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# DEFORMATIONS OF FUCHSIAN ADS REPRESENTATIONS ARE QUASI-FUCHSIAN

THIERRY BARBOT<sup>†</sup>

ABSTRACT. Let  $\Gamma$  be a finitely generated group, and let  $\text{Rep}(\Gamma, \text{SO}(2, n))$  be the moduli space of representations of  $\Gamma$  into  $\text{SO}(2, n)$  ( $n \geq 2$ ). An element  $\rho : \Gamma \rightarrow \text{SO}(2, n)$  of  $\text{Rep}(\Gamma, \text{SO}(2, n))$  is *quasi-Fuchsian* if it is faithful, discrete, preserves an acausal subset in the conformal boundary  $\text{Ein}_n$  of the anti-de Sitter space; and if the associated globally hyperbolic anti-de Sitter space is spatially compact - a particular case is the case of *Fuchsian representations*, *ie.* composition of a faithful, discrete and cocompact representation  $\rho_f : \Gamma \rightarrow \text{SO}(1, n)$  and the inclusion  $\text{SO}(1, n) \subset \text{SO}(2, n)$ .

In [BM12] we proved that quasi-Fuchsian representations are precisely representations which are Anosov as defined in [Lab06]. In the present paper, we prove that the space of quasi-Fuchsian representations is open and closed, *ie.* that it is an union of connected components of  $\text{Rep}(\Gamma, \text{SO}(2, n))$ .

The proof involves the following fundamental result: let  $\Gamma$  be the fundamental group of a globally hyperbolic spatially compact spacetime locally modeled on  $\text{AdS}_n$ , and let  $\rho : \Gamma \rightarrow \text{SO}_0(2, n)$  be the holonomy representation. Then, if  $\Gamma$  is Gromov hyperbolic, the  $\rho(\Gamma)$ -invariant achronal limit set in  $\text{Ein}_n$  is acausal.

Finally, we also provide the following characterization of representations with zero bounded Euler class: they are precisely the representations preserving a closed achronal subset of  $\text{Ein}_n$ .

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## 1. INTRODUCTION

Let  $\mathrm{SO}_0(1, n)$ ,  $\mathrm{SO}_0(2, n)$  denote the identity components of respectively  $\mathrm{SO}(1, n)$ ,  $\mathrm{SO}(2, n)$  ( $n \geq 2$ ). Let  $\Gamma$  be a cocompact torsion free lattice in  $\mathrm{SO}_0(1, n)$ . For any Lie group  $G$  we consider the moduli space of representations of  $\Gamma$  into  $G$  modulo conjugacy, equipped with the usual topology as an algebraic variety (see for example [GM88]):

$$\mathrm{Rep}(\Gamma, G) := \mathrm{Hom}(\Gamma, G)/G$$

In the case  $G = \mathrm{SO}_0(2, n)$  we distinguish the *Fuchsian representations*: they are the representations obtained by composition of the natural embedding  $\mathrm{SO}_0(1, n) \subset \mathrm{SO}_0(2, n)$  and any faithful and discrete representation of  $\Gamma$  into  $\mathrm{SO}_0(1, n)$ . The space of faithful and discrete representations of  $\Gamma$  into  $\mathrm{SO}_0(1, n)$  is the union of two connected components of  $\mathrm{Rep}(\Gamma, \mathrm{SO}_0(1, n))$ : for  $n \geq 3$ , it follows from Mostow rigidity Theorem, and for  $n = 2$ , it follows from the connectedness of the Teichmüller space - observe that there are indeed two connected components: one corresponding to representations such that  $\rho^*\xi = \xi$ , and the other to representations for which  $\rho^*\xi = -\xi$ , where  $\xi$  is a generator of  $H^n(\mathrm{SO}_0(1, n), \mathbb{Z})$ .

It follows that the space of Fuchsian representations is the union of two connected subsets of  $\mathrm{Rep}(\Gamma, \mathrm{SO}_0(2, n))$ . Therefore, one can consider the

union  $\text{Rep}_0(\Gamma, G)$  of connected components of  $\text{Rep}(\Gamma, \text{SO}_0(2, n))$  containing all the Fuchsian representations. The main result of the present paper is<sup>1</sup>:

**Theorem 1.1.** *Every deformation of a Fuchsian representation, ie. every element of  $\text{Rep}_0(\Gamma, \text{SO}_0(2, n))$  is faithful and discrete.*

If one compares this result with the *a priori* similar theory of deformations of Fuchsian representations into  $\text{SO}_0(1, n + 1)$ , one observes that the situation is at first glance completely different: it is well-known that large deformations of Fuchsian representations are **not** faithful and discrete; Fuchsian representations actually can be deformed to the trivial representation!

On the other hand, Theorem 1.1 is very similar to the principal Theorem in [Lab06] in the case  $G = \text{SL}(n, \mathbb{R})$ , and where  $\Gamma$  is a cocompact lattice in  $\text{SO}_0(1, 2)$ , ie. a closed surface group. In this situation, Fuchsian representations are induced by the inclusion  $\Gamma \subset \text{SO}_0(1, 2)$  and the morphism  $\text{SO}_0(1, 2) \rightarrow \text{SL}(n, \mathbb{R})$  corresponding to the unique  $n$ -dimensional irreducible representation of  $\text{SO}_0(1, 2)$ . The elements of  $\text{Rep}(\Gamma, \text{SL}(n, \mathbb{R}))$  in the same connected component than the Fuchsian representations are called *quasi-Fuchsian*. In [Lab06], F. Labourie proves that quasi-Fuchsian representations are *hyperconvex*, ie. that they are faithful, have discrete image, and preserve some curve in the projective space  $\mathbb{P}(\mathbb{R}^n)$  with some very strong convexity properties (in particular, this curve is strictly convex). Later, O. Guichard proved in [Gui08] that conversely hyperconvex representations are quasi-Fuchsian.

At the very heart of the theory is the notion of  $(G, P)$ -Anosov representation (or simply Anosov representation when there is no ambiguity about the pair  $(G, P)$ ), where  $G$  is a Lie group acting on any topological space  $P$ . The group  $\Gamma$  in general is a Gromov hyperbolic finitely generated group ([GW12]; see also Sect. 8 in [BM12]); typically, a closed surface group, or, more generally, a cocompact lattice in  $\text{SO}_0(1, k)$  for some  $k$ .

Unfortunately, the terminology is not uniform in the literature. For example, what is called a  $(\text{SO}_0(1, n + 1), \partial\mathbb{H}^{n+1})$ -Anosov representation in [GW12] would be called  $(G, \mathcal{Y})$ -Anosov in the terminology of [Bar10] or [BM12], where  $\mathcal{Y}$  is the space of spacelike geodesics of  $\mathbb{H}^{n+1}$ . We adopt here the definition and terminology used in [GW12].

Simple, general arguments ensure that Anosov representations are faithful, with discrete image formed by loxodromic elements, and that they form an open domain in  $\text{Rep}(\Gamma, G)$ . As a matter of fact, quasi-Fuchsian representations into  $\text{SL}(n, \mathbb{R})$  are  $(\text{SL}(n, \mathbb{R}), \mathcal{F})$ -Anosov, where  $\mathcal{F}$  is the frame variety<sup>2</sup>.

The *quasi-Fuchsian* terminology is inherited from hyperbolic geometry: a representation  $\rho : \Gamma \rightarrow \text{SO}_0(1, n + 1)$  is quasi-Fuchsian if it is faithful,

<sup>1</sup>This is a positive answer to Question 8.1 in [BM12].

<sup>2</sup>However, the converse is not necessarily true: see [Bar10] for the study of a family on non-hyperconvex  $(\text{SL}(3, \mathbb{R}), \mathcal{F})$ -Anosov representations.

discrete, and preserves a topological  $(n - 1)$ -sphere in  $\partial\mathbb{H}^{n+1}$ . It is well-known by the experts that quasi-Fuchsian representations into  $\mathrm{SO}_0(1, n + 1)$  are precisely the  $(\mathrm{SO}_0(1, n + 1), \partial\mathbb{H}^{n+1})$ -Anosov representations; and a proof can be obtained by adapting the arguments used in [BM12]. It is also a direct consequence of Theorem 1.8 in [GW12].

The anti de Sitter space  $\mathrm{AdS}_{n+1}$  is the analog of the hyperbolic space  $\mathbb{H}^{n+1}$ . It is a Lorentzian manifold, of constant sectional curvature  $-1$ . Whereas in the hyperbolic space pair of points are only distinguished by their mutual distance, in the anti-de Sitter space we have to distinguish three types of pair of points, according to the nature of the geodesic joining the two points: this geodesic may be spacelike, lightlike or timelike — in the last two cases, the points are said *causally related*. Moreover,  $\mathrm{AdS}_{n+1}$  is oriented, and admits also a *time orientation*, *ie.* an orientation of every non-spacelike geodesic. The group  $\mathrm{SO}_0(2, n)$  is precisely the group of orientation and time orientation preserving isometries of  $\mathrm{AdS}_{n+1}$ .

The anti-de Sitter space  $\mathrm{AdS}_{n+1}$  admits a conformal boundary called the *Einstein universe* and denoted by  $\mathrm{Ein}_n$ , which plays a role similar to that of the conformal boundary  $\partial\mathbb{H}^{n+1}$  for the hyperbolic space. The Einstein universe is a conformal Lorentzian spacetime, and is also subject to a causality notion: in particular, a subset  $\Lambda$  of the Einstein space  $\mathrm{Ein}_n$  is called *acausal* if any pair of distinct points in  $\Lambda$  are the extremities of a spacelike geodesic in  $\mathrm{AdS}_{n+1}$ .

Once introduced these fundamental notions, we can state the main content of [BM12]: let  $\Gamma$  be a Gromov hyperbolic group. For any representation  $\rho : \Gamma \rightarrow \mathrm{SO}_0(2, n)$  the following notions coincide:

- $\rho : \Gamma \rightarrow \mathrm{SO}_0(2, n)$  is  $(\mathrm{SO}_0(2, n), \mathrm{Ein}_n)$ -Anosov,
- $\rho : \Gamma \rightarrow \mathrm{SO}_0(2, n)$  is faithful, discrete, and preserves an **acausal** closed subset  $\Lambda$  in the conformal boundary  $\mathrm{Ein}_n$  of  $\mathrm{AdS}_{n+1}$ .

If furthermore  $\Gamma$  is isomorphic to the fundamental group of a closed manifold of dimension  $n$ , then  $\Lambda$  is a topological  $(n - 1)$ -sphere.

In particular, when  $\Gamma$  is a uniform lattice in  $\mathrm{SO}_0(1, n)$ , a representation of  $\Gamma$  into  $\mathrm{SO}_0(2, n)$  is called *quasi-Fuchsian* if it is faithful, discrete, and preserves an acausal topological  $(n - 1)$ -sphere in  $\mathrm{Ein}_n$ . In other words, Theorem 1.1 can be restated as follows: *deformations (large or small) of Fuchsian representations into  $\mathrm{SO}_0(2, n)$  are all quasi-Fuchsian*. It will be a corollary of the following more general statement:

**Theorem 1.2.** *Let  $n \geq 2$ , and let  $\Gamma$  be a Gromov hyperbolic group of cohomological dimension  $\geq n$ . Then, the moduli space  $\mathrm{Rep}_0(\Gamma, \mathrm{SO}_0(2, n))$  of  $(\mathrm{SO}_0(2, n), \mathrm{Ein}_n)$ -Anosov representations is open and closed in the modular space  $\mathrm{Rep}(\Gamma, \mathrm{SO}_0(2, n))$ .*

**Remark 1.3.** The reason for the hypothesis on the cohomological dimension is to ensure that the invariant closed acausal subset is a topological  $(n - 1)$ -sphere. It will follow from the proof that actually, under this hypothesis, if

$\text{Rep}(\Gamma, \text{SO}_0(2, n))$  is non-empty, then  $\Gamma$  is the fundamental group of a closed manifold, and its cohomological dimension is precisely  $n$ .

In order to present the ideas involved in the proof of Theorem 1.2 we need to remind a bit further a few classical definitions in Lorentzian geometry. By *spacetime* we mean here an oriented Lorentzian manifold with a time orientation given by a smooth timelike vector field. This allows to define the notion of future and past-directed causal curves. A subset  $\Lambda$  in  $(M, g)$  is *achronal* (respectively *acausal*) if there every timelike curve (respectively causal curve) joining two points in  $\Lambda$  is necessarily trivial, *ie.* reduced to one point. A *time function* is a function  $t : M \rightarrow \mathbb{R}$  which is strictly increasing along any causal curve. A spacetime  $(M, g)$  is *globally hyperbolic spatially compact* (abbreviated to *GHC*) if it admits a time function whose level sets are all compact.

Spatially compact global hyperbolicity is notoriously equivalent to the existence of a *compact Cauchy hypersurface*, that is a compact achronal set  $S$  which intersects every inextendible timelike curve at exactly one point. This set is then automatically a locally Lipschitz hypersurface (see [O’N83, Sect. 14, Lemma 29]).

Observe that all these notions are not really associated to the Lorentzian metric  $g$ , but to its conformal class  $[g]$ . Hence they are relevant to the Einstein universe, which is naturally equipped with a  $\text{SO}_0(2, n)$ -invariant conformal class of Lorentzian metric, but without any  $\text{SO}_0(2, n)$ -invariant representative.

The key fact used in [BM12] is that  $(\text{SO}_0(2, n), \text{Ein}_n)$ -Anosov representations are holonomy representations of *GHC* spacetimes locally modeled on  $\text{AdS}_{n+1}$ . Thanks to the work of G. Mess and his followers ([Mes07, ABB<sup>+</sup>07]) the classification of *GHC* locally *AdS* spacetimes has been almost completed: they are in 1 – 1 correspondance with *GHC-regular representations*.

More precisely: let  $\Gamma$  be a torsion-free finitely generated group of cohomological dimension  $n$ . A morphism  $\rho : \Gamma \rightarrow \text{SO}_0(2, n)$  is a *GHC-regular* representation if it is faithful, discrete, and preserves an **achronal** closed  $(n - 1)$ -topological sphere  $\Lambda$  in  $\text{Ein}_n$ . Define the invisible domain  $E(\Lambda)$  as the domain in  $\text{AdS}_{n+1}$  comprising points that are not causally related to any element of  $\Lambda$  (cf. Sect. 3.1). The action of  $\rho(\Gamma)$  on  $E(\Lambda)$  is then free and properly discontinuous; the quotient space, denoted by  $M_\rho(\Lambda)$ , is *GHC*. Moreover, every maximal *GHC* spacetime locally modeled on *AdS* has this form. Also observe that  $\Lambda$  only depends on  $\rho$ : there is at most one such invariant achronal sphere. Finally, if the limit set  $\Lambda$  is acausal, then the group  $\Gamma$  is Gromov hyperbolic (actually, in this case,  $\Gamma$  acts properly and cocompactly on a  $\text{CAT}(-1)$  metric space, see Proposition 8.3 in [BM12]).

Therefore, the only reason a *GHC-regular* representation may fail to be  $(\text{SO}_0(2, n), \text{Ein}_n)$ -Anosov is that the achronal sphere  $\Lambda$  might be non acausal.

The main result of the present paper, from which Theorem 1.2 follows quite directly, is:

**Theorem 1.4** (Theorem 5.3). *Let  $\rho : \Gamma \rightarrow \mathrm{SO}_0(2, n)$  be a GHC-regular representation, where  $\Gamma$  is a Gromov hyperbolic group. Then the achronal limit set  $\Lambda$  is acausal, ie.  $\rho$  is  $(\mathrm{SO}_0(2, n), \mathrm{Ein}_n)$ -Anosov.*

Even if not logically relevant to the proofs in the present paper, we point out that there are examples of GHC-regular representations with non-acausal limit set  $\Lambda$ . Let us describe briefly in this introduction the family detailed in Sect. 4.6: let  $(p, q)$  be a pair of positive integers such that  $p+q = n$ , and let  $\Gamma$  be a cocompact lattice of  $\mathrm{SO}_0(1, p) \times \mathrm{SO}_0(1, q)$ . The natural inclusion of  $\mathrm{SO}_0(1, p) \times \mathrm{SO}_0(1, q)$  into  $\mathrm{SO}_0(2, n)$  arising from the orthogonal splitting  $\mathbb{R}^{2, n} = \mathbb{R}^{1, p} \oplus \mathbb{R}^{1, q}$  induces a representation  $\rho : \Gamma \rightarrow \mathrm{SO}_0(2, n)$  which is GHC-regular, but where the invariant achronal limit set  $\Lambda$  is not acausal. The quotient space  $M_\rho(\Lambda) := \rho(\Gamma) \backslash E(\Lambda)$  is a GHC spacetime, called a *split AdS spacetime*, and the representation is a *split regular representation* (Definition 4.28).

Finally, in the last section, we give another characterization of GHC-representations. There is a fundamental bounded cohomology class  $\xi$  in  $H_b^2(\mathrm{SO}_0(2, n), \mathbb{Z})$ , the *bounded Euler class*. It can be alternatively defined as the bounded cohomology class induced by the natural Kähler form  $\omega$  of the symmetric  $2n$ -dimensional space  $\mathcal{T}_{2n} := \mathrm{SO}_0(2, n)/(\mathrm{SO}_0(2) \times \mathrm{SO}_0(n))$ , or as the one associated to the central exact sequence:

$$1 \rightarrow \mathbb{Z} \rightarrow \widetilde{\mathrm{SO}}_0(2, n) \rightarrow \mathrm{SO}_0(2, n) \rightarrow 1$$

If  $\rho : \Gamma \rightarrow \mathrm{SO}_0(2, n)$  is GH, the pull-back  $\rho^*(\xi)$  (the *Euler class*  $\mathrm{eu}_b(\rho)$ ) is necessarily trivial. Actually:

**Theorem 1.5.** *Let  $\rho : \Gamma \rightarrow \mathrm{SO}_0(2, n)$  be a faithful and discrete representation, where  $\Gamma$  is the fundamental group of a negatively curved closed manifold  $M$ . The following assertions are equivalent:*

- (1)  $\rho$  is  $(\mathrm{SO}_0(2, n), \mathrm{Ein}_n)$ -Anosov,
- (2) the bounded Euler class  $\mathrm{eu}_b(\rho)$  vanishes.

As a last comment, we recall part of the conjecture already proposed in [BM12][Conjecture 8.11]: we expect that GHC-regular representations of hyperbolic groups are all quasi-Fuchsians; in other words, that if a hyperbolic group  $\Gamma$  admits a GHC-regular representation into  $\mathrm{SO}_0(2, n)$ , then it must be isomorphic to a lattice in  $\mathrm{SO}_0(1, n)$ .

We expect actually a bit more. According to Theorem 1.2, the space of GHC-regular representations is open and closed, hence an union of connected components of  $\mathrm{Rep}(\Gamma, \mathrm{SO}_0(2, n))$ . It would be interesting to prove eventually that it coincides with  $\mathrm{Rep}_0(\Gamma, \mathrm{SO}_0(2, n))$ , ie. that quasi-Fuchsian representations are all deformations of Fuchsian representations.

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## 2. PRELIMINARIES

We assume the reader sufficiently acquainted to basic causality notions in Lorentzian manifolds like *causal* or *timelike* curves, *inextendible* causal curves, *Lorentzian length* of causal curves, *time orientation*, *future* and *past* of subsets, *time function*, *achronal* subsets, etc..., so that the brief description provided in the introduction above is sufficient. We refer to [BEE96] or [O’N83, section 14] for further details.

**Definition 2.1.** A *spacetime* is a connected, oriented, and time-oriented Lorentzian manifold.

**2.1. Anti-de Sitter space.** Let  $\mathbb{R}^{2,n}$  be the vector space of dimension  $n+2$ , with coordinates  $(u, v, x_1, \dots, x_n)$ , endowed with the quadratic form:

$$q_{2,n}(u, v, x_1, \dots, x_n) := -u^2 - v^2 + x_1^2 + \dots + x_n^2$$

We denote by  $\langle x|y \rangle$  the associated scalar product. For any subset  $A$  of  $\mathbb{R}^{2,n}$  we denote  $A^\perp$  the orthogonal of  $A$ , *ie.* the set of elements  $y$  in  $\mathbb{R}^{2,n}$  such that  $\langle y|x \rangle = 0$  for every  $x$  in  $A$ . We also denote by  $\mathcal{C}_n$  the isotropic cone  $\{w \in \mathbb{R}^{2,n} / q_{2,n}(w) = 0\}$ .

