

Rigidity in topology C^0 of the Poisson bracket for Tonelli Hamiltonians

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Abstract

We prove the following rigidity result for the Tonelli Hamiltonians.

*Let T^*M be the cotangent bundle of a closed manifold M endowed with its usual symplectic form. Let (F_n) be a sequence of Tonelli Hamiltonians that C^0 converges on the compact subsets to a Tonelli Hamiltonian F . Let (G_n) be a sequence of Hamiltonians that C^0 converges on the compact subsets to a Hamiltonian G . We assume that the sequence of the Poisson brackets $(\{F_n, G_n\})$ C^0 -converges on the compact subsets to a C^1 function \mathcal{H} . Then $\mathcal{H} = \{F, G\}$.*

Key words: Hamiltonian dynamics, Poisson bracket, Tonelli Hamiltonians, Lagrangian functions.

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1 Introduction

For two C^2 functions $H, K : N \rightarrow \mathbb{R}$ defined on a symplectic manifold (N, ω) (such functions are usually called “Hamiltonians”), their Poisson bracket $\{H, K\}$ is

$$\{H, K\} = \omega(X_H, X_K)$$

where X_H designates the symplectic gradient of H defined by $dH = \omega(X_H, \cdot)$. Hence the Poisson bracket is given by $\{H, K\} = dH(X_K)$ and describes the evolution of H along any orbit for the Hamiltonian flow of K . The Poisson bracket is clearly continuous for the topologies $(H, K) \in C^1(N, \mathbb{R}) \times C^1(N, \mathbb{R}) \mapsto \{H, K\} \in C^0(N, \mathbb{R})$.

In his PhD thesis [9], V. Humilière asks the following question concerning a possible C^0 rigidity for the Poisson bracket:

Question 1. (*V. Humilière*) *Let $(F_k), (G_k)$ be two sequences of C^∞ functions such that (F_k) C^0 -converges to $F \in C^\infty(N, \mathbb{R})$, (G_k) C^0 -converges to $G \in C^\infty(N, \mathbb{R})$ and $(\{F_k, G_k\})$ C^0 -converges to $\mathcal{H} \in C^\infty(N, \mathbb{R})$. Do we have $\{F, G\} = \mathcal{H}$?*

The answer to this question is in general negative and some counter-examples due to L. Polterovitch are provided in [10].

However, F. Cardin and C. Viterbo give a positive answer in [4] when $\mathcal{H} = 0$ and all the considered Hamiltonians have their support contained in fixed a compact set.

By introducing a notion of pseudo-representation, V. Humilière obtains in [10] results that contain the previous one. Fix a closed manifold M and a finite-dimensional Lie algebra $(\mathfrak{g}, [., .])$, a pseudo-representation is a sequence of morphisms $\rho_n : \mathfrak{g} \rightarrow C^\infty(M, \mathbb{R})$ such that for all $f, g \in \mathfrak{g}$, the sequence $(\{\rho_n(f), \rho_n(g)\} - \rho_n([f, g]))$ converges uniformly to zero. Humilière proves that the limit of a convergent pseudo-representation is a representation. In other word, assuming that there is a morphism $\rho : \mathfrak{g} \rightarrow C^\infty(M, \mathbb{R})$ such that any sequence $(\rho_n(f))$ uniformly converges to $\rho(f)$, he proves that

$$\forall f, g \in \mathfrak{g}, \{\rho(f), \rho(g)\} = \rho([f, g]);$$

i.e. that $\{\lim_{n \rightarrow \infty} \rho_n(f), \lim_{n \rightarrow \infty} \rho_n(g)\} = \lim_{n \rightarrow \infty} \{\rho_n(f), \rho_n(g)\}$.

In [2], L. Buhovski proves that if (f_n) and (g_n) are C^∞ functions that uniformly converge on every compact subset of N to the smooth functions f and g , and if the sequences $(Df_n), (Dg_n)$ are uniformly bounded on any compact subset of N , then we have

$$\lim_{n \rightarrow \infty} \{f_n, g_n\} = \{f, g\}.$$

We will describe here a new case where question 1 always has a positive question: the case where one of the considered sequence is a sequence of Tonelli Hamiltonians that

C^0 -converges to a Tonelli Hamiltonian. We will too give a simpler proof of a part of Buhovski's result.

NOTATIONS. Let M be a closed manifold and let T^*M be its cotangent bundle endowed with its usual symplectic form ω that is the derivative of the Liouville 1-form. We use the notation $(q, p) \in T^*M$ and then $\omega = dq \wedge dp$.

The first projection is denoted by $\pi : (q, p) \in T^*M \mapsto q \in M$.

DEFINITION. A C^2 function $H : T^*M \rightarrow \mathbb{R}$ is a *Tonelli Hamiltonian* if it is:

- superlinear in the fiber, i.e. $\forall A \in \mathbb{R}, \exists B \in \mathbb{R}, \forall (q, p) \in T^*M, \|p\| \geq B \Rightarrow H(q, p) \geq A\|p\|$;
- C^2 -convex in the fiber i.e. for every $(q, p) \in T^*M$, the Hessian $\frac{\partial^2 H}{\partial p^2}$ of H in the fiber direction is positive definite as a quadratic form.

We denote the Hamiltonian flow of H by (φ_t^H) and the Hamiltonian vector-field by X_H .

Note that the class of Tonelli Hamiltonian contains all the Riemannian metrics and all the mechanical systems (Riemannian metric + potential). The class of Tonelli Hamiltonians has been recently intensely studied in the setting of weak KAM theory.

