\mathfrak{a}$ of the nilradical N, the hyperbolic splitting ξ for which $\mathfrak{a}$ is Anosov is preserved by the action of G. In particular, every foliation $\mathcal{F}^{ss}[\xi]$, $\mathcal{F}^{uu}[\xi]$, $\mathcal{F}^s[\xi]$, $\mathcal{F}^u[\xi]$ is G-invariant.*

The proof of this Theorem is based on this fundamental lemma, that we will use several time in this paper:

Lemma 1. *(**Ascending chains of normalizers stabilize**) Let $\mathcal{H}_0$ be a Lie subalgebra in a nilpotent Lie algebra $\mathcal{N}$. Let $(\mathcal{H}_i)_{(i \in \mathbb{N})}$ be the infinite sequence obtained by taking the successive normalizers: for every integer i, $\mathcal{H}_{i+1}$ is the normalizer of $\mathcal{H}_i$ in $\mathcal{N}$. Then, this sequence eventually stabilizes to $\mathcal{N}$: there is an integer k such that $\mathcal{H}_k = \mathcal{N}$.*

Proof. Nilpotent groups satisfies the *normalizer condition:* for every Lie subalgebra $\mathcal{H}$, the normalizer $N(\mathcal{H})$ contains strictly $\mathcal{H}$, unless $\mathcal{H}$ is already the entire Lie algebra $\mathcal{N}$. It follows that the dimension of $\mathcal{H}_i$ increases with i, until the equality $\mathcal{H}_k = \mathcal{N}$ holds. $\square$

Lemma 2. *Let $\mathcal{H}$ be a Lie subalgebra of $\mathcal{G}$ containing a ξ-Anosov element and assume that for every $\mathfrak{a}$ in $\mathcal{H}$ the splitting ξ is $\exp(\mathfrak{a})$-invariant. Then, for every $\mathfrak{h}$ in the normalizer $N_{\mathcal{G}}(\mathcal{H})$ in $\mathcal{G}$ the splitting ξ is $\exp(\mathfrak{h})$ invariant.*

Proof. Let $a = \exp(\mathfrak{a})$ be an Anosov element in $H = \exp(\mathcal{H})$. Then, for all $\mathfrak{h}$ in $N_{\mathcal{G}}(\mathcal{H})$ the conjugate $\exp(\mathfrak{h}) a \exp(-\mathfrak{h})$ is $\exp(\mathfrak{h})(\xi)$-Anosov. If $\mathfrak{h}$ is small, the splittings ξ and $\exp(\mathfrak{h})(\xi)$ are close one to the other, in particular, transverse one to the other. Since by hypothesis ξ is $\exp(\mathfrak{h}) a \exp(-\mathfrak{h})$-invariant (since $\exp(\mathfrak{h}) a \exp(-\mathfrak{h})$ lies in $\mathcal{H}$), it follows from Remark 3 that we have:

$$\exp(\mathfrak{h})(\xi) = \xi,$$

The lemma follows since every element of $\exp(N_{\mathcal{G}}(\mathcal{H}))$ is an iterate of some $\exp(\mathfrak{h})$ with $\mathfrak{h}$ small. $\qquad\square$

Proof of Theorem 1. Let $\mathcal{H}_0$ be the subalgebra of $\mathcal{N}$ generated by the Nil-Anosov element $\mathfrak{a}$. We define, inductively, $\mathcal{H}_{i+1} = N_{\mathcal{N}}(\mathcal{H}_i)$ for all i in $\mathbb{N}$. By Lemma 2 we have that ξ is preserved by $\mathcal{H}_i$ for every i in $\mathbb{N}$. On the other hand, by Lemma 1, there exists k in $\mathbb{N}$ such that $\mathcal{N} = \mathcal{H}_k$.

Finally, by applying Lemma 2 to $N_{\mathcal{G}}(\mathcal{N})$ and using the fact that $N_{\mathcal{G}}(\mathcal{N}) = \mathcal{G}$, because $\mathcal{N}$ is the nilradical of $\mathcal{G}$, we obtain that G preserves ξ. $\qquad\square$

Proposition 1. *Let $\mathfrak{a}_0$ be an element of $\mathcal{N}$ which is ξ-Anosov. Then, the set $\mathcal{A}_\xi$ of ξ-Anosov elements of G (not necessarily in the nilradical) is an open convex cone, which is a connected component of the set $\mathcal{A}$ of Anosov elements of G.*

Proof. By Remark 1, $\mathcal{A}_\xi$ is a convex cone. According to Remark 2, $\mathcal{A}$ is an open neighborhood of $\mathcal{A}_\xi$. The map associating to an element $\mathfrak{a}$ in $\mathcal{A}$ its Anosov splitting $\xi(\mathfrak{a})$ is continuous; and it follows from Theorem 1 and Remark 3 that it is locally constant at every point in $\mathcal{A}_\xi$. It follows that $\mathcal{A}_\xi$ is open in $\mathcal{A}$. The proposition follows since $\mathcal{A}_\xi$ is clearly closed in $\mathcal{A}$. $\qquad\square$

Connected components of $\mathcal{A}$ are called *chambers*. From now we will assume that every Nil-Anosov action comes with a specified Nil-Anosov element $\mathfrak{a}_0$. We denote by ξ the splitting for which $\mathfrak{a}_0$ is ξ-Anosov, and by $\mathcal{A}_0 = \mathcal{A}_\xi$ the chamber containing the preferred Anosov element $\mathfrak{a}_0$. The splitting ξ defines the stable/unstable foliations common to every element in $\mathcal{A}_0$, that we will denote by $\mathcal{F}^s$, $\mathcal{F}^u$, $\mathcal{F}^{ss}$ and $\mathcal{F}^{uu}$, dropping the notation $[\xi]$. According to Theorem 1, all these foliations are G-invariant.

For all $\delta > 0$, $\mathcal{F}^i_\delta(x)$ denotes the open ball in $\mathcal{F}^i(x)$ centered at x with radius δ with respect to the restriction of $\|\cdot\|$ to $\mathcal{F}^i(x)$, where $i = ss, uu, s, u$.

Theorem 2 (local product structure). *Let $\phi : G \times M \to M$ be a Nil-Anosov action. There exists a $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ and for all $x \in M$, the applications*

$$[\cdot, \cdot]^u : \mathcal{F}^s_\delta(x) \times \mathcal{F}^{uu}_\delta(x) \to M; \quad [y, z]^u = \mathcal{F}^s_{2\delta}(z) \cap \mathcal{F}^{uu}_{2\delta}(y)$$

$$[\cdot, \cdot]^s : \mathcal{F}^{ss}_\delta(x) \times \mathcal{F}^u_\delta(x) \to M; \quad [y, z]^s = \mathcal{F}^{ss}_{2\delta}(z) \cap \mathcal{F}^u_{2\delta}(y)$$

are homeomorphisms onto their images.

Remark 4.

(1) Every leaf of $\mathcal{F}^{ss}$ or $\mathcal{F}^{uu}$ is a plane, i.e. diffeomorphic to $\mathbb{R}^\ell$ for some ℓ. This is a straightforward observation, since every compact domain in a leaf of $\mathcal{F}^{ss}$ (respectively of $\mathcal{F}^{uu}$) shrinks to a point under positive (respectively negative) iteration by $\phi^{\exp \mathfrak{a}_0}$.

(2) Let F be a weak leaf, let us say a weak stable leaf. For every strong stable leaf L in F, let Γ_L be the subgroup of G comprising elements a such that $\phi^a(L) = L$, and let ω_L be the saturation of L under ϕ. Thanks to Theorem 2 we have:

- ω_L is open in F,
- Γ_L is discrete.

Since F is connected, the first item implies $F = \omega_L$: the ϕ-saturation of a strong leaf is an entire weak leaf. The second item implies that the quotient $P = G/\Gamma_L$ is a manifold. For every x in F, define $p_F(x)$ as the equivalence class $a\Gamma_L$ such that x belongs to $\phi^a(L)$. The map $p_F : F \to P$ is a locally trivial fibration and the restriction of p_F to any ϕ-orbit in F is a covering map. Since the fibers are contractible (they are leaves of $\mathcal{F}^{ss}$, hence planes), the fundamental group of F is the fundamental group of P, i.e. Γ_L for any strong stable leaf L inside F.

Observe that if $\mathcal{F}^{ss}$ is oriented, then the fibration p_F is trivial: in particular, F is diffeomorphic to $P \times \mathbb{R}^p$, where p is the dimension of $\mathcal{F}^{ss}$. Of course, analogous statements hold for the strong and weak unstable leaves.

(3) In the continuation of the previous items, let F be a weak stable leaf, x a point in F and L the strong stable leaf through x. Since $p_F : F \to P$ is a locally trivial fibration and the fibers L are contractible, loops in F based at x are all homotopic to a loop $\omega^{\mathfrak{h}}_x$ obtained by composing a path from $[0, 1]$ into $\mathcal{O}_x$ of the form $t \mapsto \exp(t\mathfrak{h})$ where $\mathfrak{h} \in \mathcal{G}$ satisfies $(\exp \mathfrak{h})L = L$ with a path joining $\exp(\mathfrak{h}).x$ to x. The holonomy of such a loop does not depend on the choice of the final path joining $\exp(\mathfrak{h}).x$ to x since L is contractible; this holonomy defines a germ at x of homeomorphism of $\mathcal{F}^{uu}(x)$ which only depends on x and $\mathfrak{h}$. We denote it by $h^{\mathfrak{a}}_x$.