**Definition 2.2.** The anti-de Sitter space  $\text{AdS}_{n+1}$  is the hypersurface  $\{x \in \mathbb{R}^{2,n} / q_{2,n}(x) = -1\}$  endowed with the Lorentzian metric obtained by restriction of  $q_{2,n}$ .

At every element  $x$  of  $\text{AdS}_{n+1}$ , there is a canonical identification between the tangent space  $T_x \text{AdS}_{n+1}$  and the  $q_{2,n}$ -orthogonal  $x^\perp$

We will also consider the coordinates  $(r, \theta, x_1, \dots, x_n)$  with:

$$u = r \cos(\theta), v = r \sin(\theta)$$

We equip  $\text{AdS}_{n+1}$  with the time orientation defined by this vector field, *ie.* the time orientation such that the timelike vector field  $\frac{\partial}{\partial \theta}$  is everywhere future oriented.

Observe the analogy with the definition of hyperbolic space  $\mathbb{H}^n$ . Moreover, for every real number  $\theta_0$ , the subset  $H_{\theta_0} := \{(r, \theta, x_1, \dots, x_n) / \theta = \theta_0\} \subset \mathbb{R}^{2,n}$  is a totally geodesic copy of the hyperbolic space embedded in  $\text{AdS}_{n+1}$ . More generally, the totally geodesic subspaces of dimension  $k$  in  $\text{AdS}_{n+1}$  are connected components of the intersections of  $\text{AdS}_{n+1}$  with the linear subspaces of dimension  $(k+1)$  in  $\mathbb{R}^{2,n}$ .

**Remark 2.3.** In particular, geodesics are intersections with 2-planes. Time-like geodesics can all be described in the following way: let  $x, y$  two elements of  $\text{AdS}_{n+1}$  such that  $\langle x|y \rangle = 0$ . Then, when  $\theta$  describes  $\mathbb{R}/2\pi\mathbb{Z}$  the points  $c(\theta) := \cos(\theta)x + \sin(\theta)y$  describe a future oriented timelike geodesic containing  $x$  (for  $\theta = 0$ ) and  $y$  (for  $\theta = \pi/2$ ), parametrized by unit length: the Lorentzian length of the restriction of  $c$  to  $(0, \theta)$  is  $\theta$ .

## 2.2. Conformal model.

**Proposition 2.4.** *The anti-de Sitter space  $\text{AdS}_{n+1}$  is conformally equivalent to  $(\mathbb{S}^1 \times \mathbb{D}^n, -d\theta^2 + ds^2)$ , where  $d\theta^2$  is the standard Riemannian metric on  $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ , where  $ds^2$  is the standard metric (of curvature +1) on the sphere  $\mathbb{S}^n$  and  $\mathbb{D}^n$  is the open upper hemisphere of  $\mathbb{S}^n$ .*

*Proof.* In the  $(r, \theta, x_1, \dots, x_n)$ -coordinates the AdS metric is:

$$-r^2 d\theta^2 + ds_{hyp}^2$$

where  $ds_{hyp}^2$  is the hyperbolic metric, *ie.* the induced metric on  $H_0 = \{(r, \theta, x_1, \dots, x_n) / \theta = 0\} \approx \mathbb{H}^n$ . More precisely,  $H_0$  is a sheet of the hyperboloid  $\{(r, x_1, \dots, x_n) \in \mathbb{R}^{1,n} / -r^2 + x_1^2 + \dots + x_n^2 = -1\}$ . The map  $(r, x_1, \dots, x_n) \rightarrow (1/r, x_1/r, \dots, x_n/r)$  sends this hyperboloid on  $\mathbb{D}^n$ , and an easy computation shows that the pull-back by this map of the standard metric on the hemisphere is  $r^{-2} ds_{hyp}^2$ . The proposition follows.  $\square$

Proposition 2.4 shows in particular that  $\text{AdS}_{n+1}$  contains many closed causal curves (including all timelike geodesics, cf. Remark 2.3). But the universal covering  $\widetilde{\text{AdS}}_{n+1}$ , conformally equivalent to  $(\mathbb{R} \times \mathbb{D}^n, -d\theta^2 + ds^2)$ , contains no periodic causal curve. It is strongly causal, but not globally hyperbolic (see Definition 4.5).

**2.3. Einstein universe.** Einstein universe  $\text{Ein}_{n+1}$  is the product  $\mathbb{S}^1 \times \mathbb{S}^n$  endowed with the metric  $-d\theta^2 + ds^2$  where  $ds^2$  is as above the standard spherical metric. The universal Einstein universe  $\widetilde{\text{Ein}}_{n+1}$  is the cyclic covering  $\mathbb{R} \times \mathbb{S}^n$  equipped with the lifted metric still denoted  $-d\theta^2 + ds^2$ , but where  $\theta$  now takes value in  $\mathbb{R}$ . Observe that for  $n \geq 2$ ,  $\widetilde{\text{Ein}}_{n+1}$  is the universal covering, but it is not true for  $n = 1$ . According to this definition,  $\text{Ein}_{n+1}$  and  $\widetilde{\text{Ein}}_{n+1}$  are Lorentzian manifolds, but it is more adequate to consider them as conformal Lorentzian manifolds. We fix a time orientation: the one for which the coordinate  $\theta$  is a time function on  $\widetilde{\text{Ein}}_{n+1}$ .

In the sequel, we denote by  $p : \widetilde{\text{Ein}}_{n+1} \rightarrow \text{Ein}_{n+1}$  the cyclic covering map. Let  $\delta : \widetilde{\text{Ein}}_{n+1} \rightarrow \widetilde{\text{Ein}}_{n+1}$  be a generator of the Galois group of this cyclic covering. More precisely, we select  $\delta$  so that for any  $\tilde{x}$  in  $\widetilde{\text{Ein}}_{n+1}$  the image  $\delta(\tilde{x})$  is in the future of  $\tilde{x}$ .

Even if Einstein universe is merely a conformal Lorentzian spacetime, one can define the notion of *photons*, *ie.* (non parameterized) lightlike geodesics. We can also consider the causality relation in  $\text{Ein}_{n+1}$  and  $\widetilde{\text{Ein}}_{n+1}$ . In particular, we define for every  $x$  in  $\text{Ein}_{n+1}$  the *lightcone*  $C(x)$ : it is the union of

photons containing  $x$ . If we write  $x$  as a pair  $(\theta, x)$  in  $\mathbb{S}^1 \times \mathbb{S}^n$ , the lightcone  $C(x)$  is the set of pairs  $(\theta', y)$  such that  $|\theta' - \theta| = d(x, y)$  where  $d$  is distance function for the spherical metric  $ds^2$ .

There is only one point in  $\mathbb{S}^n$  at distance  $\pi$  of  $x$ : the antipodal point  $-x$ . Above this point, there is only one point in  $\text{Ein}_{n+1}$  contained in  $C(x)$ : the antipodal point  $-x = (\theta + \pi, -x)$ . The lightcone  $C(x)$  with the points  $x, -x$  removed is the union of two components:

- the *future cone*: it is the set  $C^+(x) := \{(\theta', y) / \theta < \theta' < \theta + \pi, d(x, y) = \theta' - \theta\}$ ,
- the *past cone*: it is the set  $C^-(x) := \{(\theta', y) / \theta - \pi < \theta' < \theta, d(x, y) = \theta - \theta'\}$ .

Observe that the future cone of  $x$  is the past cone of  $-x$ , and that the past cone of  $x$  is the future cone of  $-x$ .

According to Proposition 2.4  $\text{AdS}_{n+1}$  (respectively  $\widetilde{\text{AdS}}_{n+1}$ ) conformally embeds in  $\text{Ein}_{n+1}$  (respectively  $\widetilde{\text{Ein}}_{n+1}$ ). Observe that this embedding preserves the time orientation. Since the boundary  $\partial\mathbb{D}^n$  is an equatorial sphere, the boundary  $\partial\widetilde{\text{AdS}}_{n+1}$  is a copy of the Einstein universe  $\widetilde{\text{Ein}}_n$ . In other words, one can attach a “Penrose boundary”  $\partial\widetilde{\text{AdS}}_{n+1}$  to  $\widetilde{\text{AdS}}_{n+1}$  such that  $\widetilde{\text{AdS}}_{n+1} \cup \partial\widetilde{\text{AdS}}_{n+1}$  is conformally equivalent to  $(\mathbb{S}^1 \times \overline{\mathbb{D}}^n, -d\theta^2 + ds^2)$ , where  $\overline{\mathbb{D}}^n$  is the closed upper hemisphere of  $\mathbb{S}^n$ .

The restrictions of  $p$  and  $\delta$  to  $\widetilde{\text{AdS}}_{n+1} \subset \widetilde{\text{Ein}}_{n+1}$  are respectively a covering map over  $\text{AdS}_{n+1}$  and a generator of the Galois group of the covering; we will still denote them by  $p$  and  $\delta$ .

**2.4. Isometry groups.** Every element of  $\text{SO}(2, n)$  induces an isometry of  $\text{AdS}_{n+1}$ , and, for  $n \geq 2$ , every isometry of  $\text{AdS}_{n+1}$  comes from an element of  $\text{SO}(2, n)$ . Similarly, conformal transformations of  $\text{Ein}_{n+1}$  are projections of elements of  $\text{SO}(2, n+1)$  acting on  $\mathcal{C}_{n+1}$  (still for  $n \geq 2$ ).

In the sequel, we will only consider isometries preserving the orientation and the time orientation, *ie.* elements of the neutral component  $\text{SO}_0(2, n)$  (or  $\text{SO}_0(2, n+1)$ ).

Let  $\widetilde{\text{SO}}_0(2, n)$  be the group of orientation and time orientation preserving isometries of  $\widetilde{\text{AdS}}_{n+1}$  (or conformal transformations of  $\widetilde{\text{Ein}}_n$ ). There is a central exact sequence:

$$1 \rightarrow \mathbb{Z} \rightarrow \widetilde{\text{SO}}_0(2, n) \rightarrow \text{SO}_0(2, n) \rightarrow 1$$

where the left term is generated by the transformation  $\delta$  generating the Galois group of  $p : \widetilde{\text{Ein}}_n \rightarrow \text{Ein}_n$  defined previously. Observe that for  $n \geq 3$ ,  $\widetilde{\text{SO}}_0(2, n)$  is the universal covering of  $\text{SO}_0(2, n)$ .

**2.5. Achronal subsets.** Recall that a subset of a conformal Lorentzian manifold is *achronal* (respectively *acausal*) if there is no timelike (respectively causal) curve joining two distinct points of the subset. In  $\text{Ein}_n \approx (\mathbb{R} \times \mathbb{S}^{n-1}, -d\theta^2 + ds^2)$ , every achronal subset is precisely the graph of a 1-Lipschitz function  $f : \Lambda_0 \rightarrow \mathbb{R}$  where  $\Lambda_0$  is a subset of  $\mathbb{S}^{n-1}$  endowed with

its canonical metric  $d$ . In particular, the achronal closed topological hypersurfaces in  $\widetilde{\partial\text{AdS}}_{n+1}$  are exactly the graphs of the 1-Lipschitz functions  $f : \mathbb{S}^{n-1} \rightarrow \mathbb{R}$ : they are topological  $(n-1)$ -spheres.

Similarly, achronal subsets of  $\widetilde{\text{AdS}}_{n+1}$  are graphs of 1-Lipschitz functions  $f : \Lambda_0 \rightarrow \mathbb{R}$  where  $\Lambda_0$  is a subset of  $\mathbb{D}^n$ , and achronal topological hypersurfaces are graphs of 1-Lipschitz maps  $f : \mathbb{D}^n \rightarrow \mathbb{R}$ .

*Stricto-sensu*, there is no achronal subset in  $\text{Ein}_{n+1}$  since closed timelike curves through a given point cover the entire  $\text{Ein}_{n+1}$ . Nevertheless, we can keep track of this notion in  $\text{Ein}_{n+1}$  by defining “achronal” subsets of  $\text{Ein}_{n+1}$  as projections of genuine achronal subsets of  $\widetilde{\text{Ein}}_{n+1}$ . This definition is justified by the following results:

**Lemma 2.5** (Lemma 2.4 in [BM12]). *The restriction of  $p$  to any achronal subset of  $\widetilde{\text{Ein}}_{n+1}$  is injective.*  $\square$

**Corollary 2.6** (Corollary 2.5 in [BM12]). *Let  $\widetilde{\Lambda}_1, \widetilde{\Lambda}_2$  be two achronal subsets of  $\widetilde{\text{Ein}}_{n+1}$  admitting the same projection in  $\text{Ein}_{n+1}$ . Then there is an integer  $k$  such that:*

$$\widetilde{\Lambda}_1 = \delta^k \widetilde{\Lambda}_2$$

where  $\delta$  is the generator of the Galois group introduced above.  $\square$

**2.6. The Klein model  $\text{ADS}_{n+1}$  of the anti-de Sitter space.** We now consider the quotient  $\mathbb{S}(\mathbb{R}^{2,n})$  of  $\mathbb{R}^{2,n} \setminus \{0\}$  by positive homotheties. In other words,  $\mathbb{S}(\mathbb{R}^{2,n})$  is the double covering of the projective space  $\mathbb{P}(\mathbb{R}^{2,n})$ . We denote by  $\mathbb{S}$  the projection of  $\mathbb{R}^{2,n} \setminus \{0\}$  on  $\mathbb{S}(\mathbb{R}^{2,n})$ . For every  $x, y$  in  $\mathbb{S}(\mathbb{R}^{2,n})$ , we denote by  $\langle x | y \rangle$  the **sign** of the real number  $\langle x | y \rangle$ , where  $x, y \in \mathbb{R}^{2,n}$  are representatives of  $x, y$ . The *Klein model*  $\text{ADS}_{n+1}$  of the anti-de Sitter space is the projection of  $\text{AdS}_{n+1}$  in  $\mathbb{S}(\mathbb{R}^{2,n})$ , endowed with the induced Lorentzian metric, *ie.* :

$$\text{ADS}_{n+1} := \{x \in \mathbb{S}(\mathbb{R}^{2,n}) / \langle x | x \rangle < 0\}$$

The topological boundary of  $\text{ADS}_{n+1}$  in  $\mathbb{S}(\mathbb{R}^{2,n})$  is the projection of the isotropic cone  $\mathcal{C}_n$ ; we will denote this boundary by  $\partial\text{ADS}_{n+1}$ . The projection  $\mathbb{S}$  defines an one-to-one isometry between  $\text{AdS}_{n+1}$  and  $\text{ADS}_{n+1}$ . The continuous extension of this isometry is a canonical homeomorphism between  $\text{AdS}_{n+1} \cup \partial\text{AdS}_{n+1}$  and  $\text{ADS}_{n+1} \cup \partial\text{ADS}_{n+1}$ .

For every linear subspace  $F$  of dimension  $k+1$  in  $\mathbb{R}^{2,n}$ , we denote by  $\mathbb{S}(F) := \mathbb{S}(F \setminus \{0\})$  the corresponding projective subspace of dimension  $k$  in  $\mathbb{S}(\mathbb{R}^{2,n})$ . The geodesics of  $\text{ADS}_{n+1}$  are the connected components of the intersections of  $\text{ADS}_{n+1}$  with the projective lines  $\mathbb{S}(F)$  of  $\mathbb{S}(\mathbb{R}^{2,n})$ . More generally, the totally geodesic subspaces of dimension  $k$  in  $\text{ADS}_{n+1}$  are the connected components of the intersections of  $\text{ADS}_{n+1}$  with the projective subspaces  $\mathbb{S}(F)$  of dimension  $k$  of  $\mathbb{S}(\mathbb{R}^{2,n})$ .

**Definition 2.7.** For every  $x = \mathbb{S}(x)$  in  $\text{ADS}_{n+1}$ , we define the *affine domain* (also denoted by  $U(x)$ ):

$$U(x) := \{y \in \text{ADS}_{n+1} \mid \langle x \mid y \rangle < 0\}$$

In other words,  $U(x)$  is the connected component of  $\text{ADS}_{n+1} \setminus \mathbb{S}(x^\perp)$  containing  $x$ . Let  $V(x)$  (also denoted by  $V(x)$ ) be the connected component of  $\mathbb{S}(\mathbb{R}^{2,n}) \setminus \mathbb{S}(x^\perp)$  containing  $U(x)$ . The boundary  $\partial U(x) \subset \partial \text{ADS}_{n+1}$  of  $U(x)$  in  $V(x)$  is called the *affine boundary* of  $U(x)$ .

**Remark 2.8.** Up to composition by an element of the isometry group  $SO_0(2, n)$  of  $\mathbb{q}_{2,n}$ , we can assume that  $\mathbb{S}(x^\perp)$  is the projection of the hyperplane  $\{u = 0\}$  in  $\mathbb{R}^{2,n}$  and  $V(x)$  is the projection of the region  $\{u > 0\}$  in  $\mathbb{R}^{2,n}$ . The map

$$(u, v, x_1, x_2, \dots, x_{n+1}) \mapsto (t, \bar{x}_1, \dots, \bar{x}_n) := \left( \frac{v}{u}, \frac{x_1}{u}, \frac{x_2}{u}, \dots, \frac{x_n}{u} \right)$$

induces a diffeomorphism between  $V(x)$  and  $\mathbb{R}^{n+1}$  mapping the affine domain  $U(x)$  to the region  $\{(t, \bar{x}_1, \dots, \bar{x}_n) \in \mathbb{R}^{n+1} \mid q_{1,n}(t, \bar{x}_1, \dots, \bar{x}_n) < 1\}$ , where  $q_{1,n}$  is the Minkowski norm. The affine boundary  $\partial U(x)$  corresponds to the hyperboloid  $\{(t, \bar{x}_1, \dots, \bar{x}_n) \mid q_{1,n}(t, \bar{x}_1, \dots, \bar{x}_n) = 1\}$ . The intersections between  $U(x)$  and the totally geodesic subspaces of  $\text{ADS}_{n+1}$  correspond to the intersections of the region  $\{(t, \bar{x}_1, \dots, \bar{x}_n) \in \mathbb{R}^{n+1} \mid q_{1,n}(t, \bar{x}_1, \dots, \bar{x}_n) < 1\}$  with the affine subspaces of  $\mathbb{R}^{n+1}$ .

**Lemma 2.9** (Lemma 10.13 in [ABBZ12]). *Let  $U$  be an affine domain in  $\text{ADS}_{n+1}$  and  $\partial U \subset \partial \text{ADS}_{n+1}$  be its affine boundary. Let  $x$  be a point in  $\partial U$ , and  $y$  be a point in  $U \cup \partial U$ . There exists a causal (resp. timelike) curve joining  $x$  to  $y$  in  $U \cup \partial U$  if and only if  $\langle x \mid y \rangle \geq 0$  (resp.  $\langle x \mid y \rangle > 0$ ).  $\square$*

**Remark 2.10.** The boundary of  $U(x)$  in  $\text{ADS}_{n+1}$  is  $\mathbb{S}(x^\perp) \cap \text{ADS}_{n+1}$ . It has two boundary components: the *past component*  $H^-(x)$  and the *future component*  $H^+(x)$ . These components are characterized by the following property: timelike geodesics enter in  $U(x)$  through  $H^-(x)$  and exit through  $H^+(x)$ .

They can also be defined as follows: let  $\tilde{U}(x)$  be a lifting in  $\widetilde{\text{AdS}}_{n+1}$  of  $U(x)$ , and let  $\tilde{H}^\pm(x)$  be the lifts of  $H^\pm(x)$ . Then,  $\tilde{U}(x)$  is the intersection between the future of  $H^-(x)$  and the past of  $H^+(x)$ .

The boundary components  $H^\pm(x)$  are totally geodesic embedded copies of  $\mathbb{H}^n$ . They are also called *hyperplanes dual to  $x$* , and we distinguish the hyperplane past-dual  $H^-(x) = H^-(x)$  from the hyperplane future-dual  $H^+(x) = H^+(x)$ .

Last but not least:  $H^\pm(x)$  have also the following characteristic property: every future oriented (resp. past oriented) timelike geodesic starting at  $x$  reach  $H^+(x)$  (resp.  $H^-(x)$ ) at time  $\pi/2$  (see Remark 2.3). In other words,  $H^\pm(x)$  is the set of points at Lorentzian distance  $\pm\pi/2$  from  $x$ .