Theorem 1. *Let (F_k) be a sequence of Tonelli Hamiltonians defined on T^*M such that (F_k) C^0 -converges on the compact subsets of T^*M to a Tonelli Hamiltonian $F \in C^2(T^*M, \mathbb{R})$, let (G_k) be a sequence of C^2 Hamiltonians of T^*M that C^0 -converges on the compact subsets of T^*M to $G \in C^2(T^*M, \mathbb{R})$ and such that $(\{F_k, G_k\})$ C^0 -converges on the compact subsets of T^*M to $\mathcal{H} \in C^0(T^*M, \mathbb{R})$. Then $\{F, G\} = \mathcal{H}$.*

REMARK. We prove in lemma 1 that the C^0 convergence of (F_k) to F and the convexity hypothesis imply the C^0 -convergence of $\frac{\partial F_k}{\partial p}$ to $\frac{\partial F}{\partial p}$. But we don't have the convergence of the $\frac{\partial F_k}{\partial q}$; then this theorem is not a consequence of theorem 2.

Theorem 2. *Let (N, ω) be a (non necessarily compact) symplectic manifold. Let $(F_k), (G_k)$ be two sequences of C^2 Hamiltonians defined on N such that (G_k) C^0 -converges on the compact subsets of N to $G \in C^2(N, \mathbb{R})$, (F_k) C^1 -converges on the compact subsets of N to $F \in C^2(N, \mathbb{R})$ and $(\{F_k, G_k\})$ C^0 -converges on the compact subsets of N to $\mathcal{H} \in C^0(N, \mathbb{R})$. Then $\{F, G\} = \mathcal{H}$.*

Theorem 25.7 of [11] asserts that if (F_k) is a sequence of C^1 and convex functions defined on \mathbb{R}^n that C^0 -converge to a C^1 and convex function F on any compact subset, then the convergence is C^1 on any compact subset of \mathbb{R}^n . We deduce:

Corollary. *Let $(F_k), (G_k)$ be two sequences of C^2 Hamiltonians defined on \mathbb{R}^{2n} such that (G_k) C^0 -converges on the compact subsets of \mathbb{R}^{2n} to $G \in C^2(\mathbb{R}^{2n}, \mathbb{R})$, (F_k) C^0 -converges on the compact subsets of \mathbb{R}^{2n} to $F \in C^2(\mathbb{R}^{2n}, \mathbb{R})$, the F_k are convex and*

$(\{F_k, G_k\})$ C^0 -converges on the compact subsets of \mathbb{R}^{2n} to $\mathcal{H} \in C^0(\mathbb{R}^{2n}, \mathbb{R})$. Then $\{F, G\} = \mathcal{H}$.

Theorem 1 concerns Tonelli Hamiltonians and is new. Theorem 2 is a consequence of the result due to L. Buhovsky (see [2]) that we explained before, but we give here a simple proof, that uses the simple principle.

Ideas of proofs

Contrarily to what happens in the works that we mentioned above, we won't use here any result of symplectic topology, as symplectic capacities, displacement energy...

We use

1. the following simple principle:

Simple principle: *under the hypotheses of question 1, if for any $x \in N$, there exists a $T > 0$ and a sequence of points (x_k) such that the arcs of orbits $(\varphi_t^{F_k}(x_k))_{t \in [0, T]}$ converge in some reasonable sense to $(\varphi_t^F(x))_{t \in [0, T]}$, then the answer to question 1 is positive.*

If we have a good enough notion of convergence, the proof of this principle is not complicated and propositions 2 and 3 are some versions of this principle.

2. for theorem 1, we use some subtle variational arguments to prove that the hypothesis of the simple principle is satisfied; for theorem 2, we just use straightforward arguments coming from the theory of ordinary differential equations.

Plan

- In section 2, we give the precise statements of the propositions that we need to prove theorems 1 and 2.
- Section 3 is devoted to the proof of proposition 1 that asserts that every C^0 -convergent sequence of Tonelli Hamiltonians satisfies the hypotheses of the simple principle.
- In section 4, we prove the simple principle, i.e. propositions 2 and 3, in the considered cases.

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2 Structure of the proofs of theorems 1 and 2

2.1 How two propositions imply theorem 1

We will prove the two following propositions in sections 3 and 4. The first one uses variational arguments to state that if a sequence of Tonelli Hamiltonians (H_k) C^0 -converges to a Tonelli Hamiltonian H , then the “small” arcs of orbits for H are approximated in some sense by some small parts of orbits for the H_k .

Proposition 1. *Let $(H_k)_{k \in \mathbb{N}}$, H be some C^2 Tonelli Hamiltonians defined on T^*M such that (H_k) C^0 -converges on the compact subsets of T^*M to H .*

*Then, for every $(q, p) \in T^*M$, there exists $T > 0$ and $(q_k, p_k)_{k \in \mathbb{N}} \in T^*M$ such that a subsequence of the sequence of arcs $(Q_k, P_k) = (\varphi_t^{H_k}(q_k, p_k))_{t \in [0, T]}$ converges to $(Q, P) = (\varphi_t^H(q, p))_{t \in [0, T]}$ in the following sense (here we are in a chart because all is local and we denote the subsequence in the same way we denoted the initial sequence):*

- (Q_k) C^0 -converges to Q ;
- (P_k) converges to P in the L^2 sense;
- the (Q_k, P_k) are uniformly C^0 bounded.

The second proposition is a version of what we called previously the *simple principle*. The proof is elementary.

Proposition 2. *Let (F_k) be a sequence of C^2 Hamiltonians defined on T^*M and let F be a C^2 Hamiltonian such that:*

*for every $(q, p) \in T^*M$, there exists $T > 0$ and $(q_k, p_k)_{k \in \mathbb{N}} \in T^*M$ such that the sequence of arcs $(Q_k, P_k) = (\varphi_t^{F_k}(q_k, p_k))_{t \in [0, T]}$ converges to $(Q, P) = (\varphi_t^F(q, p))_{t \in [0, T]}$ in the following sense:*

- (Q_k) C^0 -converges to Q ;
- (P_k) converges to P in the L^2 sense;
- the (Q_k, P_k) are uniformly C^0 bounded.