Remark 5. We will also need a local description of fixed points of Anosov elements: let $a = \exp(\mathfrak{a})$ be an ξ-Anosov element of an Anosov action of G. Let x be a fixed point of ϕ^a. Let D be a small disk containing x transverse to the orbit foliation $\mathcal{O}$. There is a small neighborhood U of x in M and a neighborhood V of e (the identity element) in G such

that, for every z in U, there is one and only one element b in V such that bz belongs to D. Moreover, reducing U if necessary, we can assume that for every b in V the product ba is Anosov. Let $D' \subset D$ be a subdisk such that $\phi^a(D')$ is contained in U. Then there is a "first return map" $\varphi : D' \to D$ and a "variation of time of first return map" $\alpha : D' \to V$ uniquely defined by: $\varphi(z) = \alpha(z)a.z \in D$. Observe that the map φ could also be defined as the holonomy of $\mathcal{O}$ along the periodic orbit of the flow $\phi_{\mathfrak{a}}$ containing x.

The weak foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ are transverse to D' and thus define two foliations $\mathcal{G}^s$ and $\mathcal{G}^u$, transverse to one another, of supplementary dimensions, and preserved by the holonomy map φ. Let s_0 (respectively u_0) the leaf of $\mathcal{G}^s$ (resp. $\mathcal{G}^u$) through x. They are both φ-invariant, the action of φ on s_0 is contracting, and the action on u_0 is expanding. By considering the map from D' into $s_0 \times u_0$ associating to a point y the pair (p, q), where p is the intersection between $\mathcal{G}^u(y)$ and s_0, and q the intersection between $\mathcal{G}^s(y)$ and u_0 (cf. Theorem 2), we get that x is a saddle fixed point of φ. In particular, x is an isolated fixed point of φ.

Remark 6. *If H is a Lie subgroup of N containing a Nil-Anosov element, then every uniform lattice in H contains Anosov elements. In particular, the isotropy subgroup of a point in M whose H-orbit is compact contains Anosov elements.* Indeed, if Δ is an uniform lattice of H, then, by a result in [3, Theorem (5.1)], Δ contains a log-lattice, i.e. a finite index sub-lattice $\Delta' \subset \Delta$ such that $\log \Delta'$ is a lattice of the vector space $\mathcal{H}$. On the other hand, let $\mathfrak{h}$ be a Nil-Anosov element in $\mathcal{H}$ and let $\mathcal{A}_0$ be the chamber in $\mathcal{G}$ containing the Anosov element $\mathfrak{h}$. Then, since Proposition 1 implies that $\mathcal{H} \cap \mathcal{A}_0$ is an open convex cone in $\mathcal{H}$, the set

$$(\mathcal{H} \cap \mathcal{A}_0) \cap \log \Delta'$$

is not empty. Consequently, $\Delta' \subset \Delta$ contains Anosov elements, and thus, our assertion follows.

2.4. **Examples.** In this subsection, we give examples of general Anosov actions, not necessarily Nil-Anosov. We start showing how to produce new Anosov actions from the data of Anosov actions.

2.4.1. *Products of Anosov actions.* If $\phi_1 : G_1 \times M_1 \to M_1$ and $\phi_2 : G_2 \times M_2 \to M_2$ are two Anosov actions, then the product action of $G_1 \times G_2$ on $M_1 \times M_2$ is clearly Anosov.

The nilradical of $G_1 \times G_2$ is the product of the nilradicals of G_1 and G_2, and an element (g_1, g_2) of $G_1 \times G_2$ is Anosov for $\phi_1 \times \phi_2$ if and only if g_1 is Anosov for ϕ_1 and g_2 is Anosov for ϕ_2. It follows that the product action is Nil-Anosov if and only if $\phi_1 : G_1 \times M_1 \to M_1$ and $\phi_2 : G_2 \times M_2 \to M_2$ are both Nil-Anosov.

This remark includes the limit case where one of the action, let's say $\phi_2 : G_2 \times M_2 \to M_2$, is transitive, i.e. an action of G_2 by left translations on a compact quotient G_2/Λ_2. This is partially generalized in the next section.

2.4.2. *Nil-extensions.* We consider any extension:

$$0 \to H \to \widehat{G} \xrightarrow{p} G \to 0$$

where H is a nilpotent Lie group. A locally free action $\hat{\phi} : \widehat{G} \times \widehat{M} \to \widehat{M}$ is a *Nil-extension* of $\phi : G \times M \to M$ if there is a $\widehat{G}$-equivariant map $\pi : \widehat{M} \to M$ whose fibers are precisely the H-orbits:

$$\pi(\hat{g}.x) = p(\hat{g}).\pi(x).$$

We furthermore require that the map $x \mapsto H_x$ where H_x is the stabilizer of x for the action of H is continuous as map from $\widehat{M}$ into the space of lattices of H.

In other words, π is a locally trivial fibration over M, with fibers homeomorphic to compact quotients of H. It induces an identification between the orbit space $H\backslash\widehat{M}$ and M, and ϕ is the induced action.

A Nil-extension introduces no additional dynamical feature. In particular, elements of H have a "trivial" transversal action; they cannot be Anosov relatively to the action $\hat{\phi}$. Hence, clearly, ϕ is (Nil-)Anosov if and only if $\hat{\phi}$ is (Nil-)Anosov.

A fundamental type of Nil-extensions are the *central extensions,* i.e. Nil-extensions for which N is in the center of G. Observe that central extensions are not necessarily product; see Starkov's example described in [11, Section 3.2.3].

2.4.3. *Affine group of the real line.* Let Aff be the subgroup of $\mathrm{SL}(2, \mathbb{R})$ made of upper diagonal matrices: we have already considered previously the action of Aff on compact quotients $\mathrm{SL}(2, \mathbb{R})/\Gamma$ as an example of Anosov action for which the splitting of Anosov elements is not preserved by the entire group.

This Anosov action is not Nil-Anosov. Observe that it admits no compact orbits.

2.4.4. *Engel subalgebras.* Let G be a real Lie group, not necessarily semisimple, and let $\mathcal{G}$ be its Lie algebra. For any element $\mathfrak{h}$ of $\mathcal{G}$ we denote by $\mathrm{ad}\,\mathfrak{h}$ the adjoint action of $\mathfrak{h}$ on $\mathcal{G}$. We denote by $L_0(\mathfrak{h})$ the 0-characteristic subspace of $\mathrm{ad}\,\mathfrak{h}$: it is a subalgebra of $\mathcal{G}$, called a *Engel subalgebra.* We actually have a splitting of $\mathcal{G}$ by characteristic subspaces:

$$\mathcal{G} = L_0(\mathfrak{h}) \oplus L^s(\mathfrak{h}) \oplus L^u(\mathfrak{h}) \oplus L^i(\mathfrak{h})$$

where $L^s(\mathfrak{h})$ (resp. $L^u(\mathfrak{h})$) is the sum of characteristic subspaces associated to eigenvalues with negative real part (resp. positive real part), and where $L^i(\mathfrak{h})$ is the sum of characteristic subspaces associated to purely imaginary eigenvalues.

The sum $\mathcal{H} = L_0(\mathfrak{h}) \oplus L^i(\mathfrak{h})$ is a subalgebra. It follows immediatly that, for any cocompact lattices Γ of G the right action on $\Gamma \backslash G$ of the Lie group H admitting $\mathcal{H}$ as Lie algebra is Anosov.

Moreover, if $L_0(\mathfrak{h})$ is *minimal* (i.e. does not contain a proper Engel subalgebra) then it is a *Cartan subalgebra* (abrev. CSA) i.e. is nilpotent and equal to its own normalizer (see [28, Theorem page 80]). It happens - for example if G is real semisimple and $L_0(\mathfrak{h})$ a hyperbolic CSA, see [11, Appendix A.3] - that $L^i(\mathfrak{h})$ is trivial, then we obtain in this case an Anosov action of a nilpotent Lie group.

More generally, one can construct from CSA's many examples of *algebraic Anosov actions*, i.e. actions on quotients $\Gamma \backslash G / K$ of Lie subalgebras $\mathcal{H}$ of $\mathcal{G}$ where K is a compact subgroup commuting with $\exp(\mathcal{H})$. In [11], the authors study in detail these examples when $\mathcal{H}$ is nilpotent.

2.5. Nil-Anosov actions are Nil-faithful up to Nil-extensions.

We can reduce any action $\phi : G \times M \to M$ to a faithful one simply by replacing G by $G / \ker \phi$ and considering the induced action on M. On the other hand, many aspects of nilpotent Lie groups are simplified when they are simply connected: the exponential map is a diffeomorphism, and one can more easily manipulate subgroups of N when it is simply connected. By taking the universal covering if necessary, we can easily reduce to the case where G and its nilradical are simply connected, but then we may loose the faithfulness of the action.

Nevertheless, the following proposition shows that, up to Nil-extensions, we can keep the fact that the nilradical is simply connected.