**2.7. The Klein model of the Einstein universe.** Similarly, Einstein universe has a Klein model: the projection  $\mathbb{S}(\mathcal{C}_n)$  in  $\mathbb{S}(\mathbb{R}^{2,n})$  of the isotropic cone  $\mathcal{C}_n$  in  $\mathbb{R}^{2,n}$ . The conformal Lorentzian structure can be defined in terms of the quadratic form  $q_{2,n}$  (for more details, see [Fra05, BCD<sup>+</sup>08]).

**Remark 2.11.** In the sequel, we will frequently identify  $\text{Ein}_n$  with  $\mathbb{S}(\mathcal{C}_n)$ , since we will frequently switch from one model to the other.

An immediate corollary of Lemma 2.9 is:

**Corollary 2.12.** *For  $\Lambda \subseteq \text{Ein}_n$ , the following assertions are equivalent.*

- (1)  $\Lambda$  is achronal (respectively acausal);
- (2) when we see  $\Lambda$  as a subset of  $\mathbb{S}(\mathcal{C}_n) \approx \text{Ein}_n$  the scalar product  $\langle x | y \rangle$  is non-positive (respectively negative) for every distinct  $x, y \in \Lambda$ .

□

**Remark 2.13.** Let  $x_0$  be any element of  $\text{Ein}_n \approx \mathbb{S}(\mathcal{C}_n)$ . Then, the open domain defined by:

$$\text{Mink}(x_0) = \{x \in \mathbb{S}(\mathcal{C}_n) / \langle x_0 | x \rangle < 0\}$$

is conformally isometric to the Minkowski space  $\mathbb{R}^{1,n-1}$  (see [Fra05, BCD<sup>+</sup>08]).

In particular, the stabilizer  $G_0$  of  $x_0$  in  $\text{SO}_0(2, n)$  is isomorphic to the group of conformal isometries of  $\mathbb{R}^{1,n-1}$ , *ie.* of affine transformations whose linear part has the form  $x \mapsto \lambda g(x)$ , where  $\lambda$  is a positive real number and  $g$  an element of  $\text{SO}_0(1, n-1)$ .

### 3. REGULAR AdS MANIFOLDS

In all this section,  $\tilde{\Lambda}$  is a closed achronal subset of  $\partial \widetilde{\text{AdS}}_{n+1}$ , and  $\Lambda$  is the projection of  $\tilde{\Lambda}$  in  $\partial \text{AdS}_{n+1}$ .

**3.1. AdS regular domains.** We denote by  $\tilde{E}(\tilde{\Lambda})$  the *invisible domain* of  $\tilde{\Lambda}$  in  $\widetilde{\text{AdS}}_{n+1}$ , that is,

$$\tilde{E}(\tilde{\Lambda}) =: \widetilde{\text{AdS}}_{n+1} \setminus \left( J^-(\tilde{\Lambda}) \cup J^+(\tilde{\Lambda}) \right)$$

where  $J^-(\tilde{\Lambda})$  and  $J^+(\tilde{\Lambda})$  are the causal past and the causal future of  $\tilde{\Lambda}$  in  $\widetilde{\text{AdS}}_{n+1} \cup \partial \widetilde{\text{AdS}}_{n+1} = (\mathbb{R} \times \overline{\mathbb{D}}^{n-1}, -d\theta^2 + ds^2)$ . We denote by  $\text{Cl}(\tilde{E}(\tilde{\Lambda}))$  the closure of  $\tilde{E}(\tilde{\Lambda})$  in  $\widetilde{\text{AdS}}_{n+1} \cup \partial \widetilde{\text{AdS}}_{n+1}$  and by  $E(\Lambda)$  the projection of  $\tilde{E}(\tilde{\Lambda})$  in  $\text{AdS}_{n+1}$  (according to Corollary 2.6,  $E(\Lambda)$  only depends on  $\Lambda$ , not on the choice of the lifting  $\tilde{\Lambda}$ ).

**Definition 3.1.** A  $(n+1)$ -dimensional *AdS regular domain* is a domain of the form  $E(\Lambda)$  where  $\Lambda$  is the projection in  $\partial \text{AdS}_{n+1}$  of an achronal subset  $\tilde{\Lambda} \subset \partial \widetilde{\text{AdS}}_{n+1}$  containing at least two points. If  $\tilde{\Lambda}$  is a topological  $(n-1)$ -sphere, then  $E(\Lambda)$  is *GH-regular* (this definition is motivated by Theorem 4.12 and Proposition 4.14).

**Remark 3.2.** The invisible domain  $\widetilde{E}(\widetilde{\Lambda})$  is causally convex in of  $\widetilde{\text{AdS}}_{n+1}$ ; *ie.* every causal curve joining two points in  $\widetilde{E}(\widetilde{\Lambda})$  is entirely contained in  $\widetilde{E}(\widetilde{\Lambda})$ . This is an immediate consequence of the definitions. It follows that AdS regular domains are strongly causal.

**Remark 3.3.** Recall that  $\widetilde{\Lambda}$  is the graph of a 1-Lipschitz function  $f : \Lambda_0 \rightarrow \mathbb{R}$  where  $\Lambda_0$  is a closed subset of  $\mathbb{S}^{n-1}$  (section 2.5). Define two functions  $f^-, f^+ : \mathbb{D}^n \rightarrow \mathbb{R}$  as follows:

$$\begin{aligned} f^-(x) &:= \text{Sup}_{y \in \Lambda_0} \{f(y) - d(x, y)\}, \\ f^+(x) &:= \text{Inf}_{y \in \Lambda_0} \{f(y) + d(x, y)\}, \end{aligned}$$

where  $d$  is the distance induced by  $ds^2$  on  $\mathbb{D}^n$ . It is easy to check that

$$\widetilde{E}(\widetilde{\Lambda}) = \{(\theta, x) \in \mathbb{R} \times \mathbb{D}^n \mid f^-(x) < \theta < f^+(x)\}.$$

**Remark 3.4.** Keeping the notations in the previous remark, observe that the graph of the restriction of  $f^+$  (or  $f^-$ ) to  $\partial\mathbb{D}^n$  is a closed achronal  $(n-1)$ -sphere  $\widetilde{\Lambda}^+$  (or  $\widetilde{\Lambda}^-$ ) in  $\widetilde{\text{AdS}}_{n+1}$  which contains the initial achronal subset  $\widetilde{\Lambda}$ . They project to achronal  $(n-1)$ -spheres  $\Lambda^\pm$  in  $\partial\text{AdS}_{n+1}$  that contain  $\Lambda$ .

Furthermore, any element  $g$  of  $\text{SO}_0(2, n)$  preserving  $\Lambda$  must preserve  $E(\Lambda)$ , hence the graphs of  $f^\pm$ , and therefore must preserve  $\Lambda^+$  and  $\Lambda^-$ .

**Definition 3.5.** The graph of  $f^-$  (respectively  $f^+$ ) is a closed achronal subset of  $\widetilde{\text{AdS}}_{n+1}$ , called the *lifted past* (respectively *future*) *horizon* of  $\widetilde{E}(\widetilde{\Lambda})$ , and denoted  $\mathcal{H}^-(\widetilde{\Lambda})$  (respectively  $\mathcal{H}^+(\widetilde{\Lambda})$ ).

The projections in  $\text{AdS}_{n+1}$  of  $\mathcal{H}^\pm(\widetilde{\Lambda})$  are called *past* and *future horizons* of  $E(\Lambda)$ , and denoted  $\mathcal{H}^\pm(\Lambda)$ .

The following lemma is a refinement of Lemma 2.5:

**Lemma 3.6** (Corollary 10.6 in [ABBZ12]). *For every (non-empty) closed achronal set  $\widetilde{\Lambda} \subset \partial\widetilde{\text{AdS}}_{n+1}$ , the projection of  $\widetilde{E}(\widetilde{\Lambda})$  on  $E(\Lambda)$  is one-to-one.*  $\square$

**Definition 3.7.**  $\widetilde{\Lambda}$  is *purely lightlike* if the associated subset  $\Lambda_0$  of  $\mathbb{S}^n$  contains two antipodal points  $x_0$  and  $-x_0$  such that, for the associated 1-Lipschitz map  $f : \Lambda_0 \rightarrow \mathbb{R}$  the equality  $f(x_0) = f(-x_0) + \pi$  holds.

If  $\widetilde{\Lambda}$  is purely lightlike, for every element  $x$  of  $\mathbb{D}^n$  we have  $f^-(x) = f^+(x) = f(-x_0) + d(-x_0, x) = f(x_0) - d(x_0, x)$ , implying that  $\widetilde{E}(\widetilde{\Lambda})$  is empty. Conversely:

**Lemma 3.8** (Lemma 3.6 in [BM12]).  *$\widetilde{E}(\widetilde{\Lambda})$  is empty if and only if  $\widetilde{\Lambda}$  is purely lightlike. More precisely, if for some point  $x$  in  $\mathbb{D}^n$  the equality  $f^+(x) = f^-(x)$  holds then  $\widetilde{\Lambda}$  is purely lightlike.*  $\square$

**3.2. AdS regular domains as subsets of  $\mathbb{A}\mathbb{D}\mathbb{S}_{n+1}$ .** The canonical homeomorphism between  $\text{AdS}_{n+1} \cup \partial \text{AdS}_{n+1}$  and  $\mathbb{A}\mathbb{D}\mathbb{S}_{n+1} \cup \partial \mathbb{A}\mathbb{D}\mathbb{S}_{n+1}$  allows us to see AdS regular domains as subsets of  $\mathbb{A}\mathbb{D}\mathbb{S}_{n+1}$ .

Putting together the definition of the invisible domain  $E(\Lambda)$  of a set  $\Lambda \subset \partial \text{AdS}_{n+1}$  and Lemma 2.9, one gets:

**Proposition 3.9** (Proposition 10.14 in [ABBZ12]). *If we see  $\Lambda$  and  $E(\Lambda)$  in the Klein model  $\mathbb{A}\mathbb{D}\mathbb{S}_{n+1} \cup \partial \mathbb{A}\mathbb{D}\mathbb{S}_{n+1}$ , then*

$$E(\Lambda) = \{y \in \mathbb{A}\mathbb{D}\mathbb{S}_{n+1} \text{ such that } \langle y \mid x \rangle < 0 \text{ for every } x \in \Lambda\}$$

□

**3.3. Convex core of AdS regular domains.** In this section, we assume that  $\Lambda$  is not purely lightlike and not reduced to a single point. The following notions are classical and well-known:

**Definition 3.10.** A subset  $\Omega$  of  $\mathbb{S}(\mathbb{R}^{2,n})$  is *convex* if there is a convex cone  $J$  of  $\mathbb{R}^{2,n}$  such that  $\Omega = \mathbb{S}(J)$ . The *relative interior* of  $\Omega$ , denoted by  $\Omega^\circ$  is the convex subset  $\mathbb{S}(J^\circ)$  where  $J^\circ$  is the interior of  $J$  in the subspace spanned by  $J$ .

It is well-known that the closure of a convex subset is still convex, and that it coincides with the closure of the relative interior.

**Theorem-Definition 3.11.** Let  $\Omega = \mathbb{S}(J)$  be a convex subset of  $\mathbb{S}(\mathbb{R}^{2,n})$ . The following assertions are equivalent:

- $J$  contains no complete affine line,
- there is an affine hyperplane  $H$  in  $\mathbb{S}(\mathbb{R}^{2,n})$  such that  $H \cap J$  is relatively compact in  $H$  and such that  $\Omega = \mathbb{S}(J \cap H)$ ,
- The closure of  $\Omega$  contains no pair of opposite points.

If one of these equivalent properties hold, then  $\Omega$  is *salient*. □

**Definition 3.12.** Let  $\Omega = \mathbb{S}(J)$  a convex subset of  $\mathbb{S}(\mathbb{R}^{2,n})$ . The dual of  $\Omega$  is the closed convex subset  $\mathbb{S}(J^* \setminus \{0\})$  where:

$$J^* = \{x \in \mathbb{R}^{2,n} \mid \forall y \in J, \langle x \mid y \rangle \leq 0\}$$

**Proposition 3.13.** *Let  $\Omega$  be a convex subset of  $\mathbb{S}(\mathbb{R}^{2,n})$ . Then, the bidual  $\Omega^{**}$  is the closure  $Cl(\Omega)$  of  $\Omega$  in  $\mathbb{S}(\mathbb{R}^{2,n})$ . The relative interior  $\Omega^\circ$  is open in  $\mathbb{S}(\mathbb{R}^{2,n})$  if and only if  $\Omega^*$  is salient.* □

Let  $\hat{\Lambda}$  be the preimage of  $\Lambda \subset \text{Ein}_n = \mathbb{S}(\mathcal{C}_n)$  by  $\mathbb{S}$ . The convex hull of  $\hat{\Lambda}$  is a convex cone  $\text{Conv}(\hat{\Lambda})$  in  $\mathbb{R}^{2,n}$ , whose projection is a compact convex subset of  $\mathbb{S}(\mathbb{R}^{2,n})$ , denoted by  $\text{Conv}(\Lambda)$ , and called *the convex hull of  $\Lambda$*  and *the convex core of  $E(\Lambda)$* .

**Lemma 3.14.** *The intersection between  $\text{Conv}(\Lambda)$  and  $\text{Ein}_n$  is the union of lightlike segments in  $\text{Ein}_n$  joining two elements of  $\Lambda$ . The relative interior  $\text{Conv}(\Lambda)^\circ$  is contained in  $\mathbb{A}\mathbb{D}\mathbb{S}_{n+1}$ .*

*Proof.* Elements of  $\text{Conv}(\hat{\Lambda})$  are linear combinations  $x = \sum_{i=1}^k t_i x_i$  where  $t_i$  are non-negative real numbers and  $x_i$  elements of  $\hat{\Lambda}$ .

$$q_{2,n}(x) = \sum_{i,j=1}^k t_i t_j \langle x_i | x_j \rangle$$

Since every  $\langle x_i | x_j \rangle$  is nonpositive, we have  $q_{2,n}(x) \leq 0$ .

Moreover, if  $q_{2,n}(x) = 0$ , then every  $\langle x_i | x_j \rangle$  must be equal to 0, *ie.* the vector space spanned by the  $x_i$ 's is isotropic, hence either a line, or an isotropic plane in  $\mathcal{C}_n$ . In the first case,  $x$  is an element of  $\Lambda$ , and in the second case,  $x$  lies on a lightlike geodesic of  $\text{Ein}_n$  joining two elements of  $\Lambda$ .

Finally, assume that  $\text{Conv}(\Lambda)^\circ$  is not contained in  $\text{AdS}_{n+1}$ . Since  $q_{2,n}(x) \leq 0$  for every  $x$  in  $\hat{\Lambda}$ , it follows that  $\text{Conv}(\hat{\Lambda})$  is contained in  $\mathcal{C}_n$ , and more precisely, by the argument above, in an isotropic 2-plane. It is a contradiction since  $\Lambda$  by hypothesis is not purely lightlike.  $\square$

Actually, the case where  $\text{Conv}(\Lambda)^\circ$  is not an open subset of  $\text{AdS}_{n+1}$  is exceptional:

**Lemma 3.15** (Lemma 3.13 in [BM12]). *If  $\text{Conv}(\Lambda) \cap \text{AdS}_{n+1}$  has empty interior, then it is contained in a totally geodesic spacelike hypersurface of  $\text{AdS}_{n+1}$ .*  $\square$

Proposition 3.9 can be rewritten as follows:

**Proposition 3.16** (Proposition 10.17 in [ABBZ12]). *The domain  $E(\Lambda)$  is the intersection  $\text{AdS}_{n+1} \cap (\text{Conv}(\Lambda)^*)^\circ$ .*  $\square$

**Remark 3.17.** A corollary of Proposition 3.16 is that the invisible domain  $E(\Lambda)$  is convex, hence contains  $\text{Conv}(\Lambda)^\circ$ .

Hence, if  $x$  lies in the interior of  $\text{Conv}(\Lambda)$ , the affine domain  $U(x)$  contains the closure of  $E(\Lambda)$ . Therefore:

**Proposition 3.18.** *Assume that  $\Lambda$  is not the boundary of a totally geodesic copy of  $\mathbb{H}^n$  in  $\text{AdS}_{n+1}$ . Then, the restriction of  $\hat{p} : \widetilde{\text{AdS}}_{n+1} \rightarrow \text{AdS}_{n+1}$  to the closure of  $\widetilde{E}(\hat{\Lambda})$  is one-to-one.*

*In particular,  $\hat{p} : \widetilde{\mathcal{H}}^\pm(\hat{\Lambda}) \rightarrow \mathcal{H}^\pm(\Lambda)$  is injective.*  $\square$

The boundary of  $E(\Lambda)$  in  $\text{AdS}_{n+1}$  has two components: the past and future horizons  $\mathcal{H}^\pm(\Lambda)$  (cf. Definition 3.5). Since  $E(\Lambda)$  is convex, every point  $x$  in  $\mathcal{H}^-(\Lambda)$  lies in a support hyperplane for  $E(\Lambda)$ , *ie.* a totally geodesic hyperplane  $H$  tangent to  $\mathcal{H}^-(\Lambda)$  at  $x$ . According to Proposition 3.16,  $H$  is the hyperplane dual to an element  $p$  of  $\partial \text{Conv}(\Lambda)$ , hence  $H$  is either spacelike (if  $p \in \text{AdS}_{n+1}$ ) or degenerate (if  $p \in \text{Ein}_n$ ).

**Remark 3.19.** For every achronal subset  $\Lambda$ , the intersection  $\text{Conv}(\Lambda) \cap \text{Ein}_n$ , which is a union of lightlike geodesic segments joining elements of  $\Lambda$

is still achronal (since  $\langle \sum s_i x_i | \sum t_j y_j \rangle = \sum s_i t_j \langle x_i | y_j \rangle \leq 0$  for  $s_i, t_j \geq 0$ ,  $x_i, y_j \in \Lambda$ ). We call it the *filling* of  $\Lambda$  and denote it by  $\text{Fill}(\Lambda)$ . According to Proposition 3.16:

$$E(\text{Fill}(\Lambda)) = E(\Lambda)$$

Hence, we can always assume wlog that  $\Lambda$  is *filled*, ie.  $\Lambda = \text{Fill}(\Lambda)$ .

#### 4. GLOBALLY HYPERBOLIC ADS SPACETIMES

In all this section,  $\Lambda$  is a non-purely lightlike topological achronal  $(n-1)$ -sphere in  $\partial \text{AdS}_{n+1}$ . In particular,  $\Lambda$  is automatically filled (cf. Remark 3.19).

**Proposition 4.1** (Corollary 10.7 in [ABBZ12]). *For every achronal topological  $(n-1)$ -sphere  $\Lambda \subset \partial \text{AdS}_{n+1}$ , the intersection between the closure  $\text{Cl}(E(\Lambda))$  of  $E(\Lambda)$  in  $\text{Ein}_{n+1}$  and  $\text{Ein}_n = \partial \text{AdS}_{n+1}$  is reduced to  $\Lambda$ .  $\square$*

The meaning of Proposition 4.1 is that  $(\text{Conv}(\Lambda)^*)^\circ$  is already contained in  $\text{AdS}_{n+1}$ , so that the expression  $E(\Lambda) = \text{AdS}_{n+1} \cap (\text{Conv}(\Lambda)^*)^\circ$  is reduced to  $E(\Lambda) = (\text{Conv}(\Lambda)^*)^\circ$  when  $\Lambda$  is a topological sphere.

**Remark 4.2.** It follows from Proposition 4.1 that the GH-regular domain  $E(\Lambda)$  characterizes  $\Lambda$ , ie. invisible domains of different achronal  $(n-1)$ -spheres are different. We call  $\Lambda$  the *limit set* of  $E(\Lambda)$ .

**4.1. More on the convex hull of achronal topological  $(n-1)$ -spheres.** Recall that there are two maps  $f^-, f^+$  such that  $\tilde{E}(\tilde{\Lambda}) = \{(\theta, x)/f^-(x) < \theta < f^+(x)\}$  (cf. Definition 3.3).