*Let (G_k) be a sequence of C^2 Hamiltonians of T^*M that C^0 -converges on the compact subsets of T^*M to a C^2 Hamiltonian G and that is such that the sequence $(\{F_k, G_k\})$ C^0 -converges to some $\mathcal{H} \in C^0(T^*M, \mathbb{R})$.*

Then $\{F, G\} = \mathcal{H}$.

Theorem 1 is clearly a consequence of propositions 1 and 2.

2.2 How two propositions imply theorem 2

To prove theorem 2, we will prove that if (F_k) satisfy the hypothesis of theorem 2, then it satisfies the hypothesis of proposition 3 below .

Proposition 3. *Let (N, ω) be a (non necessarily compact) symplectic manifold. Let (F_k) be a sequence of C^2 Hamiltonians defined on N and let F be a C^2 Hamiltonian such that:*

for all $x \in N$, there exists $T > 0$ and a sequence $(x_k) \in N$ such that the sequence of arcs of orbit $(\varphi_t^{F_k}(x_k))_{t \in [0, T]}$ C^0 -converges to $(\varphi_t^F(x))_{t \in [0, T]}$.

Let (G_k) be a sequence of C^2 Hamiltonians of N that C^0 -converges on the compact subsets of N to a C^2 Hamiltonian G and that is such that the sequence $(\{F_k, G_k\})$ C^0 -converges on the compact subsets of N to some $\mathcal{H} \in C^0(N, \mathbb{R})$.

Then $\{F, G\} = \mathcal{H}$.

Let us be more precise.

DEFINITION. Let X be a locally Lipschitz vectorfield on N . A C^1 arc $\gamma : [a, b] \rightarrow N$ is an ε -solution for X if $\forall t \in [a, b], \|\gamma'(t) - X(\gamma(t))\| < \varepsilon$.

If for example the vector fields X_1 and X_2 are ε -close, every solution for X_2 is an ε -solution for X_1 .

The following proposition is classical (see for example section 6.2.2. of [8]):

Proposition 4. *Assume that the vector field X is Lipschitz on \mathbb{R}^n with Lipschitz constant K . Let $\gamma_i : [a, b] \rightarrow \mathbb{R}^n$ be a ε_i -solution for X for $i = 1, 2$. We assume that for some $\tau \in [a, b]$, we have $d(\gamma_1(\tau), \gamma_2(\tau)) \leq \delta$. Then we have:*

$$\forall t \in [a, b], \quad d(\gamma_1(t), \gamma_2(t)) \leq \delta e^{K|t-\tau|} + \frac{\varepsilon_1 + \varepsilon_2}{2}(e^{K|t-\tau|} - 1).$$

Let now $(F_k), F$ be C^2 Hamiltonians on N such that (F_k) C^1 -converges to F . Using proposition 4, we will prove:

Lemma. *for all $x \in N$, there exists $T > 0$ and a sequence $(x_k) \in N$ such that the sequence of arcs of orbit $(\varphi_t^{F_k}(x_k))_{t \in [0, T]}$ C^0 -converges to $(\varphi_t^F(x))_{t \in [0, T]}$.*

Then the conclusion of theorem 2 follows from this and proposition 3.

Proof of the lemma. Let us fix $x_0 \in N$. Then we choose a chart $U = B(x_0, r)$ at x_0 , and $T > 0$ such that $\sup_{x \in B(x_0, r)} \|X_F(x)\|.T < r$. As (F_k) C^1 -converges to F , we have

for k large enough: $\sup_{x \in B(x_0, r)} \|X_{F_k}(x)\|.T < r$. This implies that

$$\forall t \in [0, T], \varphi_t^{F_k}(x_0) \in B(x_0, r).$$

Moreover, $(\varphi_t^{F_k}(x_0))_{t \in [0, T]}$ is a ε_k -solution for X_{F_k} where $\varepsilon_k = \sup_{x \in B(x_0, r)} \|X_{F_k}(x) - X_F(x)\|$.

We deduce from proposition 4 that for k large enough

$$\forall t \in [0, T], d(\varphi_t^{F_k}(x_0), \varphi_t^F(x_0)) \leq \frac{\varepsilon_k}{2}(e^{Kt} - 1)$$

where K is a Lipschitz constant for $X_{F|B(x_0, r)}$. As $\lim_{k \rightarrow \infty} \varepsilon_k = 0$, we obtain the wanted conclusion. \square

3 Proof of proposition 1

We assume that (H_k) C^0 -converges on the compact subsets of T^*M to H . We will use the following lemma, that is in the spirit of theorem 25.7 of [11].