Definition 4. *Let $\phi : G \times M \to M$ be an action of a simply connected Lie group G. The action is Nil-faithful if the induced action of N is faithful.*

Lemma 3. *Every Nil-Anosov action of a simply connected Lie group is a Nil-extension over a Nil-faithful Nil-Anosov action of a simply connected Lie group.*

Proof. Let $\phi : G \times M \to M$ be a Nil-Anosov action which is not Nil-faithful. Let Λ be the intersection between the nilradical N and the kernel of ϕ. Then Λ is discrete in N and stable by conjugacy: it follows that Λ is in the centralizer $Z(G)$ of G. Since N is nilpotent and simply connected, $Z(N) = Z(G) \cap N$ is isomorphic to $\mathbb{R}^m$ for some m; let H be the minimal connected Lie subgroup of $Z(N)$ containing Λ. Then the torus $T := \Lambda \backslash H$ acts freely and properly on M, and the orbit space $\overline{M} := T \backslash M$ is equipped with an Anosov action of $H \backslash G$ whose nilradical $H \backslash N$ is simply connected. The quotient map $M \to \overline{M}$ is a central extension.

The Lemma is obtained by repeating the argument until the action on the quotient space is Nil-faithful. □

3. Anosov actions of nilpotent Lie groups

From now on, we restrict to the case where the Lie group G is nilpotent, i.e. equal to its nilradical N. In this section, we will observe that compact orbits play a central role in the dynamical behavior of Anosov actions of nilpotent Lie groups.

Convention: We denote by $\mathrm{Comp}(\phi)$ the set of compact ϕ-orbits.

By Lemma 3, we can assume without loss of generality that $N = G$ is simply connected (all our results are insensitive to Nil-extensions).

3.1. **Compact orbits.** Most of the content of this section and of the following section 3.2 is announced in [26], but the proofs have not been published anywhere.

Theorem 3. *Let ϕ be an Anosov action of a nilpotent Lie group N. A ϕ-orbit is compact if and only if it contains a point fixed by an Anosov element.*

The Theorem will be proved by induction involving the following Lemma.

Lemma 4. *Let H be a subgroup of N containing an Anosov element $a = \exp(\mathfrak{a})$ and let x_0 be an element of M. Let $N(H)$ denote the normalizer in N of H. If the H-orbit of x_0 is compact, then the $N(H)$-orbit of x_0 is compact.*

Proof. Assume that $\mathcal{A}_0$ is the chamber containing the Anosov element $\mathfrak{a}$ in $\mathcal{H}$. Let $\mathcal{O}_H(x_0)$ be the H-orbit of x_0, and let Δ_0 be the isotropy subgroup of x_0 by the H-action, that is Δ_0 is the uniform lattice in H which is given by $\Delta_0 = \{h \in H : \phi^h(x_0) = x_0\}$. Let x_∞ be a point in the closure of the $N(H)$-orbit of x_0, then there exists a sequence of elements z_n of $N(H)$ such that

$$\lim_{n \to \infty} \phi^{z_n}(x_0) = x_\infty.$$

As $z_n H = H z_n$, then the H-orbit of $\phi^{z_n}(x_0)$ is exactly equal to $\phi^{z_n}(\mathcal{O}_H(x_0))$, in particular, it is compact. Hence, for every n the isotropy subgroup of $\phi^{z_n}(x_0)$ by the H-action, which we denote by Δ_n, is a uniform lattice in H. Moreover, for every n we have

$$\Delta_n = z_n \Delta_0 z_n^{-1}.$$

Since N is nilpotent, the adjoint action of z_n on H is unimodular. In particular, all the lattices Δ_n have the same co-area in H. Moreover, since the action on the compact manifold M is locally free, there is a uniform bound from below for the size of elements in

Δ_n. By Mahler's criterion for nilpotent Lie groups with a $\mathbb{Q}$-structure (which is the case for H since it contains a cocompact lattice) (see [33]), up to a subsequence, we can suppose that the sequence Δ_n converges to some uniform lattice Δ_∞ in H. In particular, for every h_∞ in Δ_∞, there is a sequence of elements h_n of Δ_n converging in H to h_∞. Let γ_n be the element in Δ_0 such that $h_n = z_n \gamma_n z_n^{-1}$. Then, by continuity of the map $\phi : N \times M \to M$:

$$
\begin{aligned}
h_\infty.x_\infty &= \lim_{n \to \infty} h_n.(z_n.x_0) \\
&= \lim_{n \to \infty} z_n \gamma_n z_n^{-1} z_n.x_0 \\
&= \lim_{n \to \infty} z_n \gamma_n.x_0 \\
&= \lim_{n \to \infty} z_n.x_0 \\
&= x_\infty.
\end{aligned}
$$

This shows that Δ_∞ lies in the isotropy subgroup of x_∞ by the H-action and, consequently, the H-orbit of x_∞ is compact.

We claim that x_∞ is in the ϕ-orbit of x_0. In fact, by Remark 6, there is an Anosov element a in H such that: $\phi^a(x_\infty) = x_\infty$ (in other words, a is in Δ_∞) and $\log a$ is an element of the chamber $\mathcal{A}_0$. As defined in Remark 5, we consider the local return map $\varphi : D' \to D$ along the ϕ^a-orbit of x_∞, where D' and D are small disks transverse to the ϕ-orbit of x_∞, and x_∞ is the unique fixed point for φ, of hyperbolic type.

Let V_H be a small neighborhood of the identity in H such that the map $h \mapsto hx$ from V_H to M is injective for every element x of D. When x describes D', the pieces of orbits $V_H.x$ are disjoint one from the other, and have an area uniformly bounded from below. Since the orbits $\mathcal{O}_H(z_n.x_0)$ have all the same area, it follows that the number of intersections between D and every $\mathcal{O}_H(z_n.x_0)$ is bounded from above, uniformly in n, by some integer m.

On the other hand, the points $\phi^{z_n}(x_0)$ accumulate on x_∞, hence belong to D' for n sufficiently big. Then, if n is sufficiently big, the iterates $\varphi^k(\phi^{z_n}(x_0))$ belongs to D for all intergers k between 0 and $m+1$. By definition of m, two of these iterates must be equal, but since x_∞ is a fixed point of φ of hyperbolic type, it is possible only if $\phi^{z_n}(x_0) = x_\infty$. Hence, x_∞ belongs to the ϕ-orbit of x_0, and thus, our claim is verified.

Let g be an element of N such that $x_\infty = g.x_0$. This implies that $\Delta_\infty = g\Delta_0 g^{-1}$, hence g is an element of $N(H)$ since Δ_∞ and Δ_0 are both lattices in H.

Thus, we have proved that every point x_∞ in the closure of the $N(H)$-orbit of x_0 is in the $N(H)$-orbit of x_0: this orbit is compact.

$\square$

Proof of Theorem 3. One of the implication is a direct consequence of Remark 6. Let us prove the other implication: let x_0 be an element of M fixed by an Anosov element a in N. We will show that the ϕ-orbit of x_0 is compact. In fact, let $\mathfrak{a}$ be the element in $\mathcal{N}$ such that $\exp \mathfrak{a} = a$. We consider H_0, the subgroup of N which is the one-parameter group associated to $\mathfrak{a}$. It is clear that the H_0-orbit of x_0 is compact (a circle). We define, inductively, $H_{i+1} = N(H_i)$ for all i in $\mathbb{N}$. By Lemma 4 we have that the H_i-orbit of x_0 is compact for every i in $\mathbb{N}$. Hence, by Lemma 1, there exists k in $\mathbb{N}$ such that $N = H_k$ and the Theorem follows. □

Let us fix a Haar measure on N, so that one can estimate the area of every ϕ-orbit.

Proposition 2. *For every $C > 0$, there is only a finite number of compact ϕ-orbits of area $\leq C$.*

Proof. Assume by contradiction the existence of an infinite sequence of distinct compact orbits $\mathcal{O}_n$ of area $\leq C$. For each of them, let Δ_n be the isotropy group of $\mathcal{O}_n$: it is a lattice in N.

Since ϕ is locally free, the length of elements of Δ_n is uniformly bounded from below, independantly from n. Once more, by the Mahler's criterion extended to nilpotent Lie groups ([33]), it ensures that, up to a subsequence, the Δ_n converges to some lattice Δ_∞. In particular, for every a_∞ in Δ_∞, there is a sequence of elements a_n of Δ_n converging in N to a_∞. Furthermore, according to Remark 6, we can select a_∞ to be Anosov. Up to a subsequence, we can also pick up a sequence of elements x_n in each $\mathcal{O}_n$ converging to some x_∞ in M. Then, since $\phi^{a_n}(x_n) = x_n$, at the limit we have $\phi^{a_\infty}(x_\infty) = x_\infty$. Since a_∞ is Anosov, the ϕ-orbit $\mathcal{O}_\infty$ of x_∞ is compact. Consider a local section Σ to ϕ containing x_∞: the first return map on Σ along the orbit of ϕ^{a_∞} is hyperbolic, admitting x_∞ as an isolated fixed point. On the other hand, by pushing slightly along ϕ, we can assume without loss of generality that every x_n belongs to Σ. Since the a_n converges to a_∞, the ϕ^{a_n}-orbit of x_n approximates the ϕ^{a_∞}-orbit of x_∞, showing that the x_n are also fixed points of the first return map. It is a contradiction, since they accumulate to the isolated fixed point x_∞. □

3.2. **Spectral decomposition.**

Definition 5 (Nonwandering set of an Anosov element)**.** A point $x \in M$ is *nonwandering* with respect to an Anosov element $\mathfrak{a}$ if for any open set U containing x there is a real number $t > 1$, such that $\phi^{\exp(t\mathfrak{a})}(U) \cap U \neq \emptyset$. The set of all nonwandering points, with respect to $\mathfrak{a}$, is denoted by $\Omega(\mathfrak{a})$.