**Proposition 4.3.** *The complement of  $\Lambda$  in the boundary  $\partial \text{Conv}(\Lambda)$  has two connected components. Both are closed edgeless achronal subsets of  $\text{AdS}_{n+1}$ . More precisely, in the conformal model their liftings in  $\widetilde{\text{AdS}}_{n+1}$  are graphs of 1-Lipschitz maps  $F^+, F^-$  from  $\mathbb{D}^n$  into  $\mathbb{R}$  such that*

$$(1) \quad f^- \leq F^- \leq F^+ \leq f^+$$

*Proof.* See Proposition 3.14 in [BM12]. Observe that in [BM12], Proposition 3.14 is proved in the case where  $\Lambda$  is acausal, and not Fuchsian (the Fuchsian case being the case where  $\Lambda$  is the boundary of a totally geodesic hypersurface in  $\widetilde{\text{AdS}}_{n+1}$ ). Inequalities in equation (1) are then all strict inequalities, which is false in the general case, as we will see later<sup>3</sup>. Nevertheless, the proof of Proposition 3.14 in [BM12] can easily be adapted, providing a proof of Proposition 4.3.  $\square$

We have already observed that  $\partial E(\Lambda) \setminus \Lambda$  is the union of two achronal connected components  $\mathcal{H}^\pm(\Lambda)$ ; in a similar way,  $\partial \text{Conv}(\Lambda) \setminus \Lambda$  is the union of two achronal  $n$ -dimensional topological disks: the *past component*  $S^-(\Lambda)$  (the graph of  $F^-$ ) and the *future component*  $S^+(\Lambda)$ . Since  $E(\Lambda)$  and  $\text{Conv}(\Lambda)$  are convex and dual one to the other, for every element  $x$  in  $S^-(\Lambda)$  (respectively

<sup>3</sup>Anyway, one can already observe that in the Fuchsian case  $F^- = F^+$ .

$S^+(\Lambda)$ ) there is an element  $p$  of  $\Lambda$  or  $\mathcal{H}^+(\Lambda)$  (respectively  $\mathcal{H}^+(\Lambda)$ ) such that  $H^-(p)$  (respectively  $H^+(p)$ ) is a support hyperplane for  $S^-(\Lambda)$  (respectively  $S^+(\Lambda)$ ) at  $x$ : these support hyperplanes are either totally geodesic copies of  $\mathbb{H}^n$  (if  $p \in \text{AdS}_{n+1}$ ) or degenerate (if  $p \in \Lambda$ ).

Similarly, at every element  $x$  of  $\mathcal{H}^-(\Lambda)$  (respectively  $\mathcal{H}^+(\Lambda)$ ) there is a support hyperplane  $H^-(p)$  (respectively  $H^+(p)$ ) where  $p$  is an element of  $S^+(\Lambda) \cup \Lambda$  (respectively  $S^-(\Lambda) \cup \Lambda$ ) (see Figure 1).

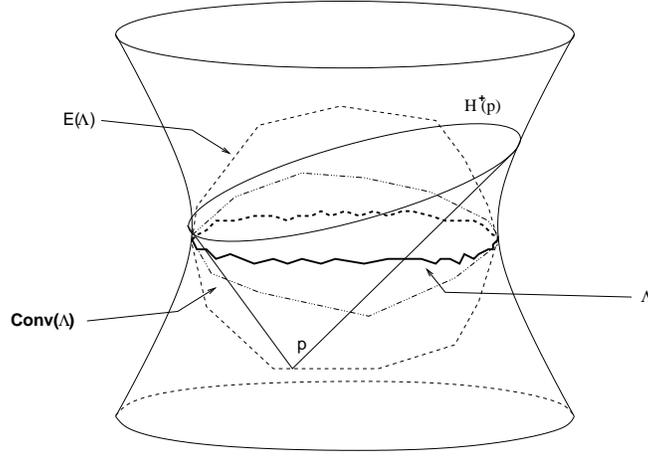


FIGURE 1. The global situation. The hyperboloid represents the boundary of an affine domain of  $\text{AdS}_{n+1}$  containing the invisible domain. The limit set  $\Lambda$  is represented by a topological circle turning around the hyperboloid, and  $\text{Conv}(\Lambda)^\circ$  is a convex subset inside the (dual) convex subset  $E(\Lambda)$ . The future-dual plane  $H^+(p)$  for  $p$  in the past boundary component  $\mathcal{H}^-(\Lambda)$  is a support hyperplane of  $S^+(\Lambda)$ .

**Remark 4.4.** For every  $p$  in  $\mathcal{H}^-(\Lambda)$ ,  $H^+(p)$  is a support hyperplane for  $\text{Conv}(\Lambda)$ , but it could be at a point in  $\Lambda$ . Elements of  $\mathcal{H}^-(\Lambda)$  that are support hyperplanes for  $\text{Conv}(\Lambda)$  at a point inside  $\text{AdS}_{n+1}$ , *ie.* in  $\mathcal{H}^+(\Lambda)$  form an interesting subset of  $\mathcal{H}^-(\Lambda)$ , the *initial singularity set* (cf. [BB09]).

#### 4.2. Global hyperbolicity.

**Definition 4.5.** A spacetime  $(M, g)$  is *globally hyperbolic* (abbreviation GH) if:

- $(M, g)$  is *causal*, *ie.* contains no timelike loop,
- for every  $p, q$  in  $M$ , the intersection  $J^+(p) \cap J^-(q)$  is empty or compact.

**Definition 4.6.** Let  $(M, g)$  be a spacetime. A *Cauchy hypersurface* is a closed acausal subset  $S \subset M$  that intersects every inextendible causal curve in  $(M, g)$  in one and only one point.

A *Cauchy time function* is a time function  $T : M \rightarrow \mathbb{R}$  such that every level set  $T^{-1}(a)$  is a Cauchy hypersurface in  $(M, g)$ .

**Theorem 4.7** ([CBG69], [BS03, BS05, BS07]). *Let  $(M, g)$  be a spacetime. The following assertions are equivalent:*

- (1)  $(M, g)$  is globally hyperbolic,
- (2)  $(M, g)$  contains a Cauchy hypersurface,
- (3)  $(M, g)$  admits a Cauchy time function,
- (4)  $(M, g)$  admits a smooth Cauchy time function.

In a GH spacetime, the Cauchy hypersurfaces are homeomorphic one to the other. In particular, if one of them is compact, all of them are compact.

**Definition 4.8.** A spacetime  $(M, g)$  is *globally hyperbolic spatially compact* (abbrev. GHC) if it contains a closed Cauchy hypersurface.

**Proposition 4.9.** *A spacetime  $(M, g)$  is GHC if and only if it contains a time function  $T : M \rightarrow \mathbb{R}$  such that every level set  $T^{-1}(a)$  is compact.  $\square$*

**4.3. Cosmological time functions.** In any spacetime  $(M, g)$ , one can define the *cosmological time function* as follows (see [AGH98]):

**Definition 4.10.** The cosmological time function of a spacetime  $(M, g)$  is the function  $\tau : M \rightarrow [0, +\infty]$  defined by

$$\tau(x) := \text{Sup}\{L(c) \mid c \in \mathcal{R}^-(x)\},$$

where  $\mathcal{R}^-(x)$  is the set of past-oriented causal curves starting at  $x$ , and  $L(c)$  is the Lorentzian length of the causal curve  $c$ .

**Definition 4.11.** A spacetime  $(M, g)$  is *CT-regular* with cosmological time function  $\tau$  if

- (1)  $M$  has *finite existence time*,  $\tau(x) < \infty$  for every  $x$  in  $M$ ,
- (2) for every past-oriented inextendible causal curve  $c : [0, +\infty) \rightarrow M$ ,  
 $\lim_{t \rightarrow \infty} \tau(c(t)) = 0$ .

**Theorem 4.12** ([AGH98]). *If a spacetime  $(M, g)$  has is CT-regular, then*

- (1)  $M$  is globally hyperbolic,
- (2)  $\tau$  is a time function, i.e.  $\tau$  is continuous and is strictly increasing along future-oriented causal curves,
- (3) for each  $x$  in  $M$ , there is at least one realizing geodesic, ie. a future-oriented timelike geodesic  $c : (0, \tau(x)] \rightarrow M$  realizing the distance from the "initial singularity", that is,  $c$  has unit speed, is geodesic, and satisfies:

$$c(\tau(x)) = x \text{ and } \tau(c(t)) = t \text{ for every } t$$

- (4)  $\tau$  is locally Lipschitz, and admits first and second derivative almost everywhere.  $\square$

However,  $\tau$  is not always a Cauchy time function (see the comment after Corollary 2.6 in [AGH98]).

A very nice feature of CT-regularity is that it is preserved by isometries (and thus, by Galois automorphisms):

**Proposition 4.13** (Proposition 4.4 in [BM12]). *Let  $(\widetilde{M}, \widetilde{g})$  be a CT-regular spacetime. Let  $\Gamma$  be a torsion-free discrete group of isometries of  $(\widetilde{M}, \widetilde{g})$  preserving the time orientation. Then, the action of  $\Gamma$  on  $(\widetilde{M}, \widetilde{g})$  is properly discontinuous. Furthermore, the quotient spacetime  $(M, g)$  is CT-regular. More precisely, if  $p : \widetilde{M} \rightarrow M$  denote the quotient map, the cosmological times  $\tilde{\tau} : \widetilde{M} \rightarrow [0, +\infty)$  and  $\tau : M \rightarrow [0, +\infty)$  satisfy:*

$$\tilde{\tau} = \tau \circ p$$

Recall that in this section  $\Lambda$  denotes a non-purely lightlike topological achronal  $(n-1)$ -sphere in  $\partial \text{AdS}_{n+1}$ .

**Proposition 4.14** (Proposition 11.1 in [ABBZ12]). *The GH-regular AdS domain  $E(\Lambda)$  is CT-regular.*  $\square$

Hence, according to Theorem 4.12, GH-regular domains are globally hyperbolic. Furthermore:

**Definition 4.15.** The region  $\{\tau < \pi/2\}$  is denoted  $E_0^-(\Lambda)$  and called the *past tight region* of  $E(\Lambda)$ .

**Proposition 4.16** (Proposition 11.5 in [ABBZ12]). *Let  $x$  be an element of the past tight region  $E_0^-(\Lambda)$ . Then, there is a unique realizing geodesic for  $x$ . More precisely, there is one and only one element  $r(x)$  in the past horizon  $\mathcal{H}^-(\Lambda)$  - called the *cosmological retract* of  $x$  - such that the segment  $(r(x), x)$  is a timelike geodesic whose Lorentzian length is precisely the cosmological time  $\tau(x)$ .*  $\square$

**Proposition 4.17** (Proposition 11.6 in [ABBZ12]). *Let  $c : (0, T] \rightarrow E_0^-(\Lambda)$  be a future oriented timelike geodesic whose initial extremity  $p := \lim_{t \rightarrow 0} c(t)$  is in the past horizon  $\mathcal{H}^-(\Lambda)$ . Then the following assertions are equivalent.*

- (1) *For every  $t \in (0, T]$ ,  $c|_{[0, t]}$  is a realizing geodesic for the point  $c(t)$ .*
- (2) *There exists  $t \in (0, T]$  such that  $c|_{[0, t]}$  is a realizing geodesic for the point  $c(t)$ .*
- (3)  *$c$  is orthogonal to a support hyperplane of  $E(\Lambda)$  at  $p := \lim_{t \rightarrow 0} c(t)$ .*

The following Proposition was known in the case  $n = 2$  ([Mes07, BB09], and was implicitly admitted in the few previous papers devoted to the higher dimensional case (for example, [ABBZ12, BM12]):

**Proposition 4.18.** *The past tight region  $E_0^-(\Lambda)$  is the past in  $E(\Lambda)$  of the future component  $S^+(\Lambda)$  of the convex core (in particular, it contains  $\text{Conv}(\Lambda)^\circ$ ). The restriction of the cosmological time to  $E_0^-(\Lambda)$  is a Cauchy time, taking all values in  $(0, \pi/2)$ .*

*Proof.* Let  $x$  be an element of  $E_0^-(\Lambda)$ . According to Propositions 4.16, 4.17 there is a realizing geodesic  $(r(x), x]$  orthogonal to a spacelike support hyperplane  $H$  tangent to  $\mathcal{H}^-(\Lambda)$  at  $r(x)$ . As described in Sect. 4.1, this support hyperplane is the hyperplane  $H^-(p)$  past-dual to an element  $p$  of  $S^+(\Lambda)$ . The realizing geodesic is contained in the geodesic  $\theta \mapsto c(\theta) = \cos(\theta)r(x) + \sin(\theta)p(x)$  (cf. Remark 2.3). For  $\theta$  in  $(0, \pi/2)$  sufficiently closed to  $\pi/2$ ,  $c(\theta)$  belongs to  $\text{Conv}(\Lambda) \subset E(\Lambda)$ , and since  $E(\Lambda)$  is convex, every  $c(\theta)$  ( $\theta \in (0, \pi/2)$ ) lies in  $E(\Lambda)$ . Moreover, according to Proposition 4.17, for every  $\theta_0$  in  $(0, \pi/2)$ , the restriction of  $c$  to  $(0, \theta_0)$  is a realizing geodesic. Hence:

$$\forall \theta \in (0, \pi/2), \quad \tau(c(\theta)) = \theta$$

Hence, every value in  $(0, \pi/2)$  is attained by  $\tau$ . Moreover,  $x$  lies in the past of  $p(x)$ , hence of  $S^+(\Lambda)$ . We have:

$$E_0^-(\Lambda) \subset I^-(S^+(\Lambda)) \cap E(\Lambda)$$

Inversely, for every  $p$  in  $I^-(S^+(\Lambda)) \cap E(\Lambda)$ , there is a (not necessarily unique) realizing geodesic  $c : (0, \tau(x)) \rightarrow E(\Lambda)$  such that  $c(\tau(x)) = x$  (cf. item (3) in Theorem 4.12). Then, the curve  $c$  being a timelike geodesic inextendible (in  $E(\Lambda)$ ) in past, for  $t \rightarrow 0$  the points  $c(t)$  converge to a limit point  $c(0)$  in  $\mathcal{H}^-(\Lambda)$ . If  $\tau(x) \geq \pi/2$ , on the one hand we observe that  $c(\pi/2)$  lies in the past of  $x = c(\tau(x))$ , hence in  $I^-(S^+(\Lambda))$ . On the other hand:

$$\langle c(\pi/2) | c(0) \rangle = 0$$

Therefore,  $c(\pi/2)$  is dual to an element of  $\mathcal{H}^-(\Lambda)$  and belongs to  $S^+(\Lambda)$ . But it is a contraction since  $S^+(\Lambda)$  is achronal and  $c(\pi/2) \in I^-(S^+(\Lambda))$ . Hence  $\tau(x) < \pi/2$ , *ie.* :

$$I^-(S^+(\Lambda)) \cap E(\Lambda) \subset E_0^-(\Lambda)$$

In order to conclude, we have to prove that  $\tau$  is a Cauchy time function. Let  $c_0 : (a, b) \rightarrow E_0^-(\Lambda)$  be an inextendible future oriented causal curve. The image of  $\tau \circ c_0$  is an interval  $(\alpha, \beta)$ . According to item (2) of Definition 4.11,  $\alpha = 0$ . We aim to prove  $\beta = \pi/2$ , hence we assume by contradiction that  $\beta < \pi/2$ . The curve  $c$  is contained in the compact subset  $Cl(E(\Lambda))$  of  $\text{AdS}_{n+1} \cup \partial \text{AdS}_{n+1} \subset \text{Ein}_{n+1}$ , hence admits a future limit point  $c(b)$  in  $\text{AdS}_{n+1} \cup \partial \text{AdS}_{n+1}$ . If  $c(b)$  lies in  $\text{Ein}_n = \partial \text{AdS}_{n+1}$ , then it is in  $\Lambda$  (cf. Proposition 4.1). Some element of  $E(\Lambda)$  (for example,  $c(\frac{a+b}{2})$ ) would be causally related to an element of  $\Lambda$ . This contradiction shows that  $c(b)$  lies in  $\text{AdS}_{n+1}$ ; more precisely, in the boundary of  $E_0^-(\Lambda)$  in  $\text{AdS}_{n+1}$ . Since  $c$  is future oriented, it follows that  $c(b)$  has to be an element of the future boundary  $S^+(\Lambda)$ .

For every  $t$  in  $(a, b)$ , we denote by  $r(t)$  the cosmological retract  $r(c(t))$  of  $c(t)$ , and we consider the unique realizing geodesic segment  $\delta_t := (r(t), c(t))$ . We extract a subsequence  $t_n$  converging to  $b$  such that  $r(t_n)$  converges to an

element  $r_0$  of  $Cl(\mathcal{H}^-(\Lambda)) = \mathcal{H}^-(\Lambda) \cup \Lambda$ . Then,  $\delta_{t_n}$  converge to a geodesic segment  $\delta_0 = (r_0, c(b))$ . Since every  $\delta_{t_n}$  is timelike,  $\delta_0$  is non-spacelike.

For every  $t$  in  $(a, b)$  we have  $c(t) = \cos \tau(c(t))r(t) + \sin \tau(c(t))p(t)$  (where  $p(t)$  is the dual of the hyperplane orthogonal to the realizing geodesic at  $r(t)$ , see above). Hence:

$$\langle r(t_n) \mid c(t_n) \rangle = -\cos \tau(c(t_n))$$

At the limit:

$$\langle r_0 \mid c(b) \rangle = -\cos(\beta) < 0 \quad (\text{since } \beta < \pi/2)$$

It follows that  $\delta_0$  is not lightlike, but timelike. Since timelike geodesics in  $\text{AdS}_{n+1}$  remain far away from  $\partial \text{AdS}_{n+1}$ , it follows that  $r_0$  lies in  $\mathcal{H}^-(\Lambda)$ .

Finally, every  $\delta_{t_n}$  is orthogonal to a support hyperplane at  $r(t_n)$ , hence at the limit  $\delta_0$  is orthogonal to a support hyperplane, which is spacelike since  $\delta_0$  is timelike. According to Proposition 4.17,  $\delta_0$  is a realizing geodesic. At the beginning of the proof, we have shown that every realizing geodesic can be extended to a timelike geodesic of length  $\pi/2$  entirely contained in  $E_0^-(\Lambda)$ , hence there is an element  $p_0$  in  $S^+(\Lambda) \cap H^+(r_0)$  such that the geodesic  $(r_0, p_0)$  contains  $\delta_0$ , in particular  $c(b)$ . Hence  $[c(b), p_0]$  is a non-trivial timelike geodesic segment joining two elements of the achronal subset  $S^+(\Lambda)$ , contradiction.

This contradiction proves  $\beta = \pi/2$ , *ie.* that the restriction of  $\tau$  to every inextendible causal curve is surjective. In other words,  $\tau$  is a Cauchy time function. The Proposition is proved.  $\square$

**Lemma 4.19.** *The restriction of  $\tau$  to  $E_0^-(\Lambda)$  is  $C^{1,1}$  (ie. differentiable with locally Lipschitz derivative), and the realizing geodesics are orthogonal to the level sets of  $\tau$ .*

*Proof.* Let  $x$  be an element of  $E_0^-(\Lambda)$ , and let  $(r(x), x]$  be the unique realizing geodesic for  $x$ . As proven during the proof of Proposition 4.18, there is an element  $p(x)$  of  $S^+(\Lambda)$  such that  $(r(x), p(x))$  is a timelike geodesic containing  $x = \cos(\tau(x))r(x) + \sin(\tau(x))p(x)$  and entirely contained in  $E_0^-(\Lambda)$ .

Let  $U$  be the affine domain  $U(p(x))$ ; the past component  $H$  of  $U$  is a support hyperplane of  $\mathcal{H}^-(\Lambda)$  at  $r(x)$  (see Definition 2.7, Remark 2.10). Let  $\tau_0 : U \rightarrow (0, \pi)$  the cosmological time function of  $U$ : for every  $y$  in  $U$ ,  $\tau_1(y)$  is the Lorentzian distance between  $y$  and  $H$ . Let  $W$  be the future of  $r(x)$  in  $U$ , and let  $\tau_0$  be the cosmological time function in  $W$ : for every  $y$  in  $W$   $\tau_0(y)$  is the the Lorentzian length of the timelike geodesic  $[r(x), y]$ . We have:

$$\tau_0(x) = \tau(x) = \tau_1(x)$$

Moreover:

$$\forall y \in W, \quad \tau_0(y) \leq \tau(y) \leq \tau_1(y)$$

A direct computation shows that  $\tau_0$  and  $\tau_1$  have the same derivative at  $x$ : by a standard argument (see for example [CC95, Proposition 1.1]) it follows that  $\tau$  is differentiable at  $x$ , with derivative  $d_x \tau = d_x \tau_0 = d_x \tau_1$ . Furthermore,

the gradient of  $\tau_0$  and  $\tau_1$  at  $x$  is  $-\nu(x)$  where  $\nu(x)$  is the future-oriented timelike vector tangent at  $x$  to the realizing geodesic  $[x, r(x))$  of Lorentzian norm  $-1$ , *ie.* :

$$\forall v \in T_x W, \quad -\langle v \mid \nu(x) \rangle = d_x \tau_0(v)x = d_x \tau(v)$$

Therefore,  $-\nu(x)$  is also the Lorentzian gradient of  $\tau$ . It follows that realizing geodesics are orthogonal to the level sets of  $\tau$ .