Lemma 1. *Let (F_k) be a sequence of C^2 functions defined on T^*M that are C^2 convex in the fiber direction and that C^0 -converges on the compact subsets of T^*M to a C^2 function $F : T^*M \rightarrow \mathbb{R}$ that is C^2 convex in the fiber direction. Then $(\frac{\partial F_k}{\partial p})$ C^0 -converges to $\frac{\partial F}{\partial p}$ on the compact subsets of T^*M .*

Proof of lemma 1. If not, by possibly extracting a subsequence, we find a compact subset $K \subset T^*M$, $\varepsilon > 0$ and a sequence $(q_k, p_k) \in K$ such that:

$$\forall k; \left\| \frac{\partial F_k}{\partial p}(q_k, p_k) - \frac{\partial F}{\partial p}(q_k, p_k) \right\| \geq 2\varepsilon.$$

Extracting a subsequence, we can assume that $\lim_{k \rightarrow \infty} (q_k, p_k) = (q_\infty, p_\infty)$ and that we are in a chart $K = K_0 \times \bar{B}(p_\infty, R) \subset M \times \mathbb{R}^n$. Then we choose $\tilde{K} = \tilde{K}_0 \times \bar{B}(p_\infty, 2R)$ that contains K in its interior. We choose $\alpha > 0$ such that $\bar{B}_\alpha = \bar{B}((q_\infty, p_\infty), 2\alpha) \subset \tilde{K}$ and:

$$\forall x \in \bar{B}_\alpha, \left\| \frac{\partial F}{\partial p}(x) - \frac{\partial F}{\partial p}(q_\infty, p_\infty) \right\| < \eta = \frac{\varepsilon}{3}.$$

As $\left\| \frac{\partial F_k}{\partial p}(q_k, p_k) - \frac{\partial F}{\partial p}(q_k, p_k) \right\| \geq 2\varepsilon$, we can find $u_k \in \mathbb{R}^n$ such that $\|u_k\| = 1$ and

$$\left(\frac{\partial F_k}{\partial p}(q_k, p_k) - \frac{\partial F}{\partial p}(q_k, p_k) \right) \cdot u_k \geq \varepsilon;$$

i.e.

$$\frac{\partial F_k}{\partial p}(q_k, p_k) \cdot u_k \geq \frac{\partial F}{\partial p}(q_k, p_k) \cdot u_k + \varepsilon.$$

Using the fact that F_k is C^2 convex in the fiber direction, we deduce:

$$\begin{aligned} F_k(q_k, p_k + \alpha \cdot u_k) &\geq F_k(q_k, p_k) + \alpha \frac{\partial F_k}{\partial p}(q_k, p_k) \cdot u_k \\ &\geq \alpha \cdot \varepsilon + F_k(q_k, p_k) + \alpha \frac{\partial F}{\partial p}(q_k, p_k) \cdot u_k \\ &\geq \alpha \varepsilon + F_k(q_k, p_k) + \alpha \frac{\partial F}{\partial p}(q_\infty, p_\infty) \cdot u_k - \alpha \eta \end{aligned}$$

Extracting a subsequence, we can assume that (u_k) converges to u_∞ . We then take the limit and obtain:

$$F(q_\infty, p_\infty + \alpha \cdot u_\infty) - F(q_\infty, p_\infty) - \alpha \frac{\partial F}{\partial p}(q_\infty, p_\infty) u_\infty \geq \alpha(\varepsilon - \eta) = \frac{2\alpha\varepsilon}{3}.$$

But we have:

$$\begin{aligned} F(q_\infty, p_\infty + \alpha \cdot u_\infty) - F(q_\infty, p_\infty) - \alpha \frac{\partial F}{\partial p}(q_\infty, p_\infty) u_\infty \\ = \alpha \cdot \int_0^1 \left(\frac{\partial F}{\partial p}(q_\infty, p_\infty + t\alpha u_\infty) - \frac{\partial F}{\partial p}(q_\infty, p_\infty) \right) u_\infty dt \leq \alpha \cdot \eta = \frac{\alpha\varepsilon}{3} \end{aligned}$$

and this is a contradiction. □

We deduce from lemma 1 that the sequence $(\frac{\partial H_k}{\partial p})$ C^0 -converges on any compact subset of T^*M .

Let us recall some well-known fact concerning Tonelli Hamiltonians that are proved for example in [7]. To any Tonelli Hamiltonian H (resp. H_k) we can associate a Lagrangian $L : TM \rightarrow \mathbb{R}$ (resp. $L_k : TM \rightarrow \mathbb{R}$) that is defined by:

$$\forall (q, v) \in TM, L(q, v) = \sup_{p \in T_q^*M} (p \cdot v - H(q, p)).$$

Then L and L_k are C^2 , C^2 -convex and superlinear in the fiber direction.

If we denote by $\mathcal{L}_q : T_q^*M \rightarrow T_qM$ the Legendre map $p \rightarrow \frac{\partial H}{\partial p}(q, p)$ (that is a C^1 -diffeomorphism), we have the equivalent formula:

$$L(q, v) = \mathcal{L}_q^{-1}(v) \cdot v - H(q, \mathcal{L}_q^{-1}(v)).$$

Let us recall that the Euler-Lagrange flow associated to L is $f_t^L = \mathcal{L} \circ \varphi_t^H \circ \mathcal{L}^{-1}$ (see the notation \mathcal{L} just below) and that if (q_t, p_t) is an orbit for the Hamiltonian H , the corresponding orbit for L is (q_t, \dot{q}_t) .

We know that $\mathcal{L}^{H_k} : (q, p) \mapsto (q, \frac{\partial H_k}{\partial p}(q, p)) = (q, \mathcal{L}_q^{H_k}(p))$ C^0 -converges on the compact subsets of T^*M to $\mathcal{L} : (q, p) \mapsto (q, \mathcal{L}_q(p))$. The following lemma implies that $((\mathcal{L}^{H_k})^{-1})$ C^0 -converges to \mathcal{L}^{-1} on any compact subset of TM and then that (L_k) C^0 -converges to L on any compact subset of TM .