Recall that we have fixed a metric $|\cdot|$ on $\mathcal{N} = \mathcal{G}$, and than an Anosov subcone $\mathcal{C}$ is strict if nonzero elements of $\overline{\mathcal{C}}$ are Anosov (Definition 3).

Definition 6 (Nonwandering set of a strict Anosov subcone). A point $x \in M$ is *nonwandering* with respect to a strict Anosov subcone $\mathcal{C}$ if for any open set U containing x there is an element $\mathfrak{a}$ of $\mathcal{C}$ such that $|\mathfrak{a}| \geq 1$ and $\phi^{\exp(\mathfrak{a})}(U) \cap U \neq \emptyset$. The set of all nonwandering points, with respect to $\mathcal{C}$, is denoted by $\Omega(\mathcal{C})$.

Let $\mathcal{C}$ be a strict Anosov subcone containing $\mathfrak{a}$. Observe that $\Omega(\mathfrak{a})$ and $\Omega(\mathcal{C})$ are closed, and the inclusion $\Omega(\mathfrak{a}) \subset \Omega(\mathcal{C})$ is obvious. If the ϕ-orbit of a point x is compact, then the restriction of the action of $\mathfrak{a}$ on this orbit is a left translation on a compact right quotient of N by a lattice; hence this action preserves a volume form on the orbit and every point is nonwandering. The inclusions $\mathrm{Comp}(\Phi) \subset \Omega(\mathfrak{a}) \subset \Omega(\mathcal{C})$ follow. Inversely:

Proposition 3. *For any strict Anosov subcone $\mathcal{C}$, the union $\mathrm{Comp}(\phi)$ of compact orbits of N is dense in $\Omega(\mathcal{C})$.*

Proof. Let $x \in \Omega(\mathcal{C})$. We denote by $\mathcal{C}^{\geq 1}$ the subset of $\mathcal{C}$ comprising elements with $|$-norm ≥ 1. For every $\varepsilon > 0$ let D_ε be a small disc of size ε centered at x and transverse to the orbit foliation $\mathcal{O}$, so that the restrictions $\mathcal{G}^s$ and $\mathcal{G}^u$ of the weak foliations to D are product; any leaf of $\mathcal{G}^s$ intersects every leaf of $\mathcal{G}^u$ in one and only one point - the existence of such a disc is a corollary of Theorem 2. We can assume that these discs are nested, i.e. that $D_\varepsilon \subset D_{\varepsilon'}$ if $\varepsilon \leq \varepsilon'$. Let U be a relatively compact symetric neighborhood of the identity in G such that the application $U \times D_\varepsilon \to M$ mapping (g, y) to $\phi^g(y)$ is a diffeomorphism onto its image.

Since $\mathcal{C}$ is an open convex cone, reducing U if necessary, one can assume that for any $\mathfrak{a} \in \mathcal{C}^{\geq 1}$ and any g in the closure of U, the product $g \exp(\mathfrak{a})$ is Anosov. Moreover, there is a uniform bound $C > 0$ such that the restriction of $d\phi^{g \exp(\mathfrak{a})}$ to E^{ss} (respectively to E^{uu}) is $Ce^{-\lambda|\mathfrak{a}|}$-contracting (respectively $C^{-1}\exp(\lambda|\mathfrak{a}|)$-expanding).

By hypothesis, for any n, there is an element x_n in $D_{1/n}$ and an element $\mathfrak{a}_n$ in $\mathcal{C}$ such that $\phi^{\mathfrak{a}_n}(x_n)$ lies in the neighborhood described above, i.e. has the form $\phi^{g_n}(y_n)$ where y_n lies in $D_{1/n}$ and g_n in U.

We consider first the case where the real numbers $|\mathfrak{a}_n|$ are bounded from above, i.e. that, up to a subsequence, $\mathfrak{a}_n$ converge to some element $\mathfrak{a}$ of $\overline{\mathcal{C}}^{\geq 1}$. Since $\mathcal{C}$ is strict, $\mathfrak{a}$ is Anosov. Up to a subsequence, x is a fixed point of $g_\infty^{-1}\exp(\mathfrak{a})$ where g_∞ is a limit of the g_n's. Hence, x fixed by an Anosov element, and thus contained in $\mathrm{Comp}(\phi)$. The Lemma is proved in this case.

We are thus reduced to the other case, i.e. the case where, up to a subsequence, the real numbers $|\mathfrak{a}_n|$ converge to $+\infty$. Consider the local unstable leaf $\mathcal{G}^s(x_n)$: its image by the first return map from a neighborhood of x_n in $D_{1/n}$ into a neighborhood of y_n in $D_{1/n}$ is contracted, and therefore this first return map is well-defined. The composition of

this map with the holonomy along $\mathcal{G}^u$ defines a contracting map from $\mathcal{G}^s(x_n)$ into itself, which therefore admits a fixed point. It means that there is an element h_n of U such that $h_n \exp(\mathfrak{a}_n)$ maps a point z_n of $D_{1/n}$ to a point p_n in its unstable leaf $\mathcal{G}^u(z_n)$.

Now apply the same argument now to the reversed subcone $-\mathcal{C}$, and replacing x_n by p_n: since the stable/unstable foliations are switched when we replace $\mathcal{C}$ by its opposite, we obtain that the first return map preserves a local stable leaf.

The intersection of theses local stable and unstable leaves is now a point fixed by an element of the form $g \exp(\mathfrak{a}_n)$ with g in $\bar{U}$, hence Anosov. Therefore $D_{1/n}$ contains a point fixed by an Anosov element, hence lying in $\mathrm{Comp}(\phi)$. The Proposition follows since n is arbitrary. $\qquad\square$

Remark 7. *It follows by proposition 3 that we have:*

$$\Omega(\mathfrak{a}) = \Omega(\mathcal{C}) = \overline{\mathrm{Comp}(\phi)}$$

In particular, the nonwandering sets $\Omega(\mathfrak{a})$ and $\Omega(\mathcal{C})$ are independent from the choice of the Anosov element $\mathfrak{a}$, the Anosov subcone $\mathcal{C}$, and even of the preferred splitting ξ and preferred Anosov chamber $\mathcal{A}_0$. Hence we can denote the nonwandering set simply by $\Omega(\phi)$ or Ω.

We will need the following refinement of Theorem 3. Recall Definition 3:

Lemma 5. *For any Anosov subcone $\mathcal{C}$, and any compact orbit $\mathcal{O}_x$, there is an element of $\mathcal{C}$ fixing an element in $\mathcal{O}_x$.*

Proof. It follows immediatly by repeating the argument used in the proof of Remark 6, replacing $\mathcal{A}_0$ by $\mathcal{C}$. $\qquad\square$

Proposition 4. *Let x, y be two elements of Ω. Assume that the intersections $\mathcal{F}^u(x) \cap \mathcal{F}^s(y)$ and $\mathcal{F}^s(x) \cap \mathcal{F}^u(y)$ are non-empty. Then, these intersections are both contained in Ω.*

Proof. The argument is classical, but the proof more oftenly involves the notion of pseudo-orbits, that we prefer to avoid here (see for exemple [39, Proposition 8.11]). Since $\mathrm{Comp}(\phi)$ is dense in Ω, and by continuity of the maps $(x,y) \mapsto \mathcal{F}^u(x) \cap \mathcal{F}^s(y)$ (cf. Theorem 2) on can assume that x and y are in $\mathrm{Comp}(\phi)$.

Let $\mathcal{C}$ be a strict Anosov subcone in our preferred Anosov chamber $\mathcal{A}_0$ (hence elements of $\mathcal{C}$ preserve the foliations $\mathcal{F}^{ss}$ and $\mathcal{F}^{uu}$). By Lemma 4, one can assume that x is fixed by some element $\mathfrak{a}$ of $\mathcal{C}$, and y by some element $\mathfrak{b}$ of $\mathcal{C}$.

Let p be a point in $\mathcal{F}^{uu}(x) \cap \mathcal{F}^s(y)$ and let U be a small product neighborhood of p. We aim to show that some non trivial element of $\mathcal{C}$ maps a point in U into U.

For k big enough, $D' = \phi_{-k\mathfrak{a}}(D)$ contains the point p. Reducing U if necessary, one can assume that U is the saturation under some arbitrarly small neighborhood V of e in N of an open neighborhood $\overline{U}$ of p in D' (see Figure 1).

Let now q be a point in $\mathcal{F}^{ss}(x) \cap \mathcal{F}^u(y)$. Similarly, some iterate $D'' = \phi_{\ell\mathfrak{a}}(D)$ with $k > 0$ contains q. Let W be a neighborhood of q as depicted in Figure 1. In this figure, we draw two hatched regions W' and W'' which have the following properties:

- $W' \subset \overline{U}$,
- $W'' \subset W$,
- $W'' = \phi_{(k+\ell)\mathfrak{a}}(W')$,
- W' is saturated by the restriction $\mathcal{G}^s_U$ of $\mathcal{F}^s$ to $\overline{U}$,
- W'' is saturated by the restriction $\mathcal{G}^u_W$ of $\mathcal{F}^u$ to W.