In order to prove that  $\tau$  is  $C^{1,1}$ , *ie.* that  $\nu$  is locally Lipschitz, we adapt the argument used in the flat case in [Bar05]. We consider first the restriction of  $\nu$  to the level set  $S_{\pi/4} = \tau^{-1}(\pi/4)$  equipped with the induced Riemannian metric. For every  $x$  in  $S_{\pi/4}$  we have  $x = \frac{r(x) + p(x)}{\sqrt{2}}$ . Observe that  $\frac{p(x) - r(x)}{\sqrt{2}}$  is then an element of  $\mathbb{R}^{2,n}$  of norm  $-1$ , orthogonal to  $x$ , hence representing an element of  $T_x \text{AdS}_{n+1}$ . This tangent vector is future-oriented and orthogonal to  $S_{\pi/4}$ : hence  $\frac{p(x) - r(x)}{\sqrt{2}}$  represents  $\nu(x)$ .

Let  $c : (-1, 1) \rightarrow S_{\pi/4}$  be a  $C^1$  curve in  $S_{\pi/4}$ . Since  $r$  is the projection on  $\mathcal{H}^-(\Lambda)$ , and since  $\mathcal{H}^-(\Lambda)$  is locally Lipschitz, the path  $r \circ c$  is differentiable almost everywhere in  $(-1, 1)$ . We denote by  $\dot{r}$ ,  $\dot{p}$ ,  $\dot{\nu}$  the derivatives of  $r$ ,  $p$ ,  $\nu = \frac{p-r}{\sqrt{2}}$  along  $c$ . Almost everywhere, we have:

$$\begin{aligned} \mathfrak{q}_{2,n}(\dot{\nu}) &= \mathfrak{q}_{2,n}\left(\frac{\dot{p} - \dot{r}}{\sqrt{2}}\right) \\ &= \frac{1}{2}(\mathfrak{q}_{2,n}(\dot{p}) + \mathfrak{q}_{2,n}(\dot{r}) - 2\langle \dot{r} \mid \dot{p} \rangle) \end{aligned}$$

But the derivative of  $c$  is:

$$\begin{aligned} \mathfrak{q}_{2,n}(\dot{c}) &= \mathfrak{q}_{2,n}\left(\frac{\dot{r} + \dot{p}}{\sqrt{2}}\right) \\ &= \frac{1}{2}(\mathfrak{q}_{2,n}(\dot{r}) + \mathfrak{q}_{2,n}(\dot{p}) + 2\langle \dot{r} \mid \dot{p} \rangle) \end{aligned}$$

Now, since  $\mathcal{H}^-(\Lambda)$  is locally convex, the quantity  $\langle \dot{r} \mid \dot{p} \rangle$ , wherever it is defined, is nonnegative. Therefore:

$$\mathfrak{q}_{2,n}(\dot{\nu}) \leq \mathfrak{q}_{2,n}(\dot{c})$$

It follows that  $\nu$  is 1-Lipschitz along  $S_{\pi/4}$ .

On other level sets  $S_t = \tau^{-1}(t)$  with  $t \in (0, \pi/2)$ , every element is of the form  $x = \cos(t)r(x) + \sin(t)p(x)$ , and  $x_{\pi/4} = \frac{r(x) + p(x)}{\sqrt{2}}$  is a point in  $S_{\pi/4}$ . Geometrically,  $x_{\pi/4}$  is the unique point in the realizing geodesic for  $x$  at cosmological time  $\pi/4$ . The unit normal vectors  $\nu(x)$  and  $\nu(x_{\pi/4})$  are parallel one to the other along the realizing geodesic  $(r(x), p(x))$ , hence, the variation of  $\nu(x)$  along  $S_t$  is controlled by the distortion of the map  $x \rightarrow x_{\pi/4}$  and the variation of  $\nu$  along  $S_{\pi/4}$ . The lemma follows.  $\square$

**4.4. GH-regular and quasi-Fuchsian representations.** Let  $\Gamma$  be a finitely generated torsion-free group, and let  $\rho : \Gamma \rightarrow \mathrm{SO}_0(2, n)$  be a faithful, discrete representation, such that  $\rho(\Gamma)$  preserves  $\Lambda$ . According to Proposition 4.13, the quotient space  $M_\rho(\Lambda) := \rho(\Gamma) \backslash E(\Lambda)$  is globally hyperbolic. Observe that moreover Cauchy hypersurfaces of  $M_\rho(\Lambda)$  are quotients of Cauchy hypersurfaces in  $E(\Lambda)$ , which are contractible (since graphs of maps from  $\mathbb{D}^n$  into  $\mathbb{R}$ ). Hence, if  $\Gamma$  has cohomological dimension  $\geq n$ , the Cauchy hypersurfaces are compact, *ie.*  $M_\rho(\Lambda)$  is spatially compact - and the cohomological dimension of  $\Gamma$  is eventually precisely  $n$ .

Inversely, in his celebrated preprint [Mes07, ABB<sup>+</sup>07], G. Mess<sup>4</sup> proved that any globally hyperbolic spatially compact AdS spacetime embeds isometrically in such a quotient space  $\Gamma \backslash E(\Lambda)$ .

**Definition 4.20.** Let  $\Gamma$  be a torsion-free discrete group. A representation  $\rho : \Gamma \rightarrow \mathrm{SO}_0(2, n)$  is *GH-regular* if it is faithful, discrete and preserves a non-empty GH-regular domain  $E(\Lambda)$  in  $\partial \mathrm{AdS}_{n+1}$ . If moreover the  $(n - 1)$ -sphere  $\Lambda$  is acausal, then the representation is *strictly GH*.

**Definition 4.21.** A (strictly) GH-regular representation  $\rho : \Gamma \rightarrow \mathrm{SO}_0(2, n)$  is (strictly) *GHC-regular* if the quotient space  $\rho(\Gamma) \backslash E(\Lambda)$  is spatially compact.

Hence a reformulation of Mess result is:

**Proposition 4.22.** *A representation  $\rho : \Gamma \rightarrow \mathrm{SO}_0(2, n)$  is GHC-regular if and only if it is the holonomy of a GHC AdS spacetime.*  $\square$

There is an interesting special case of strictly GHC-regular representations: the case of *quasi-Fuchsian representations*.

**Definition 4.23.** A strictly GHC-regular representation  $\rho : \Gamma \rightarrow \mathrm{SO}_0(2, n)$  is *quasi-Fuchsian* if  $\Gamma$  is isomorphic to a uniform lattice in  $\mathrm{SO}_0(1, n)$ .

This terminology is motivated by the analogy with the hyperbolic case.

There is a particular case: the case where  $\Lambda$  is a “round sphere” in  $\partial \mathrm{AdS}_{n+1}$ , *ie.* the boundary of a totally geodesic spacelike hypersurface  $\mathbb{S}(v^\perp) \cap \mathrm{AdS}_{n+1}$ :

**Definition 4.24.** A *Fuchsian* representation  $\rho : \Gamma \rightarrow \mathrm{SO}_0(2, n)$  is the composition of the natural inclusions  $\Gamma \subset \mathrm{SO}_0(1, n)$  and  $\mathrm{SO}_0(1, n) \subset \mathrm{SO}_0(2, n)$ , where in the latter  $\mathrm{SO}_0(1, n)$  is considered as the stabilizer in  $\mathrm{SO}_0(2, n)$  of a point in  $\mathrm{AdS}_{n+1}$ .

In other words, a quasi-Fuchsian representation is Fuchsian if and only if it admits a global fixed point in  $\mathrm{AdS}_{n+1}$ .

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<sup>4</sup>Mess only deals with the case where  $n = 2$ , but his arguments also apply in higher dimension. For a detailed proof see [Bar08, Corollary 11.2]

**4.5. The space of timelike geodesics.** Timelike geodesics in  $\text{AdS}_{n+1}$  are intersections between  $\text{AdS}_{n+1} \subset \mathbb{R}^{2,n}$  and 2-planes  $P$  in  $\mathbb{R}^{2,n}$  such that the restriction of  $q_{2,n}$  to  $P$  is negative definite. The action of  $\text{SO}_0(2, n)$  on negative 2-planes is transitive, and the stabilizer of the  $(u, v)$ -plane is  $\text{SO}(2) \times \text{SO}(n)$ . Therefore, the space of timelike geodesics is the symmetric space:

$$\mathcal{T}_{2n} := \text{SO}_0(2, n) / \text{SO}(2) \times \text{SO}(n)$$

$\mathcal{T}_{2n}$  has dimension  $2n$ . We equip it by the Riemannian metric  $g_{\mathcal{T}}$  induced by the Killing form of  $\text{SO}_0(2, n)$ . It is well known that  $\mathcal{T}_{2n}$  has nonpositive curvature, and rank 2: the maximal flats (*ie.* totally geodesic embedded Euclidean subspaces) have dimension 2. It is also naturally hermitian. More precisely: let  $\mathcal{G} = \mathfrak{so}(2, n)$  be the Lie algebra of  $G = \text{SO}_0(2, n)$ , and let  $\mathcal{K}$  be the Lie algebra of the maximal compact subgroup  $K := \text{SO}(2) \times \text{SO}(n)$ . We have the Cartan decomposition:

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{K}^{\perp}$$

where  $\mathcal{K}^{\perp}$  is the orthogonal of  $\mathcal{K}$  for the Killing form. Then,  $\mathcal{K}^{\perp}$  is naturally identified with the tangent space at the origin of  $G/K$ . The adjoint action of the  $\text{SO}(2)$  term in the stabilizer defines a  $K$ -invariant complex structure on  $\mathcal{K}^{\perp} \approx T_K(G/K)$  that propagates through left translations to an integrable complex structure  $J$  on  $\mathcal{T}_{2n} = G/K$ . Therefore,  $\mathcal{T}_{2n}$  is naturally equipped with a structure of  $n$ -dimensional complex manifold, together with a  $J$ -invariant Riemannian metric, *ie.* an hermitian structure.

Let us consider once more the achronal  $(n - 1)$ -dimensional topological sphere  $\Lambda$ . Then, it is easy to prove that every timelike geodesic in  $\text{AdS}_{n+1}$  intersects  $E(\Lambda)$  (cf. Lemma 3.5 in [BM12]), and since  $E(\Lambda)$  is convex, this intersection is connected, *ie.* is a single inextendible timelike geodesic of  $E(\Lambda)$ . In other words, one can consider  $\mathcal{T}_{2n}$  as the space of timelike geodesics of  $E(\Lambda)$ .

Let  $\rho : \Gamma \rightarrow \text{SO}_0(2, n)$  be a GH-regular representation preserving  $\Lambda$ . The (isometric) action of  $\rho(\Gamma)$  on  $\mathcal{T}_{2n}$  is free and proper, and the quotient  $\mathcal{T}_{2n}(\rho) := \rho(\Gamma) \backslash \mathcal{T}_{2n}$  is naturally identified with the space of inextendible timelike geodesics of  $M_{\rho}(\Lambda) = \rho(\Gamma) \backslash E(\Lambda)$ .

**Definition 4.25.** Let  $S$  be a differentiable Cauchy hypersurface in a GH-regular spacetime  $M_{\rho}(\Lambda)$  of dimension  $n + 1$ . The *Gauss map* of  $S$  is the map  $\nu : S \rightarrow \mathcal{T}_{2n}(\rho)$  that maps every element  $x$  of  $M_{\rho}(\Lambda)$  to the unique timelike geodesic of  $M_{\rho}(\Lambda)$  orthogonal to  $S$  at  $x$ .

When  $S$  is  $C^{1,1}$  (for example, a level set  $\tau^{-1}(t)$  of the cosmological time for  $t < \pi/2$ ), then one can define for every  $C^1$  curve  $c$  in  $S$  the *Gauss length* as the length in  $\mathcal{T}_{2n}(\rho)$  of the Lipschitz curve  $\nu \circ c$ . It defines on  $S$  a length metric, called the *Gauss metric* (of course, if  $S$  is  $C^r$  with  $r \geq 2$ , then  $\nu$  is  $C^{r-1}$ , and the Gauss metric is a  $C^{r-1}$  Riemannian metric).

Since every timelike geodesic intersects  $S$  at most once, the Gauss map is always injective. The image of the Gauss map is actually the set of timelike

geodesics that are orthogonal to  $S$ . Since every timelike geodesic intersects  $S$ , it follows easily that the image of the Gauss map is closed, and that the Gauss map is actually an embedding.

**Remark 4.26.** For every  $t < \pi/2$ , let  $\Sigma_t(\tau)$  be the image by the Gauss map of the cosmological level set  $\tau^{-1}(t)$ . According to Lemma 4.19,  $\Sigma_t(\tau)$  is the space of realizing geodesics. In particular, it does not depend on  $t$ . We will denote by  $\Sigma(\tau)$  this closed embedded submanifold, and call it the *space of cosmological geodesics*.

**4.6. Split AdS spacetimes.** Let  $(p, q)$  be a pair of positive integers such that  $p + q = n$ . Let  $(x_0, x_1, \dots, x_p, y_0, y_1, \dots, y_q)$  be a coordinate system such that the quadratic form is:

$$-x_0^2 + x_1^2 + \dots + x_p^2 - y_0^2 + y_1^2 + \dots + y_q^2$$

Let  $G_{p,q} \approx \text{SO}_0(1, p) \times \text{SO}_0(1, q)$  be the subgroup of  $\text{SO}_0(2, n)$  preserving the splitting  $\mathbb{R}^{2,n} = \mathbb{R}^{1,p} \oplus \mathbb{R}^{1,q}$  where  $\mathbb{R}^{1,q}$  is the subspace  $\{x_0 = x_1 = \dots = x_p = 0\}$  and  $\mathbb{R}^{1,p}$  the subspace  $\{y_0 = y_1 = \dots = y_q = 0\}$ .

Let  $\Lambda_p$  (respectively  $\Lambda_q$ ) be the subset  $\mathbb{S}(\mathcal{C}_p^+)$  (respectively  $\mathbb{S}(\mathcal{C}_q^+)$ ) of (the Klein model of)  $\text{Ein}_n$  where:

$$\mathcal{C}_p^+ := \{-x_0^2 + x_1^2 + \dots + x_p^2 = 0, x_0 > 0, y_0 = y_1 = \dots = y_q = 0\}$$

and

$$\mathcal{C}_q^+ := \{-y_0^2 + y_1^2 + \dots + y_q^2 = 0, y_0 > 0, x_0 = x_1 = \dots = x_p = 0\}$$

Observe that  $\Lambda_p, \Lambda_q$  are topological spheres of dimension respectively  $p - 1, q - 1$ . Moreover, for every pair of elements  $x, y$  in  $\Lambda_p \cup \Lambda_q$  the scalar product  $\langle x | y \rangle$  is nonpositive. Hence, according to Corollary 2.12,  $\Lambda_p \cup \Lambda_q$  is achronal. Moreover, every point in  $\Lambda_p$  is linked to every point in  $\Lambda_q$  by a unique lightlike geodesic segment contained in  $\text{Ein}_n$ .

**Lemma 4.27.** *The invisible domain  $E(\Lambda_p \cup \Lambda_q)$  is the interior of the convex hull of  $\Lambda_p \cup \Lambda_q$ .*

*Proof.* Clearly:

$$\text{Conv}(\mathcal{C}_p^+) = \{-x_0^2 + x_1^2 + \dots + x_p^2 \leq 0, x_0 > 0, y_0 = y_1 = \dots = y_q = 0\}$$

Similarly:

$$\text{Conv}(\mathcal{C}_q^+) = \{-y_0^2 + y_1^2 + \dots + y_q^2 \leq 0, y_0 > 0, x_0 = x_1 = \dots = x_p = 0\}$$

Therefore,  $\text{Conv}(\Lambda_p \cup \Lambda_q)$  is the projection by  $\mathbb{S}$  of the set of points  $(x_0, x_1, \dots, x_p, y_0, y_1, \dots, y_q)$  satisfying the following inequalities:

$$\begin{aligned} -x_0^2 + x_1^2 + \dots + x_p^2 &\leq 0 \\ -y_0^2 + y_1^2 + \dots + y_q^2 &\leq 0 \\ x_0 &\geq 0 \\ y_0 &\geq 0 \end{aligned}$$

According to Remark 3.17  $\text{Conv}(\Lambda_p \cup \Lambda_q)^\circ$  is contained in  $E(\Lambda_p \cup \Lambda_q)$ . Inversely, let  $z = (x_0, x_1, \dots, x_p, y_0, y_1, \dots, y_q)$  be an element of  $\mathbb{R}^{2,n}$  representing an element  $z_0$  of  $E(\Lambda_p \cup \Lambda_q)$ . Then, by definition of  $E(\Lambda_p \cup \Lambda_q)$ , the scalar product  $\langle z | x \rangle$  is negative for every element  $x$  of  $\mathcal{C}_p^+$ . It follows that  $(x_0, x_1, \dots, x_p)$  must lie in the future cone of  $\mathbb{R}^{1,p}$ , *ie.* :

$$\begin{aligned} -x_0^2 + x_1^2 + \dots + x_p^2 &< 0 \\ x_0 &> 0 \end{aligned}$$

Similarly, since  $\langle z | x \rangle < 0$  for every element  $x$  of  $\mathcal{C}_q^+$ :

$$\begin{aligned} -y_0^2 + y_1^2 + \dots + y_q^2 &< 0 \\ y_0 &> 0 \end{aligned}$$

The lemma follows.  $\square$

Let  $\Lambda_{p,q}$  be the intersection in  $\mathbb{S}(\mathbb{R}^{2,n})$  between  $\text{Conv}(\Lambda_p \cup \Lambda_q)$  and  $\mathbb{S}(\mathcal{C}_n) \approx \text{Ein}_n$ . Let  $(x_0, x_1, \dots, x_p, y_0, y_1, \dots, y_q)$  be an element of  $\mathbb{R}^{2,n}$  representing an element of  $\Lambda_{p,q}$ . According to the proof of Lemma 4.27 we must have  $-x_0^2 + x_1^2 + \dots + x_p^2 \leq 0$  and  $-y_0^2 + y_1^2 + \dots + y_q^2 \leq 0$ , and since  $(x_0, x_1, \dots, x_p, y_0, y_1, \dots, y_q)$  lies in  $\mathcal{C}_n$ , these quantities must vanish. Hence, the inequalities defining  $\Lambda_{p,q}$  are:

$$\begin{aligned} -x_0^2 + x_1^2 + \dots + x_p^2 &= 0 \\ -y_0^2 + y_1^2 + \dots + y_q^2 &= 0 \\ x_0 &\geq 0 \\ y_0 &\geq 0 \end{aligned}$$

Therefore,  $\Lambda_{p,q}$  is the union of  $\Lambda_p$ ,  $\Lambda_q$ , and every lightlike segment joining a point of  $\Lambda_p$  to a point of  $\Lambda_q$ : it is achronal, but not acausal! Topologically,  $\Lambda_{p,q}$  is the join of two spheres, therefore, its a sphere of dimension  $1 + (p - 1) + (q - 1) = n - 1$ . It is not an easy task to figure out how it fits inside  $\text{Ein}_n = \partial \text{AdS}_{n+1}$ .

For that purpose, we consider the coordinates  $(r, \theta, a_1, \dots, a_p, b_1, \dots, b_q)$  on  $\text{AdS}_{n+1}$  such that  $x_0 = r \cos \theta$ ,  $y_0 = r \sin \theta$ ,  $x_i = r a_i$ ,  $y_i = r b_i$ . According to Proposition 2.4, the  $n + 1$ -uple  $(a_1, \dots, a_p, b_1, \dots, b_q, 1/r)$  describes the upper hemisphere  $\mathbb{D}^n = \{a_1^2 + \dots + a_p^2 + b_1^2 + \dots + b_q^2 + 1/r^2 = 1\}$  in the Euclidean sphere of  $\mathbb{R}^{n+1}$  of radius 1, and  $\text{AdS}_{n+1}$  is conformally isometric to the product  $\mathbb{S}^1 \times \mathbb{D}^n$  with the metric  $-d\theta^2 + ds^2$ , where  $ds^2$  is the round metric on  $\mathbb{D}^n$ .