Lemma 2. *Let (h_k) be a sequence of homeomorphisms from T^*M to TM that C^0 -converges on any compact subset of T^*M to a homeomorphism h . Then the sequence (h_k^{-1}) C^0 -converges on any compact subset of TM to h^{-1} .*

Proof of lemma 2. If not, by possibly extracting a subsequence, we can find a sequence (y_k) in TM that converge to $y_\infty \in TM$ such that:

$$\forall k, d(h_k^{-1}(y_k), h^{-1}(y_k)) \geq 2\varepsilon > 0.$$

As h is a homeomorphism, there exists $R_2 > R_1 > 0$ such that:

$$\forall x \in T^*M, d(h^{-1}(y_\infty), x) \geq R_2 \Rightarrow d(y_\infty, h(x)) \geq 4\varepsilon$$

and

$$\forall x \in T^*M, d(h^{-1}(y_\infty), x) \leq R_1 \Rightarrow d(y_\infty, h(x)) < \varepsilon.$$

As (h_k) C^0 -converges on any compact subset of T^*M , for k large enough, we have: $h_k(B(h^{-1}(y_\infty), R_1)) \subset B(y_\infty, \varepsilon)$ and

$$h_k(B(h^{-1}(y_\infty), 2R_2) \setminus B(h^{-1}(y_\infty), R_2)) \subset TM \setminus B(y_\infty, 2\varepsilon).$$

Hence $h_k^{-1}(B(y_\infty, \varepsilon))$ is a connected subset of T^*M that meets $B(h^{-1}(y_\infty), R_1)$ but doesn't meet $B(h^{-1}(y_\infty), 2R_2) \setminus B(h^{-1}(y_\infty), R_2)$. This implies that $h_k^{-1}(B(y_\infty, \varepsilon)) \subset B(h^{-1}(y_\infty), R_2)$.

As the sequence (y_k) converges to y_∞ , we may assume that $d(y_k, y_\infty) < \varepsilon$. Hence $d(h^{-1}(y_\infty), h_k^{-1}(y_k)) \leq R_2$, the sequence $(h_k^{-1}(y_k))$ is bounded, we can extract a subsequence that converges to x_∞ . Then we have

- $d(x_\infty, h^{-1}(y_\infty)) \geq 2\varepsilon$ by taking the limit in the first inequality;
- $d(h(x_\infty), y_\infty) = \lim_{k \rightarrow +\infty} d(h_k(h_k^{-1}(y_k)), y_k) = 0$.

We have found two points $x_\infty \neq h^{-1}(y_\infty)$ that have the same image by the homeomorphism h . This is impossible. □

Let us fix $(q, p) \in T^*M$. By Weierstrass theorem (see for example [7]), there exists $\tau > 0$ such that the arc $\gamma : [0, \tau] \rightarrow M$ defined by $\gamma(t) = \pi \circ \varphi_t^H(q, p)$ is action strictly minimizing for the Lagrangian L , i.e. such that for every other absolutely continuous arc $\eta : [0, \tau] \rightarrow M$ that has the same endpoints as γ , we have:

$$A_L(\gamma) = \int_0^\tau L(\gamma(t), \dot{\gamma}(t)) dt < A_L(\eta) = \int_0^\tau L(\eta(t), \dot{\eta}(t)) dt.$$

For any $k \in \mathbb{N}$, by Tonelli theorem (see [7]), there exists an arc $\gamma_k : [0, \tau] \rightarrow M$ such that $\gamma_k(0) = q$ and $\gamma_k(\tau) = \gamma(\tau)$ that minimizes (non necessarily strictly) the action

for the Lagrangian L_k among all the absolutely continuous arcs that have the same endpoints as γ_k . Then there exists a unique $p_k \in T_q M$ such that: $\gamma_k(t) = \pi \circ \varphi_t^{H_k}(q, p_k)$ and we have $\dot{\gamma}_k(t) = \frac{\partial H_k}{\partial p}(\varphi_t^{H_k}(q, p_k))$. We define: $m_\tau = \sup_{t \in [0, \tau]} |L(\gamma(t), \dot{\gamma}(t))|$. We will prove

Lemma 3. *There exists a compact subset $K_2 = \{H \circ \mathcal{L}^{-1} \leq C + 2\}$ of TM that contains all the $(\gamma_k(t), \dot{\gamma}_k(t))$ and $(\gamma(t), \dot{\gamma}(t))$ for $t \in [0, \tau]$ and we have*

$$\lim_{k \rightarrow +\infty} A_L(\gamma_k) = A_L(\gamma).$$

Proof of lemma 3.

As (L_k) C^0 -converges to L on the compact subsets of TM , for k large enough, we have: $A_{L_k}(\gamma) \leq \tau(m_\tau + 1)$ and then $A_{L_k}(\gamma_k) \leq \tau(m_\tau + 1)$ because γ_k is minimizing for L_k . Because L is superlinear and convex in the fiber direction, there exists $R > 0$ such that

$$\forall (q, v) \in TM, \|v\| \geq R \Rightarrow L(q, v) > M_\tau = \sup \left\{ m_\tau + 3, \sup_{q \in M} |L(q, 0)| + 1 \right\}.$$

Because L_k is convex in the fiber direction, we have for $\|v\| \geq R$

$$L_k(q, v) \geq L_k(q, 0) + \left(L_k(q, R \frac{v}{\|v\|}) - L_k(q, 0) \right) \frac{\|v\| - R}{R}.$$

Because (L_k) C^0 -converges to L on the compact subsets of TM , for k large enough, we have

$$\forall (q, v), L_k(q, 0) \geq -M_\tau \quad \text{and} \quad L_k(q, R \frac{v}{\|v\|}) - L_k(q, 0) > 1.$$