We furthermore select W so that it is a product neighborhood, i.e. such that every leaf of $\mathcal{G}^s_W$ intersects every leaf of $\mathcal{G}^u_W$.
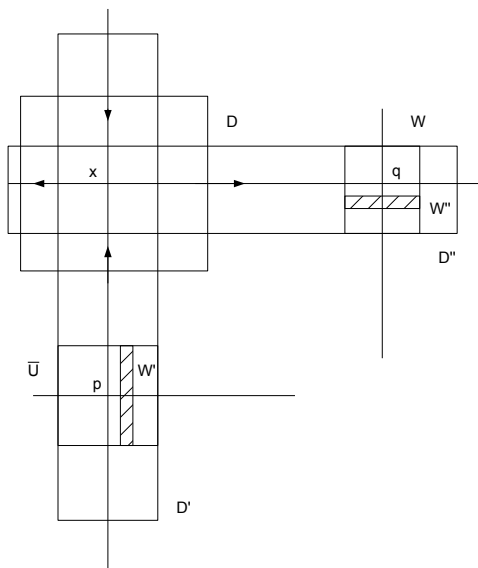

FIGURE 1. Iterations of local sections.

We now replace y by some N-iterate of itself so that q lies in $\mathcal{F}^{ss}(y)$. Then we have $p = g.p'$, where p is a point in $\mathcal{F}^{ss}(y) \cap \mathcal{F}^u(x)$ and g some element of N. By repeating the

argument above, we get a region Z' in W, saturated by the foliation $\mathcal{G}^s_W$, which is mapped by a positive iterate $\phi_{p\mathfrak{b}}$ to a region Z'' of $g^{-1}(\overline{U})$.

The closer U is, the bigger are the integers k, ℓ and p. Hence one can assume that $g.\exp(p\mathfrak{b})\exp((k+\ell)\mathfrak{a})$ is the image under $\exp$ of some element $\mathfrak{c}$ of $\mathcal{C}^{\geq 1}$. Since W'' is $\mathcal{G}^u_W$-saturated, Z' is $\mathcal{G}^s_W$-saturated, and W a product neighborhood, the intersection $W'' \cap Z'$ is non empty. It follows that any point in $W' \cap \phi_{-(k+\ell)\mathfrak{a}}(W'' \cap Z')$ is mapped by $\phi_{\mathfrak{c}}$ to a pont in $gZ'' \subset \overline{U} \subset U$ (see Figure 2). The Proposition is proved.
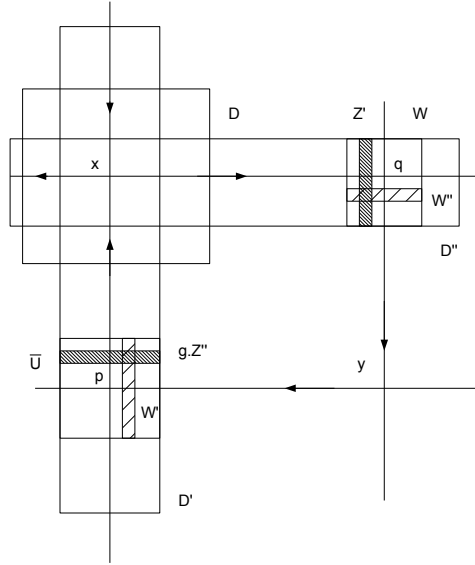

FIGURE 2. A schematic view of the argument.

$\square$

Theorem 4 (Spectral decomposition). *Let M be a closed smooth manifold and let ϕ be an Anosov action on M. The nonwandering set of ϕ can be partitioned into a finite number of ϕ-invariant closed subsets, called basic blocks:*

$$\Omega = \bigcup_{i=1}^{\ell} \Lambda_i$$

such that for every Anosov subcone $\mathcal{C}$, every Λ_i is $\mathcal{C}$-transitive, i.e. for every open subsets U, V in M intersecting Λ_i, there is an element x of $U \cap \Lambda_i$ whose $\mathcal{C}$-orbit in the meaning of Definition 3 meets $V \cap \Lambda_i$. In particular, there is an element of Λ_i whose $\mathcal{C}$-orbit is dense in Λ_i.

Proof. Let $\mathrm{Comp}(\phi)$ be the set of compact orbits of ϕ. By Proposition 3 we have $\overline{\mathrm{Comp}(\phi)} = \Omega(\phi)$. We define a relation on $\mathrm{Comp}(\phi)$ by: $x \sim y$ if and only if $\mathcal{F}^u(x) \cap \mathcal{F}^s(y) \neq \emptyset$ and $\mathcal{F}^s(x) \cap \mathcal{F}^u(y) \neq \emptyset$ with both intersections transverse in at least one point. Note that this relation is preserved by ϕ. We want to show that this is an equivalence relation and obtain each Λ_i as the closure of an equivalence class.

Note that $\sim$ is trivially reflexive and symmetric. In order to check the transitivity suppose that $x, y, z \in \mathrm{Comp}(\phi)$ and $p \in \mathcal{F}^u(x) \cap \mathcal{F}^s(y)$, $q \in \mathcal{F}^u(y) \cap \mathcal{F}^s(z)$ are transverse intersection points. There exists $\mathfrak{a} \in \mathcal{A}_0$ such that $ax = x$, where $a = \exp(\mathfrak{a})$. Since the iterations under ϕ^a of a ball around p in $\mathcal{F}^u(p) = \mathcal{F}^u(x) = \phi^a(\mathcal{F}^u(x))$ accumulate on $\mathcal{F}^u(y)$, we obtain that $\mathcal{F}^u(x)$ and $\mathcal{F}^s(z)$ have a transverse intersection. Analogously, we obtain that $\mathcal{F}^s(x)$ and $\mathcal{F}^u(z)$ have a transverse intersection.

By Theorem 2 any two sufficiently near points are equivalent, so by compactness we have finitely many equivalence classes whose (pairwise disjoint) closures we denote by $\Lambda_1, \Lambda_2, \ldots, \Lambda_\ell$.

It remains to show that every Λ_i is $\mathcal{C}$-transitive for every Anosov subcone $\mathcal{C} \subset \mathcal{A}_0$. It means that for any two open sets U and V in Λ_i there exists $\mathfrak{a} \in \mathcal{C}$ such that $\phi^{\exp(\mathfrak{a})}(U) \cap V \neq \emptyset$ - the existence of a dense $\mathcal{C}$-orbit then follows by a classical argument (see for example [15, Lemma 1]).

The density of compact orbits in Λ_i implies the existence of $x \in U$ and $\mathfrak{a} \in \mathcal{C}$ such that $\phi_{\mathfrak{a}}(x) = x$ (cf. Lemma 5).

Let y be an element in $V \cap \Lambda_i \cap \mathrm{Comp}(\phi)$. We have $x \sim y$, hence the stable leaf $\mathcal{F}^{ss}(y)$ intersects $\mathcal{F}^u(x)$. Iterating by $\phi_{\mathfrak{b}}$ where $\mathfrak{b}$ is an Anosov element in $\mathcal{A}_0$ fixing y, we observe as above that $\mathcal{F}^u(x)$ contains points arbitrarly close to y; in particular, a point p in V.

Now, according to Remark 3 item (2), there is an element $\mathfrak{g}$ such that $\exp(\mathfrak{g})p$ belongs to $\mathcal{F}^{uu}(x)$. Then, the negative iterates $p_n := a^{-n} \exp(\mathfrak{g})p$ converge to x.

Let n be big enough so that:

– $n\mathfrak{a} - \mathfrak{g}$ belongs to $\mathcal{C}$,

– $p_n := a^{-n} \exp(\mathfrak{g})p$ belongs to U.

By the Lie product formula, $\exp(-\mathfrak{g}/k + n\mathfrak{a}/k)^k$ for k going to $+\infty$ converges to $\exp(-\mathfrak{g})a^n$, hence maps the point p_n of U to an element in V. Since every $-\mathfrak{g}/k + n\mathfrak{a}/k$ is an element of $\mathcal{C}$, there is as required an element of $\mathcal{C}$ mapping an element of U to an element of V.

In order to conclude, we must show that this decomposition does not depend on our initial choice of preferred splitting ξ and the chamber $\mathcal{A}_0$. Let ξ' be another splitting for which $\mathcal{A}_{\xi'}$ is non-empty. We obtain then a spectral decomposition $\Omega = \bigcup_{i=1}^{\ell} \Lambda_i'$ such that for every Anosov subcone $\mathcal{C}' \subset \mathcal{A}_{\xi'}$ every Λ_i' is $\mathcal{C}'$-transitive. Let x and y be two elements of the same basic set Λ_i, and let Λ_j', Λ_k' be the basic sets for the splitting ξ' containing respectively x and y. Then, there are sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ in Ω, the first converging to x, the second to y, and such that y_n lies in the $\mathcal{C}$-orbit of x_n. Then, since Λ_i is isolated in Ω, every x_n and y_n lies in Λ_i for n sufficiently big. But for the same reason, x_n lies in Λ_j' and y_n lies in Λ_k' for n sufficiently big. Since Λ_j' is G-invariant, we have $\Lambda_j' = \Lambda_k'$. Therefore, every ξ-basic set Λ_i is contained in one ξ'-basic set Λ_j'. The reverse inclusion is obtained by the same argument, exchanging the roles of ξ and ξ'. The Theorem follows. $\qquad\square$

Definition 7. For every basic set Λ_i we denote by $\mathcal{F}^s(\Lambda_i)$ the union of the leaves $\mathcal{F}^s(x)$ where x describes Λ_i.