In these coordinates, the inequalities defining  $E(\Lambda_p \cup \Lambda_q)$  established in the proof of Lemma 4.27 become:

$$\begin{aligned} (2) \quad & 0 < \theta < \pi/2 \\ (3) \quad & a_1^2 + \dots + a_p^2 < \cos^2 \theta \\ (4) \quad & b_1^2 + \dots + b_q^2 < \sin^2 \theta \end{aligned}$$

Let  $\mathbb{D}_0^q$  be the subdisk of  $\mathbb{D}^n$  defined by  $a_1 = \dots = a_p = 0$ , and let  $\mathbb{D}_0^p$  be the subdisk defined by  $b_1 = \dots = b_q = 0$ . For every  $x$  in  $\mathbb{D}^n$ , let  $d_p(x)$  be the distance of  $x$  to  $\mathbb{D}_0^q$ , and define similarly the "distance to  $\mathbb{D}_0^p$ " function  $d_q : \mathbb{D}^n \rightarrow [0, +\infty)$ . Observe that since  $\mathbb{D}_0^p$  and  $\mathbb{D}_0^q$  both contain the North pole  $(0, \dots, 0, 1)$  of  $\mathbb{D}^n$ , and since every point in  $\mathbb{D}^n$  is at distance at most  $\pi/2$  of the North pole,  $d_p$  and  $d_q$  takes value in  $[0, \pi/2)$ . Now, observe that the following identities hold:

$$(5) \quad a_1^2 + \dots + a_p^2 = \sin^2 d_p(a_1, \dots, a_p, b_1, \dots, b_q, 1/r)$$

$$(6) \quad b_1^2 + \dots + b_q^2 = \sin^2 d_q(a_1, \dots, a_p, b_1, \dots, b_q, 1/r)$$

It follows that  $E(\Lambda_p \cup \Lambda_q)$  is the domain in  $\text{AdS}_{n+1} \approx \mathbb{S}^1 \times \mathbb{D}^n$  comprising points  $(\theta, x)$  such that:

$$d_q(x) < \theta < \pi/2 - d_p(x)$$

In the terminology of Definition 3.3, it means that the lifting  $\widetilde{E}(\widetilde{\Lambda_p \cup \Lambda_q})$  is defined by the functions  $f^- = d_q$  and  $f^+ = \pi/2 - d_p$ . These functions extend uniquely as 1-Lipschitz maps  $f^\pm : \overline{\mathbb{D}^n} \rightarrow [0, \pi/2]$ .

The boundary  $\partial\mathbb{D}^n = \mathbb{S}^{n-1}$  is totally geodesic in  $\overline{\mathbb{D}^n}$ , and  $\partial\mathbb{D}_0^q$ ,  $\partial\mathbb{D}_0^p$  are totally geodesic spheres of dimensions  $p$ ,  $q$ , respectively. Let  $\delta_p : \partial\mathbb{D}^n \rightarrow [0, \pi/2]$  (respectively  $\delta_q : \partial\mathbb{D}^n \rightarrow [0, \pi/2]$ ) be the function "distance to  $\partial\mathbb{D}_0^q$ " (respectively "distance to  $\partial\mathbb{D}_0^p$ "). It follows from equations<sup>5</sup> (4), (5) that every point of  $\partial\mathbb{D}_0^q$  is at distance  $\pi/2$  of  $\partial\mathbb{D}_0^p$ . Hence:

$$\delta_p + \delta_q = \pi/2$$

In other words, the restrictions of  $f^-$  and  $f^+$  to  $\partial\mathbb{D}^n$  coincide and are equal to  $\delta_q = \pi/2 - \delta_p$ . The restriction of  $f^- = f^+$  to  $\partial\mathbb{D}_0^p$  vanishes, and the graph of this restriction is  $\Lambda_p$ . The restriction of  $f^- = f^+$  to  $\partial\mathbb{D}_0^q$  is the constant map of value  $\pi/2$ , and the graph is  $\Lambda_q$ . The graph of  $f^\pm : \partial\mathbb{D}^n \rightarrow \mathbb{S}^1$  is  $\Lambda_{p,q}$ , which is therefore an achronal sphere in  $\text{Ein}_n$ .

Clearly,  $\Lambda_{p,q}$  is preserved by  $G_{p,q}$ . Let  $\Gamma$  be a cocompact lattice of  $G_{p,q} \approx \text{SO}_0(1, p) \times \text{SO}_0(1, q)$ . The inclusion  $\Gamma \subset G_{p,q} \subset \text{SO}_0(2, n)$  is a GH-regular representation, but non-strictly since the invariant achronal limit set  $\Lambda_{p,q}$  is not acausal. According to Proposition 4.13, the quotient space  $M_{p,q}(\Gamma) := \Gamma \backslash E(\Lambda_{p,q})$  is a GH spacetime. Actually, the Cauchy surfaces of  $M_{p,q}(\Gamma)$  are quotients by  $\Gamma$  of the graph of a 1-Lipschitz map  $f : \mathbb{D}^n \rightarrow \mathbb{S}^1$ , hence they are  $K(\Gamma, 1)$  (since  $\mathbb{D}^n$  is contractible). On the other hand, the quotient of  $\mathbb{H}^p \times \mathbb{H}^q$  is a  $K(\Gamma, 1)$  too. Since  $\Gamma$  is a cocompact lattice, it follows that every  $K(\Gamma, 1)$  - in particular, the Cauchy hypersurfaces in  $M_{p,q}(\Gamma)$  - are compact. The inclusion  $\Gamma \subset \text{SO}_0(2, n)$  is therefore GHC-regular.

**Definition 4.28.** The quotient space  $M_{p,q}(\Gamma)$  is a *split AdS spacetime*. The representation  $\rho : \Gamma \rightarrow \text{SO}_0(2, n)$  is a *split GHC-regular representation of type  $(p, q)$* .

<sup>5</sup>These equations naturally extend to the boundary  $\partial\mathbb{D}^n$ .

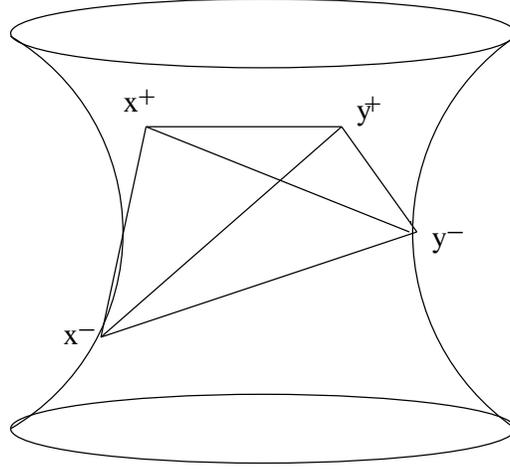


FIGURE 2. Picture of a crown. The hyperboloid represent the boundary of an affine domain of  $\text{AdS}_{n+1}$  containing the realm of the crown.

**Remark 4.29.** The split AdS spacetimes of dimension  $2 + 1$  are precisely the *Torus universes* studied in [Car03]. Observe indeed that the lattice in  $\text{SO}_0(1, 1) \times \text{SO}_0(1, 1) \approx \mathbb{R}^2$  is isomorphic to  $\mathbb{Z}^2$ , and the Cauchy surfaces are indeed tori.

**4.7. Crowns.** A particular case of split AdS spacetime is the case  $p = q = 1$  (and therefore,  $n = 2$ ). Then, the topological spheres  $\Lambda_p$  and  $\Lambda_q$  have dimension 0, *ie.*, are pair of points  $\Lambda_p = \{x^-, y^-\}$  and  $\Lambda_q = \{x^+, y^+\}$ . The topological circle  $\Lambda_{p,q}$  is then piecewise linear; more precisely, it is the union of the four lightlike segments  $[x^-, x^+]$ ,  $[x^+, y^-]$ ,  $[y^-, y^+]$ ,  $[y^+, x^-]$ . The invisible domain  $E(\Lambda_{p,q})$  is then an ideal tetrahedron, interior of the convex hull of the four ideal points  $\{x^-, y^-, x^+, y^+\}$ . This tetrahedron has six edges; four of them as the lightlike segments forming  $\Lambda_{p,q}$ , and the two others are the spacelike geodesics  $(x^-, y^-)$  and  $(x^+, y^+)$  of  $\text{AdS}_{n+1}$  (see Figure 2). Observe that  $[x^-, x^+]$  and  $[y^-, y^+]$  are future oriented, whereas  $[x^+, y^-]$  and  $[y^+, x^-]$  are past oriented.

More generally:

**Definition 4.30.** For every integer  $n \geq 2$ , a *crown* of  $\text{Ein}_n$  is 4-uple  $\mathfrak{C} = (x^-, y^-, x^+, y^+)$  in  $\text{Ein}_n$  such that:

- $\langle x^- | x^+ \rangle = \langle x^- | y^+ \rangle = 0$
- $\langle y^- | x^+ \rangle = \langle y^- | y^+ \rangle = 0$
- $\langle x^- | y^- \rangle < 0$
- $\langle x^+ | y^+ \rangle < 0$
- the lightlike segment  $[x^-, x^+]$  is future oriented.

The subset  $\{x^-, y^-, x^+, y^+\}$  is then an achronal subset of  $\text{Ein}_n$ . The invisible domain  $E(\{x^-, y^-, x^+, y^+\})$  is called the *realm of the crown*, and denoted by  $E(\mathfrak{C})$ . The convex hull of  $\{x^-, y^-, x^+, y^+\}$  is denoted by  $\text{Conv}(\mathfrak{C})$ .

**Remark 4.31.** Let  $\mathfrak{C} = (x^-, y^-, x^+, y^+)$  be a crown in  $\text{Ein}_n$ , and let  $x^-, y^-, x^+, y^+$  be elements of  $\mathbb{R}^{2,n}$  representing the vertices of the crown. Let  $V(\mathfrak{C})$  be the linear space spanned by  $x^+, x^-, y^-, y^+$ . The restriction of  $q_{2,n}$  to  $V(\mathfrak{C})$  has signature  $(2, 2)$ , and  $\mathbb{S}(V(\mathfrak{C}))$  is the unique totally geodesic copy of  $\text{Ein}_2$  in  $\text{Ein}_n$  containing  $\mathfrak{C}$ .

**Remark 4.32.** Let  $Z$  be the stabilizer in  $\text{SO}_0(2, n)$  of a crown. It preserves the orthogonal sum  $V(\mathfrak{C}) \oplus V(\mathfrak{C})^\perp$ . It is isomorphic to the product  $A \times \text{SO}(n-2)$ , where  $A$  is a maximal  $\mathbb{R}$ -split abelian subgroup of  $\text{SO}_0(2, 2)$ , hence of  $\text{SO}_0(2, n)$ . Therefore,  $Z$  is the centralizer of  $A$ , and it has finite index in the normalizer  $N$  of  $A$ . It follows that the space of crowns is naturally a finite covering over the space  $G/N$  of maximal flats in the symmetric space  $\mathcal{T}_{2n} = \text{SO}_0(2, n)/\text{SO}(2) \times \text{SO}(n)$ .

**Remark 4.33.** Up to an isometry, one can assume that the crown  $\mathfrak{C} = (x^-, y^-, x^+, y^+)$  is represented by:

$$\begin{aligned} x^+ &= (1, 0, 1, 0, 0, \dots, 0) \\ y^+ &= (1, 0, -1, 0, 0, \dots, 0) \\ x^- &= (0, 1, 0, 1, 0, \dots, 0) \\ y^- &= (0, 1, 0, -1, 0, \dots, 0) \end{aligned}$$

According to Proposition 3.9, the realm  $E(\mathfrak{C})$  is defined by the inequalities:

$$\begin{aligned} x_1 - u &< 0 \\ -x_1 - u &< 0 \\ x_2 - v &< 0 \\ -x_2 - v &< 0 \\ -u^2 - v^2 + x_1^2 + \dots + x_n^2 &< 0 \end{aligned}$$

Hence, by:

$$|x_1| < u, \quad |x_2| < v, \quad -u^2 - v^2 + x_1^2 + \dots + x_n^2 < 0$$

Observe that the last inequality is implied by the two previous inequations when  $n = 2$ .

If  $n = 2$ , the realm of a crown  $\mathfrak{C}$  coincide with the interior of  $\text{Conv}(\mathfrak{C})$  (Lemma 4.27), but this is obviously not true for  $n > 2$  since  $\text{Conv}(\mathfrak{C})$  is always 3-dimensional.

## 5. ACAUSALITY OF LIMIT SETS OF GROMOV HYPERBOLIC GROUPS

In all this section,  $\Gamma$  is a torsion-free Gromov hyperbolic group, and  $\rho : \Gamma \rightarrow \mathrm{SO}_0(2, n)$  a GHC-regular representation, with limit set  $\Lambda$ . By hypothesis,  $E(\Lambda)$  is not empty, therefore  $\Lambda$  is not purely lightlike.

### 5.1. Non-existence of crowns.

**Proposition 5.1.** *The limit set  $\Lambda$  contains no crown.*

*Proof.* Let  $\mathfrak{C} = (x^-, y^-, x^+, y^+)$  be a crown contained in  $\Lambda$ . Let  $F(\mathfrak{C})$  be the subset of  $\mathcal{T}_{2n}$  comprising timelike geodesics containing a segment  $[p^-, p^+]$  with  $p^\pm \in (x^\pm, y^\pm)$ . Let  $A$  be the maximal  $\mathbb{R}$ -split abelian subgroup stabilizing  $\mathfrak{C}$ , *ie.* the subgroup of the stabilizer  $Z$  of  $\mathfrak{C}$  acting trivially on  $V(\mathfrak{C})^\perp$  (cf. Remark 4.32). Then,  $F(\mathfrak{C})$  is an orbit of the action of  $A$  in  $\mathcal{T}_{2n}$ . Therefore,  $F(\mathfrak{C})$  is a flat in the symmetric space  $\mathcal{T}_{2n}$ .

Let  $\Sigma(\tau)$  be the space of cosmological geodesics in  $E_0^-(\Lambda)$  (cf. Remark 4.26).

*Claim:*  $\Sigma(\tau)$  contains  $F(\mathfrak{C})$ .

Let  $p^+, p^-$  be elements of  $(x^+, y^+)$ ,  $(x^-, y^-)$ . The closure of  $E(\Lambda)$  contains  $\mathrm{Conv}(\Lambda)$ , in particular, it contains  $p^\pm$ . On the other hand,  $\langle x^+ | p^- \rangle = 0$ , hence  $p^+$  does not lie in  $E(\Lambda)$ . Therefore,  $p^-$  is an element of  $\mathcal{H}^-(\Lambda)$ .

Observe that  $\langle p^- | p^+ \rangle = 0$ . Hence,  $p^-$  lies in the hyperplane  $H^-(p^+)$  past-dual to  $p^+$ . Now, since  $p^+$  lies in  $\mathrm{Conv}(\Lambda)$ , we have  $\langle p^+ | y \rangle \leq 0$  for every  $y$  in  $E(\Lambda)$ . Therefore,  $H^-(p^+)$  is a support hyperplane of  $\mathcal{H}^-(\Lambda)$  at  $p^-$ , orthogonal to the timelike geodesic  $[p^-, p^+]$ . According to Proposition 4.17,  $(p^-, p^+)$  is a realizing geodesic, hence an element of  $\Sigma(\tau)$ . The claim follows.

Consider now the Gauss metric on  $\Sigma(\tau)$  (cf. Definition 4.25). According to the claim,  $\Sigma(\tau)$  contains the Euclidean plane  $F(\mathfrak{C})$ . Since  $F(\mathfrak{C})$  is totally geodesic in  $\mathcal{T}_{2n}$ , it is also totally geodesic in  $\Sigma(\tau)$ .

On the other hand, the group  $\Gamma$  acts on  $\Sigma(\tau)$ , and the quotient of this action is compact, since this quotient is the image by the Gauss map of compact surface in  $M_\rho(\Lambda)$ . Hence,  $\Sigma(\tau)$  is quasi-isometric to  $\Gamma$ , and therefore, Gromov hyperbolic. It is a contradiction since a Gromov hyperbolic metric space cannot contain a 2-dimensional flat.  $\square$

**5.2. Compactness of the convex core.** In Sect. 3.3, we have seen that, up to a lifting in  $\widehat{\mathrm{AdS}}_{n+1}$ , the convex core  $\mathrm{Conv}(\Lambda)$  (respectively the invisible domain  $E(\Lambda)$ ) can be defined as the region between the graphs of functions  $F^\pm : \mathbb{D}^n \rightarrow \mathbb{R}$  (respectively  $f^\pm : \mathbb{D}^n \rightarrow \mathbb{R}$ ) such that (cf. Proposition 4.3):

$$(7) \quad f^- \leq F^- \leq F^+ \leq f^+$$

where the inequality  $F^- \leq F^+$  is strict as soon as  $\rho : \Gamma \rightarrow \mathrm{SO}_0(2, n)$  is not Fuchsian.

**Proposition 5.2.** *The left and right inequalities in (6) are strict, *ie.* for every  $x$  in  $\mathbb{D}^n$ , we have:*

$$f^-(x) < F^-(x) \leq F^+(x) < f^+(x)$$

*Proof.* Assume by contradiction that  $f^\pm(x) = F^\pm(x)$  for some  $x$  in  $\mathbb{D}^n$ . It means that some element  $x$  of  $\text{Conv}(\Lambda) \cap \text{AdS}_{n+1}$  is on the boundary of  $E(\Lambda)$ . This element is a linear combination  $x = t_1x_1 + \dots + t_kx_k$  where  $k \geq 2$ ,  $t_i$  are positive real numbers and  $x_i$  elements of  $\mathcal{C}_n \subset \mathbb{R}^{2,n}$  such that the projections  $\mathbb{S}(x_i)$  belong to  $\Lambda$ . Moreover, since  $x$  lies in  $\text{AdS}_n$ , we have  $\langle x_a | x_b \rangle < 0$  for some integers  $a, b$ . Since  $x$  lies in the boundary of  $E(\Lambda)$ , there is an element  $x_0$  of  $\Lambda$  such that:

$$\begin{aligned} 0 &= \langle x_0 | x \rangle \\ &= t_1 \langle x_0 | x_1 \rangle + \dots + t_k \langle x_0 | x_k \rangle \end{aligned}$$

Since  $\Lambda$  is achronal, each  $\langle x_0 | x_i \rangle$  is nonpositive, therefore vanishes. In particular:

- $\langle x_0 | x_a \rangle = \langle x_0 | x_b \rangle = 0$
- $\langle x_a | x_b \rangle < 0$

Reverting the time orientation if necessary, we can assume that  $x$  lies in the past horizon  $\mathcal{H}^-(\Lambda)$ . Moreover, we can assume without loss of generality that  $x$  is actually equal to  $x_a + x_b$ , after rescaling if necessary  $x_a, x_b$  so that  $x_a + x_b$  has norm  $-1$ , *ie.* lies in  $\text{AdS}_{n+1}$ .

Consider now any element  $y_0$  of  $E_0^-(\Lambda)$  in the future of  $x$ , *ie.* such that  $(x, y_0)$  is a future oriented timelike segment. More precisely, we can select  $y_0$  such that the timelike segment  $[x, y_0]$  is orthogonal to the segment  $[x_a, x_b]$ . Let  $t_0$  be the cosmological time at  $y_0$ , let  $S_0$  be the cosmological level set  $\tau^{-1}(t_0)$ , and let  $d_0$  the induced metric on  $S_0$ : this metric is complete since  $S_0$  admits a compact quotient.

Let  $P$  be the 3-subspace of  $\mathbb{R}^{2,n}$  spanned by  $y_0, x$  and  $x_0$ : by construction,  $P$  is orthogonal to  $x_a$  and  $x_b$ . Then,  $A := \mathbb{S}(P) \cap \text{AdS}_{n+1}$  is a totally geodesic copy of  $\text{AdS}_2$ . The restriction of  $\tau$  to  $A \cap E_0^-(\Lambda)$  is still a Cauchy time function, and  $S_0 \cap A$  is a spacelike path which contains  $\mathbb{S}(y_0)$ . Moreover, there is a sequence  $y_n$  in  $S_0 \cap A$  converging to  $\mathbb{S}(x_0)$ .