We deduce for k large enough and $\|v\| \geq R$

$$L_k(q, v) \geq \frac{\|v\| - R}{R} - M_\tau$$

and then for a certain $R^* > R$, we have

$$\forall v, \|v\| \geq R^* \Rightarrow L_k(q, v) \geq M_\tau.$$

For k large enough, we have $A_{L_k}(\gamma_k) \leq \tau(m_\tau + 1)$ and then for a $t_k \in [0, \tau]$ we have $\|\dot{\gamma}_k(t_k)\| \leq R^*$. We introduce the constant $C = \sup_{\|v\| \leq R^*} H(\mathcal{L}^{-1}(q, v))$ and the compact subsets $K_1 = \{H \circ \mathcal{L}^{-1} \leq C + 1\}$, $K_2 = \{H \circ \mathcal{L}^{-1} \leq C + 2\}$ and $K_3 = \{H \circ \mathcal{L}^{-1} \leq C + 3\}$ of TM . As H is convex on the fiber directions the intersection of K_i with a fiber is a topological ball. As $(H_k \circ (\mathcal{L}^{H_k})^{-1})$ C^0 -converges to $H \circ \mathcal{L}^{-1}$ on K_2 , for K large enough we have

$$\forall (q, v) \in K_3 \setminus K_2, H_k \circ (\mathcal{L}^{H_k})^{-1}(q, v) > C + \frac{3}{2}$$

and

$$\forall q, \exists (q, v) \in K_1, H_k \circ (\mathcal{L}^{H_k})^{-1}(q, v) < C + \frac{3}{2}.$$

As H_k is convex in the fiber direction, the set $\{H_k \circ (\mathcal{L}^{H_k})^{-1} \leq C + 1\}$ is connected and doesn't meet the boundary of K_2 ; hence $\mathcal{E}_k = \{H_k \circ (\mathcal{L}^{H_k})^{-1} \leq C + 1\} \subset K_2$.

The set \mathcal{E}_k is invariant by the Euler-Lagrange flow of L_k (the energy is invariant); we know that $\|\dot{\gamma}_k(t_k)\| \leq R^*$ and $H \circ \mathcal{L}^{-1}(\gamma_k(t_k), \dot{\gamma}_k(t_k)) \leq C$, hence for k large enough: $H_k \circ (\mathcal{L}^{H_k})^{-1}(\gamma_k(t_k), \dot{\gamma}_k(t_k)) \leq C + 1$ and then $(\gamma_k(t_k), \dot{\gamma}_k(t_k)) \in \mathcal{E}_k \subset K_2$ and

$$\forall t \in [0, \tau], (\gamma_k(t), \dot{\gamma}_k(t)) \in \mathcal{E}_k \subset K_2.$$

We introduce the notation $\varepsilon_k = \sup_{(q,v) \in K_2} |L_k(q, v) - L(q, v)|$. We have then:

$A_L(\gamma_k) \leq A_{L_k}(\gamma_k) + \tau\varepsilon_k \leq A_{L_k}(\gamma) + \tau\varepsilon_k \leq A_L(\gamma) + 2\tau\varepsilon_k$. We deduce that $\lim_{k \rightarrow +\infty} A_L(\gamma_k) = A_L(\gamma)$. Let us notice that we obtain the same result by replacing τ by any smaller τ . \square

REMARK. Let us explain the link between the previous proof and some proofs that are given by P. Bernard in [1]. He introduces a notion of “uniform families of Tonelli Hamiltonians” that satisfy in particular

- (i) there exist $h_0, h_1 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ that are superlinear such that for every Hamiltonian H of the family

$$h_0(\|p\|) \leq H(q, p) \leq h_1(\|p\|);$$

- (ii) there exists an increasing function $k : \mathbb{R}^+ \rightarrow \mathbb{R}_+$ such that if $|t| \leq 1$ and (φ_s) is the flow associated to a Hamiltonian of the family, then

$$\varphi_t(\{\|p\| \leq r\}) \subset \{\|p\| \leq k(r)\} \leq T^*M.$$

It is easy to prove that if the sequence (H_n) of Tonelli Hamiltonians C^0 -converges on the compact subsets to a Tonelli Hamiltonian H , then the family $\{H_n\}$ satisfy (i) and (ii). With these conditions, P. Bernard proves in [1] (see (B5)) a result that implies the first part of lemma 3: the minimizers in time $[0, t]$ of all the Hamiltonians of the family have a derivative that is uniformly bounded.

Let us now give the proof of proposition 1 that is

Proposition. 1 *Let $(H_k)_{k \in \mathbb{N}}$, H be C^2 Tonelli Hamiltonians defined on T^*M such that (H_k) C^0 -converges on the compact subsets of T^*M to H .*

*Then, for every $(q, p) \in T^*M$, there exists $T > 0$ and $(q_k, p_k)_{k \in \mathbb{N}} \in T^*M$ such that a subsequence of the sequence of arcs $(Q_k, P_k) = (\varphi_t^{H_k}(q_k, p_k))_{t \in [0, T]}$ converges to $(Q, P) = (\varphi_t^H(q, p))_{t \in [0, T]}$ in the following sense (here we are in a chart because all is local and we denote the subsequence in the same way we denoted the initial sequence):*

- (Q_k) C^0 -converges to Q ;
- (P_k) converges to P in the L^2 sense;
- the (Q_k, P_k) are uniformly C^0 bounded.

Lemma 3 implies that the family (γ_k) is equicontinuous (see [7]) and then C^0 -relatively compact by Ascoli theorem. A classical result (see corollary 3.2.3. in [7]) asserts that A_L is lower semi-continuous for the C^0 -topology. This implies that every limit point of the sequence (γ_k) is γ and then that (γ_k) C^0 -converges to γ . Assuming that k is large enough, we can then assume that all the γ_k 's and γ are in a same chart $B(\gamma(0), R)$. We have proved too that the $\dot{\gamma}_k$ are uniformly bounded.