Lemma 6. *For every basic set Λ_i and any Anosov element $\mathfrak{a}$ of $\mathcal{A}_0$, an element x of M lies in $\mathcal{F}^s(\Lambda_i)$ if and only if the distance between $\exp(t\mathfrak{a})x$ and Λ_i goes to zero when $t \to +\infty$.*

Proof. If x lies in $\mathcal{F}^s(\Lambda_i)$, since Λ_i is ϕ-invariant, x lies in a strong stable leaf $\mathcal{F}^{ss}(p)$ of some element in Λ_i. Then the distance $d(\exp(t\mathfrak{a})x, \exp(t\mathfrak{a})p)$ goes to 0 when $t \to +\infty$. It proves one implication of the statement.

Inversely, assume that $d(\Lambda_i, \exp(t\mathfrak{a})x)$ goes to zero. Assume by a way of contradiction that x does not belong to $\mathcal{F}^s(\Lambda_i)$. Then, for t sufficiently big, $x_t := \exp(t\mathfrak{a})x$ lies in a neighborhood of Λ_i where the local product property holds: there is a point p_t in Λ_i such that $\mathcal{F}_\delta^{uu}(x_t)$ intersects $\mathcal{F}_\delta^s(p_t)$. Since $x \notin \mathcal{F}^s(\Lambda_i)$, we have $x_t \notin \mathcal{F}^s(p_t)$. Let r_t be the distance in $\mathcal{F}^{uu}(x_t)$ between x_t and $\mathcal{F}^s(\Lambda_i)$. At one hand we have the uniform bound $r_t \leq \delta$ for any t sufficiently big. On the other hand, since $\mathcal{F}^s(\Lambda_i)$ is $\exp(\mathfrak{a})$-invariant, this distance increases exponentially with t. Contradiction. $\qquad\square$

Corollary 1. *Every point in M lies in the stable leaf (respectively unstable leaf) of a non-wandering point. In other words:*

$$M = \bigcup_{i=1}^{\ell} \mathcal{F}^s(\Lambda_i) = \bigcup_{i=1}^{\ell} \mathcal{F}^u(\Lambda_i)$$

Proof. Let $\mathfrak{a}$ be an element of $\mathcal{A}_0$. Let $V_1, V_2, \ldots, V_\ell$ be two by two disjoint open neighborhoods of the Λ_i's. The ω-limit set of the $\phi_\mathfrak{a}$-orbit of x is contained in $\Omega(\mathfrak{a})$ and thus is covered by the disjoint union of the V_i'. Moreover, it is connected, hence it is contained in one and only one open domain V_i, hence in $V_i \cap \Omega(\mathfrak{a}) = \Lambda_i$. Therefore, the distance between $\exp(t\mathfrak{a})$ and Λ_i converges to 0. Then, it follows from Lemma 6 that x lies in $\mathcal{F}^s(\Lambda_i)$.

The similar statement for the unstable foliation is proved by reversing the flow. □

Remark 8. *The decomposition $\Omega = \bigcup_{i=1}^{\ell} \Lambda_i$ is independant from the choice of the splitting ξ, but of course it is not the case for the decomposition $M = \bigcup_{i=1}^{\ell} \mathcal{F}^s(\Lambda_i) = \bigcup_{i=1}^{\ell} \mathcal{F}^u(\Lambda_i)$.*

We will need later the follwing criteria:

Proposition 5. *If every leaf of $\mathcal{F}^s$ is dense in M, then $\Omega(\phi) = M$.*

Proof. By Theorem 4, for any p in M there is $x, y \in \Omega(\phi)$ such that $p \in \mathcal{F}^u(x) \cap \mathcal{F}^s(y)$. The hypothesis implies that the stable leaf $\mathcal{F}^s(x)$ intersects $\mathcal{F}^u(y)$. Hence by Proposition 4 we have $p \in \Omega(\phi)$. □

Definition 8. *Let $\mathfrak{G}$ be the graph whose vertices are the basic sets Λ_i, and such that there is an edge connecting Λ_i and Λ_j if and only if the closure of $\mathcal{F}^u(\Lambda_i)$ contains Λ_j (hence $\mathcal{F}^u(\Lambda_j)$), or the closure of $\mathcal{F}^u(\Lambda_j)$ contains Λ_i (hence $\mathcal{F}^u(\Lambda_i)$).*

Lemma 7. *$\mathfrak{G}$ is connected.*

Proof. For every connected component C of $\mathfrak{C}$, let $F(C)$ be the union of $\mathcal{F}^u(\Lambda_i)$ where Λ_i describes the set of basic sets in C. Let x be an element in the closure of $F(C)$: since the number of basic sets is finite, x is in the closure of $\mathcal{F}^u(\Lambda_i)$ for some Λ_i in C. Let Λ_j be the basic set such that $x \in \mathcal{F}^u(\Lambda_j)$ (cf. Corollary 1). Then $\mathcal{F}^u(\Lambda_j)$ is contained in the closure of $\mathcal{F}^u(\Lambda_i)$, hence Λ_i is in C.

It follows that $F(C)$ is closed.

Let now C' be any other connected component of $\mathfrak{G}$. Assume that $F(C) \cap F(C')$ is non-empty: let x be an element in this intersection. Then x lies in some $\mathcal{F}^u(\Lambda_j)$, and one proves as above that Λ_j lies in C and in C'. Hence $C = C'$.

In other words, M is the disjoint union of the closed subsets $F(C)$. The Lemma follows from this fact and the connectedness of M. □

3.3. **Injectivity of the holonomy.** Recall that in Remark 4 item (3) we have defined, for every point x and any element $\mathfrak{h}$ of $\mathcal{N}$ satisfying $(\exp \mathfrak{h})\mathcal{F}^{ss}(x) = \mathcal{F}^{ss}(x)$ a germ at x of homeomorphism $h_x^{\mathfrak{h}}$ of $\mathcal{F}^{uu}(x)$, representing the holonomy transverse to the foliation $\mathcal{F}^s$ of a certain loop $\omega_x^{\mathfrak{h}}$ obtained by composing the orbit of x under $\phi_{\mathfrak{h}}$ during the time $[0, 1]$ and any path from $\exp(\mathfrak{h})x$ to x in $\mathcal{F}^{ss}(x)$. Moreover, every loop in the leaf $\mathcal{F}^s(x)$ is homotopic to a loop $\omega_x^{\mathfrak{h}}$, hence every element of the holonomy group of the leaf $\mathcal{F}^s(x)$ is represented by some $h_x^{\mathfrak{h}}$. We call $h_x^{\mathfrak{h}}$ the *holonomy of $\mathfrak{h}$ at x*. Observe that if $\mathfrak{h}$ belongs to $\mathcal{A}_0$, then $h_x^{\mathfrak{h}}$ is expanding, in particular, non-trivial.

Definition 9. The Anosov action ϕ has *injective stable holonomy* if for every leaf F of $\mathcal{F}^s$ the holonomy representation $\pi_1(F, x) \to \mathrm{Homeo}_{loc}(\mathcal{F}^{uu}(x))$ is injective. According to the discussion above, it is equivalent to the following requirement: for every non-trivial element $\mathfrak{h}$ of $\mathcal{G}$ and every $x \in M$ such that $\exp(\mathfrak{h})\mathcal{F}^{ss}(x) = \mathcal{F}^{ss}(x)$ the holonomy $h_x^{\mathfrak{h}}$ at x is non-trivial.

Similarly, one define injective unstable holonomy; the action *has injective holonomy* if it has stable and unstable injective holonomy.

Of course, Nil-extensions do not have injective holonomy: loops contained in the fibers are loops contained in orbits with trivial holonomy. The following proposition shows that it is the only obstruction to injective holonomy:

Proposition 6. *Faithful Anosov actions of nilpotent Lie groups have injective holonomy.*

All this section is devoted to the proof of Proposition 6. More precisely, we prove that the action has injective stable holonomy, the injectivity of the unstable holonomy is then obtained by simply replacing the Anosov element by its opposite. Actually, we will prove the reverse statement, assuming that the action has no injective stable holonomy, and proving that then it is not faithful.

Definition 10.

(1) Let $\mathfrak{h}$ be a non trivial element of $\mathcal{N}$. The set of points x satisfying $(\exp \mathfrak{h})\mathcal{F}^{ss}(x) = \mathcal{F}^{ss}(x)$ and $h_x^{\mathfrak{h}} = \mathrm{id}$ is denoted by $\mathcal{H}(\mathfrak{h})$.
(2) For every x in M, the elements $\mathfrak{h}$ of $\mathcal{N}$ satisfying $(\exp \mathfrak{h})\mathcal{F}^{ss}(x) = \mathcal{F}^{ss}(x)$ and $h_x^{\mathfrak{h}} = \mathrm{id}$ is denoted by Δ_x^0.
(3) If Δ_x^0 is non trivial, x is said *to admit trivial holonomy.*
(4) the set of points of M admitting trivial holonomy is denoted by $\mathcal{P}$.