Let  $K_0 \subset S_0$  be a compact fundamental domain for the action of  $\rho(\Gamma)$  on  $S_0$ . There is a sequence  $g_n = \rho(\gamma_n)$  in  $\rho(\Gamma)$  such  $z_n = g_n y_n$  converge to  $\bar{z}$  in  $K_0$ . We define:

$$\begin{aligned} a_n &= g_n x_a \\ b_n &= g_n x_b \\ q_n &= g_n x_0 \\ x_n &= g_n x = a_n + b_n \end{aligned}$$

Up to a subsequence, we can assume that  $\mathbb{S}(a_n), \mathbb{S}(b_n), \mathbb{S}(q_n)$  converge to elements  $\bar{a}, \bar{b}, \bar{q}$  of  $\Lambda$ , and that  $\mathbb{S}(x_n)$  converge to an element  $\bar{x}$  of the segment  $[\bar{a}, \bar{b}]$ . At this level, it could happen that this segment is reduced to one point, *ie.*  $\bar{a} = \bar{b}$ ; but we will prove that it is not the case.

*Claim:*  $\bar{x}$  lies in  $\text{AdS}_{n+1}$ .

Indeed, since every  $x_n$  belongs to  $\mathcal{H}^-(\Lambda)$ , if the limit  $\bar{x}$  does not lie in  $\text{AdS}_{n+1}$ , then it is an element of  $\Lambda$ . The segment  $[\bar{x}, \bar{z}]$ , limit of the timelike

segments  $[x_n, z_n]$ , would be causal, and the element  $\bar{z}$  of  $K_0 \subset E(\Lambda)$  would be causally related to the element  $\bar{x}$  of  $\Lambda$ : contradiction.

Therefore,  $\bar{x}$  lies in  $\mathcal{H}^-(\Lambda)$ . It follows in particular that  $\bar{a} \neq \bar{b}$ . Consider now the iterates  $p_n := g_n y_0$  of  $y_0$ . They belong to  $S_0$ . Up to a subsequence, we can assume that the sequence  $(p_n)_{n \in \mathbb{N}}$  admits a limit  $\bar{p}$ . Since  $d_0$  is complete and the  $y_n$  converge to a point in  $\partial \text{AdS}_{n+1}$ , the distance  $d_0(y_n, y_0)$  converge to  $+\infty$ . Therefore,  $d_0(z_n, p_n) = d_0(g_n y_n, g_n y_0) = d_0(y_n, y_0)$  is unbounded: the limit  $\bar{p}$  is at infinity, *ie.* an element of  $\Lambda$ .

The four points  $\bar{q}$ ,  $\bar{a}$ ,  $\bar{b}$ ,  $\bar{p}$  in  $\text{Ein}_n$  satisfy:

- $\langle \bar{q} | \bar{a} \rangle = \langle \bar{q} | \bar{b} \rangle = 0$  (since  $\langle x_0 | x_a \rangle = \langle x_0 | x_b \rangle = 0$ ),
- $\langle \bar{a} | \bar{b} \rangle < 0$  (since  $(\bar{a}, \bar{b})$  contains the element  $\bar{x}$  of  $\text{AdS}_{n+1}$ )
- $\langle \bar{p} | \bar{a} \rangle = \langle \bar{p} | \bar{b} \rangle = 0$  (since every  $p_n$  lies in  $a_n^\perp \cap b_n^\perp$ ).

Now observe that in every iterate  $A_n = g_n A_0$ , the timelike geodesic  $\Delta_0$  containing  $[x_n, z_n]$  disconnects  $A_n$ , and that the ideal points  $q_n, p_n$  lie on (the boundary of) different components of  $A \setminus \Delta_0$ .

It follows that  $\bar{p} \neq \bar{q}$ . Observe that  $\bar{q}, \bar{p}$  lies in the isotropic cone of  $\bar{a}^\perp \cap \bar{b}^\perp$ , which has signature  $(1, n-1)$ . Moreover, every  $p_n, q_n$  lies in the future of  $x_n$ : it follows that  $\bar{q}, \bar{p}$  lies in the same connected component of the isotropic cone of  $\bar{a}^\perp \cap \bar{b}^\perp$  (with the origin removed); therefore:

$$\langle \bar{p} | \bar{q} \rangle < 0$$

It follows that  $(\bar{a}, \bar{b}, \bar{p}, \bar{q})$  is a crown. It contradicts Proposition 5.1.  $\square$

**5.3. Proof of Theorem 1.4.** In this section, we prove Theorem 1.4:

**Theorem 5.3.** *Let  $\rho : \Gamma \rightarrow \text{SO}_0(2, n)$  be a GHC-regular representation, where  $\Gamma$  is a Gromov hyperbolic group. Then the achronal limit set  $\Lambda$  is acausal, *ie.*  $\rho$  is  $(\text{SO}_0(2, n), \text{Ein}_n)$ -Anosov.*

*Proof.* We equip the convex domain  $E(\Lambda)$  with its *Hilbert metric*: for every element  $x, y$  in  $E(\Lambda) \subset \text{ADS}_{n+1}$ , the hilbert distance  $d_h(x, y)$  is defined to be the cross-ratio  $[a; x; y; b]$  where  $a, b$  are the intersections between  $\partial E(\Lambda)$  and the projective line in  $\mathbb{S}(\mathbb{R}^{2, n})$  containing  $x$  and  $y$ . The Hilbert metric is of course  $\rho(\Gamma)$ -invariant.

Assume by contradiction that  $\Lambda$  is *not* acausal. Then, it contains a light-like segment  $[x, y]$  with  $x \neq y$ . We can assume wlog that this segment is maximal, *ie.* that  $[x, y]$  is precisely the intersection between  $\Lambda$  and a projective line in  $\text{Ein}_n \subset \mathbb{S}(\mathbb{R}^{2, n})$ . Let  $P$  be a projective subplane of  $\mathbb{S}(\mathbb{R}^{2, n})$  containing  $[x, y]$  and an element  $z$  of  $\text{Conv}(\Lambda)^\circ$ . The intersection  $P \cap \text{Conv}(\Lambda)^\circ$  is a convex domain containing the ideal triangle  $x, y, z$ , with a side  $[x, y]$  contained at infinity. Let  $u$  be an element in the segment  $(x, y)$ . For every  $t > 0$ , let  $x_t$  (respectively  $y_t$ ) be the element of the segment  $[z, x]$  (respectively  $[z, y]$ ) such that  $d_h(z, x_t) = t$  (respectively  $d_h(z, y_t) = t$ ), and let  $u_t$  be the intersection  $[z, u] \cap [x_t, y_t]$ . Observe that  $[z, x_t] \cup [x_t, y_t] \cup [y_t, z]$  is a geodesic triangle for  $d_h$ . Now, an elementary computation shows (see the

proof of Proposition 2.5 in [Ben04]):

$$\lim_{t \rightarrow +\infty} d_h(u_t, [z, x_t] \cup [z, y_t]) = +\infty$$

It implies that  $\text{Conv}(\Lambda) \setminus \Lambda$ , equipped with the restriction of  $d_h$ , is not Gromov hyperbolic.

But, on the other hand, the quotient of  $\text{Conv}(\Lambda) \setminus \Lambda$  by  $\rho(\Gamma)$  is compact. Indeed, according to Proposition 5.2, the future boundary  $S^+(\Lambda)$  and the past boundary  $S^-(\Lambda)$  of the convex core are contained in  $E(\Lambda)$ . Their projections in  $M_\rho(\Lambda)$  are therefore compact achronal hypersurfaces, bounding a compact region  $C$ , which is precisely the quotient of  $\text{Conv}(\Lambda) \setminus \Lambda$ .

Since  $\Gamma$  is Gromov hyperbolic,  $(\text{Conv}(\Lambda) \setminus \Lambda, d_h)$  should be Gromov hyperbolic. Contradiction.  $\square$

## 6. LIMITS OF ANOSOV REPRESENTATIONS

This section is entirely devoted to the proof of the Theorem 1.2, that we restate here for the reader's convenience:

**Theorem 1.2.** *Let  $n \geq 2$ , and let  $\Gamma$  be a Gromov hyperbolic group of cohomological dimension  $\geq n$ . Then, the modular space  $\text{Rep}_0(\Gamma, \text{SO}_0(2, n))$  of  $(\text{SO}_0(2, n), \text{Ein}_n)$ -Anosov representations is open and closed in the modular space  $\text{Rep}(\Gamma, \text{SO}_0(2, n))$ .*

We recall that one important step of the proof will be to show that under these hypothesis,  $\Gamma$  is the fundamental group of a closed manifold, and that its cohomological dimension is eventually  $n$  (cf. Remark 1.3).

Let  $\Gamma$  be as in the hypothesis of the Theorem a Gromov hyperbolic group of cohomological dimension  $\geq n$ . The fact that  $\text{Rep}_0(\Gamma, \text{SO}_0(2, n))$  is open in  $\text{Rep}_0(\Gamma, \text{SO}_0(2, n))$  is well-known (cf. Theorem 1.2 in [GW12]), hence our task is to prove that it is a closed subset.

Let  $\rho_k : \Gamma \rightarrow \text{SO}_0(2, n)$  be a sequence of  $(\text{SO}_0(2, n), \text{Ein}_n)$ -Anosov representations converging to a representation  $\rho_\infty : \Gamma \rightarrow \text{SO}_0(2, n)$ .

**Proposition 6.1.** *The limit representation  $\rho_\infty : \Gamma \rightarrow \text{SO}_0(2, n)$  is discrete and faithful.*

*Proof.* Since  $\Gamma$  is Gromov hyperbolic and non-elementary, it contains no nilpotent normal subgroup (see [GdlH90]). Hence, by a classical argument, the limit  $\rho_\infty : \Gamma \rightarrow \text{SO}_0(2, n)$  is discrete and faithful (cf. Lemma 1.1 in [GM87]).

Actually, we give a sketch of the argument, since we will need later a slightly more elaborate version of this argument. The key point is that  $\text{SO}_0(2, n)$ , as any Lie group, contains a neighborhood  $W_0$  of the identity such that every discrete subgroup generated by elements in  $W_0$  is contained in a nilpotent Lie subgroup of  $\text{SO}_0(2, n)$ . In particular, such a discrete subgroup is nilpotent, and there is a uniform bound  $N$  for the residue class (*ie.* the length of the lower central series) of these nilpotent groups.

Assume that  $\text{Ker}(\rho_\infty(\Gamma))$  is non-trivial. Then it is a normal subgroup. For any finite subset  $F$  of  $\text{Ker}(\rho_\infty(\Gamma))$ , there is an integer  $k_0$  such that  $k \geq k_0$  implies that  $\rho_k(F)$  is contained in  $W_0$ , hence nilpotent of residue class  $\leq N$ . It follows that  $\text{Ker}(\rho_\infty(\Gamma))$  is nilpotent, contradiction: the representation  $\rho_\infty$  is faithful.

Let  $\bar{G}_\infty$  be the closure of  $\rho_\infty(\Gamma)$ , and let  $\bar{G}_\infty^0$  be the identity component of  $\bar{G}_\infty$ : it is a normal subgroup of  $\bar{G}_\infty$ , and it is generated by any neighborhood of the identity. Therefore,  $\rho_\infty(\Gamma) \cap W_0$  generates a dense subgroup of  $\bar{G}_\infty^0$ . On the other hand, any expression of the form:

$$(8) \quad [\rho_\infty(\gamma_1), [\rho_\infty(\gamma_2), [\dots[\rho_\infty(\gamma_N), \rho_\infty(\gamma_{N+1})]\dots]]]$$

is the limit for  $k \rightarrow +\infty$  of:

$$(9) \quad [\rho_k(\gamma_1), [\rho_k(\gamma_2), [\dots[\rho_k(\gamma_N), \rho_k(\gamma_{N+1})]\dots]]]$$

For  $k$  sufficiently big, every  $\rho_k(\gamma_i)$  belongs to  $W_0$  and  $\rho_k(\Gamma)$  is discrete, hence (9) is trivial. The limit (8) is trivial too. It follows that  $\bar{G}_\infty^0$  is nilpotent. Then,  $\rho_\infty^{-1}(\rho_\infty(\Gamma) \cap \bar{G}_\infty^0)$  is a nilpotent normal subgroup of  $\Gamma$ . It is a contradiction, unless  $\bar{G}_\infty^0$  is trivial, *ie.* unless  $\rho_\infty(\Gamma)$  is discrete.  $\square$

An immediate consequence of the representations  $\rho_k$  being Anosov is the existence of a  $\rho_k(\Gamma)$ -equivariant map  $\xi : \partial_\infty \Gamma \rightarrow \text{Ein}_n$  whose image is a closed  $\rho_k(\Gamma)$ -invariant acausal subset  $\Lambda_k$  (cf. [GW12]). According to the Remark 3.4, for every integer  $k$ , there is a  $\rho_k(\Gamma)$ -invariant achronal topological  $(n-1)$ -sphere  $\Lambda_k^+$ , which is not pure lightlike since it contains the acausal subset  $\Lambda_k$ . Therefore, every  $\rho_k$  is a GH-regular representation. The Cauchy hypersurfaces of the associated GH spacetimes are contractible (since the universal coverings are topological disks embedded in regular domains of  $\text{AdS}_{n+1}$ ) and have fundamental groups isomorphic to  $\Gamma$ . Since  $\Gamma$  has cohomological dimension  $\geq n$ , these Cauchy hypersurfaces are compact: the  $\rho_k$  are GHC-regular representations.

The  $\rho_k(\Gamma)$ -invariant spheres  $\hat{\Lambda}_k$  are graphs of locally 1-Lipschitz maps  $f_k : \mathbb{S}^{n-1} \rightarrow \mathbb{S}^1$ . It follows easily by Ascoli-Arzelà Theorem that, up to a subsequence,  $\rho_\infty(\Gamma)$  preserves the graph of a locally 1-Lipschitz map  $f_\infty : \mathbb{S}^{n-1} \rightarrow \mathbb{S}^1$ , *ie.* an achronal sphere  $\Lambda_\infty$ .

**Lemma 6.2.**  $\Lambda_\infty$  is not purely lightlike.

*Proof.* Assume not. Then,  $\Lambda_\infty$  is the union of lightlike geodesics joining two antipodal points  $x_0$  and  $-x_0$  in  $\text{Ein}_n$ . Let  $G_0$  be the stabilizer in  $\text{SO}_0(2, n)$  of  $\pm x_0$ : the image  $\rho_\infty(\Gamma)$  is a discrete subgroup of  $G_0$ .

According to Remark 2.13, the group  $G_0$  is isomorphic to the group of conformal transformations of the Minkowski space  $\text{Mink}(x_0) \approx \mathbb{R}^{1, n-1}$ . There is an exact sequence:

$$1 \rightarrow \mathbb{R}^{1, n-1} \rightarrow G_0 \rightarrow \mathbb{R} \times \text{SO}_0(1, n-1) \rightarrow 1$$

where the left term is the subgroup of translations of  $\mathbb{R}^{1, n-1}$  and the right term the group of conformal linear transformations of  $\mathbb{R}^{1, n-1}$ . Let  $L : G_0 \rightarrow$

$\mathbb{R} \times \mathrm{SO}_0(1, n-1)$  be the projection morphism. Let  $\bar{L}$  be the closure in  $\mathbb{R} \times \mathrm{SO}_0(1, n-1)$  of  $L(\rho_\infty(\Gamma))$ , and let  $\bar{L}_0$  be the identity component of  $\bar{L}$ . Considering as in the proof of Proposition 6.1 an open domain  $V_0$  in  $G_0$  such that any discrete group generated by elements of  $V_0$  is nilpotent, and using as a trick the fact that conjugacies in  $G_0$  by homotheties in  $\mathbb{R}^{1, n-1}$  can reduce at an arbitrary small scale translations in  $\mathbb{R}^{1, n-1}$ , one proves that  $\Gamma \cap L^{-1}(L(\rho_\infty(\Gamma)) \cap \bar{L}_0)$  is a normal nilpotent subgroup of  $\Gamma$  (cf. Theorem 1.2.1 in [CD89]). Therefore, it is trivial:  $L(\rho_\infty(\Gamma))$  is a discrete subgroup of  $\mathbb{R} \times \mathrm{SO}_0(1, n-1)$ .

Now we consider  $\mathbb{R} \times \mathrm{SO}_0(1, n-1)$  as the group of isometries of the Riemannian product  $\mathbb{R} \times \mathbb{H}^{n-1}$ . By what we have just proved, the action of  $\rho_\infty(\Gamma)$  on  $\mathbb{R} \times \mathbb{H}^{n-1}$  is properly discontinuous. On the other hand,  $\Gamma$  acts properly and cocompactly on a topological disk of dimension  $n$  (a Cauchy hypersurface in  $E(\Lambda_k)$  for any  $k$ ), hence its action on  $\mathbb{R} \times \mathbb{H}^{n-1}$  is cocompact. It is a contradiction since  $\mathbb{R} \times \mathbb{H}^{n-1}$  is not Gromov hyperbolic (it contains flats of dimension 2).  $\square$

*Proof of Theorem 1.2.* According to Proposition 6.1 and Lemma 6.2,  $\rho_\infty : \Gamma \rightarrow \mathrm{SO}_0(2, n)$  is a GH-regular representation. It is actually a GHC-regular representation since Cauchy surfaces in  $\rho_\infty(\Gamma) \backslash E(\Lambda_\infty)$  are  $K(\Gamma, 1)$  and thus compact since Cauchy surfaces in every  $\rho_k(\Gamma) \backslash E(\Lambda_k)$  are compact. According to Theorem 5.3, the representation  $\rho_\infty : \Gamma \rightarrow \mathrm{SO}_0(2, n)$  is  $(\mathrm{SO}_0(2, n), \mathrm{Ein}_n)$ -Anosov.  $\square$

## 7. BOUNDED COHOMOLOGY

This section is devoted to the proof of:

**Theorem 1.4.** *Let  $\rho : \Gamma \rightarrow \mathrm{SO}_0(2, n)$  be a faithful and discrete representation, where  $\Gamma$  is the fundamental group of a negatively curved closed manifold  $M$ . The following assertions are equivalent:*

- (1)  $\rho$  is  $(\mathrm{SO}_0(2, n), \mathrm{Ein}_n)$ -Anosov,
- (2) the bounded Euler class  $\mathrm{eu}_b(\rho)$  vanishes.

For a friendly introduction to bounded cohomology, close to our present concern, see [Ghy01, Section 6].

**7.1. The bounded Euler class.** We have the following central exact sequence:

$$(10) \quad 1 \rightarrow \mathbb{Z} \rightarrow \widetilde{\mathrm{SO}}_0(2, n) \rightarrow \mathrm{SO}_0(2, n) \rightarrow 1$$

where  $\mathbb{Z}$  is the group<sup>6</sup> of deck transformations of the covering  $\hat{p} : \widetilde{\mathrm{Ein}}_n \rightarrow \mathrm{Ein}_n$ , generated by the transformation  $\delta$  (cf section 2.3). Fix the element  $x_0 = (0, x_0)$  in  $\widetilde{\mathrm{Ein}}_n \approx \mathbb{R} \times \mathbb{S}^{n-1}$ . In these coordinates,  $\delta$  is the transformation  $(\theta, x) \mapsto (\theta + 2\pi, x)$ . Hence, we can define a section  $\sigma : \mathrm{SO}_0(2, n) \rightarrow \widetilde{\mathrm{SO}}_0(2, n)$ , called *canonical section*, which maps every element  $g$  of  $\mathrm{SO}_0(2, n)$

<sup>6</sup>Observe that  $\mathbb{Z}$  is not always the center of  $\widetilde{\mathrm{SO}}_0(2, n)$ , since  $-\mathrm{Id}$  is an element of  $\mathrm{SO}_0(2, n)$  when  $n$  is even.

to the unique element  $\sigma(g)$  of  $\widetilde{\text{SO}}_0(2, n)$  above  $g$  and such that  $\sigma(g(x_0))$  lies in the domain:

$$\mathcal{W}_0 := \{(\theta, \mathbf{x}) \in \mathbb{R} \times \mathbb{S}^{n-1} / -\pi \leq \theta < \pi\}$$

Observe that  $\mathcal{W}_0$  is a fundamental domain for the action of  $\langle \delta \rangle = \mathbb{Z}$  on  $\widetilde{\text{Ein}}_n$ .