Let us now prove that $(\dot{\gamma}_k)$ converges to $\dot{\gamma}$ for $\|\cdot\|_2$. The method that we use was suggested by Patrick Bernard.

We use the notation $p(t) = (\mathcal{L}^H)^{-1}(\gamma(t), \dot{\gamma}(t))$. Using the method of characteristics (see for example [7]), it is easy to build in some neighbourhood of $\gamma(0)$ a C^2 local solution $u_t : B(\gamma(0), r) \rightarrow \mathbb{R}$ of the Hamilton-Jacobi equation such that $du(\gamma(0)) = p(0)$ (with $t \in [-\tau, \tau]$ with an possibly smaller τ); we have then

$$\forall q \in B(\gamma(0), r), \frac{\partial u_t}{\partial t}(q) + H(q, du_t(q)) = 0.$$

As γ is a solution of the Euler-Lagrange equations for L , note that we have the equality in the Young inequality

$$H(\gamma(t), du_t(\gamma(t))) + L(\gamma(t), \dot{\gamma}(t)) = du_t(\gamma(t))\dot{\gamma}(t);$$

integrating the two previous equalities, we deduce that for every $t \in (0, \tau]$, we have

$$A_L(\gamma|_{[0,t]}) = u_t(\gamma(t)) - u_0(\gamma(0)).$$

Now, Young inequality gives

$$L(q, v) + H(q, du_t(q)) - du_t(q)v \geq 0.$$

The second derivative with respect to v of the above function is bounded from below on every compact subset. Hence there exists $C > 0$ such that

$$\forall (q, v) \in K_2 \cap \pi^{-1}(B(\gamma(0), r)), L(q, v) + H(q, du_t(q)) - du_t(q)v \geq C \|v - (\mathcal{L}^H)^{-1}(du_t(q))\|^2;$$

i.e.

$$\forall (q, v) \in K_2 \cap \pi^{-1}(B(\gamma(0), r)), L(q, v) - \frac{\partial u_t}{\partial t}(q) - du_t(q)v \geq C \|v - (\mathcal{L}^H)^{-1}(du_t(q))\|^2.$$

We take $(q, v) = (\gamma_k(t), \dot{\gamma}_k(t))$ and integrate between 0 and τ

$$A_L(\gamma_k) - (u_\tau(\gamma_k(\tau)) - u_0(\gamma_k(0))) \geq C \int_0^\tau \|\dot{\gamma}_k(t) - (\mathcal{L}^H)^{-1}(du_t(\gamma_k(t)))\|^2 dt.$$

As (γ_k) uniformly converge to γ , the left hand term converges to zero when k tends to $+\infty$ and $t \mapsto (\mathcal{L}^H)^{-1}(du_t(\gamma_k(t)))$ uniformly converges to $\dot{\gamma}(t) = (\mathcal{L}^H)^{-1}(du_t(\gamma(t)))$. We deduce that

$$\lim_{k \rightarrow +\infty} \|\dot{\gamma}_k - \dot{\gamma}\|_2 = 0.$$

REMARK. As suggested by the referee, we suggest another possible proof of the convergence for $\|\cdot\|_2$. We don't assume that H has any polynomial growth in the fiber direction. But by using [5], we can easily modify H outside a large enough compact to obtain a H with quadratic growth. In this case, it is possible to use proposition 3.10 p 123 of the book [3] by G. Buttazzo, M. Giaquinta & S. Hildebrandt to obtain the convergence for $\|\cdot\|_2$.

The Lagrangians L_k, L are C^2 -convex in the fiber direction and the sequence (L_k) C^0 -converges on any compact subset of TM . Hence, by lemma 1, the sequence $(\frac{\partial L_k}{\partial v})$ C^0 -converges on any compact subset of TM to $\frac{\partial L}{\partial v}$. We deduce that $(\varphi_t^{H_k}(q, p_k))_{t \in [0, T]} = (\gamma_k(t), \frac{\partial L_k}{\partial v}(\gamma_k(t), \dot{\gamma}_k(t)))_{t \in [0, T]}$ is uniformly bounded and converges to $(\varphi_t^H(q, p))_{t \in [0, T]} = (\gamma(t), \frac{\partial L}{\partial v}(\gamma(t), \dot{\gamma}(t)))_{t \in [0, T]}$ for the distance that is the sum (we are in a chart) of the uniform distance of M and the $\|\cdot\|_2$ distance in the fiber direction. Indeed, if we use the notation $\varepsilon_k = \sup_{K_2} \|\frac{\partial L}{\partial v} - \frac{\partial L_k}{\partial v}\|$ and if ℓ is a Lipschitz constant for $\frac{\partial L}{\partial v}$ on K_2 , we have (in a chart)

$$\begin{aligned} \|\frac{\partial L_k}{\partial v}(\gamma_k, \dot{\gamma}_k) - \frac{\partial L}{\partial v}(\gamma, \dot{\gamma})\|_2 &\leq \|\frac{\partial L_k}{\partial v}(\gamma_k, \dot{\gamma}_k) - \frac{\partial L}{\partial v}(\gamma_k, \dot{\gamma}_k)\|_2 + \|\frac{\partial L}{\partial v}(\gamma_k, \dot{\gamma}_k) - \frac{\partial L}{\partial v}(\gamma, \dot{\gamma})\|_2 \\ &\leq \sqrt{\tau} \varepsilon_k + \ell (\sqrt{\tau} \|\gamma_k - \gamma\|_{C^0} + \|\dot{\gamma}_k - \dot{\gamma}\|_2) \end{aligned}$$

□

4 Proof of propositions 2 and 3

4.1 Proof of proposition 2

We recall proposition 2

Proposition. 2 *Let (F_k) be a sequence of C^2 Hamiltonians defined on T^*M and let F be a C^2 Hamiltonian such that:*

*for every $(q, p) \in T^*M$, there exists $T > 0$ and $(q_k, p_k)_{k \in \mathbb{N}} \in T^*M$ such that the sequence of arcs $(Q_k, P_k) = (\varphi_t^{F_k}(q_k, p_k))_{t \in [0, T]}$ converges to $(Q, P) = (\varphi_t^F(q, p))_{t \in [0, T]}$ in the following sense:*

- (Q_k) C^0 -converges to Q ;
- (P_k) converges to P in the L^2 sense;
- the arcs (Q_k, P_k) are uniformly C^0 bounded.