Lemma 8. *The set $\mathcal{H}(\mathfrak{h})$ is $\mathcal{F}^{ss}$-invariant.*

Proof. Let x be an element of $\mathcal{H}(\mathfrak{h})$ and y an element in $\mathcal{F}^{ss}(x)$. If y is sufficiently close to x, the loop $\omega_y^{\mathfrak{h}}$ considered in Remark 4 in order to define $h_y^{\mathfrak{h}}$ can be chosen close to the loop $\omega_x^{\mathfrak{h}}$. These loops are then freely homotopic one to the other in $\mathcal{F}^s(x)$. Hence we have $h_y^{\mathfrak{h}} = h_x^{\mathfrak{h}} = \mathrm{id}$ and $y \in \mathcal{H}(\mathfrak{h})$.

The case for general y in $\mathcal{F}^{ss}(x)$ is obtained by iterating under $\exp(t\mathfrak{a}_0)$. $\qquad\square$

Lemma 9. *The subset $\mathcal{P}$ is either empty, or the entire M.*

Proof. Let x be an element of $\mathcal{P}$, i.e. an element of $\mathcal{H}(\mathfrak{h})$ for some non-trivial element $\mathfrak{h}$. For every g in N, we have $gx \in \mathcal{H}(\mathrm{ad}(g)\mathfrak{h})$ hence $\mathcal{P}$ is N-invariant. It follows, together with Lemma 8, that $\mathcal{P}$ is $\mathcal{F}^s$-invariant.

Let now y be an element of $\mathcal{F}^{uu}(x)$. Since the holonomy $h_x^{\mathfrak{h}}$ is trivial, if y is sufficiently close to x in $\mathcal{F}^{uu}(x)$, the image $(\exp \mathfrak{h})y$ lies in the same stable leaf $\mathcal{F}^s(y)$ than y. Moreover $\phi_{\mathfrak{h}}$ maps $\mathcal{F}^{ss}(y)$ close to itself (since it preserves $\mathcal{F}^{ss}(x)$) and therefore we have $\mathcal{F}^{ss}(y) = (\exp \mathfrak{h}')\mathcal{F}^{ss}(y)$ for some element $\mathfrak{h}'$ close to $\mathfrak{h}$. The loop $\omega_y^{\mathfrak{h}'}$ can be chosen so that it remains close to $\omega_x^{\mathfrak{h}}$; hence to be the lifting in $\mathcal{F}^s(y)$ of $\omega_x^{\mathfrak{h}}$. Since $h_x^{\mathfrak{h}}$ is trivial, it follows that $h_y^{\mathfrak{h}'}$ is also trivial, hence $y \in \mathcal{H}(\mathfrak{h}) \subset \mathcal{P}$.

Now for a general y in $\mathcal{F}^{uu}(x)$, not necessarily close to x, some iterate $\exp(t\mathfrak{a}_0)y$ will be close to $\exp(t\mathfrak{a}_0)x$ in $\mathcal{F}^{uu}(\exp(t\mathfrak{a}_0)x)$, hence in $\mathcal{P}$ anyway.

The set $\mathcal{P}$ is therefore invariant by the foliations $\mathcal{F}^s$ and $\mathcal{F}^u$. The lemma follows. $\qquad \square$

From now on, we assume $\mathcal{P} = M$.

Lemma 10. *Let x_0 be an element of a compact ϕ-orbit $\mathcal{O}_0$. Then $\exp(\Delta_x^0)$ is a normal subgroup of the N-stabilizer Δ_x.*

Proof. The key point is that the intersection between the strong stable leaf $\mathcal{F}^{ss}(x_0)$ and $\mathcal{O}_0$ is reduced to x_0. Indeed, by the local product structure the intersection between $\mathcal{O}_0$ and $\mathcal{F}_\delta^{ss}(x)$ is reduced to x_0, and the general claim follows by iterating under (the inverse of) a ξ-Anosov element fixing x_0.

Therefore we have $\exp(\Delta_x^0) \subset \Delta_x$. It is straightforward to check that $\exp(\Delta_x^0)$ is a subgroup.

The fact that $\exp(\Delta_x^0)$ is normal in the stabilizer follows easily from the fact that the holonomy $h_x^{\mathrm{ad} g(\mathfrak{h})}$ is the conjugate under ϕ_g of $h_x^{\mathfrak{h}}$, hence trivial if $h_x^{\mathfrak{h}}$ is trivial. $\qquad \square$

Let $0 = N^{r+1} \lhd N^r \lhd \ldots \lhd N^1 = [N, N] \lhd N^0 = N$ the lower central serie for N. For every x in $\mathrm{Comp}(\phi)$, let $k(x)$ be the biggest integer k for which $\exp(\Delta_x^0) \cap N^k$ is not trivial. Observe that k is N-invariant. We denote by Δ_x^1 the intersection $\exp(\Delta_x^0) \cap N^{k(x)}$.

Lemma 11. *For every x in $\mathrm{Comp}(\phi)$ the weak unstable leaf $\mathcal{F}^u(x)$ is fixed pointwise by Δ_x^1.*

Proof. Then, for every g in Δ_x and every γ in Δ_x^1, the commutator $g\gamma g^{-1}\gamma^{-1}$ lies in $\exp(\Delta_x^0) \cap N^{k+1}$, hence is trivial. It follows that Δ_x^1 lies in the centralizer of Δ_x. Since N is nilpotent and Δ_x a lattice in it, Δ_x^1 is contained in the center Z of N (indeed, Δ_x^1 lies in the center of the Zariski closure of Δ_x, which is the entire N, see Theorem 2.10 in [38]).

It follows that for every $h = \exp(\mathfrak{h})$ in Δ_x^1, the entire orbit $\mathcal{O}(x)$ is fixed pointwise by h. Moreover, for every y in $\mathcal{O}(x)$, the unstable leaf $\mathcal{F}^{uu}(y)$ is preserved, and fixed pointwise since the holonomy $h_y^{\mathfrak{h}}$ is trivial. The lemma follows. $\qquad \square$

Lemma 12. *Let x_1 and x_2 be two points in $\mathrm{Comp}(\phi)$. If x_1 and x_2 belongs to the same basic set Λ_i, then we have $k(x_1) = k(x_2)$ and $\Delta_{x_1}^1 = \Delta_{x_2}^1$.*

Proof. Let y be a point in $\mathcal{F}^u(x_1) \cap \mathcal{F}^{ss}(x_2)$. Let $\mathfrak{h}$ be an element of $\Delta_{x_1}^0$ whose projection by exp is in $\Delta_{x_1}^1$. According to Lemma 11, y is a fixed point of $\exp(\mathfrak{h})$, hence belongs to $\mathcal{H}(\mathfrak{h})$. According to Lemma 8, x_2 lies in $\mathcal{H}(\mathfrak{h})$. Hence, $\mathfrak{h}$ lies in $\Delta_{x_2}^0$. Moreover, since $\exp(\mathfrak{h}) \in N^{k(x_1)}$, we have:

$$k(x_2) \geq k(x_1).$$

In a symetric way, one shows that every element in $\Delta_{x_2}^0$ with exponential in $\Delta_{x_2}^1$ is contained in $\Delta_{x_1}^0$ and that $k(x_1) \geq k(x_2)$. The lemma follows. $\square$

It follows that for every basic set Λ_i, there is a non trivial discrete subgroup Δ_i of the center Z of N fixing Λ_i pointwise. Actually, by Lemma 11, the entire $\mathcal{F}^u(\Lambda_i)$ is contained in $\mathrm{Fix}(\Delta_i)$.

More precisely, there is an integer k_i (the common value of k on $\mathrm{Comp}(\phi) \cap \Lambda_i$) such that:

– no non trivial element of N^{k_i+1} fixes pointwise $\mathcal{F}^u(\Lambda_i)$ (indeed, such an element would be an element of $\Delta_x^0 \cap N^{k(x)+1}$ for any x in $\mathrm{Comp}(\phi) \cap \Lambda_i$),

– Δ_i is exactly the subgroup of N^{k_i} comprising the elements fixing pointwise $\mathcal{F}^u(\Lambda_i)$,

Moreover, Δ_i is contained in the center Z of N.

It follows from these properties that if Λ_j is any basic set contained in the closure of $\mathcal{F}^u(\Lambda_i)$ then we have $k_j = k_i$ and $N^{k_i} = N^{k_j}$. Since the graph $\mathfrak{G}$ is connected (Definition 8, Lemma 7), all the basic sets have the same integer k_i and the same N^{k_i}. It follows that elements of N^{k_i} acts trivially on the entire M. It is impossible if ϕ is faithful.

The proof of Proposition 6 is complete.