For any pair  $(g_1, g_2)$  of elements of  $\text{SO}_0(2, n)$ , we define  $c(g_1, g_2)$  as the unique integer  $k$  such that  $\sigma(g_1 g_2) = \delta^k \sigma(g_1) \sigma(g_2)$ .

**Lemma 7.1** (Compare with Lemma 6.3 in [Ghy01]). *The 2-cocycle  $c$  takes only the values  $-1, 0$  or  $1$ .*

*Proof.* Let  $x_1 = (\theta_1, \mathbf{x}_1)$  and  $x_2 = (\theta_2, \mathbf{x}_2)$  be the images of  $x_0$  by  $\sigma(g_1)$ ,  $\sigma(g_2)$ , respectively. Let  $x_3 = (\theta_3, \mathbf{x}_3)$  be the image of  $x_2$  by  $\sigma(g_1)$ .

– (1) *If  $|\theta_2| \leq d(\mathbf{x}_2, \mathbf{x}_0)$ .* It means that  $x_2$  is not in  $I^\pm(x_0)$ . Then,  $x_3 = \sigma(g_1)(x_2)$  is not in  $I^\pm(x_1)$ . Therefore:

$$|\theta_3 - \theta_1| \leq d(\mathbf{x}_3, \mathbf{x}_1) \leq \pi$$

implying  $|\theta_3| \leq 2\pi$ . It follows that if  $x_3 = \sigma(g_1)\sigma(g_2)(x_0)$  is not already in  $\mathcal{W}_0$ ,  $\delta^\epsilon(x_3)$  for  $\epsilon = \pm 1$  does. Hence  $c(g_1, g_2) = \epsilon$  is  $0, -1$  or  $1$  as required.

– (2) *If  $\theta_2 > d(\mathbf{x}_2, \mathbf{x}_0)$ .* Then,  $0 < \pi - \theta_2 < \pi - d(\mathbf{x}_2, \mathbf{x}_0) = d(\mathbf{x}_2, -\mathbf{x}_0)$  where  $-\mathbf{x}_0$  is the antipodal point in  $\mathbb{S}^{n-1}$  at distance  $\pi$  from  $\mathbf{x}_2$ . The point  $x_2$  is not in  $J^\pm((\pi, -\mathbf{x}_0))$ , hence its image  $x_3$  by  $\sigma(g_1)$  is not in  $J^\pm((\pi + \theta_1, -\mathbf{x}_1))$ . It follows:

$$|\theta_3 - (\pi + \theta_1)| < d(\mathbf{x}_3, -\mathbf{x}_1) \leq \pi$$

Therefore:

$$|\theta_3| < 3\pi$$

Hence, for some  $\epsilon = 0$  or  $\pm 1$  we have that  $\delta^\epsilon(x_3)$  lies in  $\mathcal{W}_0$ , and  $c(g_1, g_2) = \epsilon$  is  $0, -1$  or  $1$ .

– (3) *If  $-\pi \leq \theta_2 < -d(\mathbf{x}_2, \mathbf{x}_0)$ .* We apply the same argument that in case (2), by observing that  $\mathbf{x}_2$  is then non causally related to  $(-\pi, -\mathbf{x}_0)$ . Details are left to the reader.  $\square$

**Definition 7.2.**  $c$  is a bounded 2-cocycle. It represents an element of the bounded cohomology space  $H_b^2(\text{SO}_0(2, n), \mathbb{Z})$  called the bounded Euler class.

For any representation  $\rho : \Gamma \rightarrow \text{SO}_0(2, n)$ , the pull-back  $\rho^*([c])$  is an element of  $H_b^2(\Gamma, \mathbb{Z})$ , denoted by  $\text{eu}_b(\rho)$ .

Of course,  $c$  also represents an element of the "classical" cohomological space  $H^2(\text{SO}_0(2, n), \mathbb{Z})$ . The associated 2-cocycle  $\text{eu}(\rho)$  represents the obstruction to lift  $\rho$  to a representation  $\tilde{\rho} : \Gamma \rightarrow \widetilde{\text{SO}}_0(2, n)$ . Indeed,  $\text{eu}(\rho) = 0$  means that there is a 1-cochain  $a : \Gamma \rightarrow \mathbb{Z}$  such that for every  $\gamma_1, \gamma_2$  in  $\Gamma$  we have:

$$c(\rho(\gamma_1), \rho(\gamma_2)) = a(\gamma_1 \gamma_2) - a(\gamma_1) - a(\gamma_2)$$

Then, the map  $\gamma \rightarrow \delta^{a(\gamma)} \sigma(\rho(\gamma))$  is a morphism, *ie.* a representation  $\tilde{\rho} : \Gamma \rightarrow \widetilde{\text{SO}}_0(2, n)$  which is a lift of  $\rho$ .

Now  $\text{eu}_b(\rho) = 0$  means that  $\text{eu}(\rho) = 0$ , but also that one can select the 1-cochain  $a$  so that it is *bounded*. The following proposition is a natural generalization of the fact a group of orientation-preserving homeomorphisms of the circle has a vanishing bounded Euler class if and only if it has a global fixed point (see the end of section 6.3 in [Ghy01]):

**Proposition 7.3.** *The bounded Euler class  $\text{eu}_b(\rho)$  vanishes if and only if  $\rho$  lifts to a representation  $\tilde{\rho} : \Gamma \rightarrow \widetilde{\text{SO}}_0(2, n)$  such that  $\tilde{\rho}(\Gamma)$  preserves a closed  $(n - 1)$ -dimensional achronal topological sphere in  $\widetilde{\text{Ein}}_n$ .*

*Proof.* *Invariant achronal sphere  $\Rightarrow \text{eu}_b(\rho) = 0$ .*

Assume that  $\rho$  lifts to a representation  $\tilde{\rho} : \Gamma \rightarrow \widetilde{\text{SO}}_0(2, n)$  (ie. that  $\text{eu}(\rho) = 0$ ) and that  $\tilde{\rho}(\Gamma)$  preserves a closed  $(n - 1)$ -dimensional achronal topological sphere  $\Lambda$  in  $\widetilde{\text{Ein}}_n$ , ie. the graph of a 1-Lipschitz map  $f : \mathbb{S}^{n-1} \rightarrow \mathbb{R}$ . Let  $a : \Gamma \rightarrow \mathbb{Z}$  the map associating to  $\gamma$  the unique integer  $k$  such that:

$$\tilde{\rho}(\gamma) = \delta^k \sigma(\rho(\gamma))$$

$a$  is the 1-cochain whose coboundary represents the Euler class of  $\rho$ , the point is to prove that  $a$  is bounded.

The invariant achronal sphere  $\Lambda$  is contained in the closure of an affine domain of  $\widetilde{\text{Ein}}_n$  (cf. Lemma 2.5), ie. in a domain of the form  $\{\theta_0 - \pi \leq \theta \leq \theta_0 + \pi\}$ . More precisely, either it is contained in a domain  $\delta^q \mathcal{W}_0$  for some integer  $q$ , or it contains a point  $(q\pi, x)$ , in which case  $\Lambda$  is contained in the domain  $\{(q - 1)\pi \leq \theta < (q + 1)\pi\}$ . In both cases, there is an integer  $q$  such that  $\Lambda$  is contained in the union  $\mathcal{Z}_q := \delta^{q-1} \mathcal{W}_0 \cup \delta^q \mathcal{W}_0$ .

For every  $\gamma$  in  $\Gamma$ , the image of  $x_0 = (0, x_0)$  by  $\sigma(\rho(\gamma))$  is a point  $(\theta, y_0)$  with  $|\theta| \leq \pi$ , hence the intersection between  $\mathcal{W}_0$  and  $\sigma(\rho(\gamma))(\mathcal{W}_0)$  is non-trivial. Since  $\delta$  commutes with  $\sigma(\rho(\gamma))$ , the intersection  $\mathcal{W}_q \cap \sigma(\rho(\gamma))(\mathcal{W}_q)$  is non-empty. *A fortiori*, the same is true for the intersection  $\mathcal{Z}_q \cap \sigma(\rho(\gamma))(\mathcal{Z}_q)$ . However, since  $\delta$  acts by adding  $2\pi$  on the coordinate  $\theta$ , the intersection  $\mathcal{Z}_q \cap \delta^r \sigma(\rho(\gamma))(\mathcal{Z}_q)$  is empty as soon as  $r$  is an integer of absolute value  $> 2$ .

On the other hand, we know that  $\mathcal{Z}_q \cap \tilde{\rho}(\gamma)\mathcal{Z}_q$  is non-empty since  $\mathcal{Z}_q$  contains the invariant sphere  $\Lambda$ . It follows that the integer  $a(\gamma)$  has absolute value at most 2.

$\text{eu}_b(\rho) = 0 \Rightarrow$  *Invariant achronal sphere*

Assume now that  $\text{eu}_b(\rho)$  vanishes, ie. that there is a bounded map  $a : \Gamma \rightarrow \mathbb{Z}$  such that  $\gamma \rightarrow \delta^{a(\gamma)} \sigma(\rho(\gamma))$  is a representation  $\tilde{\rho} : \Gamma \rightarrow \widetilde{\text{SO}}_0(2, n)$ . Let  $\alpha$  be an upper bound for  $|a(\gamma)|$  ( $\gamma \in \Gamma$ ). Let  $f_{id} : \mathbb{S}^n \rightarrow \mathbb{R}$  be the null map, and for every element  $\gamma$  of  $\Gamma$ , let  $f_\gamma : \mathbb{S}^n \rightarrow \mathbb{R}$  be the 1-Lipschitz map whose graph is the image by  $\tilde{\rho}(\gamma)$  of the graph of  $f_0$ . The graph of  $f_\gamma$  contains  $\delta^{a(\gamma)} \sigma(\rho(\gamma))(0, x_0)$ , hence a point of  $\theta$ -coordinate of absolute value bounded from above by  $|a(\Gamma)| + \pi$ . Since every  $f_\gamma$  is 1-Lipschitz and since the sphere has diameter  $\pi$ , there is a uniform upper bound for all the  $f_\gamma$ . For every  $x$  in  $\mathbb{S}^n$  define:

$$f_\infty(x) := \text{Sup}_{\gamma \in \Gamma} f_\gamma(x)$$

Then  $f_\infty$  is a 1-Lipschitz map, whose graph is clearly  $\rho(\Gamma)$ -invariant.  $\square$

**7.2. Proof of Theorem 1.5.** Let  $\rho : \Gamma \rightarrow \mathrm{SO}_0(2, n)$  be a faithful and discrete representation, where  $\Gamma$  is the fundamental group of a negatively curved closed manifold  $M$ .

According to the Proposition 7.3, the bounded Euler class  $\mathrm{eu}_b(\rho)$  vanishes if and only if  $\rho$  lifts to a representation  $\tilde{\rho} : \Gamma \rightarrow \widetilde{\mathrm{SO}}_0(2, n)$  such that  $\tilde{\rho}(\Gamma)$  preserves a closed  $(n - 1)$ -dimensional achronal topological sphere in  $\widetilde{\mathrm{Ein}}_n$ . According to Theorem 1.4, such a sphere, if it exists, must be acausal. The equivalence between items (1) and (2) follows.

**7.3. The case  $n = 2$ .** In this last section, we explain in which way one can deduce from Proposition 7.3 the following classical result:

**Proposition 7.4.** *Let  $\rho_1, \rho_2$  be two representations of  $\Gamma$  into  $\mathrm{PSL}(2, \mathbb{R})$  such that  $\mathrm{eu}_b(\rho_1) = \mathrm{eu}_b(\rho_2)$ . Then,  $\rho_1$  and  $\rho_2$  are semi-conjugated, ie. there is a monotone map  $f : \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$  such that:*

$$\forall \gamma \in \Gamma, \rho_1(\gamma) \circ f = f \circ \rho_2(\gamma)$$

Let us first recall the definition of the bounded Euler class for a representation  $\rho : \Gamma \rightarrow \mathrm{PSL}(2, \mathbb{R})$ : it is completely similar to definition we have presented above.

Let  $p : \widetilde{\mathrm{SL}}(2, \mathbb{R}) \rightarrow \mathrm{PSL}(2, \mathbb{R})$  be the universal covering. It acts naturally on the universal covering  $\widetilde{\mathbb{RP}}^1$  of the projective line  $\mathbb{RP}^1$ , so that the kernel of  $p$  is the center of  $\widetilde{\mathrm{SL}}(2, \mathbb{R})$  and also the Galois group of  $\widetilde{\mathbb{RP}}^1$ . We fix a total order  $<$  on  $\widetilde{\mathbb{RP}}^1 \approx \mathbb{R}$  and a generator  $\tau$  of  $\ker p$  so that  $\tau(x) > x$  for every  $x$  in  $\widetilde{\mathbb{RP}}^1$ . Once fixed an element  $x_0$  of  $\widetilde{\mathbb{RP}}^1$ , there is still a canonical section  $\sigma : \mathrm{PSL}(2, \mathbb{R}) \rightarrow \widetilde{\mathrm{SL}}(2, \mathbb{R})$ , which is not a homomorphism, which associates to any element  $g$  of  $\mathbb{RP}^1$  the unique element  $\tilde{g}$  such that:

$$x_0 \leq \tilde{g}x_0 < \tau(x_0)$$

Then, the Euler class of the representation  $\rho$  is the bounded cohomology class represented by the cocycle  $c$  defined by:

$$\sigma(\rho(\gamma_1\gamma_2)) = \tau^{c(\gamma_1, \gamma_2)}\sigma(\rho(\gamma_1))\sigma(\rho(\gamma_2))$$

Now let  $\rho_1, \rho_2$  be two representations of  $\Gamma$  into  $\mathrm{PSL}(2, \mathbb{R})$  satisfying the statement of Proposition 7.4: they have the same bounded cohomology class, meaning that, if  $c_1, c_2$  are the two cocycles defined as above representing the bounded Euler class, we have:

$$(11) \quad c_2(\gamma_1, \gamma_2) = c_1(\gamma_1, \gamma_2) + a(\gamma_1\gamma_2) - a(\gamma_1) - a(\gamma_2)$$

where  $a : \Gamma \rightarrow \mathbb{Z}$  is some bounded map.

It has the following consequence: consider the map  $\Gamma \times \widetilde{\mathrm{SL}}(2, \mathbb{R}) \rightarrow \widetilde{\mathrm{SL}}(2, \mathbb{R})$  which associates to  $(\gamma, \tilde{g})$  the element:

$$\gamma * \tilde{g} := \tau^{-a(\gamma)}\sigma(\rho_2(\gamma))\tilde{g}\sigma(\rho_1(\gamma))^{-1}$$

Then:

$$\begin{aligned}
(\gamma_1 \gamma_2) * \tilde{g} &= \tau^{-a(\gamma_1 \gamma_2)} \sigma(\rho_2(\gamma_1 \gamma_2)) \tilde{g} [\sigma(\rho_1(\gamma_1 \gamma_2))]^{-1} \\
&= \tau^{-a(\gamma_1 \gamma_2) + c_2(\gamma_1, \gamma_2)} \rho_2(\gamma_1) \rho_2(\gamma_2) \tilde{g} [\tau^{c_1(\gamma_1, \gamma_2)} \sigma(\rho_1(\gamma_1)) \sigma(\rho_1(\gamma_2))]^{-1} \\
&= \tau^{-a(\gamma_1) - a(\gamma_2)} \tilde{g} [\sigma(\rho_1(\gamma_1)) \sigma(\rho_1(\gamma_2))]^{-1} \text{ (see (11))} \\
&= \gamma_1 * (\gamma_2 * \tilde{g})
\end{aligned}$$

Now the key point is that  $\widetilde{\text{SL}}(2, \mathbb{R})$  is a model for the universal anti-de Sitter space  $\widetilde{\text{AdS}}_3$ . Indeed,  $-\det$  defines on the space  $\text{Mat}(2, \mathbb{R})$  of two-by-two matrices a quadratic form of signature  $(2, 2)$ , which is preserved by the following action of  $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ :

$$\forall g_1, g_2 \in \text{SL}(2, \mathbb{R}), \forall A \in \text{Mat}(2, \mathbb{R}), (g_1, g_2).A := g_1 A g_2^{-1}$$

The kernel of this action is the group  $I$  of order two generated by  $(-\text{Id}, -\text{Id})$ , where  $\text{Id}$  denote the identity matrix. Hence there is a natural isomorphism between  $\text{SO}_0(2, 2)$  and  $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})/I$ .

Therefore, the action  $*$  we have defined is an isometric action of  $\Gamma$  on  $\widetilde{\text{AdS}}_3$ , hence induces a representation  $\tilde{\rho} : \Gamma \rightarrow \widetilde{\text{SO}}_0(2, 2)$ . Furthermore, the fact that the map  $a$  involved in the coboundary is bounded implies that this representation  $\tilde{\rho}$  is the lifting of a representation into  $\text{SO}_0(2, 2)$  whose bounded Euler class vanishes, *ie.* that the group  $\tilde{\rho}(\Gamma)$  preserves a closed achronal circle in  $\widetilde{\text{Ein}}_2$ .

We claim that the existence of such an invariant achronal circle is equivalent to the existence of a semi-conjugacy between  $\rho_1$  and  $\rho_2$  as stated in the conclusion of Proposition 7.4.

For the proof of this claim, it is convenient to consider the *projectivized* anti-de Sitter and Einstein spaces, *ie.* the quotients of  $\text{AdS}_3$  and  $\text{Ein}_2$  by  $-\text{Id}$ . The projectivized anti-de Sitter space is then naturally identified with  $\text{PSL}(2, \mathbb{R})$ . According to the identification between  $(\text{Mat}(2, \mathbb{R}), -\det)$  and  $(\mathbb{R}^{2,2}, q_{2,2})$ , we obtain an identification between the projectivized Klein model  $\overline{\text{Ein}}_2$  and the space of non-zero non-invertible 2-by-2 matrices up to a non-zero factor. Such a class is characterized by the kernel and the image of its elements, *ie.* two lines in  $\mathbb{R}^2$ . In other words,  $\overline{\text{Ein}}_2$  is naturally isomorphic to the product  $\mathbb{RP}^1 \times \mathbb{RP}^1$ . The conformal action of  $\text{PO}(2, 2) \approx \text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$  on  $\mathbb{RP}^1 \times \mathbb{RP}^1$  is the obvious one:

$$(g_1, g_2).(x, y) = (g_1 x, g_2 y)$$

since the image of  $g_1 A g_2^{-1}$  is the image by  $g_1$  of the image of  $A$ , and its kernel is the image under  $g_2$  of the kernel of  $A$ . The isotropic circles in  $\overline{\text{Ein}}_2 \approx \mathbb{RP}^1 \times \mathbb{RP}^1$  are the circles  $\{*\} \times \mathbb{RP}^1$  and  $\mathbb{RP}^1 \times \{*\}$ . It follows quite easily that **acausal** circles in  $\overline{\text{Ein}}_2$  are graphs in  $\mathbb{RP}^1 \times \mathbb{RP}^1$  of homeomorphisms from  $\mathbb{RP}^1 \rightarrow \mathbb{RP}^1$ . **Achronal** circles are allowed to follow during some time one segment in  $\{*\} \times \mathbb{RP}^1$  or  $\mathbb{RP}^1 \times \{*\}$ . It follows that they are fillings (cf. Remark 3.19) of graphs of maps  $f : \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$  which are *monotone*, *ie.* of degree 1, preserving the cyclic order on  $\mathbb{RP}^1$ , but which can be constant on

some intervals and which can be non-continuous at certain points. In other words,  $f$  lifts to a non-decreasing map  $\tilde{f} : \widetilde{\mathbb{R}\mathbb{P}^1} \rightarrow \widetilde{\mathbb{R}\mathbb{P}^1}$ . For more details on this well-known geometric feature, we refer to [Mes07] or [BBZ07].

In summary, we have proved that the representation  $(\rho_1, \rho_2) : \Gamma \rightarrow \mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R}) \approx \mathrm{PO}(2, 2)$  preserves a closed achronal circle  $\Lambda$  in  $\overline{\mathrm{Ein}_2} \approx \mathbb{R}\mathbb{P}^1 \times \mathbb{R}\mathbb{P}^1$ , which is the filling of the graph of a monotone map  $f : \mathbb{R}\mathbb{P}^1 \rightarrow \mathbb{R}\mathbb{P}^1$ . The invariance of  $\Lambda$  means precisely that  $f$  is  $\Gamma$ -equivariant: Proposition 7.4 is proved.

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