Let (G_k) be a sequence of C^2 Hamiltonians of T^*M that C^0 -converges on the compact subsets of T^*M to a C^2 Hamiltonian G and that is such that the sequence $(\{F_k, G_k\})$ C^0 -converges on the compact subsets of T^*M to some $\mathcal{H} \in C^0(T^*M, \mathbb{R})$. Then $\{F, G\} = \mathcal{H}$.

Let $(F_k), F, (G_k), G$ as in the hypotheses of proposition 2.

Let $(q, p) \in T^*M$ be a point. We want to prove that $\{F, G\}(q, p) = \mathcal{H}(q, p)$.

We associate $T, (q_k, p_k), (Q, P), (Q_k, P_k)$ to (q, p) as above. As (P_k) converges to P in the L^2 sense, (P_k) has a subsequence (also denoted by (P_k)) that converges almost everywhere to P . Because (G_k) converge to G on the compact subsets and the P_k are uniformly bounded, we deduce that for almost every $s < t$ in $[0, T]$, we have

$$\begin{aligned} G(Q(t), P(t)) - G(Q(s), P(s)) &= \lim_{k \rightarrow +\infty} (G_k(Q_k(t), P_k(t)) - G_k(Q_k(s), P_k(s))) \\ &= \lim_{k \rightarrow +\infty} \int_s^t \{G_k, F_k\} \circ (Q_k(\sigma), P_k(\sigma)) d\sigma. \end{aligned}$$

Note that

$$\begin{aligned} &\left| \int_s^t (\{G_k, F_k\} \circ (Q_k(\sigma), P_k(\sigma)) + \mathcal{H}(Q(\sigma), P(\sigma))) d\sigma \right| \\ &\leq \left| \int_s^t (\{G_k, F_k\} + \mathcal{H})(Q_k(\sigma), P_k(\sigma)) d\sigma \right| + \left| \int_s^t (\mathcal{H}(Q_k(\sigma), P_k(\sigma)) - \mathcal{H}(Q(\sigma), P(\sigma))) d\sigma \right|. \end{aligned}$$

The first term tends to 0 because of the convergence of $(\{F_k, G_k\})$ to \mathcal{H} on the compact subsets and the second one tends to 0 by the Lebesgue dominated convergence theorem.

We finally obtain for almost every s, t

$$G(Q(t), P(t)) - G(Q(s), P(s)) = - \int_s^t \mathcal{H}(Q(\sigma), P(\sigma)) d\sigma$$

and by continuity the result for all s, t . Differentiating at 0 with respect to t , we obtain

$$\{G, F\}(q, p) = -\mathcal{H}(q, p).$$

□

4.2 Proof of proposition 3

The idea is exactly the same as for proposition 2. the proof is simpler because we assume a C^0 convergence. Let us recall proposition 3.

Proposition. 3 *Let (N, ω) be a (non necessarily compact) symplectic manifold. Let (F_k) be a sequence of C^2 Hamiltonians defined on N and let F be a C^2 Hamiltonian such that:*

for all $x \in N$, there exists $T > 0$ and a sequence $(x_k) \in N$ such that the sequence of arcs of orbit $(\varphi_t^{F_k}(x_k))_{t \in [0, T]}$ C^0 -converges to $(\varphi_t^F(x))_{t \in [0, T]}$.

Let (G_k) be a sequence of C^2 Hamiltonians of N that C^0 -converges on the compact subsets of N to a C^2 Hamiltonian G and that is such that the sequence $(\{F_k, G_k\})$ C^0 -converges to some $\mathcal{H} \in C^0(N, \mathbb{R})$.

Then $\{F, G\} = \mathcal{H}$.

Because of the C^0 -convergence of $(\varphi_t^{F_k}(x_k))_{t \in [0, T]}$ to $(\varphi_t^F(x))_{t \in [0, T]}$, we can choose a compact subset K of N that contains all the $\varphi_t^{F_k}(x_k)$. As (G_k) C^0 -converges to G on K , we have

$$G(\varphi_T^F(x)) - G(x) = \lim_{k \rightarrow +\infty} G_k(\varphi_T^{F_k}(x)) - G(x) = \lim_{k \rightarrow +\infty} \int_0^T \{G_k, F_k\}(\varphi_t^{F_k}(x)) dt.$$

As $\{G_k, F_k\}$ C^0 -converges to $-\mathcal{H}$ on K and $(\varphi_t^{F_k}(x_k))_{t \in [0, T]}$ C^0 -converges to $(\varphi_t^F(x))_{t \in [0, T]}$, we have

$$\lim_{k \rightarrow +\infty} \int_0^T \{G_k, F_k\}(\varphi_t^{F_k}(x)) dt = - \int_0^T \mathcal{H}(\varphi_t^F(x)) dt.$$

Differentiating $G(\varphi_T^F(x)) - G(x) = - \int_0^T \mathcal{H}(\varphi_t^F(x)) dt$ with respect to T , we obtain $\{G, F\}(x) = -\mathcal{H}(x)$. \square

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