4. ACTIONS OF CODIMENSION ONE

In this section, we keep considering an Anosov action $\phi : N \times M \to M$ of a simply connected nilpotent Lie group N, but we know add the additional hypothesis:

$$E^{uu} \textbf{ has dimension one.}$$

4.1. Affine structures along the strong unstable leaves.

Most of the content of this section is classical. We refer to [10, section 4.1] for more details. An affine structure of class C^2 on $\mathbb{R}$ is equivalent to the data of differential 1-form on $\mathbb{R}$, hence once fixed a differential 1-form ω_0 on M which does not vanish on E^{uu}, any continuous function $f : M \to \mathbb{R}$ induces an affine structure along leaves of $\mathcal{F}^{uu}$: the one induced by the restriction of $f\omega_0$ on the

leaf. We say that such an affine structure is ϕ-*invariant* if for every $a \in N$ and every $x \in M$ the restriction of ϕ^a induces an affine diffeomorphism bewtween $\mathcal{F}^{uu}(x)$ and $\mathcal{F}^{uu}(\phi^a(x))$.

Proposition 7. *There exists an unique affine structure along $\mathcal{F}^{uu}$ depending continuously on the points and invariant by the action ϕ.*

Proof. Let $\mathfrak{a}_0$ be an Anosov element of $\mathcal{N}$ expanding the leaves of $\mathcal{F}^{uu}$. Then, a contracting fixed-point Theorem on a suitable Banach space ensures the existence of a unique affine structure along $\mathcal{F}^{uu}$ invariant by the one-parameter subgroup H_0 generated by $\mathfrak{a}_0$ (see [4] or [24, Proposition 3.4.1, Remarque 3.4.2]). Since it is unique, this affine structure is also $N(H_0)$-invariant. By an induction involving Lemma 1 we finally get that this affine structure is N-invariant.

For a proof of the completeness of the leaves, see [24, §1.3]. □

Corollary 2. *Let $\mathfrak{a}$ be an element of $\mathcal{N}$, and x an element of M such that $\phi_{\mathfrak{a}}(x) = x$. Then the restriction of $\phi_{\mathfrak{a}}$ to $\mathcal{F}^{uu}(x)$ is differentially linearizable. In particular, either $\mathcal{F}^{uu}(x)$ pointwise fixed by a, or x is an isolated fixed point of $\phi_{\mathfrak{a}}$ in $\mathcal{F}^{uu}(x)$.* □

4.2. **Anosov actions of codimension one are abelian.** An important corollary of the previous section is:

Theorem 5. *Every codimension one Anosov action of a nilpotent Lie group is a Nil-extension over a codimension one Anosov action of $\mathbb{R}^k$.*

Proof. According to Lemma 3, such an Anosov action is a Nil-extension over a faithful Anosov action. Consider a compact orbit $\Lambda \backslash N$: the holonomy of the stable leaf containing this orbit is linearizable (Corollary 2): it provides a morphism from Λ into $\mathbb{R}^*$. According to proposition 6, this morphism is injective. The theorem follows. □

It follows that many results in [10], insensitive to Nil-extensions, still hold for Anosov actions of nilpotent Lie groups of codimension one, not necessarily abelian. More precisely:

Let $\pi : \widetilde{M} \to M$ be the universal covering map of M and $\tilde{\phi}$ be the lift of ϕ on $\widetilde{M}$. The foliations $\mathcal{F}^{ss}$, $\mathcal{F}^{uu}$, $\mathcal{F}^s$ and $\mathcal{F}^u$ lift to foliations $\widetilde{\mathcal{F}}^{ss}$, $\widetilde{\mathcal{F}}^{uu}$, $\widetilde{\mathcal{F}}^s$ and $\widetilde{\mathcal{F}}^u$ in $\widetilde{M}$. We denote by Q^ϕ be the orbit space of $\tilde{\phi}$ and $\pi^\phi : \widetilde{M} \to Q^\phi$ be the canonical projection. We have the following properties:

- The orbits of ϕ are incompressible: every loop in a ϕ-orbit $\mathcal{O}$ which is homotopically non-trivial in $\mathcal{O}$ is homotopically non-trivial in M.
- The foliations $\widetilde{\mathcal{F}}^{uu}$, $\widetilde{\mathcal{F}}^{ss}$, $\widetilde{\mathcal{F}}^u$, $\widetilde{\mathcal{F}}^s$ and the foliation defined by $\tilde{\phi}$ are by closed planes. The intersection between a leaf of $\widetilde{\mathcal{F}}^u$ and a leaf of $\widetilde{\mathcal{F}}^s$ is at most an orbit of $\tilde{\phi}$. Every orbit of $\tilde{\phi}$ meets a leaf of $\widetilde{\mathcal{F}}^{uu}$ or $\widetilde{\mathcal{F}}^{ss}$ at most once.

- The universal covering of M is diffeomorphic to $\mathbb{R}^{n+k}$, and the orbit space Q^ϕ is is homeomorphic to $\mathbb{R}^n$.

Last but not least:

Theorem 6. *Let ϕ be a codimension one Anosov action of a nilpotent Lie group N of dimension k on a closed manifold M of dimension $n + k$ with $n \geq 3$. Then, $\Omega(\phi) = M$.*

Proof. This is a consequence of Theorem 1 in [10] and of the observation that a Nil-extension of a topologically transitive action is still topologically transitive. However, we propose here another proof, assuming the theorem to be known for flows (cf. [45]) and proving it in the case $n \geq 2$.

According to Theorem 5 one can assume that the group N is $\mathbb{R}^k$ with $n \geq 2$. Then, since any subgroup of $\mathbb{R}$ is either cyclic or dense, it follows from Corollary 2 that any stable leaf F intersecting an unstable leaf $\mathcal{F}^{uu}(\theta_0)$ of a periodic orbit θ_0 but with $\theta_0 \notin F$ is locally dense; more precisely, F is contained in the interior of the closure $\overline{F}$ of F. On the other hand, it follows from Corollary 1 that every leaf of $\mathcal{F}^s$ satisfies this property.

Let F be any leaf of $\mathcal{F}^s$, and G be a leaf in the closure $\overline{F}$: this leaf is contained in the interior of its closure $\overline{G}$, which is contained in $\overline{F}$. Hence G is contained in the interior of $\overline{F}$. Therefore, $\overline{F}$ is closed and open, i.e. the entire M. The Theorem then follows from Proposition 5. $\qquad\square$

4.3. **Anosov actions of codimension two.** In the previous section we have seen that if $n \geq 3$, Anosov actions of codimension one are topologically transitive. In this section, elaborated with R. Varão, we consider the remaining case $n = 2$. We show:

Theorem 7. *Let ϕ be a codimension one Anosov action of a nilpotent Lie group N of dimension k on a closed manifold M of dimension $k + 2$. Then ϕ is a Nil-extension over an Anosov flow on a 3-manifold.*

This Theorem is a direct consequence of Theorem 5 and the main result of V. Arakawa in his (unpublished) Ph. D Thesis ([1]). However, we propose here a proof much shorter than the one appearing in [1].

Proof. By a way of contradiction, assume that we have a codimension one Anosov action of a nilpotent Lie group N of dimension k on a closed manifold M of dimension $k+2$ that is not a Nil-extension over an Anosov flow on a 3-manifold. By Theorem 5 one can assume that N is $\mathbb{R}^k$ with $k \geq 2$. Let θ_0 be an element of the orbit space Q^ϕ which is a lift of a compact orbit: θ_0 is fixed by a subgroup Λ of $\pi_1(M)$ isomorphic to a lattice in $\mathbb{R}^k$, i.e. isomorphic to $\mathbb{Z}^k$. According to Corollary 2 and the local product structure near θ_0, the action of Λ in Q^ϕ

near θ_0 is linearizable. More precisely: there is a neighborhood U of θ_0 in Q^ϕ, morphisms $\rho_{1,2} : \Lambda \to]0, +\infty[$ and a homeomorphism $\psi : U \to V \subset \mathcal{G}^s(\theta_0) \times \mathcal{G}^u(\theta_0) \approx \mathbb{R}^2$ such that:

$$\forall \gamma \in \Lambda, \forall \theta \in U \ \ \psi(\gamma.\theta) = (e^{\rho_1(\gamma)}\psi_1(\theta), e^{\rho_2(\gamma)}\psi_2(\theta))$$

where $\psi(\theta) = (\psi_1(\theta), \psi_2(\theta))$.

We denote by $\rho(\Gamma)$ the subgroup of $\mathbb{R}^2$ comprising elements of the form $(\rho_1(\gamma), \rho_2(\gamma))$ with $\gamma \in \Lambda$. We have two cases to consider:

  (1) Either $\rho(\Gamma)$ is discrete in $\mathbb{R}^2$ (hence a lattice since $k \geq 2$),
  (2) Or $\rho(\Gamma)$ is not closed in $\mathbb{R}^2$.

In the first case, θ_0 lies in the closure of the orbit under Λ of any point in $U \setminus \{\theta_0\}$. In the second case, the orbit under Λ of any point θ' in $U \setminus \{\theta_0\}$ accumulates non-trivially to θ'.

In both situations, we conclude that no element of $U \setminus \{\theta_0\}$ can be a lift of a compact orbit of ϕ. It follows that the basic set containing the projection of θ_0 is reduced to one compact orbit.

Since θ_0 was arbitrary, we conclude that the nonwandering set $\Omega(\phi)$ is a finite union of compact orbits. According to Corollary 1 the foliation $\mathcal{F}^s$ has only finitely many leaves: contradiction. $\qquad\square$

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